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Banach-Lie groupoids associated to W^*-algebras

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We investigate the Banach Lie groupoids naturally associated to W^*-algebras. We also present statements describing the relationship between these groupoids and the Banach Poisson geometry which follows in the canonical way from the W^*-algebra structure.

1. Introduction

During recent decades the notion of groupoid entered many branches of mathematics including topology [4], differential geometry in general [12] and Poisson geometry [5], [6] in particular as well as the theory of operator algebras [15]. Let us recall shortly that a groupoid is a small category all of whose morphisms are invertible. In accordance with [12] they are "the natural formulation of a symmetry for objects which have bundle structure". Nevertheless the role of groupoids is not so widely accepted as that of groups. On the other hand the theory of W^*-algebras (von Neumann algebras) occupies an outstanding place in mathematics and mathematical physics [17], [19].

Motivated by the existence of the canonically defined Banach Lie-Poisson structure on the predual $\mathfrak{M}_*$ of any W^*-algebra $\mathfrak{M}$, see [13], and by the importance of this structure in the theory of infinite dimensional Hamiltonian systems, see [14], we clarify here some natural connections between Banach Poisson geometry and groupoid theory from one side and W^*-algebras from the other.

In Section 2 we show that the structure of any W^*-algebra $\mathfrak{M}$ naturally defines two important groupoids $\mathcal{U}(\mathfrak{M})$ and $\mathcal{G}(\mathfrak{M})$ the first of which consists of the partial isometries in $\mathfrak{M}$ and the second, being the "complexification" of $\mathcal{U}(\mathfrak{M})$, consists of the partially invertible elements of $\mathfrak{M}$. In this section we also discuss canonical actions of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ on the lattice of projections $\mathcal{L}(\mathfrak{M})$ and on the cone $\mathfrak{M}_*^+$ of the positive normal states on $\mathfrak{M}$. In Theorem 2.16 we show that the action groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ has a faithful representation on the GNS bundle $\overline{\mathbb{E}} \to \mathfrak{M}_*^+$. Theorem 2.17 shows that one can consider $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ as a subgroupoid of the groupoid of partial isometries $\mathcal{U}(\overline{\rho}(\mathfrak{M})')$ for the commutant $\overline{\rho}(\mathfrak{M})'$ of the W^*-representation $\overline{\rho} : \mathfrak{M} \to L^\infty(L^2\Gamma(\mathbb{E}, \mathfrak{M}_*^+))$ of $\mathfrak{M}$ in the Hilbert space of the square-summable sections for the bundle $\overline{\mathbb{E}} \to \mathfrak{M}_*^+$.

The topology of groupoids $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ are described in Section 3. We show here that $\mathcal{G}(\mathfrak{M})$ is not a topological groupoid with respect to any natural topology of $\mathfrak{M}$. However the groupoid $\mathcal{U}(\mathfrak{M})$ is a topological groupoid with respect to the uniform topology as well as the $s^*(\mathcal{U}(\mathfrak{M}), \mathfrak{M}_*)$-topology. Theorem 3.3 describes the topological structure of the action groupoids related to $\mathcal{U}(\mathfrak{M})$.

In Section 4 we investigate the complex Banach manifold structure on the lattice $\mathcal{L}(\mathfrak{M})$ and the groupoid $\mathcal{G}(\mathfrak{M})$ and show that $\mathcal{G}(\mathfrak{M})$ is a Banach-Lie groupoid with $\mathcal{L}(\mathfrak{M})$ as its base manifold, see Theorem 4.5. The last statement is also true for the groupoid $\mathcal{U}(\mathfrak{M})$ when we consider it as a real Banach manifold, see Theorem 4.6.

In Section 5 we present several, essential in the present context, statements describing relationship between groupoids $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ and the canonical Poisson structure defined on the Banach vector bundles $\mathcal{A}_*\mathcal{G}(\mathfrak{M})$ and $\mathcal{A}_*\mathcal{U}(\mathfrak{M})$ predual to the algebroids $\mathcal{A}\mathcal{G}(\mathfrak{M})$ and $\mathcal{A}\mathcal{U}(\mathfrak{M})$ of Banach-Lie groupoids $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$, respectively. From that we conclude that in the framework of W^*-algebras theory there exists a natural illustration of the deep ideas connecting finite dimensional Poisson geometry and Lie groupoids theory which was investigated in [6], [9], [12], [21], [22].

2. Groupoids associated to W^*-algebras and their representations

In this section we introduce various groupoids defined in a canonical way by the given W^*-algebra $\mathfrak{M}$. We will also describe representations of these groupoids on vector bundles related to the algebra $\mathfrak{M}$ as well as to its predual $\mathfrak{M}_*$. The basic facts from the theory of groupoids can be found in the appendix to this paper. The detailed presentation of the groupoid theory can be found in [12]. The part of the theory of W^*-algebras indispensable for the subsequent investigations is given in [17] and [19].

2.1. Groupoid $\mathcal{G}(\mathfrak{M})$ of partially invertible elements of a W^*-algebra $\mathfrak{M}$

By $\mathcal{U}(\mathfrak{M})$ we shall denote the set of all partial isometries in $\mathfrak{M}$, i.e. $u \in \mathcal{U}(\mathfrak{M})$ if and only if $u^*u \in \mathcal{L}(\mathfrak{M})$, where $\mathcal{L}(\mathfrak{M})$ is the lattice of orthogonal projections $p = p^* = p^2 \in \mathfrak{M}$. Note that the condition $u^*u \in \mathcal{L}(\mathfrak{M})$ is equivalent to the following conditions $uu^* \in \mathcal{L}(\mathfrak{M})$, $uu^*u = u$ and $u^*uu^* = u^*$, e.g. see Corollary 1.1.9 in [17]. We also recall that $p \leqslant q$ if and only if $pq = p$ and $\mathcal{L}(\mathfrak{M})$ is a complete lattice under the order $\leqslant$. More facts about $\mathcal{L}(\mathfrak{M})$ can be found in Section 1.10 in [17], see also Appendix A.

For $x \in \mathfrak{M}$ one defines the left support $l(x) \in \mathcal{L}(\mathfrak{M})$ (respectively right support $r(x) \in \mathcal{L}(\mathfrak{M})$) as the smallest projection in $\mathfrak{M}$ such that $l(x)x = x$ (respectively $x\ r(x) = x$). If $x = x^*$ then $l(x) = r(x) =: s(x)$ and one calls $s(x)$ the support of x. Let

$$x = u|x|, \tag{2.1}$$

where

$$u^*u = s(|x|),$$

be the polar decomposition of x, where $u \in \mathcal{U}(\mathfrak{M})$ and $|x| \in \mathfrak{M}^+ := \{x \in \mathfrak{M} : \ x^* = x > 0\}$, see Theorem 1.12.1 in [17] . Then one has

$$\begin{aligned} l(x) = s(|x^*|) = uu^*, \\ r(x) = s(|x|) = u^*u. \end{aligned} \tag{2.2}$$

In this paper we will denote by $G(p\mathfrak{M}p)$ the group of all invertible elements of the W^*-subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$. In particular if $p = \mathbf{1}$ then $G(\mathfrak{M})$ will be the group of all invertible elements of $\mathfrak{M}$ and if $p = 0$ then $G(p\mathfrak{M}p) = \{0\}$. Similarly by $U(p\mathfrak{M}p)$ and $U(\mathfrak{M})$ we will denote the groups of unitary elements of $p\mathfrak{M}p$ and $\mathfrak{M}$.

For any $x \in \mathfrak{M}$ one has $|x| \in p\mathfrak{M}p$, where $p = s(|x|)$. Let us define the subset $\mathcal{G}(\mathfrak{M}) \subset \mathfrak{M}$ by

$$\mathcal{G}(\mathfrak{M}) := \{x \in \mathfrak{M} : \ |x| \in G(p\mathfrak{M}p), \text{ where } p = s(|x|)\}. \tag{2.3}$$

Equivalently, x belongs to $\mathcal{G}(\mathfrak{M})$ if the left multiplication $L_{|x|}$ by $|x|$ defines a right $\mathfrak{M}$-module isomorphism

$$L_{|x|} : p\mathfrak{M} \ \tilde{\to} \ p\mathfrak{M} \tag{2.4}$$

of the right ideal $p\mathfrak{M}$.

