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Augmentations and rulings of Legendrian knots

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For any Legendrian knot Λ in $(\mathbb{R}^3, \ker(dz - ydx))$, we show that the existence of an augmentation to any field of the Chekanov-Eliashberg differential graded algebra over $\mathbb{Z}[t, t^{-1}]$ is equivalent to the existence of a ruling of the front diagram, generalizing results of Fuchs, Ishkhanov, and Sabloff. We also show that any even graded augmentation must send t to -1.

1. Introduction

A Legendrian knot in $(\mathbb{R}^3, \xi_{\text{std}})$ is an embedding $\Lambda : S^1 \to \mathbb{R}^3$ which is everywhere tangent to the contact planes. In [4] (see related [6]), Chekanov introduced a combinatorial way to associate a non-commutative differential graded algebra (DGA) over $\mathbb{Z}/2$ to a Lagrangian diagram of a Legendrian knot Λ in $\mathbb{R}^3$. The DGA is generated by crossings of Λ and the differential is determined by a count of immersed polygons whose edges lie on the knot and whose corners lie at crossings of Λ. In the literature, this DGA is called the Chekanov-Eliashberg DGA. Chekanov showed that the homology of the DGA is invariant under Legendrian isotopy. He also showed that a linearized version of the homology of the DGA could be used to distinguish between two Legendrian 5_2 knots in $\mathbb{R}^3$ which could not be distinguished by the rotation and Thurston-Bennequin numbers. In the early 2000's, Etnyre, Ng, and Sabloff gave a lift of the Chekanov-Eliashberg DGA to a DGA $(\mathcal{A}, \partial)$ over $R = \mathbb{Z}[t, t^{-1}]$ which has a full $\mathbb{Z}$-grading (see [10]). One can recover the Chekanov-Eliashberg DGA by setting $t = 1$, which requires one to consider the grading mod $2r(\Lambda)$, and considering the coefficients mod 2 (where $r(\Lambda)$ is the rotation number, defined in §2).

Another Legendrian knot invariant uses generating families, functions whose critical values generate front diagrams of Legendrian knots. Following ideas introduced by Eliashberg in [5], Fuchs [11] and Chekanov-Pushkar [3] gave invariants involving decompositions of the generating families, which are now called "normal rulings" and can also be used to distinguish between Chekanov's 5_2 knots.

Remarkably, there is a close connection between the Chekanov-Eliashberg DGA and rulings. Fuchs [11], Fuchs-Ishkhanov [12], and Sabloff [17] showed that the existence of a ruling is equivalent to the existence of an augmentation to $\mathbb{Z}/2$ of the Chekanov-Eliashberg DGA, where an augmentation to a ring S is an algebra map $\epsilon : \mathcal{A} \to S$ such that $\epsilon \circ \partial = 0$ and $\epsilon(1) = 1$.

The main result of this paper gives a generalization of these results using an extension of Sabloff's construction in [17]. Let F be a field and $R = \mathbb{Z}[t, t^{-1}]$. Given a ρ-graded augmentation $\epsilon : \mathcal{A} \to F$ of the $\mathbb{Z}[t, t^{-1}]$-differential graded algebra $(\mathcal{A}, \partial)$ of a knot Λ, we will find a ρ-graded normal ruling of the knot diagram. Conversely, given a ρ-graded normal ruling of the knot diagram, we will define a ρ-graded augmentation $\epsilon : \mathcal{A} \to F$ of the DGA over $\mathbb{Z}[t, t^{-1}]$ with $\epsilon(t) = -1$. (For $\rho = 0$, this is the so called graded case and for $\rho = 1$, the ungraded case.) Terminology will be introduced in §2.

In §3 and §4, we will show:

Theorem 1.1. *Let Λ be a Legendrian knot in $\mathbb{R}^3$. Given a field F, $(\mathcal{A}, \partial)$ has a ρ-graded augmentation $\epsilon : \mathcal{A} \to F$ if and only if any front diagram of Λ has a ρ-graded normal ruling. Furthermore, if ρ is even, then $\epsilon(t) = -1$.*

Note that this generalizes Fuchs, Fuchs-Ishkhanov, and Sabloff's results, giving a correspondence between normal rulings and augmentations to any field F of the DGA over $\mathbb{Z}[t, t^{-1}]$. This does not contradict the result in [15] that there are augmentations to matrix algebras which do not send t to -1 as the matrix algebras are not fields.

Theorem 1.1 can be extended and interpreted in terms of the augmentation variety for a Legendrian knot. Define

$$\mathrm{Aug}_\rho(\Lambda) = \{\epsilon(t) : \epsilon \text{ a } \rho\text{-graded augmentation of } (\mathcal{A}, \partial)\} \subset F^*$$

the **augmentation variety** of Λ, where $F^* = F \backslash \{0\}$.

In higher dimensions, understanding the augmentation variety is interesting and useful (see [1] and [14]), so there has been some question as to whether we can determine the augmentation variety in $\mathbb{R}^3$ with the standard contact structure. In §3, we prove:

Theorem 1.2. *If ρ is odd and $\rho | 2r(\Lambda)$, then*

$$\mathrm{Aug}_\rho(\Lambda) = \begin{cases} \{-x^2 : x \in F^*\} & \text{if there exists a } \rho\text{-graded normal ruling of } \Lambda \\ & \text{which is not oriented (introduced in §3)} \\ \{-1\} & \text{if there exists a } \rho\text{-graded normal ruling of } \Lambda \\ & \text{and all rulings are oriented} \\ \emptyset & \text{if there are no } \rho\text{-graded normal rulings of } \Lambda. \end{cases}$$

For example, the right handed trefoil Λ in Figure 1 has DGA $(\mathcal{A}, \partial)$ with $|c_i| = 0$ for $1 \le i \le 3$, $|c_4| = |c_5| = 1$, and $|t| = 0$. Then $\mathcal{A} = \mathcal{A}(c_1, \dots, c_5)$ with differential

$$\begin{aligned} \partial c_1 &= \partial c_2 = \partial c_3 = 0 \\ \partial c_4 &= t + c_1 + c_3 + c_1 c_2 c_3 \\ \partial c_5 &= 1 - c_1 - c_3 - c_3 c_2 c_1. \end{aligned}$$

Let F be a field. If $\epsilon : \mathcal{A} \to F$ is a 1-graded (ungraded) augmentation, then

$$\begin{aligned} 0 &= \epsilon(t) + \epsilon(c_1) + \epsilon(c_3) + \epsilon(c_1)\epsilon(c_2)\epsilon(c_3) \\ 0 &= 1 - \epsilon(c_1) - \epsilon(c_3) - \epsilon(c_3)\epsilon(c_2)\epsilon(c_1) \end{aligned}$$

and so $\epsilon(t) = -1$. Thus $\mathrm{Aug}_1(\Lambda) = \{-1\}$.

Now consider the left handed trefoil Λ' depicted in Figure 1. The associated DGA is $(\mathcal{A}', \partial')$ with $|c_1| = |c_2| = |c_4| = -1$, $|c_3| = |c_5| = |c_6| = 1$, and

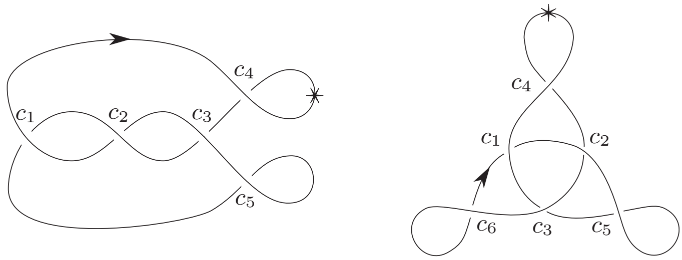

Figure 1: The left figure is a Legendrian right handed trefoil and the right is a Legendrian left handed trefoil with crossings labeled. The $*$ indicates the placement of the base point corresponding to t.

$|t| = 2$. Then $\mathcal{A} = \mathcal{A}(c_1, \ldots, c_6)$ with differential

$$\begin{aligned} \partial' c_1 &= \partial' c_2 = \partial' c_3 = 0 \\ \partial' c_4 &= t + c_1 c_2 \\ \partial' c_5 &= 1 + c_2 c_3 \\ \partial' c_6 &= 1 + c_3 c_1. \end{aligned}$$

Let F be a field. If $\epsilon : \mathcal{A}' \to F$ is a 1-graded (ungraded) augmentation, then

$$\begin{aligned} 0 &= \epsilon(t) + \epsilon(c_1)\epsilon(c_2) \\ 0 &= 1 + \epsilon(c_2)\epsilon(c_3) \\ 0 &= 1 + \epsilon(c_3)\epsilon(c_1). \end{aligned}$$

Therefore $\epsilon(c_2) = -(\epsilon(c_3))^{-1} = \epsilon(c_1)$ and so $\epsilon(t) = -(\epsilon(c_3))^{-2}$. So any non-zero choice of $\epsilon(c_3)$ yields an augmentation and thus $\mathrm{Aug}_1(\Lambda') = \{-x^2 : x \in F^*\}$.

This result complements the recent work of Henry and Rutherford [13]. Henry and Rutherford show that counts of the augmentations to any finite field, without restrictions on where the augmentation sends t, are Legendrian knot invariants and that they can be related to the ruling polynomials of the knot, thus showing that the Chekanov-Eliashberg algebra determines the ruling polynomial. Our result shows that if ρ is even, one can restrict the count of ρ-graded augmentations to augmentations which send t to -1, as there are not any which do not.

Theorem 1.1 tells us that if there exists an augmentation to $\mathbb{Z}/2$, then there exists an augmentation to any field. In §5, we will show that given an augmentation to $\mathbb{Z}/2$ of the Chekanov-Eliashberg DGA, we can use constructions similar to those in the proof of Theorem 1.1 to define an augmentation to any ring. In particular:

Theorem 1.3. *Let Λ be a Legendrian knot in $\mathbb{R}^3$. Let $(\mathcal{A}_{\mathbb{Z}/2}, \partial)$ be the Chekanov-Eliashberg DGA over $\mathbb{Z}/2$ and let $(\mathcal{A}, \partial)$ be the DGA over $R = \mathbb{Z}[t, t^{-1}]$. If $\epsilon' : \mathcal{A}_{\mathbb{Z}/2} \to \mathbb{Z}/2$ is an augmentation of $(\mathcal{A}_{\mathbb{Z}/2}, \partial)$, then one can find a lift of ϵ' to an augmentation $\epsilon : \mathcal{A} \to \mathbb{Z}$ of $(\mathcal{A}, \partial)$ such that $\epsilon(t) = -1$.*

In other words, we will define ϵ so that the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{A}, \partial) & \xrightarrow{\ \epsilon\ } & \mathbb{Z} \\ {\scriptstyle t=1}\downarrow & & \downarrow \\ (\mathcal{A}_{\mathbb{Z}/2}, \partial) & \xrightarrow[\ \epsilon'\ ]{} & \mathbb{Z}/2 \end{array}$$

This theorem tells us that given an augmentation to $\mathbb{Z}/2$ of $(\mathcal{A}_{\mathbb{Z}/2}, \partial)$, there exists an augmentation to any ring S of $(\mathcal{A}, \partial)$ which sends t to -1.

1.1. Outline of the article

In §2 we recall background on Legendrian knots and give definitions of the Chekanov-Eliashberg DGA, including sign conventions for defining the algebra over $\mathbb{Z}[t, t^{-1}]$, and a normal ruling. §3 gives the proof that given an augmentation one can define a normal ruling. §4 finishes the proof of Theorem 1.1 by proving that given a normal ruling one can define an augmentation. §4 goes to prove Theorem 1.2, giving the augmentation variety in the odd graded case. The paper concludes with the proof of Theorem 1.3 in §5.

1.2. Acknowledgements

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2. Background material

2.1. Diagrams of knots

In this section, we will briefly review necessary ideas of Legendrian knot theory. For further references on this subject, see [8].

A **contact structure** on a 3-manifold M is a completely nonintegrable 2-plane field ξ. Locally, a contact structure is the kernel of a 1-form α which satisfies the non-degeneracy condition

$$\alpha \wedge d\alpha \neq 0$$

at every point in M. We will be concerned with the **standard contact structure** on $\mathbb{R}^3$, which is the completely nonintegrable 2-plane field $\xi_0 = \ker \alpha_0$, where $\alpha_0 = dz - ydx$. A **Legendrian knot** is an embedding $\Lambda : S^1 \to \mathbb{R}^3$ which is everywhere tangent to the contact planes. A **Legendrian isotopy** is an ambient isotopy of Λ through Legendrian knots. We are interested in Legendrian isotopy classes of Legendrian knots in $\mathbb{R}^3$.

The classical invariants for Legendrian isotopy classes of knots are the topological knot type, Thurston-Bennequin number, and rotation number (see [2]). The **Thurston-Bennequin number** measures the self-linking of a Legendrian knot Λ. If Λ' is a knot that is a push off of Λ in a direction tangent to the contact structure, then $tb(\Lambda)$ is the linking number of Λ and Λ'. The **rotation number** r of an oriented Legendrian knot Λ is the rotation of its tangent vector field with respect to any global trivialization of ξ_0, for example, $\{\partial_y, \partial_x + y\partial_z\}$. A natural question is then whether these invariants with the topological knot type alone classify Legendrian knots, in other words, whether all Legendrian knots are "Legendrian simple." Eliashberg and Fraser [7] show that Legendrian unknots are Legendrian simple and Etnyre and Honda [9] show that Legendrian torus and figure eight knots are as well.

Two particularly useful projections of Legendrian knots are the Lagrangian projection and the front projection. The **Lagrangian projection** is the map

$$\pi_\ell : (x, y, z) \mapsto (x, y).$$

The **front projection** is the map

$$\pi_f : (x, y, z) \mapsto (x, z).$$

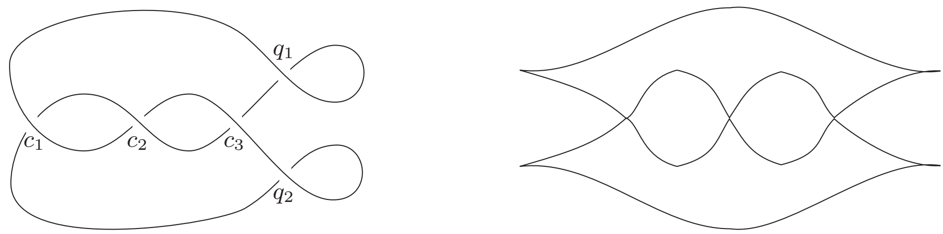

Figure 2: The left figure gives a Lagrangian projection of a Legendrian right handed trefoil with crossings labeled and the right figure gives a front projection.

In general, we will call the Lagrangian projection (resp. front projection) of a Legendrian knot a **Lagrangian diagram** (resp. **front diagram**). Figure 2 gives Lagrangian (left) and front (right) projections of a Legendrian version of a right handed trefoil.

Note that one can recover the y coordinate of a knot from the slope of the front diagram (see [8]):

$$y = \frac{dz}{dx}.$$

This implies that lines tangent to a front diagram of a Legendrian knot are never vertical. Front diagrams instead have semicubical cusps. It also implies that at a double point the strand with the smaller (more negative) slope has a smaller y coordinate and so passes in front of the strand with larger (more positive) slope. For a front diagram of an oriented Legendrian knot, the rotation number is half of the difference between the number of downward-pointing cusps and the number of upward-pointing cusps.

In particular, we will find that front diagrams in plat position will be easier to manipulate. A front diagram is in **plat position** if all of the left cusps have the same x coordinate, all of the right cusps have the same x coordinate, and there do not exist crossings in the diagram which have the same x coordinate. One can use Legendrian versions of the Reidemeister II moves and planar isotopy to put any front diagram into plat position. The diagram of the trefoil given in Figure 2 is an example of a diagram in plat position.

Note. Label the crossings of a front diagram of Λ in plat position by $\{c_1, \ldots, c_n, q_1, \ldots, q_m\}$ with $q_1, \ldots, q_m$ the crossings at the right cusps labeled from the top to the bottom and $c_1, \ldots, c_n$ the remaining crossings labeled from left to right (see Figure 6).

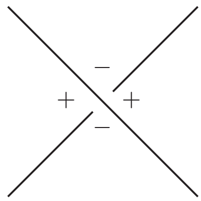

Figure 3: A labeling of the Reeb signs of the quadrants around a crossing.

2.2. Definition of the DGA and augmentations

This section contains a brief overview of the differential graded algebra presented by Etnyre, Ng, Sabloff in [10] which lifts the Chekanov-Eliashberg differential graded algebra over $\mathbb{Z}/2$ in [4] to a DGA over $\mathbb{Z}[t, t^{-1}]$.

Given a front diagram of an oriented Legendrian knot Λ in $\mathbb{R}^3$ with the standard contact structure, Ng's resolution process [16] gives a Lagrangian diagram for a knot Legendrian isotopic to Λ by smoothing left cusps, replacing right cusps with a loop, and resolving crossings so that the over crossing strand has smaller (more negative) slope.

Note. Label the crossings of the Lagrangian resolution of a front diagram of Λ by $\{c_1, \dots, c_n, q_1, \dots, q_m\}$ with $q_1, \dots, q_m$ the crossings from resolving the right cusps and $c_1, \dots, c_n$ the remaining crossings. Label each quadrant around a crossing as shown in Figure 3. We will refer to these labels as the **Reeb signs** and will call a quadrant at a crossing **positive** or **negative** depending on its Reeb sign.

Definition 2.1. Let Λ be an oriented Legendrian knot decorated with $*$ for the base point. The algebra $\mathcal{A}_R(c_1, \dots, c_n, q_1, \dots, q_m)$ is the noncommutative graded free associative unital algebra over $R = \mathbb{Z}[t, t^{-1}]$ generated (as an algebra) by $\{c_1, \dots, c_n, q_1, \dots, q_m\}$. We will sometimes shorten this to $\mathcal{A}_R$.