Proposition 2.1. *The subset $\mathcal{G}(\mathfrak{M}) \subset \mathfrak{M}$ has a canonical structure of a groupoid with $\mathcal{L}(\mathfrak{M})$ as the base set. The groupoid structure of $\mathcal{G}(\mathfrak{M})$ is defined as follows:*

(i) the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ is the inclusion;

(ii) the source and target maps: $\boldsymbol{s}, \boldsymbol{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ are defined by

$$\boldsymbol{s}(x) := r(x) = u^*u \quad \text{and} \quad \boldsymbol{t}(x) := l(x) = uu^*; \tag{2.5}$$

(iii) the product

$$\mathcal{G}(\mathfrak{M})^{(2)} \ \ni \ (x, y) \mapsto \ xy \ \in \mathcal{G}(\mathfrak{M}) \tag{2.6}$$

on the set of composable pairs

$$\mathcal{G}(\mathfrak{M})^{(2)} := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M}); \quad \boldsymbol{s}(x) = \boldsymbol{t}(y)\}$$

is given by the product in the W^*-algebra $\mathfrak{M}$;

(iv) the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ is given by

$$\iota(x) := |x|^{-1}u^*, \tag{2.7}$$

where u and $|x|$ are defined in the unique way by the polar decomposition (2.1).

Proof. Since $\varepsilon : \mathcal{L}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ is inclusion the maps $\mathbf{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ and $\mathbf{s} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ are surjective.

From $x\mathbf{s}(x) = x$ one has $yx\mathbf{s}(x) = yx$. This gives $\mathbf{s}(yx) \leqslant \mathbf{s}(x)$, where "$\leqslant$" means lattice order in $\mathcal{L}(\mathfrak{M})$. Using $r(y) = \iota(y)y = l(x)$ and $\iota(y)yx\ \mathbf{s}(yx) = \iota(y)yx$ we obtain $x\ \mathbf{s}(yx) = x$. Thus we have $\mathbf{s}(x) \leqslant \mathbf{s}(yx)$. This shows that $r(yx) = \mathbf{s}(yx) = \mathbf{s}(x) = r(x)$. In a similar way we show that $l(yx) = l(y)$.

The associativity of the product (2.6) follows from the associativity of the algebra product.

Using (2.1) and (2.2) we get

$$\begin{aligned} \iota(x)x &= \varepsilon(\mathbf{s}(x)), \\ x\iota(x) &= \varepsilon(\mathbf{t}(x)), \\ \mathbf{s}(x) &= \mathbf{t}(\iota(x)), \\ \mathbf{t}(x) &= \mathbf{s}(\iota(x)) \end{aligned} \tag{2.8}$$

for $x \in \mathcal{G}(\mathfrak{M})$. The above proves the groupoid structure of $\mathcal{G}(\mathfrak{M})$. □

From now on we will call $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ the groupoid of partially invertible elements of the W^*-algebra $\mathfrak{M}$.

A groupoid $G \rightrightarrows B$ is called a transitive groupoid if and only if for any elements $b_1, b_2 \in B$ there exists $x \in G$ such that $\mathbf{s}(x) = b_1$ and $\mathbf{t}(x) = b_2$.

Remark 2.2. The groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is not transitive.

It is better to see this in the case when $\mathfrak{M} = Mat_{N\times N}(\mathbf{C}) \cong End(\mathbf{C}^N)$. Let $\mathcal{L}_n(End(\mathbf{C}^N)) \subset End(\mathbf{C}^N)$ denote the subset of projections of rank n, where $0 \leqslant n \leqslant N$. The groupoid $\mathcal{G}(End(\mathbf{C}^N)) \rightrightarrows \mathcal{L}(End(\mathbf{C}^N))$ is the disjoint union of transitive groupoids $\mathcal{G}_n(End(\mathbf{C}^N)) \rightrightarrows \mathcal{L}_n(End(\mathbf{C}^N))$, where

$$\mathcal{G}_n(End(\mathbf{C}^N)) := \mathbf{s}^{-1}(\mathcal{L}_n(End(\mathbf{C}^N))) \cap \mathbf{t}^{-1}(\mathcal{L}_n(End(\mathbf{C}^N))).$$

The component of this union corresponding to $n = 0$ is the groupoid (one element group) $\{0\} \rightrightarrows \{0\}$. For $n = N$ one has $\mathcal{G}_N(End(\mathbf{C}^N)) \rightrightarrows \{\mathbf{1}\}$, i.e. the groupoid reduces to the group $GL(N, \mathbf{C})$.

Remark 2.3. The groupoid $\mathcal{G}_n(End(\mathbf{C}^N)) \rightrightarrows \mathcal{L}_n(End(\mathbf{C}^N))$ is a gauge groupoid of the Stiefel principal bundle $\pi_n : Stief_n(n, \mathbf{C}^N) \to Grass(n, \mathbf{C}^N) \cong \mathcal{L}(End(\mathbf{C}^N))$ over Grassmannian of n-dimensional subspaces in $\mathbf{C}^N$.

For the definition of the gauge groupoid see e.g. [12].

Remark 2.4. For the finite dimension case we have $\mathcal{G}(\mathfrak{M}) = \mathfrak{M}$. However, it is not true in general.

In order to see the above fact we take the W^*-algebra of bounded operators $L^\infty(L^2(\mathbb{R}, dt))$ on the Hilbert space $L^2(\mathbb{R}, dt)$. As an example of $x \in L^\infty(L^2(\mathbb{R}, dt))$ such that $l(x) = r(x) = \mathbf{1}$ we can take the operator

$$(M_{|\sin|}\psi)(t) := |\sin t|\psi(t),$$

of multiplication by the function $|\sin|$. The inverse $M^{-1}_{|\sin|}$ of $M_{|\sin|}$ is an unbounded operator. So, $M^{-1}_{|\sin|} \notin L^\infty(L^2(\mathbb{R}, dt))$, and thus we find that $M_{|\sin|} \notin \mathcal{G}(L^\infty(L^2(\mathbb{R}, dt)))$.

2.2. Groupoid $\mathcal{U}(\mathfrak{M})$ of partial isometries of a W^*-algebra $\mathfrak{M}$

We show that the set of partially isometries $\mathcal{U}(\mathfrak{M})$ has a structure of a subgroupoid of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partially invertible elements of the W^*-algebra $\mathfrak{M}$.

Proposition 2.5. *The set $\mathcal{U}(\mathfrak{M})$ of partial isometries in $\mathfrak{M}$ is a wide subgroupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.*

Proof. Since

$$|x^*| = u|x|u^* \tag{2.9}$$

we see that the groupoid $\mathcal{G}(\mathfrak{M})$ is invariant with respect to $*$-involution. Thus from the definition of the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ follows that the involution $J : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ defined by

$$J(x) := \iota(x)^* = \iota(x^*) \tag{2.10}$$

is an automorphism of the groupoid $\mathcal{G}(\mathfrak{M})$. We note also that the set of fixed points of $J : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$, i.e.

$$\{u \in \mathcal{G}(\mathfrak{M}) : \quad J(u) = u\} \tag{2.11}$$

is the set $\mathcal{U}(\mathfrak{M})$ of all partial isometries of the W^*-algebra $\mathfrak{M}$. Assuming sources and target maps for $\mathcal{U}(\mathfrak{M})$ as in (2.5) we immediately see that $\mathcal{U}(\mathfrak{M})$ is closed with respect the groupoid product (2.6) and the inverse map $\mathcal{G}(\mathfrak{M})$ defined in (2.7). So one has the groupoid structure $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on $\mathcal{U}(\mathfrak{M})$ with $\mathcal{L}(\mathfrak{M})$ as the base set. □

2.3. Inner groupoid $\mathcal{J}(\mathfrak{M})$ and inner action

The other important wide subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is the inner subgroupoid $\mathcal{J}(\mathfrak{M}) \subset \mathcal{G}(\mathfrak{M})$ defined by

$$\mathcal{J}(\mathfrak{M}) := \bigcup_{p \in \mathcal{L}(\mathfrak{M})} (\mathbf{s}^{-1}(p) \cap \mathbf{t}^{-1}(p)). \tag{2.12}$$

It is a totally intransitive subgroupoid and one can consider it as a bundle $s : \mathcal{J}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of groups $\mathbf{s}^{-1}(p) \cap \mathbf{t}^{-1}(p) = G(p\mathfrak{M}p)$ indexed by $p \in \mathcal{L}(\mathfrak{M})$. One has also the action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) \to \mathcal{J}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M})$ on $\mathcal{J}(\mathfrak{M})$ defined by

$$I_x y := xy\ \iota(x) \tag{2.13}$$

for $(x, y) \in \mathcal{G}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{J}(\mathfrak{M}) : r(x) = s(y)\}$. This action is called the inner action. Note that the moment map for the inner action is the support map $s : \mathcal{J}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$. Since for $y \in \mathcal{J}(\mathfrak{M})$ one has $s(y) = l(y) = r(y)$ one can consider the lattice of projections $\mathcal{L}(\mathfrak{M})$ as a wide subgroupoid of $\mathcal{J}(\mathfrak{M})$.

Proposition 2.6. *The inner action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) \to \mathcal{J}(\mathfrak{M})$ restricts to an action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ which is the canonical left action of the groupoid $\mathcal{G}(\mathfrak{M})$ on its base set.*

Proof. Let us observe that $(x, p) \in \mathcal{G}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M})$ if and only if $p = \mathbf{s}(x) = u^*u \in \mathcal{L}(\mathfrak{M})$, where $x = u|x| \in \mathcal{G}(\mathfrak{M})$. One has

$$I_x(p) = xp\iota(x) = u|x|p|x|^{-1}u^* = uu^* \in \mathcal{L}(\mathfrak{M}). \tag{2.14}$$

The last equality in (2.14) follows from the fact that the projection p is the identity element of the group $G(p\mathfrak{M}p)$ and $|x| \in G(p\mathfrak{M}p)$. □

2.4. Order relation on the set of inner orbits

Here we discus two relations canonically defined by the inner action of the groupoid $\mathcal{G}(\mathfrak{M})$ on the lattice $\mathcal{L}(\mathfrak{M})$.

The groupoid structure of $\mathcal{G}(\mathfrak{M})$ allows us to define the principal bundles:

$$\begin{aligned} \mathbf{s} : \mathbf{t}^{-1}(p) \to \mathcal{O}_p \\ \mathbf{t} : \mathbf{s}^{-1}(p) \to \mathcal{O}_p \end{aligned} \tag{2.15}$$

over the orbit $\mathcal{O}_p := \{xp\iota(x) : \ x \in \mathbf{s}^{-1}(p)\}$ of the inner action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M})$ on the lattice $\mathcal{L}(\mathfrak{M})$. The structure group for the principal bundles (2.15) is the group $G(p\mathfrak{M}p)$.

The inner action (2.13) defines an equivalence relation on $\mathcal{L}(\mathfrak{M})$:

$$p \sim q \quad \text{iff} \quad q \in \mathcal{O}_p, \tag{2.16}$$

for which the equivalence class $[p]$ is the orbit $\mathcal{O}_p$ generated from the projection $p \in \mathcal{L}(\mathfrak{M})$.

Proposition 2.7. *The equivalence relation (2.16) is the same as the Murray-von Neumann equivalence relation (see e.g. Definition 2.1.1 in [17]), i.e. $q \in \mathcal{O}_p$ if and only if there exists a partial isometry $u \in \mathcal{U}(\mathfrak{M})$ such that $p = u^*u$ and $q = uu^*$.*

Proof. If $q \in \mathcal{O}_p$ then there exists $x \in \mathcal{G}(\mathfrak{M})$ such that $I_x(p) = q$. One has $x = u|x|$, where $|x| \in G(p\mathfrak{M}p)$ and $p = s(|x|) = u^*u$. From (2.14) one finds that $q = uu^*$.

Now let us assume that there exists $u \in \mathcal{U}(\mathfrak{M}) \subset \mathcal{G}(\mathfrak{M})$ such that $p = u^*u$ and $q = uu^*$. Putting in (2.14) $x = u$ we find that

$$I_u(p) = upu^* = uu^* = q, \tag{2.17}$$

i.e. $q \in \mathcal{O}_p$. □

From Propositions 2.5 and 2.7 it follows immediately:

Corrolary 2.8. *The inner actions of the groupoids $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on $\mathcal{L}(\mathfrak{M})$ have the same orbits.*

Remark 2.9. If $\mathfrak{M}$ is a finite W^*-algebra then the Murray-von Neumann equivalence relation becomes the unitary equivalence relation, i.e. $q \sim p$ if and only if there exists $u \in U(\mathfrak{M})$ such that $q = upu^*$ (see Proposition 1.38 in [19]).

The above is not valid for an infinite W^*-algebra $\mathfrak{M}$.

Let $\mathcal{L}(p) := \{q \in \mathcal{L}(\mathfrak{M}) : \ q \leqslant p\} \subset \mathcal{L}(\mathfrak{M})$ be the lattice ideal of the subprojections of the projection $p \in \mathcal{L}(\mathfrak{M})$. One has the canonically defined relation $\prec$ on the set of equivalence classes of the equivalence relation (2.16), i.e.

$$[q] \prec [p] \quad \text{iff} \bigcup_{q' \in [q]} \mathcal{L}(q') \ \subset \bigcup_{p' \in [p]} \mathcal{L}(p'). \tag{2.18}$$

Proposition 2.10. *The relation $\prec$ defined in (2.18) is a order relation on the set of the inner action orbits of groupoid $\mathcal{G}(\mathfrak{M})$ on $\mathcal{L}(\mathfrak{M})$. If $\mathfrak{M}$ is a factor then this order is total.*

Proof. The proof of reflexivity and transitivity of the relation $\prec$ is trivial. Now let us show that if $[p] \prec [q]$ and $[q] \prec [p]$ then $[p] = [q]$. For this reason it is enough to show that $[p] \cap [q] \neq \emptyset$. Firstly let us observe that

$$\bigcup_{p' \in [p]} \mathcal{L}(p') = \bigcup_{p' \in max[p]} \mathcal{L}(p'), \tag{2.19}$$

where $max[p]$ is the set of maximal elements of the orbit $[p] = \mathcal{O}_p$. Thus one has $[p] \prec [q]$ and $[q] \prec [p]$ if and only if

$$\bigcup_{p' \in max[p]} \mathcal{L}(p') = \bigcup_{q' \in max[q]} \mathcal{L}(q'). \tag{2.20}$$

It follows from (2.20) that for $p' \in max[p]$ there exists $q' \in max[q]$ such that $p' \leqslant q'$. For the same reason there exists $p'' \in max[p]$ such that $q' \leqslant p''$. Since p' and p'' are maximal elements of the orbit $[p]$ we find that $p' = q' = p'' \in [p] \cap [q] \neq \emptyset$. So, the relation $\prec$ is antisymmetric.
In the factor case the linearity of order relation $\prec$ follows from the Comparability Theorem, e.g. see [17], [19]. □

The equivalence relation (2.16) and the order relation (2.18) are fundamental for the classification of W^*-algebras. So, the problem of classification of $\mathcal{U}(\mathfrak{M})$-orbits on $\mathcal{L}(\mathfrak{M})$ is strictly related to the Murray and von Neumann classification of W^*-algebras, see e.g. [17], [19]. The reason is that the inner action $I : \mathcal{U}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ preserves the lattice structure of $\mathcal{L}(\mathfrak{M})$, i.e. for $(u,p) \in \mathcal{U}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M})$ the maps

$$I_u : \mathcal{L}(p) \to \mathcal{L}(upu^*) \tag{2.21}$$

are isomorphisms of the lattice ideals. In particular if a projection $z \in \mathcal{L}(\mathfrak{M}) \cap \mathcal{Z}(\mathfrak{M})$ is central, where $\mathcal{Z}(\mathfrak{M})$ is the center of $\mathfrak{M}$, then the lattice $\mathcal{L}(z) = \mathcal{L}(z\mathfrak{M})$ is preserved by the inner action. This allows us to reduce the classification of $\mathcal{U}(\mathfrak{M})$-orbits on $\mathcal{L}(\mathfrak{M})$ to the classification of $\mathcal{U}(\mathfrak{M})$-orbits for the case when $\mathfrak{M}$ is a factor.

2.5. Left and right action of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on $\mathfrak{M}$

Let us consider the following two actions of $\mathcal{G}(\mathfrak{M})$ on the W^*-algebra $\mathfrak{M}$:

(i) the left action $L : \mathcal{G}(\mathfrak{M}) *_l \mathfrak{M} \to \mathfrak{M}$ defined by

$$L_x y := xy \tag{2.22}$$

for $(x,y) \in \mathcal{G}(\mathfrak{M}) *_l \mathfrak{M} := \{(x,y) \in \mathcal{G}(\mathfrak{M}) \times \mathfrak{M}; \quad r(x) = l(y)\}$;

(ii) the right action $R : \mathcal{G}(\mathfrak{M}) *_r \mathfrak{M} \to \mathfrak{M}$ defined by

$$R_x y := yx \tag{2.23}$$

for $(x,y) \in \mathcal{G}(\mathfrak{M}) *_r \mathfrak{M} := \{(x,y) \in \mathcal{G}(\mathfrak{M}) \times \mathfrak{M}; \quad l(x) = r(y)\}$.

The moment map $\mu : \mathfrak{M} \to \mathcal{L}(\mathfrak{M})$ (see Appendix C) for the left action L (the right action R) is the left support map $\mu := l : \mathfrak{M} \to \mathcal{L}(\mathfrak{M})$ (the right support map $\mu := r : \mathfrak{M} \to \mathcal{L}(\mathfrak{M})$) defined in (2.2). Since both actions are intertwined by the inverse map, i.e.

$$\iota \circ L_x = R_{\iota(x)} \circ \iota \tag{2.24}$$

we will restrict ourselves to the left action only. All statements concerning the right action $R : \mathcal{G}(\mathfrak{M}) *_r \mathfrak{M} \to \mathfrak{M}$ we obtain converting statements for the left action L by (2.24).