The grading for t is defined to be $-2r(\Lambda)$. To give c_i a grading, we first must specify a capping path γ_{c_i}. The **capping path** γ_{c_i} is the unique path in Λ which begins at the under crossing of c_i, ends at the over crossing of c_i, and does not go through the base point $*$ (note that this may mean the capping path has the opposite orientation of the knot), as seen in Figure 4.

Define the rotation number $r(\gamma_{c_i})$ to be the fractional number of counterclockwise revolutions made by the tangent vector to γ_{c_i} in the Lagrangian

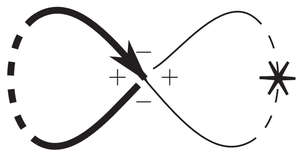

Figure 4: The choice of capping path for a crossing, where the capping path is denoted by a heavy line and the arrow gives the orientation of the capping path. The signs are the Reeb signs.

projection as we follow the path. One can perturb the diagram of Λ so that all crossings are orthogonal and thus $r(\gamma_{c_i})$ is an odd multiple of $1/4$. Define the grading on c_i by

$$|c_i| = -2r(\gamma_{c_i}) - \frac{1}{2}.$$

(Note that by setting $t = 1$ we recover Chekanov's grading from [4], though we then need to consider the grading mod $2r(\Lambda)$.)

Since we are working with front projections of knots, we can assign the gradings mod $2r(\Lambda)$ of crossings at right cusps: $|q_k| = 1$. Let $C(\Lambda)$ be the set of points on Λ corresponding to cusps of the front projection of Λ. A **Maslov potential function** is a locally constant function

$$\mu : \Lambda \backslash C(\Lambda) \to \mathbb{Z}/2r(\Lambda)$$

such that for two strands meeting at a cusp (either left or right), the upper strand has Maslov potential one higher than the lower strand. Such a function is well-defined up to a constant. Near a crossing c_k, let α_k be the strand in the front diagram with more negative slope and let β_k be the strand with more positive slope. The grading defined earlier now becomes

$$|c_k| \equiv \mu(\alpha_k) - \mu(\beta_k) \mod 2r(\Lambda).$$

Label a point on the diagram $*$. This will be the base point corresponding to t. In §2.5 we will discuss the case when we have multiple base points. We define the differential ∂ on $\mathcal{A}_R(c_1, \dots, c_n, q_1, \dots, q_m)$ by appropriately counting immersed disks in the Lagrangian resolution of the front projection of Λ. (Later we will make assumptions about the form of the diagrams so that all the disks are embedded.)

Given a generator a and an ordered set of generators $\{b_1, \dots, b_k\}$, let $\Delta(a; b_1, \dots, b_k)$ be the set of orientation-preserving immersions

$$f : D^2 \to \mathbb{R}^2$$

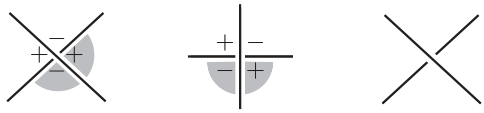

Figure 5: The signs in the figure are Reeb signs. The orientation signs are -1 for the shaded quadrants and $+1$ everywhere else. For crossings of odd degree, all orientation signs are $+1$. For crossings of even degree, we use the convention indicated in the left figure if the crossing comes from the front projection and the convention in the right figure if the crossing is in a dip, which will be discussed in §2.4. Note that a crossing has even/odd degree precisely when it is positive/negative in the sense of writhe.

(up to smooth reparametrization) that map ∂D^2 to the Lagrangian resolution of $\pi_f(\Lambda)$, such that

1) f is a smooth immersion except at $a, b_1, \dots, b_k$,
2) $a, b_1, \dots, b_k$ are encountered in counter-clockwise order along $f(\partial D^2)$,
3) near $a, b_1, \dots, b_k$, $f(D^2)$ covers exactly one quadrant, specifically, a quadrant with positive Reeb sign near a and a quadrant with negative Reeb sign near b_i for $1 \leq i \leq k$.

We can assign a word in $\mathcal{A}$ to each embedded disk by starting with the first corner after the one covering the $+$ quadrant and listing the crossing labels of all negative corners as encountered while following the boundary of the immersed polygon counter-clockwise. We associate a sign to each immersed disk by associating an **orientation sign** $\delta_{Q,a}$ to each quadrant Q in the neighborhood of a crossing a, determined by Figure 5, and defining the sign of a disk $f(D^2)$, the product of the orientation signs over all the corners of the disk, denoted $\delta(f(D^2))$. In practice, we can define $\delta(a; b_1 \cdots b_k)$ to be the sign of the unique disk with positive corner at a (with respect to Reeb signs) and negative corners at $b_1, \dots, b_k$, the product of the orientation signs over all corners of the disk. (In exceptional cases there may be more than one disk with negative corners at $b_1, \dots, b_k$.) Note that our convention for assigning orientation signs differs from [10]. At any crossing c where our convention differs from that in [10], one can recover the convention in [10] by sending c to $-c$.

Define $n_*(a; b_1, \dots, b_k)$ to be the signed count of the number of times one encounters the base point $*$ while following $f(\partial D^2)$ in the counter-clockwise direction, where the sign is determined by whether one encounters

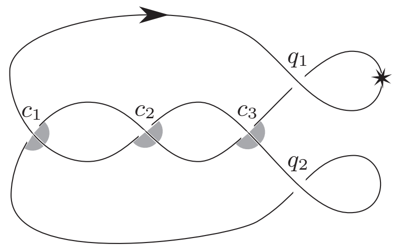

Figure 6: The Lagrangian resolution of the front diagram of the right trefoil in plat position. Crossings are labeled and $*$ indicates the base point corresponding to t. The shaded regions are quadrants with orientation sign -1. All other quadrants have orientation sign $+1$.

the base point while following the orientation of the knot or going against the orientation of the knot.

Definition 2.2. The algebra $\mathcal{A}_R$ is a differential graded algebra (DGA) whose differential ∂ is defined as follows:

$$\partial a = \sum_{(b_1,\ldots,b_k)} \delta(a; b_1 \cdots b_k) t^{n_*(a;b_1,\ldots,b_k)} b_1 \cdots b_k,$$

where the sum is taken over $(b_1, \ldots, b_k)$ where $\Delta(a; b_1, \ldots, b_k)$ is nonempty. Extend ∂ to $\mathcal{A}_R$ via $\partial(\mathbb{Z}[t, t^{-1}]) = 0$ and the signed Leibniz rule:

$$\partial(vw) = (\partial v)w + (-1)^{|v|} v(\partial w).$$

From Theorem 3.7 in [10], the differential ∂ has degree -1 and satisfies $\partial^2 = 0$.

For example, the right handed trefoil depicted in Figure 6 with $r = 0$ and $tb = 1$ has $|c_i| = 0$ and $|q_i| = 1$. We have $\mathcal{A}_R = \mathcal{A}_R(c_1, c_2, c_3, q_1, q_2)$ with differential

$$\begin{aligned}
\partial c_1 &= \partial c_2 = \partial c_3 = 0 \\
\partial q_1 &= t + c_1 + c_3 + c_1 c_2 c_3 \\
\partial q_2 &= 1 - c_1 - c_3 - c_3 c_2 c_1.
\end{aligned}$$

Definition 2.3. A graded algebra isomorphism

$$\phi : \mathcal{A}(a_1, \ldots, a_n) \to \mathcal{A}(b_1, \ldots, b_n)$$

is **elementary** if there exists $j \in \{1, \ldots, n\}$ such that

$$\phi(a_i) = \begin{cases} b_i & i \neq j \\ ub_j + v & v \in \mathcal{A}(b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n), u \text{ a unit in } R, i = j. \end{cases}$$

A composition of elementary isomorphisms is called **tame**.

Definition 2.4. Define the algebra $\mathcal{E}_i = \mathcal{A}(e_1^i, e_2^i)$ by setting $|e_1^i| = i - 1$, $|e_2^i| = i$, $\partial e_2^i = e_1^i$, and $\partial e_1^i = 0$.

This algebra models the second Reidemeister move, which produces two new crossings.

Definition 2.5. Given a DGA $(\mathcal{A}(a_1, \ldots, a_n), \partial)$, the **degree** i **stabilization** of $(\mathcal{A}(a_1, \ldots, a_n), \partial)$ is defined to be $\mathcal{A}(a_1, \ldots, a_n, e_1^i, e_2^i)$. The grading and the differential are inherited from $\mathcal{A}$ and $\mathcal{E}_i$. Two DGA's $(\mathcal{A}, \partial)$ and $(\mathcal{A}', \partial')$ are **stable tame isomorphic** if there exist two sequences of stabilizations $S_{i_1}, \ldots, S_{i_n}$ and $S_{j_1}, \ldots, S_{j_m}$ and a tame isomorphism

$$\phi : S_{i_n}(\cdots(S_{i_1}(\mathcal{A}))\cdots) \to S_{j_m}(\cdots(S_{j_1}(\mathcal{A}'))\cdots),$$

which is also a chain map.

In fact, the stable tame isomorphism class of the DGA is invariant under Legendrian isotopy. Chekanov proved this result over $\mathbb{Z}/2$ in [4] and Etnyre, Ng, and Sabloff proved this result over $\mathbb{Z}[t, t^{-1}]$ in [10].

Now that we have the DGA associated with the projection of Λ, we can discuss the augmentations.

Definition 2.6. Let F be a field. An **augmentation** of $(\mathcal{A}, \partial)$ to F is a ring map $\epsilon : \mathcal{A} \to F$ such that $\epsilon \circ \partial = 0$ and $\epsilon(1) = 1$. If $\rho | 2r(\Lambda)$ and ϵ is supported on generators of degree divisible by ρ, then ϵ is ρ**-graded**. In particular, if $\rho = 0$, we say it is **graded** and if $\rho = 1$, we say it is **ungraded.** We call a generator a **augmented** if $\epsilon(a) \neq 0$.

For example, if we recall the DGA over $\mathbb{Z}[t, t^{-1}]$ for the right handed trefoil, then we can classify the augmentations to any field F as follows: Let $\epsilon : \mathcal{A}_R \to F$ be an augmentation. Then $\epsilon(t) = -1$ and

- if $\epsilon(c_1) = 0$, then $\epsilon(c_3) = 1$ and $\epsilon(c_2) \in F$
- if $\epsilon(c_3) = 0$, then $\epsilon(c_1) = 1$ and $\epsilon(c_2) \in F$

- if $\epsilon(c_1), \epsilon(c_3) \neq 0$, then

$$\epsilon(c_2) = (1 - \epsilon(c_1) - \epsilon(c_3))(\epsilon(c_1))^{-1}(\epsilon(c_3))^{-1}.$$

Note that if F is a finite field, as in [13], and $|F|$ is the number of elements in F, then we see that there are $|F|$ augmentations of the first type, $|F|$ augmentations of the second type, and $|F^*|^2$ augmentations of the third type, where $F^* = F \backslash \{0\}$. In fact,

$$(1) \quad \{(\epsilon(c_1),\epsilon(c_2), \epsilon(c_3), \epsilon(q_1), \epsilon(q_2), \epsilon(t)) : \epsilon \text{ an augmentation to } F\} = F \coprod F \coprod (F^*)^2.$$

In [13], this is called the augmentation variety of $(\mathcal{A}(\Lambda), \partial)$. Comparing this with possible rulings of the trefoil, definition given in §2.3, one sees that (1) coincides with Theorem 3.4 of [13].

For example, the following are examples of graded augmentations to $\mathbb{R}$.

	c_1	c_2	c_3	q_1	q_2	t
ϵ_1	1	$\frac{1}{2}$	0	0	0	-1
ϵ_2	0	$\frac{1}{2}$	1	0	0	-1
ϵ_3	2	$\frac{3}{4}$	$-\frac{2}{5}$	0	0	-1
ϵ_4	$-\frac{2}{5}$	$\frac{3}{4}$	2	0	0	-1
ϵ_5	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	-1

Note that any augmentation of a stabilization $S(\mathcal{A})$ restricts to an augmentation of the smaller algebra $\mathcal{A}$ and any augmentation of the algebra $\mathcal{A}$ extends to an augmentation of the stabilization $S(\mathcal{A})$ where the augmentation sends e_1^i to 0 and e_2^i to an arbitrary element of F if $\rho|i$ and 0 otherwise.

2.3. Rulings

This paper will show that there is a way to construct an augmentation from a normal ruling and a normal ruling from an augmentation.

Definition 2.7. Consider a front diagram in plat position of a Legendrian knot Λ. A **ruling** of this diagram consists of a one-to-one correspondence between the set of left cusps and the set of right cusps where, for each pair of corresponding cusps, two paths in the front diagram join them. These **ruling paths** must satisfy the following:

1) Any two paths in the ruling only meet at crossings or cusps;

2) The interiors of the two paths joining corresponding cusps are disjoint. Thus each pair of paths bound a topological disk.

The first condition tells us the ruling paths never overlap at more than a finite number of points. The second condition tells us that there are disks similar to those in the differential ∂, but possibly with "obtuse" corners. As noted in [11], these imply that the ruling paths cover the front diagram and the x-coordinate of each path in the ruling is monotonic.

Near a crossing, the two ruling paths which intersect at the crossing are called **crossing paths**. The two paths paired with the crossing paths are called **companion paths**.

Given a ruling, at any crossing, we either have that the crossing paths pass through each other, or one path lies entirely above (has z-coordinate strictly greater than) the other. In the latter case, we say the ruling is **switched** at the crossing. If all of the switched crossings in the ruling are of the form (a), (b), or (c), as seen in Figure 7, then we say the ruling is **normal**. Thus, the possible configurations near a crossing in a normal ruling are shown in Figure 7.

If all of the switched crossings have grading divisible by ρ for some ρ such that $\rho|2r(\Lambda)$, then we say the ruling is ρ**-graded**. In particular, if $\rho = 0$, then we say the ruling is **graded** and if $\rho = 1$, then we say the ruling is **ungraded**.

For example, the right handed trefoil has three graded normal rulings as seen in Figure 8.

In [3], Chekanov and Pushkar showed that the number of ρ-graded normal rulings is invariant under Legendrian isotopy.

2.4. Dips

We will construct a normal ruling of the diagram by using the augmentation to construct an augmentation ϵ of the dipped diagram satisfying Property (R), as called in [17]. However, the notation in the following section will be necessary to write down Property (R).

Given a Legendrian knot Λ in plat position, we construct a **dip** between two crossings by a sequence of Reidemeister II moves, as seen in Figure 9 in the front projection and Lagrangian projection. In the front projection, it is clear that the diagram with the dip is isotopic to the original diagram. To construct a dip, number the $2m$ strands from bottom to top. Using a type II Reidemeister move, push strand 2 over strand 1, then strand 3 over strand 1, then strand 3 over strand 2, and so on. So that strand k is pushed over

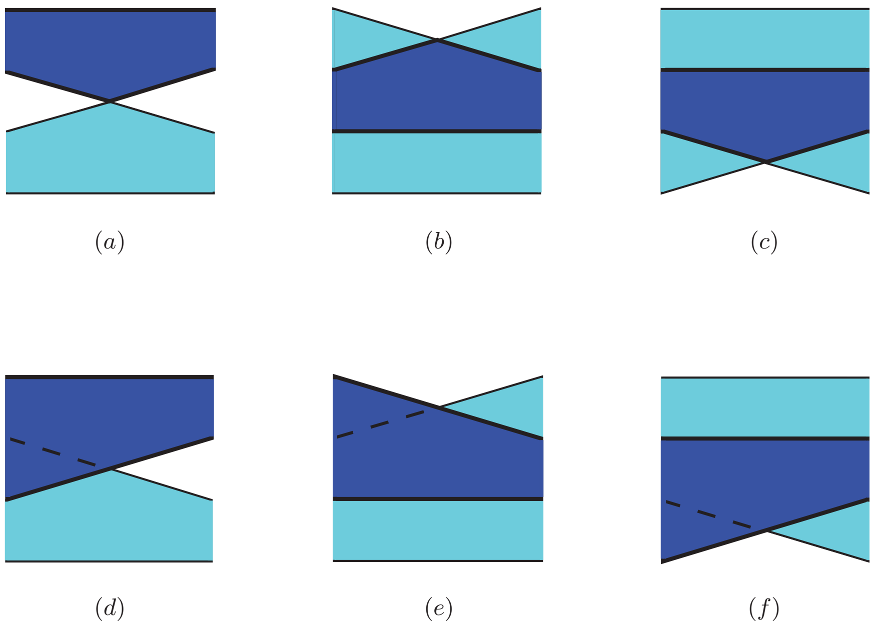

Figure 7: By including vertical reflections of (d), (e), and (f), these are all possible configurations of crossings appearing in a normal ruling. The top row contains all possible configurations for switched crossings in a normal ruling.

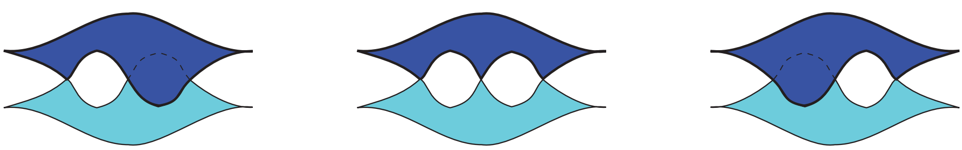

Figure 8: The graded normal rulings of the right handed trefoil.

strand ℓ in lexicographic order. If strand k crosses strand ℓ after strand i crosses strand j, we write $(i, j) < (k, \ell)$.