Theorem 2.11. *The actions $L : \mathcal{U}(\mathfrak{M}) *_l \mathfrak{M} \to \mathfrak{M}$ and $R : \mathcal{U}(\mathfrak{M}) *_r \mathfrak{M} \to \mathfrak{M}$ defined by (2.22) and (2.23) are free. Their orbits are indexed by elements of the cone $\mathfrak{M}^+$ of positive selfadjoint elements of $\mathfrak{M}$.*

Proof. Since left and right actions are intertwined by the inverse map (2.7) it is enough to consider the case of the left action L. Let us assume that there are elements $u_1,\ u_2 \in \mathcal{U}(\mathfrak{M})$ such that $y := u_1 x = u_2 x$ for $u_1^* u_1 = vv^* = u_2^* u_2$, where $v \in \mathcal{U}(\mathfrak{M})$ is defined by

$$x = v|x|. \tag{2.25}$$

Since $|y|^2 = y^*y = x^* u_1^* u_1 x = x^* vv^* x = x^* x = |x|^2$ we have $y = u_1 v|y| = u_2 v|y|$. Thus from the uniqueness of the polar decomposition(2.1) we obtain $u_1 v = u_2 v$. The above gives $u_1 = u_1 u_1^* u_1 = u_1 vv^* = u_2 vv^* = u_2 u_2^* u_2 = u_2$. So, the left action L is free.

Taking the polar decomposition $x = v|x|$ of $x \in \mathfrak{M}$ we obtain that $v^*x = v^*v|x| = |x| \in \mathfrak{M}^+$. So any orbit $\mathcal{O}_x$ of $\mathcal{U}(\mathfrak{M})$ intersects $\mathfrak{M}^+$. If $x, y \in \mathcal{O}_x \cap \mathfrak{M}^+$ then $x = |x|,\ y = |y|$ and $|y| = u|x|$ for some $u \in \mathcal{U}(\mathfrak{M})$. Thus from uniqueness of the polar decomposition we obtain $x = |x| = |y| = y$. □

2.6. Inner representation of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on the bundle $\mathcal{A}(\mathfrak{M})$ of W^*-subalgebras of $\mathfrak{M}$

In this subsection we investigate the representations of the groupoids on the vector bundles which are given by the structure of the W^*-algebra.

Let us begin by briefly explaining that what one understands by representation of a groupoid is a direct generalization of the notion of group representation in a vector space. However, for groupoids one takes a vector bundle instead of a vector space. For the purposes of this paper as a rule we assume that the fibres $\pi^{-1}(m)$, $m \in M$, of vector bundle $(\mathbb{E}, M, \pi : \mathbb{E} \to M)$ under consideration will be not necessary isomorphic. In consequence of that the structural groupoid $\mathcal{G}(\mathbb{E})$ of this bundle would be not necessary transitive on base M.

Recall, see also [12], that the structural groupoid $\mathcal{G}(\mathbb{E})$ consists of linear isomorphisms $e^n_m : \mathbb{E}_m \tilde{\to} \mathbb{E}_n$ between the fibers of $\pi : \mathbb{E} \to M$. The base of $\mathcal{G}(\mathbb{E})$ is the base set M of the bundle. The source map $\mathbf{s} : \mathcal{G}(\mathbb{E}) \to M$ and the target map $\mathbf{t} : \mathcal{G}(\mathbb{E}) \to M$ are defined by $\mathbf{s}(e^n_m) := m$ and $\mathbf{t}(e^n_m) := n$ respectively. The inverse map is given by $\iota(e^n_m) := (e^n_m)^{-1}$ and the identity section by $\varepsilon(m) := id^m_m$. Finally the product of isomorphisms $e^m_l : \mathbb{E}_l \tilde{\to} \mathbb{E}_m$ and $e^n_m : \mathbb{E}_m \tilde{\to} \mathbb{E}_n$ is given by their composition $e^n_m \circ e^m_l : \mathbb{E}_l \tilde{\to} \mathbb{E}_n$.

Usually one investigates vector bundles with some additional structures. In the sequel we will consider cases when the fibres of $\pi : \mathbb{E} \to M$ will be provided with these structures. For example a Hilbert space structure, a W^*-algebra structure, a lattice structure or a W^*-algebra module structure.

Definition 2.12. Let G be a groupoid with base set B. One defines a representation of G on the vector bundle $\pi : \mathbb{E} \to M$ as a groupoid morphism:

$$\begin{array}{ccc} G & \xrightarrow{\ \ \phi\ \ } & \mathcal{G}(\mathbb{E}) \\ \mathbf{s} \downarrow\downarrow \mathbf{t} & & \mathbf{s} \downarrow\downarrow \mathbf{t} \\ B & \xrightarrow{\ \ \varphi\ \ } & M \end{array} \tag{2.26}$$

of G into the structural groupoid $\mathcal{G}(\mathbb{E})$ of the bundle.

Let us take $p,\ q \in \mathcal{L}(\mathfrak{M})$. According to the commonly accepted notation by $\mathcal{G}(\mathfrak{M})^q_p$ we denote the set $\mathbf{t}^{-1}(q) \cap \mathbf{s}^{-1}(p)$. For any $p \in \mathcal{L}(\mathfrak{M})$ one has the following inclusions:

$$\begin{array}{c} \mathbf{s}^{-1}(p) \subset r^{-1}(p) \subset \mathfrak{M}p \\ \mathbf{t}^{-1}(p) \subset l^{-1}(p) \subset p\mathfrak{M} \end{array} \tag{2.27}$$

where $\mathfrak{M}p$ ($p\mathfrak{M}$) is the left (right) W^*-ideal generated by p. Recall that the left support $l(x)$ and right support $r(x)$ are defined for any element $x \in \mathfrak{M}$, but the source map $\mathbf{s}$ and target map $\mathbf{t}$ are defined only for elements in $\mathcal{G}(\mathfrak{M})$.

Now we consider the bundle $\pi : \mathcal{M}_R(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of right $\mathfrak{M}$-modules over the lattice $\mathcal{L}(\mathfrak{M})$ with total space defined by

$$\mathcal{M}_R(\mathfrak{M}) := \{(y, p) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : \quad p\, l(y) = l(y)\} \tag{2.28}$$

and bundle map $\pi := pr_2$ as the projection on the second component of the product $\mathfrak{M} \times \mathcal{L}(\mathfrak{M})$. The fibre $\pi^{-1}(p)$ over $p \in \mathcal{L}(\mathfrak{M})$ is the right ideal $p\mathfrak{M}$ of $\mathfrak{M}$ generated by the projection p. Any element $x \in \mathcal{G}(\mathfrak{M})^q_p$ defines by the left multiplication an isomorphism $L_x : p\mathfrak{M} \xrightarrow{\sim} q\mathfrak{M}$ of the right $\mathfrak{M}$-modules, i. e.

$$L_x(ay) = L_x(a)y \tag{2.29}$$

for $a \in p\mathfrak{M}$ and $y \in \mathfrak{M}$. The $\mathfrak{M}$-module isomorphisms $L : p\mathfrak{M} \xrightarrow{\sim} q\mathfrak{M}$, where $p, q \in \mathcal{L}(\mathfrak{M})$, form the groupoid $\mathcal{G}(\mathcal{M}_R(\mathfrak{M}))$ of structural isomorphisms of the fibers of the bundle $\pi : \mathcal{M}_R(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$. One can show that $L = L_x$ for some $x \in \mathcal{G}(\mathfrak{M})^q_p$. Thus we have the following statement:

Proposition 2.13. *The structural groupoid $\mathcal{G}(\mathcal{M}_R(\mathfrak{M}))$ of the bundle $\pi : \mathcal{M}_R(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ is isomorphic to $\mathcal{G}(\mathfrak{M})$.*

Replacing $\mathcal{M}_R(\mathfrak{M})$ by $\mathcal{M}_L(\mathfrak{M})$ and the action $x \to L_x$ by the right action $x \to R_x$, where $x \in \mathcal{G}(\mathfrak{M})$, we obtain the anti-isomorphism of $\mathcal{G}(\mathfrak{M})$ with $\mathcal{G}(\mathcal{M}_L(\mathfrak{M}))$. Using the above two representations we obtain a representation of $\mathcal{G}(\mathfrak{M})$ on the bundle $\pi : \mathcal{A}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of the W^*-subalgebras of $\mathfrak{M}$ with total space $\mathcal{A}(\mathfrak{M})$ defined by

$$\mathcal{A}(\mathfrak{M}) := \{(y,p) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : \quad y \in p\mathfrak{M}p\} \tag{2.30}$$

and the bundle map by $\pi := pr_2$. The morphism $I : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathcal{A}(\mathfrak{M}))$ of $\mathcal{G}(\mathfrak{M})$ into the structural groupoid $\mathcal{G}(\mathcal{A}(\mathfrak{M}))$ of the bundle $\pi : \mathcal{A}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ is defined as follows

$$I_x := R_{\iota(x)} \circ L_x : p\mathfrak{M}p \to q\mathfrak{M}q, \tag{2.31}$$

where $x \in \mathcal{G}(\mathfrak{M})_p^q$.