The **dipped diagram** involves introducing a dip between each crossing in the plat position diagram and between the left, respectively right, cusps and the first, respectively last, crossing (see Figure 13). Each Reidemeister II move introduces two new variables. For the dip immediately after crossing c_k, we will use a^k_{rs} and b^k_{rs} to denote the new crossings introduced when strand r is passed over strand s ($r > s$), with b^k_{rs} being the leftmost and

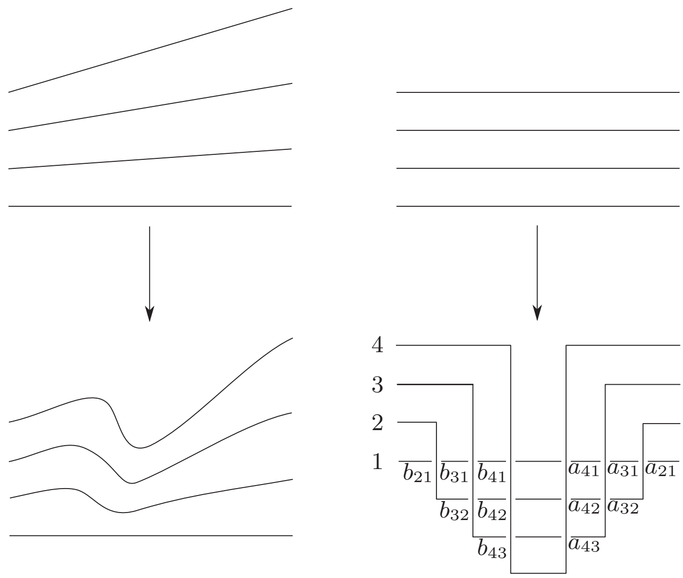

Figure 9: The left diagram gives the modification of the front projection when creating a dip, while the right diagram gives the modification of the Lagrangian projection. In the Lagrangian projection, the b^k-lattice is made up of the crossings on the left and the a^k-lattice is made up of the crossings on the right. The crossings in the b^k-lattice are labeled down and to the left, while the crossings in the a^k-lattice to the right, with k's suppressed.

a^k_{rs} being the rightmost new crossing (see Figure 9). We will say the b^k_{rs} generators belong to the b^k**-lattice** and the a^k_{rs} belong to the a^k**-lattice**. Thus we will have a^k/b^k-lattices for $0 \leq k \leq n$. While dipped diagrams have many more crossings than the original knot diagram, the differential ∂ on $\mathcal{A}_R$ is generally much simpler. We note that if μ is a Maslov potential function on the front diagram, then

$$|b^k_{rs}| = \mu(r) - \mu(s).$$

Since the differential ∂ lowers degree by one,

$$|a^k_{rs}| = |b^k_{rs}| - 1.$$

Orientation sign assignments are given in Figure 5. We can reduce possible disks, and thus possible terms in the differential, further in certain

Figure 10: Possible disks contributing to ∂ with a negative corner at a.

cases. As the disks in the computation of $\mathcal{A}_{\mathbf{Z}/2}$ are the same disks in the computation of $\mathcal{A}$, we have the following lemma from [17].

Lemma 2.8 ([17] Lemma 3.1). *If a and b are the new crossings created by a type II move during the creation of a dip and y is any other crossing, then a appears at most once in any term of ∂y, and if a appears in any term of ∂y, then b does not.*

This follows from considering the disks which have a negative corner at a as seen in Figure 10.

Through consideration of the dipped diagram, we see

- the differential of crossings in the b^k-lattice involve at most
  - c_k,
  - base points (we will discuss the case when we have more than one in the next section),
  - crossings in the a^{k-1}-lattice,
  - crossings in the a^k-lattice,
  - crossings in the b^k-lattice,
- the differential of crossings in the a^k-lattice only involve
  - base points,
  - crossings in the a^k-lattice,
- the differential of c_k only involves
  - base points,
  - $a^{k-1}_{i+1,i}$ if strands i and $i+1$ cross at c_k

for all $1 \leq k \leq n$. This greatly reduces the types of totally augmented disks for which to look to compute whether we have an augmentation, where a

totally augmented disk is a disk which contributes to the differential, all of whose negative corners are augmented.

Notation 2.9. $a^k_{\{r,s\}} = a^k_{\max(r,s),\min(r,s)}$

2.5. Augmentations before and after a base point move

As we create dips, we will find that the signs are simpler if, in certain cases, we add in a few extra base points. In [15], Ng and Rutherford give the DGA isomorphisms induced by adding a base point and by moving one base point around a knot. First, we need to extend our definition of the DGA over $\mathbb{Z}[t, t^{-1}]$ to a DGA over $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$, which we will call $\mathcal{A}(\Lambda, *)$. To this end, label s points on the Lagrangian resolution of the front diagram of Λ by the base points $*_1, \ldots, *_s$ respectively associated to $t_1, \ldots, t_s$.

Definition 2.10. The algebra $\mathcal{A}$ is a DGA whose grading is defined analogously to the case when there is only one base point: We define $|t_1| = -2r(\Lambda)$ and $|t_i| = 0$ for $1 < i \leq s$. Given a crossing c, let γ_c be the unique path following the under strand of c to the over strand of c while avoiding $*_1$ and define $|c| = -2r(\gamma_c) - \frac{1}{2}$. The differential ∂ is defined as follows:

$$\partial a = \sum_{(b_1,\ldots,b_k)} \delta(a; b_1 \cdots b_k) t_1^{n_{*_1}(a;b_1,\ldots,b_k)} \cdots t_s^{n_{*_s}(a;b_1,\ldots,b_k)} b_1 \cdots b_k,$$

where the sum is taken over $(b_1, \ldots, b_k)$ where $\Delta(a; b_1, \ldots, b_k)$ is nonempty. Extend ∂ to $\mathcal{A}$ via $\partial(\mathbb{Z}[t, t^{-1}]) = 0$ and the signed Leibniz rule:

$$\partial(vw) = (\partial v)w + (-1)^{|v|} v(\partial w).$$

Theorem 2.11 ([15] Thm. 2.19). *The map $\partial : \mathcal{A}(\Lambda, *) \to \mathcal{A}(\Lambda, *)$ lowers degree by 1 and is a differential: $\partial^2 = 0$. Up to stable tame isomorphism, the differential graded algebra $(\mathcal{A}(\Lambda, *), \partial)$ is an invariant of Λ under Legendrian isotopy (and choice of base point(s)).*

Theorem 2.12 ([15] Thm. 2.20). *Let $*_1, \ldots, *_k$ and $*'_1, \ldots, *'_k$ denote two collections of base points on the Lagrangian resolution of the front diagram of a Legendrian knot Λ, each of which is cyclically ordered along Λ. Let $(\mathcal{A}(\Lambda, *_1, \ldots, *_k), \partial)$ and $\mathcal{A}(\Lambda, *'_1, \ldots, *'_k), \partial')$ denote the corresponding multi-pointed DGAs. Then there is a DGA isomorphism $\Psi : (\mathcal{A}(\Lambda, *_1, \ldots, *_k), \partial) \to (\mathcal{A}(\Lambda, *'_1, \ldots, *'_k), \partial')$ such that $\Psi(t_i) = t_i$ for all i.*

In the proof of this theorem, the isomorphism Ψ is defined so that $\Psi(c_j) = c_j$ if no base point is pushed over or under the crossing c_j. If, however, the base point $*_i$ is pushed over crossing c_j, then $\Psi(c_j) = t_i^{\pm 1} c_j$, the sign depending on whether the base point is pushed along the knot in the direction of the orientation or against the orientation of the knot. If the base point $*_i$ is pushed under the crossing c_j, then $\Psi(c_j) = c_j t_i^{\pm 1}$, again, the sign depending on the orientation of the knot.

Theorem 2.13 ([15] Thm. 2.21). *Let $*_1, \ldots, *_k$ be a cyclically ordered collection of base points along Λ, and let $*$ be a single base point on Λ. Then there is a DGA homomorphism $\phi : (\mathcal{A}(\Lambda, *), \partial) \to (\mathcal{A}(\Lambda, *_1, \ldots, *_k), \partial)$ such that $\phi \circ \partial = \partial \circ \phi$ and $\phi(t) = t_1 \cdots t_k$.*

Thus, we can assume there is one base point on each of the right cusps. Also, this shows us that if ϵ' is an augmentation on the diagram after moving the base point $*_i$ over the crossing c_j, then $\epsilon = \epsilon' \Psi$ is an augmentation on the diagram before moving the base point.

Remark 2.14. In summary, if $\epsilon(t_i) = -1$, then moving the base point $*_i$ over or under a crossing only changes the augmentation by changing the sign of the augmentation on that crossing, no matter the orientation of the strand.

Note that these theorems tell us that we have the following relationship between augmentations to a field F on a diagram with one base point and the augmentations to F on the same diagram but with mutliple base points: Suppose t is the variable associated to the single base point $*$, and $t_1, \ldots, t_s$ are the variables associated to the base points $*_1, \ldots, *_s$. By moving all the base points $*_1, \ldots, *_s$ to the location of $*$ and then using Theorem 2.13, we see that for every augmentation ϵ to F of the multiple base point diagram, there exists an augmentation ϵ' to F of the single base point diagram such that $\epsilon'(a) = x_a \epsilon(a)$ for some $x_a \in F^*$ for all crossings a and

$$\epsilon'(t) = \epsilon(t_1 \cdots t_s) = \prod_{i=1}^{s} \epsilon(t_i).$$

In particular, if $\epsilon(t_i) = \pm 1$ for all i, then $\epsilon'(a) = \pm \epsilon(a)$ for all crossings a. One can also easily check that for all augmentations ϵ' to F of the single base point diagram and $x_1, \ldots, x_s \in F^*$ such that $x_1 \cdots x_s = \epsilon'(t)$, there exists an augmentation ϵ to F of the multiple base point diagram with $\epsilon(a) = x_a \epsilon'(a)$

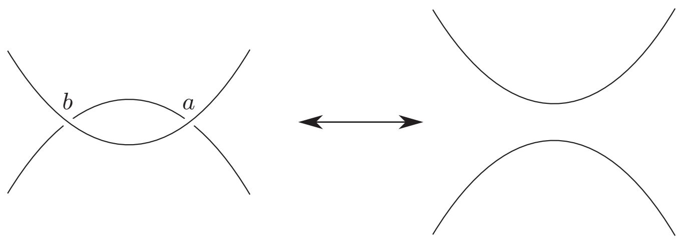

Figure 11: A type II Reidemeister move with crossings a and b.

for some $x_a \in F^*$ for all crossings and $\epsilon(t_i) = x_i$. In particular, if $\epsilon(t_i) = \pm 1$ for all i, then $x_a = \pm 1$ for all crossings a.

2.6. Augmentations before and after type II moves

To understand how augmentations before the addition of a dip relate to augmentations after, we need to consider the stable DGA isomorphism induced by a type II move. Suppose $(\mathcal{A}'_Z, \partial')$ is the DGA over $Z = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_q, t_q^{-1}]$ for a knot diagram before a type II move and that $(\mathcal{A}_Z, \partial)$ is the DGA over Z afterward. So

$$\begin{aligned} \mathcal{A}_Z &= \mathcal{A}_Z(a, b, a_1, \dots, a_r, b_1, \dots, b_s; \partial) \\ \mathcal{A}'_Z &= \mathcal{A}_Z(a_1, \dots, a_r, b_1, \dots, b_s; \partial'). \end{aligned}$$

Suppose that the other crossings are ordered by height:

$$h(b_s) \geq \cdots \geq h(b_1) \geq h(b) > h(a) \geq h(a_1) \geq \cdots \geq h(a_r).$$

From [17] we know it is possible to construct a dip in the plat diagram so that this ordering takes the following form: Suppose strand k is pushed over strand ℓ. Each a_j either lies to the left of the dip or $a_j = a_{mn}$ or b_{mn} with $m - n \leq k - \ell$. Similarly, b_j either lies to the right of the dip or $b_j = a_{mn}$ or b_{mn} with $m - n > k - \ell$.

Recall the algebra $\mathcal{E}_i = \mathcal{A}_Z(e_1, e_2)$ with $|e_1| = i - 1$, $|e_2| = i$, $\partial e_2 = e_1$, and $\partial e_1 = 0$. Define the vector space map $H : S(\mathcal{A}'_Z) \to S(\mathcal{A}'_Z)$ by

$$H(w) = \begin{cases} 0 & w \in \mathcal{A}'_Z \\ 0 & w = Qe_2R \text{ with } Q \in \mathcal{A}'_Z, R \in S(\mathcal{A}'_Z) \\ (-1)^{|Q|+1} Qe_2R & w = Qe_1R \text{ with } Q \in \mathcal{A}'_Z, R \in S(\mathcal{A}'_Z). \end{cases}$$

Note that either crossing a or b is a positive crossing, so $\partial b = -a + v$, where v is a sum of terms in the a_i and $t_i^{\pm 1}$. Define $\Phi_0 : \mathcal{A}_Z \to S_{|b|}(\mathcal{A}'_Z)$ by

$$\Phi_0(w) = \begin{cases} e_2 & w = b \\ -e_1 + v & w = a \\ w & \text{otherwise.} \end{cases}$$

[10] tells us Φ_0 is a grading-preserving elementary isomorphism. Inductively define maps Φ_i on the generators of $\mathcal{A}_Z$ on generators by:

$$\Phi_i(w) = \begin{cases} b_i + H(\partial' b_i - \Phi_{i-1}\partial b_i) & w = b_i \\ \Phi_{i-1}(w) & \text{otherwise.} \end{cases}$$

In [10], it is shown that $\Phi := \Phi_s$ is a DGA isomorphism between $\mathcal{A}_Z$ and $S_{|b|}(\mathcal{A}'_Z)$.

If there is an augmentation ϵ' on $S(\mathcal{A}'_Z)$, then $\epsilon = \epsilon'\Phi$ is an augmentation on $\mathcal{A}_Z$. One can check that

$$\epsilon(a_i) = \epsilon'(a_i), \qquad \epsilon(a) = \epsilon'(v), \qquad \epsilon(b) = \epsilon'(e_2). \tag{2}$$

Recall that if $|e_2| = 0$, then $\epsilon'(e_2)$ can be chosen arbitrarily.

Analogous to the result for the $\mathbb{Z}/2$ case in [17], we have:

Lemma 2.15. *After a type II Reidemeister move involved in making a dip in a plat diagram, suppose $\epsilon(b_i)$ has been determined for $i < j$. Then*

$$\epsilon(b_j) = \epsilon'(b_j) - \sum_p \delta(b_j; Q_p a R_p)(-1)^{|\Phi(Q_p)|}\epsilon(Q_p b R_p)$$

for $Q_p, R_p \in \mathcal{A}'_Z$ such that $\partial b_j = P + \sum_p \delta(b; Q_p a R_p) Q_p a R_p$ where P is the sum of the terms in ∂b_j which do not contain a.

Proof. We know

$$\Phi(b_i) = b_i + H(\partial' b_i - \Phi\partial b_i).$$

We will prove the result by inducting on j. For the base case, suppose $j = 1$. Since ∂ lowers height, we know $\partial b_1 \in \mathcal{A}_Z(a, b, a_1, \dots, a_r)$ and $\partial' b_1 \in \mathcal{A}_Z(a_1, \dots, a_r)$. By Lemma 2.8, we know if P is the sum of terms in ∂b_1

which do not contain a, then ∂b_1 has the form

$$\partial b_1 = P + \sum_p \delta(b_1; Q_p a R_p) Q_p a R_p,$$

where $Q_p, R_p \in \mathcal{A}_Z(a_1, \ldots, a_r)$. Therefore

$$\begin{aligned} H(\partial' b_1 - \Phi\partial b_1) &= H\left(\partial' b_1 - \Phi\left(P + \sum_p \delta(b_1; Q_p a R_p) Q_p a R_p\right)\right) \\ &= H\left(\partial' b_1 - \Phi(P) - \sum_p \delta(b_1; Q_p a R_p) Q_p(-e_1 + v) R_p\right). \end{aligned}$$

We know $\partial' b_1 \in \mathcal{A}_Z(a_1, \ldots, a_r)$, so $H(\partial' b_1) = 0$. Since $P \in \mathcal{A}_Z(b, a_1, \ldots, a_r)$, we know $\Phi(P) \in \mathcal{A}_Z(e_2, a_1, \ldots, a_r)$ and so $H(\Phi(P)) = 0$. Thus

$$\begin{aligned} H(\partial' b_1 - \Phi\partial b_1) &= -\sum_p \delta(b_1; Q_p a R_p) H(Q_p(-e_1 + v) R_p) \\ &= \sum_p (-1)^{|Q_p|+1} \delta(b_1; Q_p a R_p) Q_p e_2 R_p. \end{aligned}$$

So

$$\begin{aligned} \epsilon(b_1) &= \epsilon'(\Phi(b_1)) \\ &= \epsilon'(b_1 + H(\partial' b_1 - \Phi\partial b_1)) \\ &= \epsilon'(b_1) + \epsilon'\left(\sum_p (-1)^{|Q_p|+1} \delta(b_1; Q_p a R_p) Q_p e_2 R_p\right) \\ &= \epsilon'(b_1) - \sum_p (-1)^{|Q_p|} \delta(b_1; Q_p a R_p) \epsilon(Q_p b R_p). \end{aligned}$$

Since

$$\begin{aligned} \Phi(b_1) &= b_1 + H(\partial' b_1 - \Phi\partial b_1) \\ &= b_1 - \sum_p (-1)^{|Q_p|} \delta(b_1; Q_p a R_p) Q_p e_2 R_p, \end{aligned}$$

we have also shown that e_1 does not appear in $\Phi(b_1)$.

Now suppose the equation is satisfied for b_i and that e_1 does not appear in $\Phi(b_i)$ for $i < j$. As before, since ∂ is height decreasing, we know that $\partial b_j \in \mathcal{A}_Z(a, b, a_1, \ldots, a_r, b_1, \ldots, b_{j-1})$ and $\partial' b_j \in \mathcal{A}_Z(a_1, \ldots, a_r, b_1, \ldots, b_{j-1})$.

By Lemma 2.15 we know that if P is the sum of terms in ∂b_j which do not contain a, then

$$\partial b_j = P + \sum_p \delta(b_j; Q_p a R_p) Q_p a R_p,$$

where $Q_p, R_p \in \mathcal{A}_Z(a_1, \ldots, a_r, b_1, \ldots, b_{j-1})$. By the inductive assumption, $\Phi(b_i)$ does not contain e_1 for $i < j$ and so $\Phi(Q_p), \Phi(R_p)$, and $\Phi(P)$ do not contain e_1. So

$$\begin{aligned} H(\Phi(Q_p a R_p)) &= H(\Phi(Q_p)(-e_1 + v)\Phi(R_p)) \\ &= (-1)^{|\Phi(Q_p)|}\Phi(Q_p) e_2 \Phi(R_p). \end{aligned}$$

Therefore

$$H(\partial' b_j - \Phi\partial b_j) = -\sum_p (-1)^{|\Phi(Q_p)|}\delta(b_j; Q_p a R_p)\Phi(Q_p) e_2 \Phi(R_p).$$

Thus $\Phi(b_j) = b_j + H(\partial' b_j - \Phi\partial b_j)$ does not contain e_1.

We then see

$$\begin{aligned} \epsilon(b_j) &= \epsilon'\Phi(b_j) \\ &= \epsilon'(b_j + H(\partial' b_j - \Phi\partial b_j)) \\ &= \epsilon'(b_j) - \sum_p (-1)^{|\Phi(Q_p)|}\delta(b_j; Q_p a R_p)\epsilon(Q_p b R_p), \end{aligned}$$

as desired. □

Therefore, after a type II move involved in making a dip, if $\epsilon(b_i)$ has been determined for $i < j$, then

$$\epsilon(b_j) = \epsilon'(b_j) - \sum (-1)^{|\Phi(Q_p)|}\delta(b_j; Q_p a R_p)\epsilon(Q_p b R_p),$$

where the sum is over totally augmented disks with positive corner at b_j and a negative corner at b.

3. Augmentation to ruling

In this section, we will use a construction similar to that of Sabloff's in [17] to construct a ρ-graded normal ruling from a ρ-graded augmentation to a fixed field F. This shows the forward direction of Theorem 1.1. Suppose that D is the front diagram of a Legendrian knot Λ in plat position. By the discussion in §2.5 we can assume that there are base points $*_1, \ldots, *_m$, one on each right cusp, labeled from top to bottom corresponding to $t_1, \dots, t_m$. Let $\epsilon' : \mathcal{A}_Z \to F$ be a ρ-graded augmentation of the DGA $(\mathcal{A}_Z, \partial)$ over $Z = \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ of D. (Note that then $\epsilon'(t) = \prod_{i=1}^m \epsilon'(t_i)$ for the corresponding augmentation over $\mathbb{Z}[t, t^{-1}]$.) We will construct a ρ-graded normal ruling for the knot diagram while simultaneously extending the augmentation to an augmentation ϵ of the dipped diagram by adding one dip at a time from left to right. We will add base points to the diagram as we go to simplify the augmentation.

Start the ruling at the left of the diagram, pairing strands $2k$ and $2k-1$ for $1 \leq k \leq m$. We will extend the ruling from left to right along the diagram such that Property (R), stated below, is satisfied. We can ensure Property (R) is satisfied because when introducing new crossings in the creation of the dips, the a/b-lattices, we get to choose where the augmentation sends the crossings in the b-lattice. We have enumerated the conditions we will need to check to ensure we end up with a ρ-graded augmentation of the dipped diagram and a ρ-graded normal ruling.

Property (R): At any dip, the generator a_{rs}^j is augmented if and only if the strands r and s are paired in the ruling between c_j and c_{j+1}.

Recall that the crossings from the resolution of the right cusps are labeled $q_1, \dots, q_m$ from top to bottom and that the remaining crossings are labeled $c_1, \dots, c_n$ from left to right. Also, the strands are labeled from bottom to top. It will also be important to recall that the orientation signs at positive original crossings are given by the left most diagram in Figure 5, while orientation signs at positive crossings in the a/b-lattices are given in the right diagram.

We will inductively define augmentations on partially dipped diagrams by adding dips one at a time from left to right and defining augmentations on these diagrams. In particular, if ϵ_j is an augmentation on the diagram with dips added up to the crossing c_j, we will extend the ruling and construct ϵ_{j+1}, an augmentation on the diagram with dips added up to the crossing c_{j+1}:

1) Extend the ruling over c_j by a switch if $\epsilon_j(c_j) \neq 0$ and just to the left of c_j, the ruling matches configuration (a), (b), or (c) in Figure 7. Otherwise, no switch.

2) Consult Figure 12 to determine whether any base points will be added between c_j and c_{j+1}. For each added base point, follow the strand it will end up on to the right all the way to a right cusp and add a base

point $*_\alpha$ at the right cusp. Fix $\epsilon_{j+1}(t_\alpha) = -1$ and recall from §2.5 that we must then set $\epsilon_{j+1}(t_i) = -\epsilon_j(t_i)$, where $*_i$ is the base point already at the right cusp ($1 \leq i \leq m$). Move the base point $*_\alpha$ along the strand to between c_j and c_{j+1}, modifying the augmentation on any crossing the base point goes over or under by a factor of -1 according to Remark 2.14.

3) Place a dip between crossings c_j and c_{j+1}, making sure to place the dip so that the new base points are to the right if they end up in the dip according to Figure 12 and to the left if not. Between each Reidemeister II move involved in making the dip:
   a) Extend the augmentation ϵ' of the DGA of the diagram before the Reidemeister II move to an augmentation ϵ of the DGA of the new diagram satisfying Property (R) by defining ϵ on the two new crossings by Figure 12 and modifying ϵ from ϵ' by Lemma 2.15.
   b) Move base points to location specified by Figure 12 and modify ϵ using Remark 2.14.

Note that ϵ_{j+1} will agree with ϵ_j on the diagram to the left of c_j though, according to Lemma 2.15, they may differ on $c_{j+1}, \ldots, c_n$.

When we complete this process and have a fully dipped diagram, the augmentation $\epsilon_n = \epsilon$ is a ρ-graded augmentation of the dipped diagram, and we have a normal ruling of the original diagram. We will also see that the resulting augmentation has restrictions on what $\epsilon(t)$ equals depending on whether ρ is even or odd, yielding Theorem 3.1 and Theorem 1.2.

For example, Figure 13 gives an augmentation to $\mathbb{R}$ of the right handed trefoil and the resulting ruling and augmentation of the dipped diagram from following this process.

3.1. Left cusps

Let ϵ_0 be the ρ-graded augmentation of the original diagram. We know the ruling must pair strand $2k$ with strand $2k-1$ for $1 \leq k \leq m$ (where m is the number of right cusps) at the left end of the diagram. Now add a dip between the left cusps and c_1. We must now extend ϵ_0 to an augmentation ϵ_1 of the new diagram. This will require successively extending the augmentation ϵ' of the diagram before the Reidemeister II move to the augmentation ϵ of the diagram after one of the moves involved in constructing a dip. We will compute how the augmentation ϵ_0 changes as we complete each Reidemeister II move in constructing the dip.

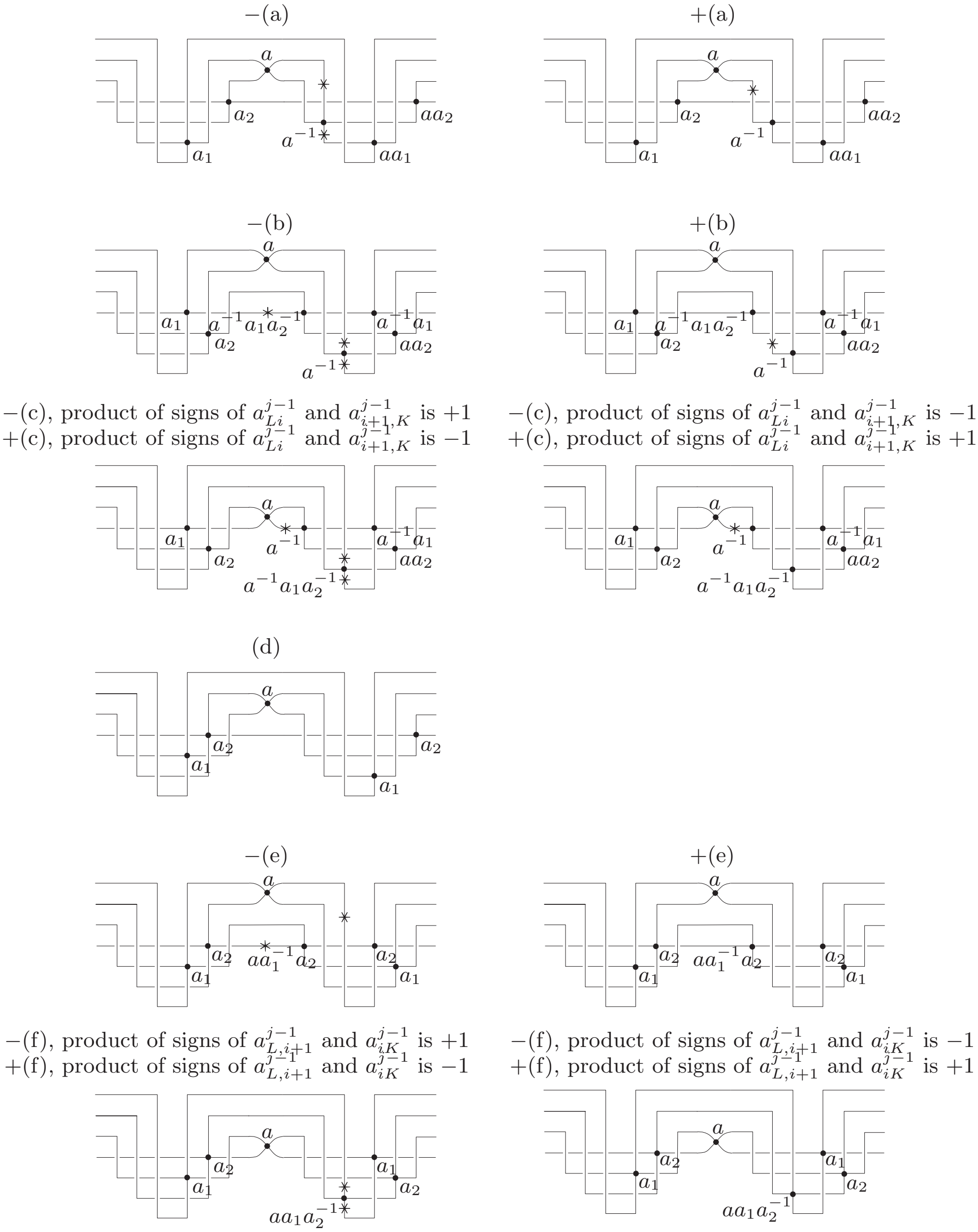

Figure 12: In the diagrams, $*$ denotes a base point. A dot denotes the specified crossing is augmented and the augmentation sends the crossing to the label. Here −/+(a) denotes a negative/positive crossing where the ruling has configuration (a) and the rest are defined analogously.

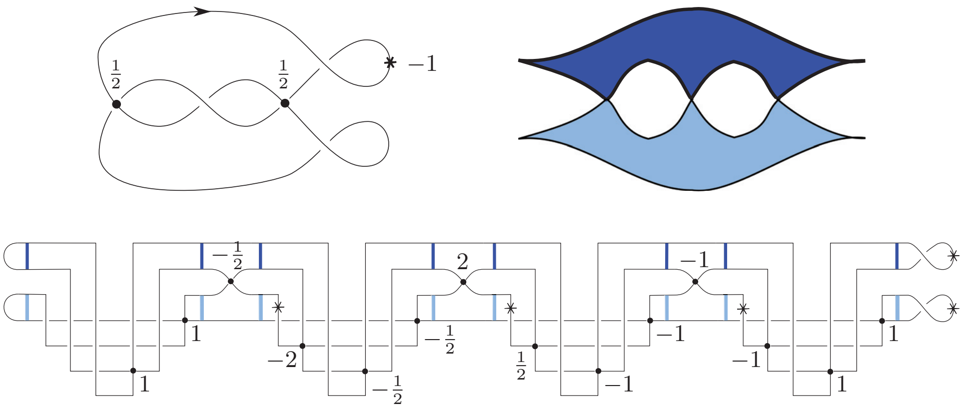

Figure 13: The top left diagram gives an augmentation of the right trefoil. The top right diagram gives the normal ruling and the bottom diagram gives the augmentation of the dipped diagram resulting from following the process of the proof. The dots denote that the crossing is augmented and the label on the dot gives where the augmentation sends the crossing. The $*$ gives the placement of the base points. All base points are sent to -1 by the augmentation. (In general, it may not be the case that all base points are sent to -1.)

Consider the type II Reidemeister move which pushes strand k over strand ℓ. We must consider the following when extending ϵ', the augmentation before pushing strand k over strand ℓ, to ϵ, the augmentation of the resulting diagram.

1) We must choose $\epsilon'(e_2)$. In this case, choose $\epsilon'(e_2) = 0$. Thus, equation (2) tells us

$$\epsilon(b^0_{k\ell}) = \epsilon'(e_2) = 0.$$

2) By equation (2),

$$\epsilon(a^0_{k\ell}) = \epsilon'(v_{k\ell}),$$

where

$$\partial b^0_{k\ell} = a^0_{k\ell} + v_{k\ell}.$$

From Figure 14, we know $v_{k\ell}$ is a sum of words in b^0_{ij} for $(i,j) < (k,\ell)$ and contains a 1 if $(k,\ell) = (2r, 2r-1)$ for some $1 \leq r \leq m$. Since

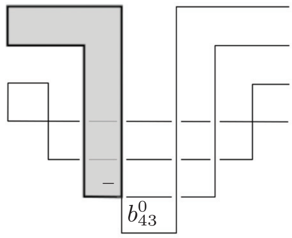

Figure 14: Shaded region gives the disk which contributes 1 to ∂b^0_{43}.

$\epsilon'(b^0_{ij}) = 0$ for all $(i, j) < (k, \ell)$, by step (1), we see that

$$\epsilon(a^0_{k\ell}) = \epsilon'(v_{k\ell}) = \begin{cases} 1 & (k, \ell) = (2r, 2r-1) \text{ for some } 1 \leq r \leq m \\ 0 & \text{otherwise.} \end{cases}$$

3) We must now check whether any "corrections" need to be made to ϵ' to get ϵ. In particular, whether there are any "corrections" which need to be made to ϵ' on the a^0_{ij} generators with $(i, j) < (k, \ell)$ but $i - j \geq k - \ell$. As $\epsilon'(e_2) = 0$, Lemma 2.15 tells us there are no corrections.

We must now check that the resulting augmentation is ρ-graded. Since ϵ' is ρ-graded and ϵ is constructed from ϵ' by a grading preserving Reidemeister II DGA isomorphism, we only need to check that any new crossings which are augmented by ϵ have grading divisible by ρ. We know

$$|b^0_{2r,2r-1}| = \mu(2r) - \mu(2r-1) = (\mu(2r-1) + 1) - \mu(2r-1) = 1$$

for $1 \leq r \leq m$ and so

$$|a^0_{2r,2r-1}| = |b^0_{2r,2r-1}| - 1 = 0$$

for $1 \leq r \leq m$ since ∂ lowers degree by 1. So ϵ is a ρ-graded augmentation satisfying Property (R).

3.2. Extending across original crossings

Consider the crossing c_j, the crossing of strands i and $i + 1$. Let us extend the ruling across the crossing c_j and use ϵ_j, the augmentation of the diagram with dips added up to the crossing c_j, to define ϵ_{j+1}, the diagram with dips added up the crossing c_{j+1}. Note that ϵ_{j+1} will agree with ϵ_j on crossings to the left of the dip added between c_j and c_{j+1}.

First we need to extend the ruling; extend the ruling across c_j by a switch if $\epsilon_j(c_j) \neq 0$ and just to the left of c_j, the ruling so far matches configuration (a), (b), or (c). Otherwise, there is no switch. Let $1 \leq L, K \leq n$ such that strand i is paired with strand L and strand $i+1$ is paired with strand K in the ruling between c_j and c_{j+1}.

We will now construct a dip between between c_j and c_{j+1}, move base points into place, and extend ϵ_j to an augmentation ϵ_{j+1} in the process.

It will be useful to note that Table 1 gives all possibly totally augmented disks in the various configurations of the ruling near crossings, up to base points. Since the way we extend the ruling across c_j depends on $\epsilon_j(c_j)$ and the ruling immediately to the left of c_j, we will need to consider when $\epsilon_j(c_j) = 0$ and $\epsilon_j(c_j) \neq 0$.

(Case 1: $\epsilon_j(c_j) = 0$) In this case, extend the ruling across c_j without a switch. As with adding a dip between the left cusps and c_1, we will compute how the augmentation ϵ' of the diagram before a Reidemeister II move changes to an augmentation ϵ after each move involved in the making the dip. Consider the type II move that pushes strand k over strand ℓ. Let ϵ' be the augmentation on the diagram before the move and let ϵ be the augmentation on the resulting diagram. We will proceed as follows:

1) Define ϵ on the b^j-lattice.
2) Define ϵ on the a^j-lattice.
3) Make corrections to ϵ using Lemma 2.15.
4) Make corrections due to moving base points into place.

Following this process, we have:

1) Choose $\epsilon'(e_2) = 0$.
2) From equation (2), we know

$$\epsilon(a^j_{k\ell}) = \epsilon'(v_{k\ell}).$$

Since neither c_j nor any crossing in the b^j-lattice is augmented, the only totally augmented disks in $v_{k\ell}$ have a positive corner at $b^j_{k\ell}$ and a single augmented negative corner in the a^{j-1}-lattice.

If such a disk exists, by Property (R), the negative corner in the a^{j-1}-lattice must be where two paired strands in the ruling cross as seen in Figure 15. Since this is the only negative corner of the disk, we know k and ℓ are paired in the ruling between c_j and c_{j+1} as well. So,

Table 1: All possible totally augmented disks.