Note here that $\mathcal{J}(\mathfrak{M}) \subset \mathcal{A}(\mathfrak{M})$ and the action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{A}(\mathfrak{M}) \to \mathcal{A}(\mathfrak{M})$ is an extension of the inner action $I : \mathcal{G}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) \to \mathcal{J}(\mathfrak{M})$. For $u \in \mathcal{U}(\mathfrak{M})_p^q$ we find that $I_u : p\mathfrak{M}p \to q\mathfrak{M}q$ is an isomorphism of W^*-subalgebras of $\mathfrak{M}$. Thus we have

Proposition 2.14. *The inner action $I : \mathcal{U}(\mathfrak{M}) * \mathcal{A}(\mathfrak{M}) \to \mathcal{A}(\mathfrak{M})$ of the partial isometries groupoid $\mathcal{U}(\mathfrak{M})$ on $\mathcal{A}(\mathfrak{M})$ preserves the positivity, normality, selfadjointness and the norm of the elements of the fibres of $\mathcal{A}(\mathfrak{M})$, i.e.:*

(i) $|I_u x| = I_u |x|$,

(ii) $xx^* = x^*x$ *iff* $(I_u x)^*(I_u x) = (I_u x)(I_u x)^*$

(iii) $x = x^*$ *iff* $(I_u x)^* = I_u x$

(iv) $\|I_u x\| = \|x\|$

for $(u, x) \in \mathcal{U}(\mathfrak{M}) * \mathcal{A}(\mathfrak{M})$.

Let $\mathfrak{M}^+, \mathfrak{M}^h$ and $\mathfrak{M}^n$ denote the sets of positive, selfadjoint and normal elements of $\mathfrak{M}$ respectively. Let S be the sphere in $\mathfrak{M}$, i. e. $x \in S$ if and only if $\|x\| = 1$. We conclude from Proposition 2.14 that the subsets $\mathcal{J}(\mathfrak{M}) \cap \mathfrak{M}^+$, $\mathcal{J}(\mathfrak{M}) \cap \mathfrak{M}^h$, $\mathcal{J}(\mathfrak{M}) \cap \mathfrak{M}^n$, $\mathcal{J}(\mathfrak{M}) \cap S$ and $\mathcal{J}(\mathfrak{M}) \cap \mathcal{U}(\mathfrak{M})$ are invariant with respect to the inner action $I : \mathcal{U}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M}) \to \mathcal{J}(\mathfrak{M})$. Let us also note that the lattice of projections $\mathcal{L}(\mathfrak{M})$ consists of the extreme points in $\mathfrak{M}^+ \cap S$, e.g. see Proposition 1.6.2 in [17].

2.7. Left and right predual actions of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on $\mathfrak{M}_*$

For the sake of completeness and further applications let us consider the actions

$$\begin{array}{l} L_*: \ \mathcal{G}(\mathfrak{M}) *_{l_*} \mathfrak{M}_* \to \mathfrak{M}_* \\ R_*: \ \mathcal{G}(\mathfrak{M}) *_{r_*} \mathfrak{M}_* \to \mathfrak{M}_* \end{array} \tag{2.32}$$

of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on $\mathfrak{M}_*$ which are predual to the actions $L : \mathcal{G}(\mathfrak{M}) *_l \mathfrak{M} \to \mathfrak{M}$ and $R : \mathcal{G}(\mathfrak{M}) *_r \mathfrak{M} \to \mathfrak{M}$ respectively.

For this reason, refering to Section 2 in [19], we recall that the left predual action $L_* : \ \mathfrak{M} \times \mathfrak{M}_* \to \mathfrak{M}_*$ (respectively the right predual action $R_* : \ \mathfrak{M} \times \mathfrak{M}_* \to \mathfrak{M}_*$) of W^*-algebra $\mathfrak{M}$ on the predual Banach space $\mathfrak{M}_*$ is defined by:

$$\langle x, L_{*a}\omega\rangle := \langle xa, \omega\rangle \quad \text{(respectively } \langle x, R_{*a}\omega\rangle := \langle ax, \omega\rangle) \tag{2.33}$$

for any $x \in \mathfrak{M}$, where $a \in \mathfrak{M}$ and $\omega \in \mathfrak{M}_*$. So, one has

$$(L_{*a})^* = R_a, \quad \text{and} \quad (R_{*a})^* = L_a. \tag{2.34}$$

For any element $\omega \in \mathfrak{M}_*$ one takes the closed left invariant subspace $[\mathfrak{M}\omega] \subset \mathfrak{M}_*$ (respectively the right invariant subspace $[\omega\mathfrak{M}] \subset \mathfrak{M}_*$) generated from ω by the left (respectively right) action of $\mathfrak{M}$ (2.33). The annihilator $[\mathfrak{M}\omega]^0 \subset \mathfrak{M}$ of the Banach subspace $[\mathfrak{M}\omega] \subset \mathfrak{M}_*$ is a right W^*-ideal in $\mathfrak{M}$. Similarly the annihilator $[\omega\mathfrak{M}]^0 \subset \mathfrak{M}$ of the Banach subspace $[\omega\mathfrak{M}] \subset \mathfrak{M}_*$ is a left W^*-ideal in $\mathfrak{M}$. Thus there exist orthogonal projections $e, f \in \mathfrak{M}$ such that $[\mathfrak{M}\omega]^0 = e\mathfrak{M}$ and $[\omega\mathfrak{M}]^0 = \mathfrak{M}f$. The projection e is the greatest one of all the projections $q \in \mathfrak{M}$ such that $R_{*q}\omega = 0$. Similarly the projection f is the greatest one of all the projections $q \in \mathfrak{M}$ such that $L_{*q}\omega = 0$. Thus one defines the maps

$$r_*(\omega) := 1 - e \tag{2.35}$$

and

$$l_*(\omega) := 1 - f, \tag{2.36}$$

where $(1 - e)$ and $(1 - f)$ are the least projection with the property $R_{*(1-e)}\omega = \omega$ and $L_{*(1-f)}\omega = \omega$ respectively. The projections $r_*(\omega)$ and $l_*(\omega)$ are called, respectively, the right support projection and the left support projection of $\omega \in \mathfrak{M}_*$. It follows from the polar decomposition (see e.g. Theorem 4.2 in [19])

$$\omega = L_{*v}|\omega| \tag{2.37}$$

of $\omega \in \mathfrak{M}_*$, where $v \in \mathcal{U}(\mathfrak{M})$ and $|\omega| \in \mathfrak{M}_*^+$, that

$$r_*(\omega) = v^*v \quad \text{and} \quad l_*(\omega) = vv^*.$$

The $*$-operation $\mathfrak{M}_* \ni \omega \mapsto \omega^* \in \mathfrak{M}_*$ is defined by the equality

$$\langle x, \omega^* \rangle := \overline{\langle x^*, \omega \rangle},$$

where $x \in \mathfrak{M}$, and $\omega \in \mathfrak{M}_*^+$ iff $\omega = \omega^*$ and $\langle x, \omega^* \rangle \geqslant 0$ for any $x \in \mathfrak{M}^+$. For details see Section 1.5 in [17].

Considering $r_* : \mathfrak{M}_* \to \mathcal{L}(\mathfrak{M})$ and $l_* : \mathfrak{M}_* \to \mathcal{L}(\mathfrak{M})$ as moment maps we define the actions (2.32) by

$$\mathcal{G}(\mathfrak{M}) *_{r_*} \mathfrak{M}_* \ni (x, \omega) \mapsto R_{*x}\omega \in \mathfrak{M}_* \tag{2.38}$$

and

$$\mathcal{G}(\mathfrak{M}) *_{l_*} \mathfrak{M}_* \ni (x, \omega) \mapsto L_{*x}\omega \in \mathfrak{M}_*, \tag{2.39}$$

respectively, where

$$\mathcal{G}(\mathfrak{M}) *_{r_*} \mathfrak{M}_* = \{(x, \omega) \in \mathcal{G}(\mathfrak{M}) \times \mathfrak{M}_*; \quad \mathbf{t}(x) = x\iota(x) = r_*(\omega)\}$$

and

$$\mathcal{G}(\mathfrak{M}) *_{l_*} \mathfrak{M}_* = \{(x, \omega) \in \mathcal{G}(\mathfrak{M}) \times \mathfrak{M}_*; \quad \mathbf{s}(x) = \iota(x)x = l_*(\omega)\}.$$

One obtains the left and right predual actions of the subgroupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on $\mathfrak{M}_*$ as the restrictions of (2.38) and (2.39), respectively.

2.8. Inner representation of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ on the predual bundle $\mathcal{A}_*(\mathfrak{M})$

Let us also define the bundle $\pi_* : \mathcal{A}_*(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ predual to the bundle of the W^*-algebras $\pi : \mathcal{A}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$. In this case the total space is the following

$$\begin{aligned}\mathcal{A}_*(\mathfrak{M}) := \{(\omega, p) \in \mathfrak{M}_* \times \mathcal{L}(\mathfrak{M}) : \\ r_*(\omega) = p\, r_*(\omega) \text{ and } l_*(\omega) = l_*(\omega)p\}\end{aligned} \tag{2.40}$$

and the bundle map π_* is the projection of $(\omega, p) \in \mathfrak{M}_* \times \mathcal{L}(\mathfrak{M})$ on the second component. Note that one can identify the fibre $\pi_*^{-1}(p) = (R_{*p} \circ L_{*p})(\mathfrak{M}_*)$, $p \in \mathcal{L}(\mathfrak{M})$, with the Banach space $(p\mathfrak{M}p)_*$ predual to subalgebra $p\mathfrak{M}p$.

We define the predual inner action $\tilde{I}_* : \mathcal{G}(\mathfrak{M}) * \mathcal{A}_*(\mathfrak{M}) \to \mathcal{A}_*(\mathfrak{M})$

$$\tilde{I}_{*x}(\omega, p) := (I_{*x}\omega, I_x p) \tag{2.41}$$

of the groupoid $\mathcal{G}(\mathfrak{M})$ on the bundle $\mathcal{A}_*(\mathfrak{M})$ where $\mathcal{G}(\mathfrak{M}) * \mathcal{A}_*(\mathfrak{M}) := \{(x, (\omega, p)) : \ \mathbf{s}(x) = p\}$, the bundle map $\pi_* : \mathcal{A}_*(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ is the moment map and

$$I_{*x} = L_{*x} \circ R_{*\iota(x)}.$$

Now we define the following subbundles of $\pi_* : \mathcal{A}_*(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$. The subbundle $\pi_* : \mathcal{J}_*(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ whose total space is defined by

$$\mathcal{J}_*(\mathfrak{M}) := \{(\omega, p) \in \mathcal{A}_*(\mathfrak{M}) : \quad l_*(\omega) = r_*(\omega) = p\}. \tag{2.42}$$

The subbundle of selfadjoint normal functionals $\pi_* : \mathcal{A}_*^h(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ for which the total space is

$$\mathcal{A}_*^h(\mathfrak{M}) := \{(\omega, p) \in \mathcal{A}_*(\mathfrak{M}) : \ \omega^* = \omega\}. \tag{2.43}$$

The subbundle of positive normal functionals $\pi_* : \mathcal{A}_*^+(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ with the set

$$\mathcal{A}_*^+(\mathfrak{M}) := \{(\omega, p) \in \mathcal{A}_*(\mathfrak{M}) : \ \omega^* = \omega > 0\}. \tag{2.44}$$

as the total space.

When we restrict the action (2.41) to subgroupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ we obtain the following statement.

Proposition 2.15. *For the predual inner action* $I_* : \mathcal{U}(\mathfrak{M}) * \mathcal{A}_*(\mathfrak{M}) \to \mathcal{A}_*(\mathfrak{M})$ *of the partial isometries groupoid* $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ *on* $\mathcal{A}_*(\mathfrak{M})$ *one* *has:*

$$\omega = \omega^* \quad \text{iff} \quad (I_{*u}\omega)^* = I_{*u}\omega \tag{2.45}$$

$$\|I_{*u}\omega\| = \|\omega\| \tag{2.46}$$

$$|I_{*u}\omega| = I_{*u}|\omega| \tag{2.47}$$

for $(u,(\omega,p)) \in \mathcal{U}(\mathfrak{M}) * \mathcal{A}_*(\mathfrak{M})$, *i.e. the subbundles (2.42), (2.43), (2.44) are invariant with respect to this action.*

Proof. In order to prove (2.45) we note that for $\omega \in \pi_*^{-1}(u^*u)$ we have $\langle \omega^*, x\rangle := \overline{\langle \omega, x^*\rangle}$, where $x \in \mathfrak{M}$, so we obtain

$$(I_{*u}\omega)^* = I_{*u}\omega^*.$$

Thus and from $I_{*u^*u}\omega = \omega$ we have that $\omega = \omega^*$ iff $(I_{*u}\omega)^* = I_{*u}\omega$. Since $\|u\| = 1$ and $L_{*u^*u}\omega = \omega$ one has

$$\|L_{*u}\omega\| \leqslant \|\omega\| \leqslant \|L_{*u}\omega\|\,. \tag{2.48}$$

Similarly we prove that $\|R_{*u}\omega\| = \|\omega\|$. Thus we have (2.46).

Let us assume that $\mathfrak{M}_* \ni \omega \geqslant 0$ then for any $x \in \mathfrak{M}^+$ one has $u^*xu \in \mathfrak{M}^+$ and

$$\langle I_{*u}\omega, x\rangle = \langle \omega, u^*xu\rangle \geqslant 0.$$

Thus we find that $I_{*u}\omega \geqslant 0$ iff $\omega \geqslant 0$. If $\omega = \omega^*$ then for any $x \in \mathfrak{M}$ we have

$$\langle (I_{*u}\omega)^*, x\rangle = \overline{\langle \omega, u^*xu\rangle} = \langle \omega^*, (u^*xu)^*\rangle = \langle I_{*u}\omega^*, x\rangle.$$

The above shows that inner action commutes with conjugation operation.

Let us take the polar decomposition of $\omega \in \pi_*^{-1}(u^*u)$

$$\omega = L_{*v}|\omega|. \tag{2.49}$$

We note that the polar decomposition of $I_{*u}\omega \in \pi_*^{-1}(u^*u)$, where $v^*v \leqslant u^*u$ and $vv^* \leqslant u^*u$, is given by

$$I_{*u}\omega = L_{uvu^*}|I_{*u}\omega|. \tag{2.50}$$

From (2.50) it follows that

$$|I_{*u}\omega| = L_{(uvu^*)^*}I_{*u}\omega.$$

Thus for any $x \in \mathfrak{M}$ we have

$$\begin{aligned}\langle |I_{*u}\omega|, x\rangle &= \langle L_{(uvu^*)^*} I_{*u}\omega, x\rangle = \langle I_{*u}\omega, xuv^*u^*\rangle = \langle \omega, u^*xuv^*u^*u\rangle \\ &= \langle \omega, u^*xuv^*\rangle = \langle L_{*v^*}\omega, u^*xu\rangle = \langle |\omega|, u^*xu\rangle = \langle I_{*u}|\omega|, x\rangle.\end{aligned}$$

Thus we obtain (2.47) □

2.9. GNS-bundle

Below we will consider the action groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \rightrightarrows \mathfrak{M}_*^+$ over $\mathfrak{M}_*^+$ (for the definition of an action groupoid see Appendix E), with $s_* : \mathfrak{M}_*^+ \to \mathcal{L}(\mathfrak{M})$ as a moment map and the action $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \to \mathfrak{M}_*^+$ given by

$$I_{*u}\omega = u\omega u^*, \tag{2.51}$$

where $s_*(\omega) = l_*(\omega) = r_*(\omega)$, since $\omega \in \mathfrak{M}_*^+$. Let us note that from (2.47) the cone $\mathfrak{M}_*^+$ is invariant with respect to the action (2.51).

Now basing on GNS construction, see [7], [18], we define the pre-Hilbert bundle $\pi : \mathbb{E} \to \mathfrak{M}_*^+$ over the cone of the positive normal states $\mathfrak{M}_*^+$. The total space $\mathbb{E}$ and bundle projection we define as follows

$$\mathbb{E} := \{(x, \omega) \in \mathfrak{M} \times \mathfrak{M}_*^+ : \quad xs_*(\omega) = x\} \tag{2.52}$$

and $\pi := pr_2|_{\mathbb{E}}$.

Since for $\omega \in \mathfrak{M}_*^+$ one has $\mathbb{E}_\omega = \pi^{-1}(\omega) = \mathfrak{M}s_*(\omega)$ the scalar product

$$\mathbb{E}_\omega \times \mathbb{E}_\omega \ \ni \ (x, y) \mapsto \langle x|y\rangle_\omega := \langle \omega, x^*y\rangle \ \in \ \mathbb{C} \tag{2.53}$$

is non degenerate. Thus it defines the pre-Hilbert space structure on $\mathbb{E}_\omega$. Note here that $\langle \omega, x^*x\rangle = 0$ if and only if $x \in \mathfrak{M}(1 - s_*(\omega))$.

Completing $\mathbb{E}_\omega$ with respect to the norm $\|x\|_\omega := \langle \omega, x^*x\rangle^{\frac{1}{2}}$ we obtain the bundle $\overline{\pi} : \overline{\mathbb{E}} \to \mathfrak{M}_*^+$ of Hilbert spaces. For clear reasons we will call this bundle the GNS bundle.