Configuration of c_j	Positive corner	Terms in ∂ corresp. to totally aug. disks up to base pts.
not augmented	b^j_{rs}, r, s paired, $r, s \notin \{i, i+1\}$ $b^j_{\{i,L\}}$ $b^j_{\{i+1,K\}}$	a^{j-1}_{rs}, a^j_{rs} $a^{j-1}_{\{i+1,L\}}, a^j_{\{i,L\}}$ $a^{j-1}_{\{i,K\}}, a^j_{\{i+1,K\}}$
(a)	b^j_{rs}, r, s paired, $r, s \notin \{i, i+1\}$ b^j_{iL} $b^j_{i+1,L}$ b^j_{Ki} $b^j_{K,i+1}$	a^{j-1}_{rs}, a^j_{rs} $c_j a^{j-1}_{iL}, a^j_{iL}$ $a^{j-1}_{iL}, b^j_{i+1,i} a^j_{iL}$ $a^{j-1}_{K,i+1}, a^{j-1}_{K,i+1} c_j b^j_{i+1,i}$ $a^j_{K,i+1} c_j, a^j_{K,i+1}$
(b)	b^j_{rs}, r, s paired, $r, s \notin \{i, i+1\}$ b^j_{iK} $b^j_{i+1,K}$ b^j_{iL} $b^j_{i+1,L}$	a^{j-1}_{rs}, a^j_{rs} $a^{j-1}_{i+1,K}, c_j a^{j-1}_{iL} b^j_{LK}$ $a^{j-1}_{iL} b^j_{LK}, a^j_{i+1,K}$ $c_j a^{j-1}_{iL}, a^j_{iL}$ $a^{j-1}_{iL}, b^j_{i+1,i} a^j_{iL}$
(c)	b^j_{rs}, r, s paired, $r, s \notin \{i, i+1\}$ b^j_{Ki} b^j_{Li} $b^j_{K,i+1}$ $b^j_{L,i+1}$	a^{j-1}_{rs}, a^j_{rs} $a^{j-1}_{K,i+1}, a^{j-1}_{K,i+1} c_j b^j_{i+1,i}$ $a^{j-1}_{Li} b^j_{i+1,i}, a^j_{Li}$ $a^{j-1}_{K,i+1} c_j, a^j_{K,i+1}$ $a^{j-1}_{Li}, b^j_{LK} a^j_{K,i+1}$
(d)	b^j_{rs}, r, s paired, $r, s \notin \{i, i+1\}$ b^j_{iL} $b^j_{K,i+1}$	a^{j-1}_{rs}, a^j_{rs} $a^{j-1}_{i+1,L}, a^j_{iL}$ $a^{j-1}_{Ki}, a^j_{K,i+1}$
(e)	b^j_{rs}, r, s paired, $r, s \notin \{i, i+1\}$ b^j_{iL} b^j_{iK} $b^j_{i+1,K}$	a^{j-1}_{rs}, a^j_{rs} $a^{j-1}_{i+1,L}, a^j_{iL}$ $c_j a^{j-1}_{iK}, a^{j-1}_{i+1,L} b^j_{LK}$ $a^{j-1}_{iK}, a^j_{i+1,K}$
(f)	b^j_{rs}, r, s paired, $r, s \notin \{i, i+1\}$ b^j_{Li} $b^j_{K,i+1}$ $b^j_{L,i+1}$	a^{j-1}_{rs}, a^j_{rs} $a^{j-1}_{L,i+1}, a^j_{Li}$ $a^{j-1}_{Ki}, a^j_{K,i+1}$ $a^{j-1}_{Li} c_j, b^j_{LK} a^j_{K,i+1}$

if we recall that $a^j_{\{k,\ell\}} = a^j_{\max(k,\ell),\min(k,\ell)}$, then

$$\epsilon(a^j_{k\ell}) = \epsilon'(v_{k\ell}) = \epsilon'(\delta(b^j_{k\ell}; a^{j-1}_{k\ell}) a^{j-1}_{k\ell})$$

$$= \begin{cases} \epsilon(a^{j-1}_{\{i,K\}}) & (k,\ell) = \{i+1, K\} \\ \epsilon(a^{j-1}_{\{i+1,L\}}) & (k,\ell) = \{i, L\} \\ \epsilon(a^{j-1}_{k\ell}) & \text{if } k,\ell \text{ paired and } k,\ell \neq i, i+1 \\ 0 & \text{otherwise.} \end{cases}$$

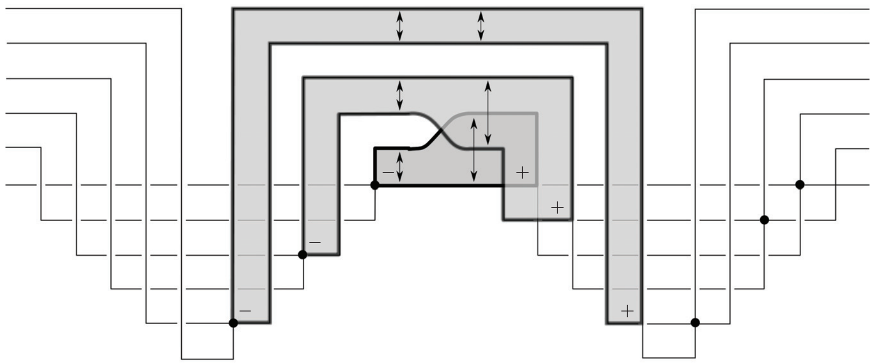

Figure 15: The disks with one negative corner in the a^{k-1}-lattice which contribute terms to the differential of crossings in the b^k-lattice if $\epsilon_j(c_j) = 0$.

3) Since $\epsilon'(e_2) = 0$, by Lemma 2.15, we know there are no "corrections" to $\epsilon(a^j_{rs})$ for $(r,s) < (k,\ell)$.
4) As there are no base points to move into place, no modifications to the augmentation are needed.

We must now check that the resulting augmentation is ρ-graded. Since ϵ' satisfies Property (R), we know $a^{j-1}_{\{i,K\}}$, $a^{j-1}_{\{i+1,L\}}$, and $a^{j-1}_{k\ell}$ are augmented if strands k and ℓ are paired between c_j and c_{j+1}. Thus, if ϵ' is a ρ-graded augmentation, then each has degree divisible by ρ. Since ∂ lowers degree by one,

$$|b^j_{\{i+1,K\}}| = |a^{j-1}_{\{i,K\}}| + 1, \qquad |b^j_{\{i,L\}}| = |a^{j-1}_{\{i+1,L\}}| + 1, \qquad |b^j_{k\ell}| = |a^{j-1}_{k\ell}| + 1$$

and since $|a^j_{rs}| = |b^j_{rs}| - 1$,

$$|a^j_{\{i+1,K\}}| = |a^{j-1}_{\{i,K\}}|, \qquad |a^j_{\{i,L\}}| = |a^{j-1}_{\{i+1,L\}}|, \qquad |a^j_{k\ell}| = |a^{j-1}_{k\ell}|.$$

So ϵ is a ρ-graded augmentation satisfying Property (R) if ϵ' is ρ-graded.

(Case 2: $\epsilon_j(c_j) \neq 0$) Now suppose c_j is augmented. This breaks into six cases, one for each possible configuration of c_j seen in Figure 7. In each case, while creating the dip, we will extend the augmentation ϵ_j of the knot diagram before adding the dip between crossings c_j and c_{j+1} over the dip, move the base points into place and modify the augmentation accordingly to end up with an augmentation ϵ_{j+1} of the modified diagram. As in the case where c_j was not augmented, we will compute how the augmentation changes as we do each Reidemeister II move involved in making a dip between c_j and c_{j+1}.

Configuration (a): By considering Figure 12, we see that if c_j is a negative crossing, we add two base points at the right cusp to the right on strand $i+1$ and move them along strand $i+1$ to between c_j and c_{j+1}, modifying the augmentation on any crossings we push the base points over/under according to Remark 2.14. Note that as we are moving two base points along the same strand, no modification of the augmentation is necessary. If c_j is a positive crossing, we add one base point on strand i and follow the same process, though, in this case, modification of the augmentation by a factor of -1 on the crossings we push the base point over/under is necessary by Remark 2.14. Note that whether c_j is a positive or negative crossing, one base point will be to the left of the dip we are adding, and, if c_j is a negative crossing, we will also have one base point to the right.

Consider the Reidemeister II move where strand k is pushed over strand ℓ. Let ϵ' be the augmentation on the diagram before the move and let ϵ be the augmentation of the diagram after. Note that by our strand labeling convention $L < i < i+1 < K$.

As before, we must consider the following:

$(k, \ell) < (i+1, i)$:

1) Choose $\epsilon'(e_2) = 0$.
2) We know $\epsilon(a^j_{k\ell}) = \epsilon'(v_{k\ell})$. If $k \neq i, i+1$, then Table 1 tells us

$$\epsilon'(v_{k\ell}) = \epsilon'(\delta(b^j_{k\ell}; a^{j-1}_{k\ell})a^{j-1}_{k\ell}).$$

So, in this case, $v_{k\ell}$ has a totally augmented disk if and only if $\epsilon'(a^{j-1}_{k\ell}) \neq 0$ if and only if k and ℓ are paired between c_{j-1} and c_{j+1} by Property (R). Otherwise $(k, \ell) = (i+1, L)$ or $(k, \ell) = (i, L)$. In these cases

$$\begin{aligned}\epsilon(a^j_{iL}) &= \epsilon'(v_{iL}) \\ &= \begin{cases} \epsilon'(\delta(b^j_{iL}; c_j a^{j-1}_{iL})c_j a^{j-1}_{iL}) & c_j \text{ negative crossing} \\ \epsilon'(\delta(b^j_{iL}; c_j a^{j-1}_{iL})t^{\pm 1}_{\alpha} c_j a^{j-1}_{iL}) & c_j \text{ positive crossing} \end{cases} \\ &= \epsilon(c_j a^{j-1}_{iL})\end{aligned}$$

and

$$\begin{aligned}\epsilon(a^j_{i+1,L}) &= \epsilon'(v_{i+1,L})\\ &= \begin{cases}\epsilon'(\delta(b^j_{i+1,L}; a^{j-1}_{iL})t^{\pm 1}_\alpha a^{j-1}_{iL}) & c_j \text{ negative crossing}\\ \epsilon'(\delta(b^j_{i+1,L}; a^{j-1}_{iL})a^{j-1}_{iL}) & c_j \text{ positive crossing}\end{cases}\\ &= \begin{cases}-\epsilon(a^{j-1}_{iL}) & c_j \text{ negative crossing}\\ \epsilon(a^{j-1}_{iL}) & c_j \text{ positive crossing.}\end{cases}\end{aligned}$$

3) Since $\epsilon'(e_2) = 0$ by Lemma 2.15, there are no "corrections" to the augmentation of the previously constructed portion of the a^j-lattice.

4) In the case where c_j is a negative crossing, according to Figure 12, we move a base point over $a^j_{i+1,L}$ to get

$$\epsilon(a^j_{i+1,L}) = \left\{\begin{array}{ll} -(-\epsilon(a^{j-1}_{iL})) & c_j \text{ negative crossing}\\ \epsilon(a^{j-1}_{iL}) & c_j \text{ positive crossing}\end{array}\right\} = \epsilon(a^{j-1}_{iL}).$$

Note that we do not need to move the other base points as they are to the left of the dip and so no more modifications are necessary.

$(k, \ell) = (i+1, i)$:

1) According to Figure 12, choose $\epsilon'(e_2) = (\epsilon(c_j))^{-1}$. Then

$$\epsilon(b^j_{i+1,i}) = \epsilon'(e_2) = (\epsilon(c_j))^{-1}.$$

2) From looking at Table 1, we see that $v_{i+1,i} = 0$ and so

$$\epsilon(a^j_{i+1,i}) = \epsilon'(v_{i+1,i}) = 0.$$

3) As $\epsilon'(e_2) = 1$ we need to check for "corrections." In particular, the disk in Figure 16 contributes the term $a^j_{i+1,i}a^j_{iL}$ to $\partial a^j_{i+1,L}$ and is the only disk with negative corner at $a^j_{i+1,i}$ whose other negative corners are augmented since a^j_{iL} is the only crossing of strand L which is

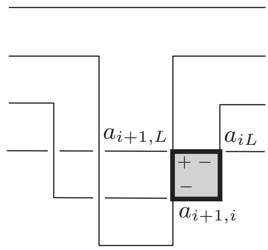

Figure 16: The disk contributing to $\partial a^j_{i+1,L}$, which requires "correcting" the augmentation. Crossings are labeled.

augmented by Property (R). Thus Lemma 2.15 tells us

$$\begin{aligned}
\epsilon(a^j_{i+1,L}) &= \begin{cases} \epsilon'(a^j_{i+1,L}) - (-1)^{|t^{\pm 1}_\alpha|}\delta(a^j_{i+i,L}; a^j_{i+1,i}a^j_{iL})\epsilon(t^{\pm 1}_\alpha b^j_{i+1,i}a^j_{iL}) \\ \hfill c_j \text{ negative crossing} \\ \epsilon'(a^j_{i+1,L}) - (-1)^{|1|}\delta(a^j_{i+i,L}; a^j_{i+1,i}a^j_{iL})\epsilon(b^j_{i+1,i}a^j_{iL}) \\ \hfill c_j \text{ positive crossing} \end{cases} \\
&= \begin{cases} \epsilon(a^{j-1}_{iL}) + \epsilon(t^{\pm 1}_\alpha b^j_{i+1,i}a^j_{iL}) & c_j \text{ negative crossing} \\ \epsilon(a^{j-1}_{iL}) - \epsilon(b^j_{i+1,i}a^j_{iL}) & c_j \text{ positive crossing} \end{cases} \\
&= \epsilon(a^{j-1}_{iL}) - (\epsilon(c_j))^{-1}\epsilon(c_j a^{j-1}_{iL}) \\
&= 0,
\end{aligned}$$

where t_α is associated with the base point $*$, since

$$\delta(a^j_{i+1,L}; a^j_{i+1,i}a^j_{iL}) = \begin{cases} -1 & c_j \text{ negative crossing} \\ 1 & c_j \text{ positive crossing.} \end{cases}$$

Thus ϵ satisfies Property (R).

4) By Remark 2.14, moving a base point over $a^j_{i+1,i}$ will not change the augmentation since $\epsilon(a^j_{i+1,i}) = 0$ in the case where c_j is a negative crossing.

$(k,\ell) > (i+1,i)$:

1) According to Figure 12, choose $\epsilon'(e_2) = 0$.
2) As before, if neither strands k nor ℓ is a crossing strand, then $a^j_{k\ell}$ is augmented if and only if k and ℓ are paired in the ruling between c_j

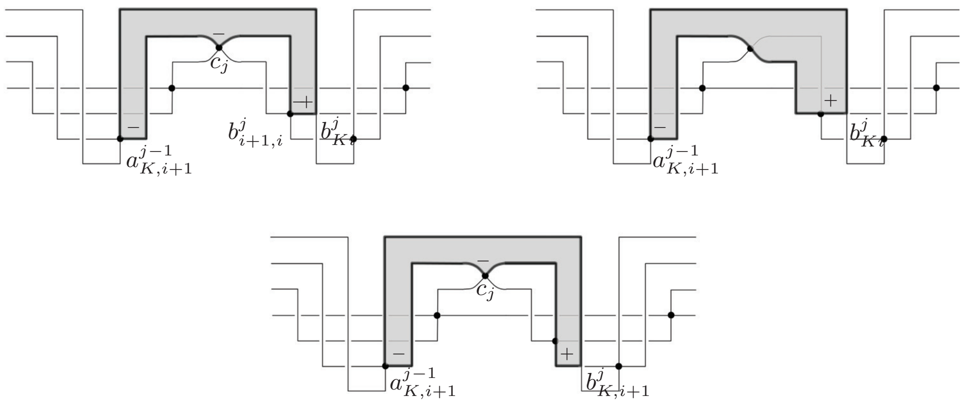

Figure 17: Totally augmented disks with one negative corner in the a^{j-1}-lattice contributing to the differential of crossings in the b^j-lattice. The crossings at corners of the disks are labeled.

and c_{j+1}. Note that this tells us the augmentation on the a^j-lattice is the same as the a^{j-1}-lattice. We do, however, see in Figure 17 that there is one totally augmented disk in $v_{K,i+1}$ and two in v_{Ki}.

Thus

$$\begin{aligned}
\epsilon(a^j_{Ki}) &= \epsilon'(v_{Ki}) \\
&= \begin{cases} \epsilon'(\delta(b^j_{Ki};a^{j-1}_{K,i+1})a^{j-1}_{K,i+1}) + \epsilon'(\delta(b^j_{Ki};a^{j-1}_{K,i+1}c_jb^j_{i+1,i})a^{j-1}_{K,i+1}c_jt^{\pm1}_\alpha b^j_{i+1,i}) \\ \hfill c_j \text{ negative crossing} \\ \epsilon'(\delta(b^j_{Ki};a^{j-1}_{K,i+1})a^{j-1}_{K,i+1}t^{\pm1}_\alpha) + \epsilon'(\delta(b^j_{Ki};a^{j-1}_{K,i+1}c_jb^j_{i+1,i})a^{j-1}_{K,i+1}c_jb^j_{i+1,i}) \\ \hfill c_j \text{ positive crossing} \end{cases} \\
&= \begin{cases} \epsilon(a^{j-1}_{K,i+1}) - \epsilon(a^{j-1}_{K,i+1}c_jb^j_{i+1,i}) & c_j \text{ negative crossing} \\ -\epsilon(a^{j-1}_{K,i+1}) + \epsilon(a^{j-1}_{K,i+1}c_jb^j_{i+1,i}) & c_j \text{ positive crossing} \end{cases} \\
&= 0,
\end{aligned}$$

since

$$\epsilon(a^j_{K,i+1}c_jb^j_{i+1,i}) = \epsilon(a^{j-1}_{K,i+1}c_j)(\epsilon(c_j))^{-1} = \epsilon(a^{j-1}_{K,i+1}).$$

And,

$$\begin{aligned}\epsilon(a^j_{K,i+1}) &= \epsilon'(v_{K,i+1})\\ &= \begin{cases}\epsilon'(\delta(b^j_{K,i+1}; a^{j-1}_{K,i+1}c_j)a^{j-1}_{K,i+1}c_j t_\alpha^{\pm1}t_\beta^{\pm1}) & c_j \text{ negative crossing}\\ \epsilon'(\delta(b^j_{K,i+1}; a^{j-1}_{K,i+1}c_j)a^{j-1}_{K,i+1}c_j) & c_j \text{ positive crossing}\end{cases}\\ &= \epsilon(a^{j-1}_{K,i+1}c_j).\end{aligned}$$

3) Since $\epsilon'(e_2) = 0$, by Lemma 2.15, no "corrections."

4) By Figure 12, no base points to move.

If ϵ' is a ρ-graded augmentation, then $\rho \big| |c_j|$ since c_j is augmented. Thus, the ruling is ρ-graded so far. Since ϵ is constructed from ϵ' by a grading-preserving Reidemeister II DGA isomorphisms and ϵ' is ρ-graded, we only need to check that a few crossings have degree divisible by ρ. We see that $|b^j_{i+1,i}| = \mu(i+1) - \mu(i) = |c_j|$ and, since ∂ lowers degree by one,

$$\begin{aligned}|a^j_{K,i+1}| &= |b^j_{K,i+1}| - 1 = |a^{j-1}_{K,i+1}|\\ |a^j_{iL}| &= |b^j_{iL}| - 1 = |a^{j-1}_{iL}|.\end{aligned}$$

So ϵ is ρ-graded.

As in the nonaugmentated case, if strands k and ℓ are paired in the ruling between c_{j-1} and c_{j+1}, then $a^{j-1}_{k\ell}$ is augmented and $|a^j_{k\ell}| = |a^{j-1}_{k\ell}|$. So ϵ is a ρ-graded augmentation which satisfies Property (R).

Configuration (b): Now suppose the ruling has configuration (b) near c_j. Note that with our strand assignments $i+1 > i > L > K$. According to Figure 12, if c_j is a negative crossing, then follow strand K to the right to a right cusp and add a base point and follow strand $i+1$ to the right to a right cusp and add two base points. Move these base points back along their respective strands to between c_j and c_{j+1}, modifying the augmentation according to Remark 2.14. If c_j is a positive crossing, then follow strand i to the right to a right cusp, add a base point, and move it back to between c_j and c_{j+1}, modifying the augmentation as necessary.

As before, we will compute how the augmentation ϵ_j changes as we complete Reidemeister II moves involved in the construction of a dip, to yield the extended augmentation ϵ_{j+1}.

Consider the augmentation ϵ extension of the augmentation ϵ' where strand k is pushed over strand ℓ in the creation of a dip between c_j and c_{j+1}.

$(k,\ell)<(L,K)$: This case follows in the way of the first case of configuration (a) so that setting $\epsilon'(e_2)=0$, we transfer the augmentation on the a^{j-1}-lattice to that a^j-lattice.

$(k,\ell)=(L,K)$:

1) According to Figure 12, set $\epsilon'(e_2)=(\epsilon(c_j a_{iL}^{j-1}))^{-1}\epsilon(a_{i+1,K}^{j-1})$ to obtain

$$\epsilon(b_{LK}^j)=\epsilon'(e_2)=(\epsilon(c_j a_{iL}^{j-1}))^{-1}\epsilon(a_{i+1,K}^{j-1}).$$

2) We see that $\epsilon'(v_{LK})=0$, since K and L are neither paired nor crossing strands in the ruling between c_j and c_{j+1}. Thus

$$\epsilon(a_{LK}^j)=\epsilon'(v_{LK})=0.$$

3) There are no "corrections" as any disk in the a_j-lattice with negative corner at a_{LK}^j must have an augmented negative corner of the form a_{L*}^j, but strand L is paired with strand i in the ruling between c_j and c_{j+1}, so the only such crossing has not been made in the dip yet.

4) No base points to move, so no corrections.

$(L,K)<(k,\ell)<(i+1,i)$:

1) According to Figure 12, set $\epsilon'(e_2)=0$.

2) In Figure 18, we see all the totally augmented disks contributing to $v_{k\ell}$ in $\partial b_{k\ell}^j$.
Therefore

$$\begin{aligned}\epsilon(a_{iK}^j)&=\epsilon'(v_{iK})\\&=\begin{cases}\epsilon'(\delta(b_{iK}^j;a_{i+1,K}^{j-1})a_{i+1,K}^{j-1}t_\alpha^{\pm1})+\epsilon'(\delta(b_{iK}^j;c_j a_{iL}^{j-1}b_{LK}^j)c_j a_{iL}^{j-1}b_{LK}^j) & c_j\text{ negative crossing}\\ \epsilon'(\delta(b_{iK}^j;a_{i+1,K}^{j-1})a_{i+1,K}^{j-1})+\epsilon'(\delta(b_{iK}^j;c_j a_{iL}^{j-1}b_{LK}^j)c_j a_{iL}^{j-1}b_{LK}^j) & c_j\text{ positive crossing}\end{cases}\\&=\left\{\begin{array}{ll}-\epsilon(a_{i+1,K}^{j-1})+\epsilon(c_j a_{iL}^{j-1}b_{LK}^j) & c_j\text{ negative crossing}\\-\epsilon(a_{i+1,K}^{j-1})-\epsilon(c_j a_{iL}^{j-1}b_{LK}^j) & c_j\text{ positive crossing}\end{array}\right\}=0,\end{aligned}$$

since

$$\epsilon(c_j a_{iL}^{j-1}b_{LK}^j)=\epsilon(c_j a_{iL}^{j-1})(\epsilon(c_j a_{iL}^{j-1}))^{-1}\epsilon(a_{i+1,K}^{j-1})=\epsilon(a_{i+1,K}^{j-1}).$$

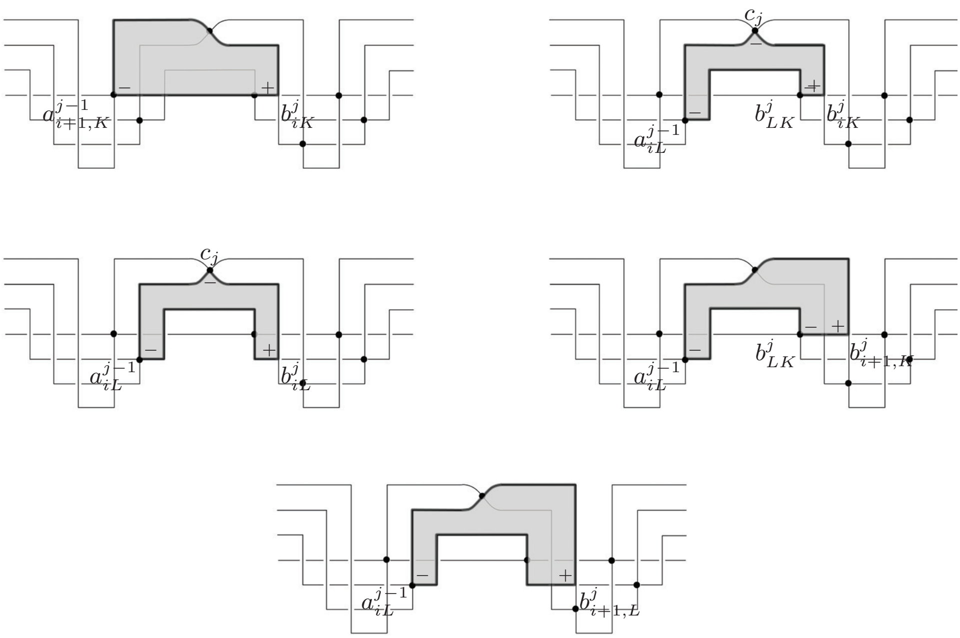

Figure 18: Totally augmented disks with one negative corner in the b^{j-1}-lattice which contribute to the differential of a crossing in the a^j-lattice. All crossings at corners of disks are labeled.

We also have

$$\begin{aligned}
\epsilon(a^j_{iL}) &= \epsilon'(v_{iK}) = \epsilon'(\delta(b^j_{iL}; c_j a^{j-1}_{iL}) c_j a^{j-1}_{iL}) \\
&= \begin{cases} \epsilon(c_j a^{j-1}_{iL}) & c_j \text{ negative crossing} \\ -\epsilon(c_j a^{j-1}_{iL}) & c_j \text{ positive crossing,} \end{cases} \\
\epsilon(a^j_{i+1,K}) &= \epsilon'(v_{i+1,K}) = \epsilon'(\delta(b^j_{i+1,K}; a^{j-1}_{iL} b^j_{LK}) c_j a^{j-1}_{iL}) \\
&= \epsilon(a^{j-1}_{iL} b^j_{LK}) \\
&= \epsilon(a^{j-1}_{iL})(\epsilon(c_j a^{j-1}_{iL}))^{-1} \epsilon(a^j_{i+1,K}), \\
\epsilon(a^j_{i+1,L}) &= \epsilon'(v_{i+1,L}) = \epsilon'(\delta(b^j_{i+1,L}; a^{j-1}_{iL}) a^{j-1}_{iL}) = \epsilon(a^{j-1}_{iL}).
\end{aligned}$$

3) Since $\epsilon'(e_2) = 0$, by Lemma 2.15, there are no "corrections."

4) Note that if c_j is a negative crossing, according to Figure 12, we need to move two base points over $a^j_{i+1,K}$ and $a^j_{i+1,L}$, so no changes. However, if c_j is a positive crossing, then we need to move one base point over

a^j_{iK} and a^j_{iL} to give $\epsilon(a^j_{iK}) = 0$ and

$$\epsilon(a^j_{iL}) = \left\{ \begin{array}{ll} \epsilon(c_j a^{j-1}_{iL}) & c_j \text{ negative crossing} \\ -(-\epsilon(c_j a^{j-1}_{iL})) & c_j \text{ positive crossing} \end{array} \right\} = \epsilon(c_j a^{j-1}_{iL}).$$

$(k, \ell) = (i, i+1)$:

1) According to Figure 12, set $\epsilon'(e_2) = (\epsilon(c_j))^{-1}$ and so $\epsilon(b^j_{i+1,i}) = (\epsilon(c_j))^{-1}$.
2) As before, $\epsilon(a^j_{i+1,i}) = \epsilon'(v_{i+1,i}) = \epsilon'(0) = 0$.
3) We do have one correction: the disk $a^j_{i+1,i} a^j_{iL}$ in $\partial a^j_{i+1,L}$. Lemma 2.15 tells us

$$\begin{aligned} \epsilon(a^j_{i+1,L}) &= \begin{cases} \epsilon'(a^j_{i+1,L}) - (-1)^{|t^{\pm 1}_\alpha|} \delta(a^j_{i+1,L}; a^j_{i+1,i} a^j_{iL}) \epsilon(t^{\pm 1}_\alpha b^j_{i+1,i} a^j_{iL}) \\ \hfill c_j \text{ negative crossing} \\ \epsilon'(a^j_{i+1,L}) - (-1)^{|1|} \delta(a^j_{i+1,L}; a^j_{i+1,i} a^j_{iL}) \epsilon(b^j_{i+1,i} a^j_{iL}) \\ \hfill c_j \text{ positive crossing} \end{cases} \\ &= \begin{cases} \epsilon(a^{j-1}_{iL}) + \epsilon(t^{\pm 1}_\alpha b^j_{i+1,i} a^j_{iL}) & c_j \text{ negative crossing} \\ \epsilon(a^{j-1}_{iL}) - \epsilon(b^j_{i+1,i} a^j_{iL}) & c_j \text{ positive crossing} \end{cases} \\ &= 0, \end{aligned}$$

since

$$\epsilon(b^j_{i+1,i} a^j_{iL}) = (\epsilon(c_j))^{-1} \epsilon(c_j a^{j-1}_{iL}) = \epsilon(a^{j-1}_{iL}).$$

4) As $\epsilon(a^j_{i+1,i}) = 0$, no corrections are needed when moving the base point $*_\alpha$ over $a^j_{i+1,i}$.

Since

$$\begin{aligned} |b^j_{i+1,i}| &= \mu(i+1) - \mu(i) = |c_j| \\ |b^j_{LK}| &= \mu(L) - \mu(K) \\ &= \mu(L) - \mu(i) + \mu(i) - \mu(i+1) + \mu(i+1) - \mu(K) \\ &= -|a^{j-1}_{iL}| - |c_j| + |a^{j-1}_{i+1,K}|, \end{aligned}$$

$a^{j-1}_{iL}, a^{j-1}_{i+1,K}$, and c_j are augmented by ϵ', and ϵ' is ρ-graded, we know ϵ is ρ-graded.

Since ϵ' satisfies Property (R) on the a^{j-1}-lattice, we know ϵ is a ρ-graded augmentation which satisfies Property (R). In fact, ϵ is just $a^j_{i+1,i}$, a^j_{iL}

augmented with the rest of the augmentation on the a^j-lattice transferred from the a^{j-1}-lattice.

Configuration (c), (d), (e), (f): Similarly, one can extend ϵ_j over a crossing c_j with the ruling having configuration (c), (d), (e), or (f) near c_j to an augmentation ϵ_{j+1} satisfying Property (R) by defining it on new crossings as specified in Figure 12. We omit the calculations.

3.3. Right cusps

By construction and Lemma 2.15, $\epsilon = \epsilon_n$ is an augmentation. In this section, we will show that we do in fact have a ruling. Recall that $q_1, \dots, q_m$ are the crossings at the right cusps numbered from top to top. Then

$$\partial q_k = t_k^{\pm 1} + a^n_{2m-2k+2,2m-2k+1}$$

for $1 \leq k \leq m$, where the power of t_k is determined by the orientation of the knot at the right cusp, since strands $2m-2k+2$ and $2m-2k+1$ are incident to the k-th right cusps from the bottom. Since ϵ is an augmentation,

$$\begin{aligned} 0 &= \epsilon \partial q_k \\ &= \epsilon(t_k^{\pm 1} + a^n_{2m-2k+2,2m-2k+1}) \\ &= (\epsilon(t_k))^{\pm 1} + \epsilon(a^n_{2m-2k+2,2m-2k+1}). \end{aligned}$$

Since $0 \neq \epsilon(t) = \prod_{i=1}^{s} \epsilon(t_i)$,

$$\epsilon(a^n_{2m-2k+2,2m-2k+1}) = -(\epsilon(t_k))^{\pm 1} \neq 0.$$

Since ϵ satisfies Property (R), this tells us strands $2m-2k+2$ and $2m-2k+1$ are paired at the right cusps for all $1 \leq k \leq m$ and so this construction does give a ruling.

This concludes the proof of the forward direction of Theorem 1.1. This construction also gives restrictions on $\epsilon(t)$ for any augmentation ϵ. In particular, the final statement in Theorem 1.1:

Theorem 3.1. *If ρ is even with $\rho \big| 2r(K)$, then any ρ-graded augmentation ϵ satisfies $\epsilon(t) = -1$.*

Proof. Consider the associated ρ-graded ruling. If ρ is even, then any ρ-graded ruling is only switched at crossings c_k with $\rho \big| |c_k|$ and so $2 \big| |c_k|$. Thus any paired strands in the ruling have opposite orientation. If strand i is

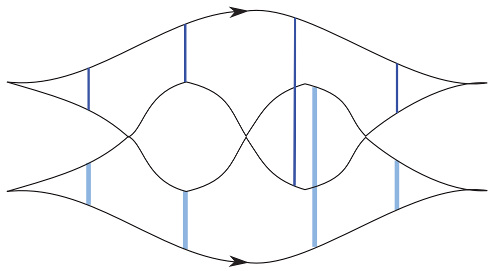

Figure 19: Oriented right handed trefoil with a normal graded ruling indicated.

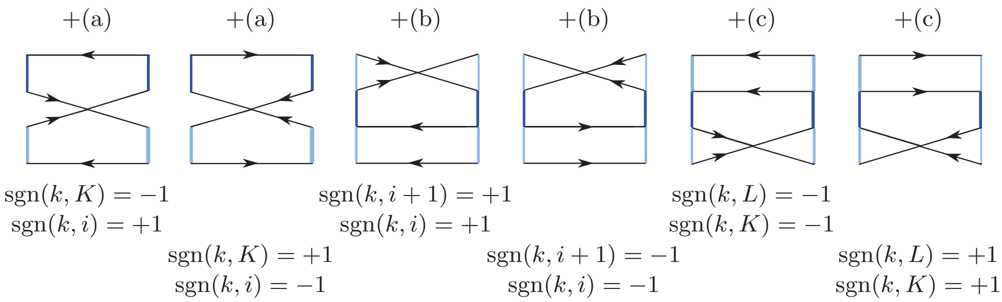

Figure 20: All possible ruling configurations and orientations near a crossing which is switched in a ρ-graded normal ruling when ρ is even with signs of ruled pairs given with our strand labeling convention.

oriented to the right, we assign that portion of the ruling, the sign $+1$ and if it is instead oriented to the left, we assign -1. Define $\mathrm{sgn}(i,k)$ to be the sign for strands $i > j$ paired in the ruling between c_k and c_{k+1}. Note that this sign can only change going over a switched crossing.

For example, if we have the trefoil with the orientation given in Figure 19, then

k	0	0	1	1	2	2	3	3
i	4	2	4	2	4	3	4	2
$\mathrm{sgn}(i,k)$	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1

Given a ρ-graded ruling with ρ even, we also see that we cannot have switched crossings which are negative crossings. So all switched crossings have one of the configurations appearing in Figure 20.

Note that in these switch configurations the signs of ruling pairs do not change. Thus, each ruling path is an oriented unknot. The important part of this is that if a ruling pair has sign +1, respectively −1, at the left cusp, then it has sign +1, respectively −1, at the right cusp.

We will show that for any k such that $0 \leq k \leq n$,

$$\prod (\epsilon(a^k_{rs}))^{\operatorname{sgn}(r,k)} = 1, \tag{3}$$

where the product is taken over all paired strands r and s in the ruling between c_k and c_{k+1}.