Theorem 2.16. *One has a faithful representation*

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ & \xrightarrow{\ \phi\ } & \mathcal{G}(\overline{\mathbb{E}}) \\ s \downarrow\downarrow t & & s \downarrow\downarrow t \\ \mathfrak{M}_*^+ & \xrightarrow{\ id\ } & \mathfrak{M}_*^+ \end{array} \tag{2.54}$$

of the right action of groupoid $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ *on the GNS bundle* $\overline{\pi} : \overline{\mathbb{E}} \to \mathfrak{M}_*^+$ *with the fibres isomorphisms* $\phi(u,\omega) : \overline{\mathbb{E}}_\omega \to \overline{\mathbb{E}}_{I_{*u}\omega}$ *defined as follows*

$$\phi(u,\omega)(xs_*(\omega),\omega) := (xs_*(\omega)u^*, I_{*u}\omega) \tag{2.55}$$

Proof. For $(u,\omega) \in \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ we have $u^*u = s_*(\omega)$. Thus the following sequence of equalities

$$\begin{aligned} \langle \phi(u,\omega)xs_*(\omega)|\phi(u,\omega)ys_*(\omega)\rangle_{I_{*u}\omega} &= \langle I_{*u}\omega, (xs_*(\omega)u^*)^* ys_*(\omega)u^*\rangle \\ &= \langle \omega, u^*us_*(\omega)x^*ys_*(\omega)u^*u\rangle \\ &= \langle \omega, (xs_*(\omega))^* ys_*(\omega)\rangle \\ &= \langle xs_*(\omega)|ys_*(\omega)\rangle_\omega \end{aligned} \tag{2.56}$$

shows that $\phi(u,\omega) : \mathbb{E}_\omega \to \mathbb{E}_{I_{*u}\omega}$ extends to an isomorphism of Hilbert spaces.

For elements $(u,\omega),\ (v,\lambda) \in \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ such that $\mathbf{t}(v) = \mathbf{s}(u)$, i.e. $\omega = I_{*v}\lambda$ we have

$$\begin{aligned} \phi((u,\omega)(v,\lambda))(xs_*(\lambda),\lambda) &= \phi(uv,\lambda)(xs_*(\lambda),\lambda) = (xs_*(\lambda)(uv)^*, I_{*uv}\lambda) \\ &= \phi(u,\omega)(xs_*(\lambda)v^*, I_{*v}\lambda) \\ &= (\phi(u,\omega)\circ(\phi(v,\lambda))(xs_*(\lambda),\lambda)) \end{aligned}$$

for any $(xs_*(\lambda),\lambda) \in \mathbb{E}_\lambda$. Thus we obtain

$$\phi((u,\omega)(v,\lambda)) = \phi(u,\omega)\circ\phi(v,\lambda). \tag{2.57}$$

We recall that $(u,\omega)(v,\lambda)$ is the product of $(\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+)^{(2)}$ defined by (E.7).

One can easily check that for $\phi(u,\omega)$ and $\phi(u^*, I_{*u}\omega)$ we have

$$\begin{aligned} &(\phi(u^*, I_{*u}\omega)\circ\phi(u,\omega))(xs_*(\omega),\omega) \\ &= \phi(u^*, I_{*u}\omega)(xs_*(\omega)u^*, I_{*u}\omega) \\ &= (xs_*(\omega)u^*u, I_{*u^*u}\omega) = (xs_*(\omega),\omega) \end{aligned}$$

for any $(xs_*(\omega),\omega) \in \mathbb{E}_\omega$. The above shows that

$$\phi(u^*, I_{*u}\omega)\circ\phi(u,\omega) = id|_{\mathbb{E}_\omega}.$$

In the similar way we prove that

$$\phi(u,\omega)\circ\phi(u^*, I_{*u}\omega) = id|_{\mathbb{E}_{I_{*u}\omega}}.$$

Thus we get

$$(\phi(u,\omega))^{-1} = \phi(u^*, I_{*u}\omega). \tag{2.58}$$

For any $(u,\omega) \in \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ one has

$$(id \circ \mathbf{s})(u,\omega) = id(\omega) = \omega,$$
$$(id \circ \mathbf{t})(u,\omega) = id(I_{*u}\omega) = I_{*u}\omega$$

and

$$(\mathbf{s} \circ \phi)(u,\omega)(xs_*(\omega),\omega) = \mathbf{s}(\phi(u,\omega)(xs_*(\omega),\omega)) = \omega,$$
$$(\mathbf{t} \circ \phi)(u,\omega)(xs_*(\omega),\omega) = \mathbf{t}(\phi(u,\omega)(xs_*(\omega),\omega)) = I_{*u}\omega,$$

which shows that $id \circ \mathbf{s} = \mathbf{s} \circ \phi$ and $id \circ \mathbf{t} = \mathbf{t} \circ \phi$, i. e. the diagram (2.54) is commutative. The above shows that ϕ is a groupoid morphism.

From

$$\phi(u,\omega) = \phi(u',\omega'). \tag{2.59}$$

we find that

$$\omega = \omega'$$
$$s_*(\omega) = u^*u = u'^*u' = s_*(\omega') \tag{2.60}$$

and

$$xs_*(\omega)u^* = xs_*(\omega)u'^* \tag{2.61}$$

for any $x \in \mathfrak{M}$. Setting $x = s_*(\omega)$ in (2.61) we prove that from (2.59) it follows

$$(u,\omega) = (u',\omega').$$

Thus ϕ is a faithful morphism of groupoids. □

In order to obtain a faithful W^*-representation of $\mathfrak{M}$, see [17], we recall that $\mathbb{E}_\omega$ is a left W^*-ideal of $\mathfrak{M}$. Hence one has a W^*-representation $\overline{\rho_\omega} : \mathfrak{M} \to L^\infty(\overline{\mathbb{E}_\omega})$ of $\mathfrak{M}$ in the W^*-algebra of bounded operators on Hilbert space $\overline{\mathbb{E}_\omega}$, defined by the continuous extension of

$$\rho_\omega(x)ys_*(\omega) := xys_*(\omega), \tag{2.62}$$

where $x \in \mathfrak{M}$ and $ys_*(\omega) \in \mathbb{E}_\omega$.

Let us denote by $L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+)$ the Hilbert space of square summable sections $\psi:\mathfrak{M}_*^+\to\overline{\mathbb{E}}$

$$\sum_{\omega\in\mathfrak{M}_*^+}\|\psi(\omega)\|_\omega^2<\infty \tag{2.63}$$

of the GNS bundle.

The direct sum

$$\overline{\rho}:=\bigoplus_{\omega\in\mathfrak{M}_*^+}\overline{\rho_\omega} \tag{2.64}$$

is a faithful W^*-representation $\overline{\rho}:\mathfrak{M}\to L^\infty(L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+))$ of $\mathfrak{M}$ in the Hilbert space $L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+)$, see for example Theorem 1.16.7 in [17]. So, one has a $*$-isomorphism $\mathfrak{M}\cong\overline{\rho}(\mathfrak{M})$ of $\mathfrak{M}$ with the W^*-subalgebra

$$\overline{\rho}(\mathfrak{M})\subset L^\infty(L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+)).$$

Recall that a *-homomorphism of W^*-algebras is a W^*-homomorphism if it is a map $\phi:\mathfrak{M}_1\to\mathfrak{M}_2$ continuous with respect to the $\sigma(\mathfrak{M}_1,\mathfrak{M}_{1*})$-topology and $\sigma(\mathfrak{M}_2,\mathfrak{M}_{2*})$-topology.

Theorem 2.17. *There exists a groupoid monomorphism*

$$\Lambda:\mathcal{U}(\mathfrak{M})*\mathfrak{M}_*^+\to\mathcal{U}(\overline{\rho}(\mathfrak{M})') \tag{2.65}$$

of the action groupoid $\mathcal{U}(\mathfrak{M})*\mathfrak{M}_*^+$ into the groupoid of partial isometries $\mathcal{U}(\overline{\rho}(\mathfrak{M})')$ of the W^*-algebra $\overline{\rho}(\mathfrak{M})'$, where $\overline{\rho}(\mathfrak{M})'$ is the commutant of $\overline{\rho}(\mathfrak{M})$ in the operator algebra $L^\infty(L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+))$.