Clearly this is true for $k = 0$. Induct on k. Suppose equation (3) is true for $k - 1$. We will show that equation (3) holds for k. If the ruling is not switched at c_k, then the result is clear. If c_k has configuration type $+(a)$, then, by Figure 12,

$$\epsilon(a^k_{rs}) = \begin{cases} \epsilon(c_k)\epsilon(a^{k-1}_{K,i+1}) & (r,s) = (K, i+1) \\ \epsilon(c_k)\epsilon(a^{k-1}_{iL}) & (r,s) = (i, L) \\ \epsilon(a^{k-1}_{rs}) & \text{otherwise} \end{cases}$$

and

$$\operatorname{sgn}(K,k) = -\operatorname{sgn}(i,k), \quad \operatorname{sgn}(r,k) = \operatorname{sgn}(r,k-1)$$

for all strands r and s paired in the ruling between c_k and c_{k+1}. Thus

$$\begin{aligned}
&\prod_{r,s} (\epsilon(a^k_{rs}))^{\operatorname{sgn}(r,k)} \\
&= (\epsilon(c_k)\epsilon(a^{k-1}_{K,i+1}))^{\operatorname{sgn}(K,k)} (\epsilon(c_k)\epsilon(a^{k-1}_{iL}))^{\operatorname{sgn}(i,k)} \\
&\quad \cdot \prod_{(r,s)\neq(K,i+1),(i,L)} (\epsilon(a^{k-1}_{rs}))^{\operatorname{sgn}(r,k)} \\
&= (\epsilon(c_k))^{-\operatorname{sgn}(i,k)} (\epsilon(a^{k-1}_{K,i+1}))^{\operatorname{sgn}(K,k)} (\epsilon(c_k))^{\operatorname{sgn}(i,k)} (\epsilon(a^{k-1}_{iL}))^{\operatorname{sgn}(i,k)} \\
&\quad \cdot \prod_{(r,s)\neq(K,i+1),(i,L)} (\epsilon(a^{k-1}_{rs}))^{\operatorname{sgn}(r,k-1)} \\
&= (\epsilon(a^{k-1}_{K,i+1}))^{\operatorname{sgn}(K,k-1)} (\epsilon(a^{k-1}_{iL}))^{\operatorname{sgn}(i,k-1)} \prod_{(r,s)\neq(K,i+1),(i,L)} (\epsilon(a^{k-1}_{rs}))^{\operatorname{sgn}(r,k-1)} \\
&= \prod_{r,s} (\epsilon(a^{k-1}_{rs}))^{\operatorname{sgn}(r,k-1)} \\
&= 1.
\end{aligned}$$

Similarly, we can see the same is true if c_k has configuration $+(b)$ or $+(c)$ since $\operatorname{sgn}(r,k) = \operatorname{sgn}(r,k-1)$ for all strands r and s which are paired in the ruling between c_k and c_{k+1}.

In particular, the result is true for $k = n$. Since

$$\partial q_\ell = t_\ell^{\mathrm{sgn}(2m-2\ell+2,2m-2\ell+1)} + a^n_{2m-2\ell+2,2m-2\ell+1},$$

we know

$$0 = \epsilon \partial q_\ell = (\epsilon(t_\ell))^{\mathrm{sgn}(2m-2\ell+2,n)} + \epsilon(a^n_{2m-2\ell+2,2m-2\ell+1})$$

for all $1 \leq \ell \leq m$. Thus

$$\epsilon(t_\ell) = -(\epsilon(a^n_{2m-2\ell+2,2m-2\ell+1}))^{\mathrm{sgn}(2m-2\ell+2,n)}$$

and so, if s is the number of base points, then

$$\begin{aligned}\epsilon(t) = \prod_{\ell=1}^{s} \epsilon(t_\ell) &= (-1)^{s-m} \prod_{\ell=1}^{m} \left(-(\epsilon(a^n_{2m-2\ell+2,2m-2\ell+1}))^{\mathrm{sgn}(2m-2\ell+2,n)} \right) \\ &= (-1)^s \prod_{\ell=1}^{m} (\epsilon(a^n_{2m-2\ell+2,2m-2\ell+1}))^{\mathrm{sgn}(2m-2\ell+2,n)} \\ &= (-1)^s \\ &= -1\end{aligned}$$

as by Lemma 3.2 we know we have an odd number of base points. □

Recall that we add an even number of base points if a crossing c_k has configuration (d), (e), (f), or not augmented, two for each $-$(a) crossing, an odd number for each $+$(a), $\pm$(b), $\pm$(c), and one for each right cusp. Thus, to show there are an odd number of base points, it suffices to show the following: (The following argument was communicated to the author by Lenhard Ng.)

Lemma 3.2. *If c gives the number of right cusps, s is the number of switches in the ruling, and a_- is the number of $-$(a) crossings, then*

$$c + s + a_- \equiv 1 \mod 2.$$

Proof. We will prove this result by showing each of the following statements:

(4) $$tb + r \equiv \#\text{ components} \mod 2$$
(5) $$tb \equiv c + cr \mod 2$$
(6) $$cr \equiv s \mod 2$$
(7) $$r \equiv a_- \mod 2$$

where r is the rotation number and cr is the number of crossings. Note that if we add these four equations together, we get that

$$c + s + a_- \equiv \# \text{ components} \mod 2.$$

Since in our case we have a knot, this gives the desired result.

Statement 4 is a standard result. Statement 5 results from the fact that the Thurston-Bennequin number is the number of right cusps plus the number of crossings counted with sign. To prove statement 6, we will count the number of interlaced pairs from left to right.

We say that two pairs of points are **interlaced** if we encounter the pairs alternately as we move vertically. In other words, if a_i denotes one pair of companion strands and b_i denotes the other, then they appear from top to bottom as $a_1 b_1 a_2 b_2$.

Note that the number of interlaced pairs does not change as we go from left to right over a switched crossing and changes by ± 1 as we go from left to right over a nonswitched crossing. We also know that we have zero interlaced pairs at the left and right cusps. Thus, the number of nonswitched crossings, which is equal to the number of crossings minus the number of switched crossings, is even, which gives

$$cr \equiv s \mod 2.$$

The proof of statement 7 will be a little more involved. First, at any vertical segment of the dipped diagram which does not include a crossing, if r and s ($r > s$) are paired, assign the pair the number 0 if they are oriented the same way and $\operatorname{sgn}(r, k)$ as defined in Theorem 3.1 otherwise. To any such vertical slice of the diagram, associate the sum of these numbers over the ruled pairs in that slice. For example, Figure 21 gives the assignments for the given ruling of the left handed trefoil.

One can check that this count goes up by ± 2 as you go over a $-(a)$ crossing and otherwise does not change. At the left cusps, we compute the sum to be $u_L - d_L$, where u_L is the number of up cusps and d_L the number of down. At the right cusps, we compute the sum to be $d_R - u_R$, where u_R and d_R are defined analogously. Therefore we have

$$\begin{aligned} (d_R - u_R) &\equiv (u_L - d_L) + 2a_- \mod 4, \\ 2r &\equiv 2a_- \mod 4, \\ r &\equiv a_- \mod 2. \end{aligned}$$

□

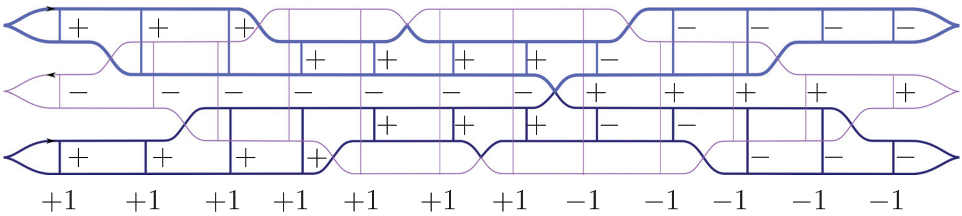

Figure 21: The diagram gives a normal ruling of the left handed trefoil. At each vertical slice of the diagram, the paired strands in the ruling are decorated with $+/-$, denoting the assignment of $+1/-1$ to the corresponding paired strands. The number below the vertical slice gives the assigned sum over the ruled pairs.

The augmentation variety is more complicated when ρ is odd. Given a ρ-graded augmentation to a field F, once again, consider the associated ρ-graded ruling.

Remark 3.3. Since

$$\partial q_\ell = t_\ell^{\mathrm{sgn}(2m-2\ell+2,2m-2\ell+1)} + a^n_{2m-2\ell+2,2m-2\ell+1},$$

we know

$$\epsilon(t_\ell) = -(\epsilon(a^n_{2m-2\ell+2,2m-2\ell+1}))^{\mathrm{sgn}(2m-2\ell+2,n)} = -x_\ell^2\epsilon(a^n_{2m-2\ell+2,2m-2\ell+1}),$$

where

$$x_\ell = \begin{cases} 1 & \text{if } \mathrm{sgn}(2m-2\ell+2,n)=1 \\ \epsilon(a^n_{2m-2\ell+2,2m-2\ell+1}) & \text{if } \mathrm{sgn}(2m-2\ell+2,n)=-1. \end{cases}$$

If s is the number of base points, then

$$\begin{aligned} \epsilon(t) = \prod_{\ell=1}^{s}\epsilon(t_\ell) &= (-1)^{s-m}\prod_{\ell=1}^{m}\left(-x_\ell^2\epsilon(a^n_{2m-2\ell+2,2m-2\ell+1})\right) \\ &= (-1)^s x^2\prod_{\ell=1}^{m}\epsilon(a^n_{2m-2\ell+2,2m-2\ell+1}) \\ &= -x^2, \end{aligned}$$

for $x=\prod x_\ell \in F^*$, since there are an odd number of base points by Lemma 3.2.

It is clear that

$$1 = \prod_{r,s \text{ paired}} \epsilon(a^0_{rs}).$$

Looking at the various configurations for the switched crossings c_k (see Figure 12), we see that

$$\frac{\prod_{r,s \text{ paired}}(\epsilon(a^{i+1}_{rs}))}{\prod_{r,s \text{ paired}}(\epsilon(a^{i}_{rs}))} = \begin{cases} (\epsilon(c_k))^2 & \text{if the ruling has configuration (a) near } c_k \\ 1 & \text{otherwise} \end{cases}$$

for $1 \leq i < n$. So

$$\prod_{k=1}^{m} \epsilon(a^n_{2k,2k-1}) = \prod_{r,s \text{ paired}} \epsilon(a^n_{rs}) = x^2$$

for some $x \in F^*$. Therefore there exists $x \in F^*$ such that

$$\epsilon(t) = \prod_{k=1}^{s} \epsilon(t_k) = -x^2.$$

The following theorem, restated from the introduction, gives a slightly more precise result for when there exists a ρ-graded normal ruling for the diagram which is not oriented, meaning a ruling for which not all ruling strands are oriented unknots.

Theorem 1.2. If ρ is odd and $\rho | 2r(\Lambda)$, then

$$\mathrm{Aug}_\rho(\Lambda) = \begin{cases} \{-x^2 : x \in F^*\} & \text{if there exists a } \rho\text{-graded normal ruling of } \Lambda \\ & \text{which is not oriented} \\ \{-1\} & \text{if there exists a } \rho\text{-graded normal ruling of } \Lambda \\ & \text{and all rulings are oriented} \\ \emptyset & \text{if there are no } \rho\text{-graded normal rulings of } \Lambda. \end{cases}$$

Proof. Suppose there exists a ρ-graded normal ruling for Λ which is not oriented. Fix $0 \neq x \in F$. Since every ruling is oriented on the portion at the left cusps, for it to be an unoriented ruling, there has to be a crossing which takes the ruling from oriented to unoriented going from left to right. The only configurations for the ruling which do this are the crossings with configuration $-$(a), $-$(b), or $-$(c). Thus, a normal ruling of Λ is not oriented if and only if it has a crossing with configuration $-$(a), $-$(b), or $-$(c). In fact, any ruling is also oriented at the right cusps and so must have at least two crossings where the ruling has configuration −(a),−(b), or −(c).

Consider Λ from the last crossing with configuration −(a),−(b), or −(c), which we will denote c_k, to the right cusps. Note that any crossing with configuration +(a), +(b), +(c), ±(d), ±(e), ±(f), or not switched preserves the orientation of the paired strands in the ruling. In other words, whatever orientation the strands in the ruling have just to the right c_k is the orientation they have all the way through to the right cusps. Let $\sigma \in S_{2m}$ be the permutation of the strands so that if strands r and s with $r > s$ are paired in the ruling immediately to the right of the crossing c_k, then strand $\sigma(r)$ is the strand with higher label and $\sigma(s)$ is the strand with lower label if we follow the ruled pair to the right cusps. (Note that $\sigma(r) = \sigma(s) + 1$ and $2|\sigma(r)$.)

As in the ρ even case, set the orientation $\operatorname{sgn}(r, j) = 1$ if strand r is oriented to the right immediately after crossing c_j and $\operatorname{sgn}(r, j) = -1$ if strand r is oriented to the left for $k \le j \le n$. Labeling strands as before, this gives us

$$\begin{aligned}\text{(8)}\qquad &\operatorname{sgn}(\max(i+1, K), k)\operatorname{sgn}(\max(i, L), k)\\ &= \begin{cases} +1 & \text{if } c_k \text{ has configuration } -\text{(a)}\\ -1 & \text{if } c_k \text{ has configuration } -\text{(b) or } -\text{(c).}\end{cases}\end{aligned}$$

Set $\ell_r = m + 1 - \frac{\sigma(r)}{2}$. Note that ℓ is chosen so that $\sigma(r)$ and $\sigma(s)$ are the strands crossing at q_ℓ. Thus

$$\partial q_{\ell_r} = t_{\ell_r}^{\operatorname{sgn}(r,k)} + a^n_{\sigma(r),\sigma(s)}$$

and so

$$\epsilon(t_{\ell_r}) = -(\epsilon(a^n_{\sigma(r),\sigma(s)}))^{\operatorname{sgn}(r,k)}$$

since $\operatorname{sgn}(r, k) = \pm 1$.

Define ϵ, an augmentation to F of the DGA $(\mathcal{A}(\Lambda'), \partial)$ of the dipped diagram Λ' of Λ, satisfying Property (R), by

$$\epsilon(c_j) = \begin{cases} x^{\operatorname{sgn}(K,k)} & j = k \text{ and } c_j \text{ has configuration } -\text{(a)}, -\text{(c)}\\ x^{\operatorname{sgn}(i,k)} & j = k \text{ and } c_j \text{ has configuration } -\text{(b)}\\ 1 & \text{if the ruling is switched at } c_j \text{ and } j \neq k\\ 0 & \text{otherwise.}\end{cases}$$

Note that Property (R) tells us that

$$\epsilon(a^k_{rs}) = \epsilon(a^n_{\sigma(r),\sigma(s)})$$

for all strands r and s paired in the ruling between c_k and c_{k+1}. We also note that ϵ must be a ρ-graded augmentation, since it was defined using a ρ-graded normal ruling.

We see that if c_j has configuration $-$(a), then

$$\epsilon(a^n_{\sigma(r),\sigma(s)}) = \begin{cases} x^{\operatorname{sgn}(K,k)} & (r,s) = (K, i+1) \\ 1 & r, s \text{ paired in ruling} \\ 0 & \text{otherwise.} \end{cases}$$

If c_j has configuration $-$(b), then

$$\epsilon(a^n_{\sigma(r),\sigma(s)}) = \begin{cases} x^{\operatorname{sgn}(i,k)} & (r,s) = (i, L) \\ x^{-\operatorname{sgn}(i,k)} & (r,s) = (i+1, K) \\ 1 & r, s \text{ paired in ruling} \\ 0 & \text{otherwise.} \end{cases}$$

If c_j has configuration $-$(c), then

$$\epsilon(a^n_{\sigma(r),\sigma(s)}) = \begin{cases} x^{\operatorname{sgn}(K,k)} & (r,s) = (K, i+1) \\ x^{-\operatorname{sgn}(K,k)} & (r,s) = (L, i) \\ 1 & r, s \text{ paired in ruling} \\ 0 & \text{otherwise.} \end{cases}$$

We know

$$\begin{aligned} \epsilon(t) = \prod_{j=1}^{s} \epsilon(t_j) &= (-1)^{s-m} \prod_{j=1}^{m} \epsilon(t_{m-j+1}) \\ &= (-1)^{s-m} \prod_{j=1}^{m} (-(\epsilon(a^n_{2j,2j-1}))^{\operatorname{sgn}(2j,n)}) \end{aligned}$$

$$
\begin{aligned}
&= (-1)^s (\epsilon(a^n_{\{\sigma(K),\sigma(i+1)\}}))^{\operatorname{sgn}(\max(K,i+1),k)} (\epsilon(a^n_{\{\sigma(i),\sigma(L)\}}))^{\operatorname{sgn}(\max(i,L),k)} \\
&\quad \cdot \prod_{\substack{j=1 \\ j\neq m-\ell_{\max(K,i+1)}+1, m-\ell_{\max(L,i)}+1}}^{m} (\epsilon(a^n_{2j,2j-1}))^{\operatorname{sgn}(2j,n)} \\
&= -(\epsilon(a^n_{\{\sigma(K),\sigma(i+1)\}}))^{\operatorname{sgn}(\max(K,i+1),k)} (\epsilon(a^n_{\{\sigma(i),\sigma(L)\}}))^{\operatorname{sgn}(\max(i,L),k)} \\
&= \begin{cases} -(x^{\operatorname{sgn}(K,k)})^{\operatorname{sgn}(K,k)} (x^{\operatorname{sgn}(K,k)})^{\operatorname{sgn}(i,k)} & c_k \text{ has configuration } -(\text{a}) \\ -(x^{-\operatorname{sgn}(i,k)})^{\operatorname{sgn}(i+1,k)} (x^{\operatorname{sgn}(i,k)})^{\operatorname{sgn}(i,k)} & c_k \text{ has configuration } -(\text{b}) \\ -(x^{\operatorname{sgn}(K,k)})^{\operatorname{sgn}(K,k)} (x^{-\operatorname{sgn}(K,k)})^{\operatorname{sgn}(L,k)} & c_k \text{ has configuration } -(\text{c}) \end{cases} \\
&= -x^2
\end{aligned}
$$

by equation (8).

By Remark 3.3,

$$\operatorname{Aug}_\rho(\Lambda) \subset \{-x^2 : x \in F^*\},$$

so $\operatorname{Aug}_\rho(\Lambda) = \{-x^2 : x \in F^*\}$.

Now suppose there exists a ρ-graded normal ruling for Λ and all ρ-graded normal rulings of Λ are oriented. In this case, the ruling must only have switched crossings with configuration +(a), +(b), +(c), (d), (e), or (f). Note that the proof of Theorem 3.1 only required this be the case for the ruling, so the augmentation associated to the normal ruling must have $\epsilon(t) = -1$ and so $\operatorname{Aug}_\rho(\Lambda) = \{-1\}$.

If there do not exist any ρ-graded rulings for Λ, then clearly $\operatorname{Aug}_\rho(\Lambda) = \emptyset$. ☐

4. Ruling to augmentation

To show the backward direction of Theorem 1.1, that given a ρ-graded normal ruling of a front diagram of a Legendrian knot, we can find a ρ-graded augmentation of $\mathcal{A}$, it suffices to show that given a ρ-graded normal ruling of a front diagram, there exists a ρ-graded augmentation ϵ of the dipped diagram. We will do this by, in some sense, following the same argument as previously, but backwards. This includes the condition that the augmentation of the dipped diagram satisfies Property (R).

In particular, we will be able to find an augmentation ϵ of the dipped diagram satisfying Property (R) for which, if a crossing c_k is augmented, $\epsilon(c_k) = 1$ and such that $\epsilon(t_1 \cdots t_s) = -1$ where $*_1, \ldots, *_s$ are the base points in the final diagram.

4.1. Definition of augmentation

As previously, we can assume the base point $*$ corresponding to t is in the loop of the top right cusp. We can then add one base point to each right cusp. We will set $\epsilon(t_i) = -1$ $(1 \leq i \leq m)$, this will also be true for the base points added subsequently. Note that we will not need to do any of the "correction" calculations for disks and base points as we are defining the map this way.

4.1.1. Left. For any ruling, at the left end of the diagram, we have strand $2k$ paired with $2k-1$ for $1 \leq k \leq m$, where m is the number of right cusps. For ϵ to satisfy Property (R), we must set

$$\epsilon(b^0_{rs}) = 0$$

for all k and ℓ and

$$\epsilon(a^0_{rs}) = \begin{cases} 1 & \text{there exists } k \text{ s. t. } r = 2k, s = 2k-1, 1 \leq k \leq m \\ 0 & \text{otherwise.} \end{cases}$$

4.1.2. Original crossings. Consider a crossing c_j. If the ruling is switched at c_j, set $\epsilon(c_j) = 1$. If not, set $\epsilon(c_j) = 0$. (Note that we can augment the switched crossings to any nonzero element of F and still get an augmentation, but we may end up with an augmentation where $\epsilon(t) \neq -1$.)

Add base points and augment crossings in the dips, following Figure 12.

4.2. Properties of the augmentation

By the proof that augmentations imply rulings, ϵ is an augmentation and by the following, the resulting augmentation ϵ on the original undipped diagram with one base point $*$ associated to t satisfies $\epsilon(t) = -1$.

Since we have set $\epsilon(t_i) = -1$ for all $1 \leq i \leq s$ and Lemma 3.2 tells us s is odd,

$$\epsilon(t) = \prod_{i=1}^{s} \epsilon(t_i) = (-1)^s = -1.$$

5. Lifting augmentations

Given an augmentation to $\mathbb{Z}/2$ of the Chekanov-Eliashberg DGA over $\mathbb{Z}/2$. We will now use constructions similar to those in the proof of Theorem 1.1

to construct a lift of the augmentation to an augmentation to $\mathbb{Z}$ of the lift of the Chekanov-Eliashberg DGA and thus that one can construct an augmentation to any ring. Restating from the introduction:

Theorem 1.3. Let Λ be a Legendrian knot in $\mathbb{R}^3$. Let $(\mathcal{A}_{\mathbf{z}/2}, \partial)$ be the Chekanov-Eliashberg DGA over $\mathbb{Z}/2$ and let $(\mathcal{A}, \partial)$ be the DGA over $R = \mathbb{Z}[t, t^{-1}]$. If $\epsilon' : \mathcal{A}_{\mathbf{z}/2} \to \mathbb{Z}/2$ is an augmentation of $(\mathcal{A}_{\mathbf{z}/2}, \partial)$, then one can find a lift of ϵ' to an augmentation $\epsilon : \mathcal{A} \to \mathbb{Z}$ of $(\mathcal{A}, \partial)$ such that $\epsilon(t) = -1$.

Proof. Recall that $\mathcal{E}_i = \mathcal{A}(e_1, e_2)$ where $|e_1| = i - 1$, $|e_2| = i$, $\partial(e_2) = e_1$, and $\partial(e_1) = 0$ and $S_i(\mathcal{A}_R(a_1, \ldots, a_n)) = \mathcal{A}_R(a_1, \ldots, a_n, e_1, e_2)$.

Note that, for any augmentation ϵ on $\mathcal{A}_R$ to $\mathbb{Z}$, there exists an augmentation $\widehat{\epsilon}$ on $S(\mathcal{A}_R)$ to $\mathbb{Z}$ which agrees with ϵ on $\mathcal{A}_R \subset S(\mathcal{A}_r)$ and for any augmentation $\widehat{\epsilon}$ on $S(\mathcal{A}_R)$ to $\mathbb{Z}$, there exists an augmentation ϵ on $\mathcal{A}_R$ to $\mathbb{Z}$ which agrees with ϵ on $\mathcal{A}_R \subset S(\mathcal{A}_R)$. And, we have the analogous result for any augmentation of $\mathcal{A}_{\mathbf{z}/2}$. Thus, clearly one can find a lift $\epsilon : \mathcal{A}_R \to \mathbb{Z}$ of $\epsilon' : \mathcal{A}_{\mathbf{z}/2} \to \mathbb{Z}/2$ if and only if one can find a lift $\epsilon : S(\mathcal{A}_R) \to \mathbb{Z}$ of $\epsilon : S(\mathcal{A}_{\mathbf{z}/2}) \to \mathbb{Z}/2$.

So, if there exists a lift for $\mathcal{A}$, then there exists a lift for any stable tame isomorphic differential graded algebra. Therefore, to show the result, it suffices to show there exists a lift of the augmentation to $\mathbb{Z}/2$ of differential graded algebras of knots in plat position. So we may assume Λ is in plat position.

Given an augmentation $\epsilon' : \mathcal{A}_{\mathbf{z}/2} \to \mathbb{Z}/2$ of the Chekanov-Eliashberg DGA over $\mathbb{Z}/2$. Using Lemma 2.15 modulo 2 and the definition given in Figure 12 mod 2, we can extend ϵ' to an augmentation $\widehat{\epsilon} : \widehat{\mathcal{A}_{\mathbf{z}/2}} \to \mathbb{Z}/2$ of the DGA over $\mathbb{Z}/2$ for the dipped diagram of Λ. We saw that if we know $\widehat{\epsilon}(c_i)$ and the augmentation on the a^i/b^i-lattices for $i < j$, then

$$\widehat{\epsilon}(c_j) \equiv \epsilon'(c_j) + \sum_{i=0}^{j-1} \sum_{k,\ell} \sum_{p} \widehat{\epsilon}(Q_p b^i_{k\ell} R_p) \mod 2$$

where, for $0 \le i < j$, $\partial b^i_{k\ell} = P + \sum_p Q_p a^i_{k\ell} R_p$ before passing strand k over strand ℓ in the creation of the dip between c_i and c_{i+1} and P is the sum of terms which do not contain $a^i_{k\ell}$ with our labeling convention. This is the same as the construction introduced in [17]. From [17] we know that this augmentation satisfies Property (R).

Let $(\widetilde{\mathcal{A}_Z}, \widetilde{\partial})$ be the lift of the Chekanov-Eliashberg DGA over $\mathbb{Z}/2$ to a DGA over $Z = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ of the DGA over $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$ of the dipped diagram of Λ. Define $\widetilde{\epsilon}: \widetilde{\mathcal{A}_Z} \to \mathbb{Z}$ by

$$\widetilde{\epsilon}(c_j) = \begin{cases} 1 & \text{if } \widehat{\epsilon}(c_j) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

on the original crossings, define $\widetilde{\epsilon}$ as given by Figure 12 for all other crossings, add base points where indicated in Figure 12, and define

$$\widetilde{\epsilon}(t_i) = \begin{cases} -\widetilde{\epsilon}(a^n_{2m-2i+2,2m-2i+1}) & \text{if } 1 \leq i \leq m \\ -1 & \text{if } m < i \leq s. \end{cases}$$

Note that all crossings and base points are augmented to 0 or ± 1. One can check that with this definition, $\widetilde{\epsilon}$ is an augmentation of the dipped diagram of Λ. Note that as the same original crossings are augmented in the dipped diagram, this augmentation must correspond to the same ruling as $\widehat{\epsilon}$ and by definition, satisfies Property (R). So, clearly,

$$\widetilde{\epsilon}(c) \equiv \widehat{\epsilon}(c) \mod 2$$

for all crossings c in the dipped diagram of Λ.

We will use induction on k to show that

$$\prod \widetilde{\epsilon}(a^k_{pq}) = 1,$$

where the product is taken over all paired strands p and q, for all $1 \leq k \leq n$ and thus, that

$$\prod_{i=1}^{s} \widetilde{\epsilon}(t_i) = -1.$$

Since $\widetilde{\epsilon}(a^0_{pq}) = 1$ for $(p,q) = (2m-2k+2, 2m-2k+1)$ for some k such that $1 \leq k \leq m$, we know

$$\prod_{p,q} \widetilde{\epsilon}(a^0_{pq}) = 1.$$

Looking at Figure 12, we see that

$$\frac{\prod_{p,q} \widetilde{\epsilon}(a^k_{pq})}{\prod_{p,q} \widetilde{\epsilon}(a^{k-1}_{pq})} = \left\{ \begin{array}{ll} (\widetilde{\epsilon}(c_{k-1}))^2 & \text{if the ruling has config. (a) near } c_{k-1} \\ 1 & \text{otherwise} \end{array} \right\} = 1,$$

since $\widetilde{\epsilon}(c_{k-1}) = \pm 1$. Thus, if $\prod \widetilde{\epsilon}(a_{pq}^{k-1}) = 1$, then $\prod \widetilde{\epsilon}(a_{pq}^{k}) = 1$. So, in particular, $\prod \widetilde{\epsilon}(a_{pq}^{n}) = 1$. Thus

$$\begin{aligned}\prod_{i=1}^{s} \widetilde{\epsilon}(t_i) &= (-1)^{s-m} \prod_{i=1}^{m} \widetilde{\epsilon}(t_i) = (-1)^{s-m} \prod_{i=1}^{m} (-\widetilde{\epsilon}(a^n_{2m-2i+2,2m-2i+1})) \\ &= (-1)^s \prod_{i=1}^{m} \widetilde{\epsilon}(a^n_{2m-2i+2,2m-2i+1}) = (-1)^s = -1,\end{aligned}$$

since Lemma 3.2 tells s is odd.

Lemma 2.15 in its original form also gives us a method to define an augmentation of the original diagram from an augmentation of the dipped diagram of Λ. Thus we have the augmentation $\epsilon : \mathcal{A}_Z \to \mathbb{Z}$ of the original diagram, defined by

$$\epsilon(c_j) = \widetilde{\epsilon}(c_j) + \sum_{i=0}^{j-1} \sum_{k,\ell} \sum_{p} \epsilon(b^i_{k\ell}; Q'_p a^i_{k\ell} R'_p)(-1)^{|\Phi(Q'_p)|} \widetilde{\epsilon}(Q'_p b^i_{k\ell} R'_p)$$

where, for $0 \le i < j$, $\partial b^i_{k\ell} = P + \sum_p \epsilon(b^i_{k\ell}; Q'_p a^i_{k\ell} R'_p) Q'_p a^i_{k\ell} R'_p$ before passing strand k over strand ℓ in the creation of the dip between c_i and c_{i+1} and P is the sum of terms which do not contain $a^i_{k\ell}$ with our labeling convention. Note that the "correction" disks in the $\mathbb{Z}/2$ case are the same as the "correction" disks in the $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ case, but the $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ "correction" disks may be counted with negative sign and the disk may have extra corners at base points. Recall that $\widetilde{\epsilon}(t_i) = -1$ for $m < i \le s$. Thus

$$\widetilde{\epsilon}(Q'_p) \equiv \widetilde{\epsilon}(Q_p) \mod 2, \qquad \widetilde{\epsilon}(R'_p) \equiv \widetilde{\epsilon}(R_p) \mod 2,$$

since the disk which contributes Q'_p (resp. R'_p) to the differential may have extra corners at base points t_i for $m < i \le s$ (base points not occurring at right cusps) which the disk which contributes Q_p (resp. R_p) to the differential does not have.

We will now show that ϵ is, in fact, a lift of ϵ'.

$$\begin{aligned}\epsilon(c_j) &= \widetilde{\epsilon}(c_j) + \sum_{i=0}^{j-1} \sum_{k,\ell} \sum_{p} \delta(b^i_{k\ell}; Q'_p a^i_{k\ell} R'_p)(-1)^{|\Phi(Q'_p)|} \widetilde{\epsilon}(Q'_p b^i_{k\ell} R'_p) \\ &\equiv \widetilde{\epsilon}(c_j) + \sum_{i=0}^{j-1} \sum_{k,\ell} \sum_{p} \widetilde{\epsilon}(Q_p b^i_{k\ell} R_p) \mod 2\end{aligned}$$

$$\equiv \left(\epsilon'(c_j) + \sum_{i=0}^{j-1}\sum_{k,\ell}\sum_{p}\widehat{\epsilon}(Q_p b^i_{k\ell} R_p)\right) + \sum_{i=0}^{j-1}\sum_{k,\ell}\sum_{p}\widetilde{\epsilon}(Q_p b^i_{k\ell} R_p) \mod 2$$

$$\equiv \epsilon'(c_j) + 2\sum_{i=0}^{j-1}\sum_{k,\ell}\sum_{p}\widehat{\epsilon}(Q_p b^i_{k\ell} R_p) \mod 2$$

$$\equiv \epsilon'(c_j) \mod 2,$$

since $\widetilde{\epsilon}$ is a lift of $\widehat{\epsilon}$. Note that this shows that the resulting augmentation of the DGA over $\mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ is a lift and so, by the discussion of moving and adding base points in §2.5, the augmentation of the DGA over $\mathbb{Z}[t, t^{-1}]$ is a lift as well, and

$$\epsilon(t) = \prod_{i=1}^{s} \epsilon(t_i) = -1.$$

And, since there is a unital homomorphism from $\mathbb{Z}$ to any unital ring S, we can also use ϵ' to define an augmentation $\epsilon : \mathcal{A} \to S$ with $\epsilon(t) = -1$. □

References

[1] Mina Aganagic, Tobias Ekholm, Lenhard Ng, and Cumrun Vafa, *Topolocial strings, D-model, and knot contact homology*, Adv. Theor. Math. Phys. **18** (2014), no. 4, 827–956.

[2] Daniel Bennequin, *Entrelacements et équations de Pfaff*, In Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982), volume 107 of Astérisque, pages 87–161, Soc. Math. France, Paris, 1983.

[3] Yu. V. Chekanov and P. E. Pushkar′, *Combinatorics of fronts of Legendrian links, and Arnol′d's 4-conjectures*, Uspekhi Mat. Nauk **60** (2005), no. 1(361), 99–154.

[4] Yuri Chekanov, *Differential algebra of Legendrian links*, Invent. Math. **150** (2002), no. 3, 441–483.

[5] Ya. M. Eliashberg, *A theorem on the structure of wave fronts and its application in symplectic topology*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 3, 65–72, 96.

[6] Yakov Eliashberg, *Invariants in contact topology*, in Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), number Extra Vol. II, pages 327–338, 1998.

[7] Yakov Eliashberg and Maia Fraser, *Classification of topologically trivial Legendrian knots*, in Geometry, topology, and dynamics (Montreal, PQ, 1995), volume 15 of CRM Proc. Lecture Notes, pages 17–51. Amer. Math. Soc., Providence, RI, 1998.

[8] John B. Etnyre, *Legendrian and transversal knots*, in Handbook of knot theory, pages 105–185. Elsevier B. V., Amsterdam, 2005.

[9] John B. Etnyre and Ko Honda, *Knots and contact geometry. I. Torus knots and the figure eight knot*, J. Symplectic Geom. **1** (2001), no. 1, 63–120.

[10] John B. Etnyre, Lenhard L. Ng, and Joshua M. Sabloff, *Invariants of Legendrian knots and coherent orientations*, J. Symplectic Geom. **1** (2002), no. 2, 321–367.

[11] Dmitry Fuchs, *Chekanov-Eliashberg invariant of Legendrian knots: existence of augmentations*, J. Geom. Phys. **47** (2003), no. 1, 43–65.

[12] Dmitry Fuchs and Tigran Ishkhanov, *Invariants of Legendrian knots and decompositions of front diagrams*, Mosc. Math. J. **4** (2004), no. 3, 707–717, 783.

[13] Michael B. Henry and Dan Rutherford, *Ruling polynomials and augmentations over finite fields*, Journal of Topology **8** (2015), no. 1, 1–37.

[14] Lenhard Ng, *Framed knot contact homology*, Duke Math. J. **141** (2008), no. 2, 365–406.

[15] Lenhard Ng and Daniel Rutherford, *Satellites of Legendrian knots and representations of the Chekanov–Eliashberg algebra*, Algebr. Geom. Topol. **13** (2013), no. 5, 3047–3097.

[16] Lenhard L. Ng, *Computable Legendrian invariants*, Topology **42** (2003), no. 1, 55–82.

[17] Joshua M. Sabloff, *Augmentations and rulings of Legendrian knots*, Int. Math. Res. Not. **19** (2005), 1157–1180.

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