Proof. We note that Hilbert subspaces $\overline{\mathbb{E}}_{\omega_1}$ and $\overline{\mathbb{E}}_{\omega_2}$ of $L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+)$ are invariant with respect to the representation $\overline{\rho}$. We note also that any element $e_{\omega_1}^{\omega_2}\in\mathcal{G}(\overline{\mathbb{E}})_{\omega_1}^{\omega_2}\subset\mathcal{G}(\overline{\mathbb{E}})$ can be extended to a partial isometry of the Hilbert space $L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+)$ with the kernel equals $(\overline{\mathbb{E}}_{\omega_1})^\perp\subset L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+)$. The map $\phi(u,\omega):\mathbb{E}_\omega\to\mathbb{E}_{I_{*u}\omega}$, defined in (2.55), intertwining representations $\overline{\rho}_\omega$ and $\overline{\rho}_{I_{*u}\omega}$, i.e.

$$\phi(u,\omega)\cdot\overline{\rho}_\omega(y)=\overline{\rho}_{I_{*u}\omega}(y)\cdot\phi(u,\omega) \tag{2.66}$$

for $y\in\mathfrak{M}$, extends to the partial isometry $\overline{\phi(u,\omega)}$ of the Hilbert space $L^2\Gamma(\mathbb{E},\mathfrak{M}_*^+)$. By (2.66) this partial isometry belongs to the commutant

$\overline{\rho}(\mathfrak{M})'$ of the W^*-algebra $\overline{\rho}(\mathfrak{M})$. Using the notation

$$\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \ \ni \ (u,\omega) \mapsto \Lambda(u,\omega) := \overline{\phi(u,\omega)}$$

we obtain from the Proposition (2.16) that (2.65) is a groupoid monomorphism. □

Finally let us note that the projection

$$pr_1 : \mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+ \to \mathcal{U}(\mathfrak{M}) \ \cong \ \mathcal{U}(\overline{\rho}(\mathfrak{M})'') \tag{2.67}$$

of $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ on the first component of the product $\mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_*^+$ defines a covering morphism of the groupoids, see Appendix C. The groupoid isomorphism $\mathcal{U}(\mathfrak{M}) \ \cong \ \mathcal{U}(\overline{\rho}(\mathfrak{M})'')$ in (2.67) follows from the isomorphism

$$\overline{\rho}(\mathfrak{M})'' = \overline{\rho}(\mathfrak{M}) \cong \mathfrak{M}$$

of W^*-algebras, where the equality $\overline{\rho}(\mathfrak{M})'' = \overline{\rho}(\mathfrak{M})$ is a consequence of the bicommutant theorem, see e.g. Theorem 1.20.3 in [17], and one has $\overline{\rho}(\mathfrak{M}) \cong \mathfrak{M}$ due to the fact that $\overline{\rho}$ is a faithful W^*-representation.

3. Topologies on the groupoids $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

The following locally convex topologies are considered on a W^*-algebra $\mathfrak{M}$: the uniform topology, the Arens-Mackey topology $\tau(\mathfrak{M}, \mathfrak{M}_*)$, the strong *-topology $s^*(\mathfrak{M}, \mathfrak{M}_*)$, the strong topology $s(\mathfrak{M}, \mathfrak{M}_*)$, the σ-weak topology $\sigma(\mathfrak{M}, \mathfrak{M}_*)$, see e. g.[17]. All these topologies define the corresponding topologies on the groupoids $\mathcal{G}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$. Hence, the natural question arises for which of the topologies listed above the groupoids are topological groupoids.

Let us start from the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Proposition 3.1. *For a infinite-dimensional W^*-algebra $\mathfrak{M}$ the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is not a topological groupoid with respect to any topology of $\mathfrak{M}$ mentioned above.*

Proof. Let us take $p \in \mathcal{L}(\mathfrak{M})$ and define $x_n \in \mathcal{G}(\mathfrak{M})$ by

$$x_n = p + \frac{1}{n}(1-p), \quad n \in \mathbb{N}.$$

One has

$$\mathbf{s}(x_n) = \mathbf{t}(x_n) = 1 \quad \text{and} \quad \mathbf{s}(p) = \mathbf{t}(p) = p.$$

Since the uniform limit of x_n is

$$p = \lim_{n\to\infty} x_n,$$

we see that source and target maps of $\mathcal{G}(\mathfrak{M})$ are not continuous. Thus we obtain that $\mathcal{G}(\mathfrak{M})$ is not a topological groupoid. Note that the above consideration does not depend on the choice of topology on $\mathfrak{M}$. □

The case of the groupoid $\mathcal{U}(\mathfrak{M})$ is much better than that of $\mathcal{G}(\mathfrak{M})$. Let us begin our considerations from the uniform topology. Since all algebraic operations in $\mathfrak{M}$ and the *-involution are uniformly continuous and groupoid maps are expressed by these operations we conclude that the groupoid $\mathcal{U}(\mathfrak{M})$ is a topological groupoid with respect to the uniform topology. Let us remark also that $\mathcal{U}(\mathfrak{M})$ is uniformly closed in $\mathfrak{M}$ and $\mathcal{L}(\mathfrak{M})$ is uniformly closed in $\mathcal{U}(\mathfrak{M})$. Note also that the set $\mathcal{U}(\mathfrak{M})^{(2)} = (\mathbf{s}\times\mathbf{t})^{-1}(\{(p,p):\ p\in\mathcal{L}(\mathfrak{M})\})$ is closed in $\mathcal{U}(\mathfrak{M})\times\mathcal{U}(\mathfrak{M})$.

The groupoid of partial isometries $\mathcal{U}(\mathfrak{M})$ is not topological with respect to the $\sigma(\mathfrak{M},\mathfrak{M}_*)$-topology (the weak *-topology) and with respect to the $s(\mathfrak{M},\mathfrak{M}_*)$-topology (the strong topology). The reason is that the product map (2.6) is not continuous with respect to $\sigma(\mathfrak{M},\mathfrak{M}_*)$-topology and the involution (2.7) is not continuous with respect to the $s(\mathfrak{M},\mathfrak{M}_*)$-topology.

The Arens-Mackey topology $\tau(\mathfrak{M},\mathfrak{M}_*)$ coincides with the s^*-strong topology $s^*(\mathfrak{M},\mathfrak{M}_*)$ on the bounded parts of $\mathfrak{M}$, see [17]. So both of them induce on $\mathcal{U}(\mathfrak{M})$ the same topology. Hence without loss of generality we can restrict our consideration to the $s^*(\mathcal{U}(\mathfrak{M}),\mathfrak{M}_*)$-topology of $\mathcal{U}(\mathfrak{M})$.

Let us take the closed unit ball $\mathcal{B}=\{x\in\mathfrak{M}:\ \|x\|\leqslant 1\}$ in $\mathfrak{M}$. The product map $\mathcal{B}\times\mathcal{B}\ \ni\ (x,y)\ \mapsto\ xy\ \in\ \mathcal{B}$ restricted to $\mathcal{B}$ as well as the *-involution are continuous with respect to $s^*(\mathcal{B},\mathfrak{M}_*)$-topology. From the above we conclude:

Proposition 3.2. *The groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partial isometries is a topological groupoid with respect to the $s^*(\mathcal{U}(\mathfrak{M}),\mathfrak{M}_*)$-topology.*

Let us define on $\mathfrak{M}_*\cong\{(p,\omega)\in\mathcal{L}(\mathfrak{M})\times\mathfrak{M}_*;\ \ p=r_*(\omega)\}$ (respectively $\mathfrak{M}_*\cong\{(p,\omega)\in\mathcal{L}(\mathfrak{M})\times\mathfrak{M}_*;\ \ p=l_*(\omega)\}$) the topology $\mathfrak{T}_{\mathfrak{M}_*}$ as the topology inherited from the product topology of $\mathcal{L}(\mathfrak{M})\times\mathfrak{M}_*$. The moment map $r_*:\mathfrak{M}_*\to\mathcal{L}(\mathfrak{M})$ (respectively $l_*:\mathfrak{M}_*\to\mathcal{L}(\mathfrak{M})$) is continuous with respect to $\mathfrak{T}_{\mathfrak{M}_*}$. Since the topology $\mathfrak{T}_{\mathfrak{M}_*}$ of $\mathfrak{M}_*$ is stronger than the uniform topology of $\mathfrak{M}_*$ the action (2.38) (respectively (2.39)) is also continuous with respect to $\mathfrak{T}_{\mathfrak{M}_*}$.

Let us define the set

$$\mathcal{P}(\mathfrak{M}_*) := \{\omega \in \mathfrak{M}_* : \quad l_*(\omega) = r_*(\omega)\}. \tag{3.1}$$

We conclude from the Proposition (2.15) that subsets $\mathfrak{M}_*^+ \subset \mathfrak{M}_*^h \subset \mathcal{P}(\mathfrak{M}_*) \subset \mathfrak{M}_*$ of positive normal functionals, selfadjoint functionals and $\mathcal{P}(\mathfrak{M}_*)$ are invariant with respect to the predual inner action $I_* : \mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_* \to \mathfrak{M}_*$. The groupoid $\mathcal{U}(\mathfrak{M})$ acts continuously on $\mathfrak{M}_*^h$, $\mathfrak{M}_*^+$ and $\mathcal{P}(\mathfrak{M}_*)$ with respect to their $\mathfrak{T}_{\mathfrak{M}_*}$-topology. Since

$$s_*(I_{*u}\omega) = s_*(u\omega u^*) = uu^* = \mathbf{t}(u)$$
$$I_{*u}(I_{*v}\omega) = I_{*u}(v\omega v^*) = uv\omega v^* u^* = uv\omega(uv)^* = I_{*uv}\omega$$
$$I_{*\varepsilon(s_*(\omega))}\omega = I_{*s_*(\omega)}\omega = u^* u\omega u^* u = \omega$$

we see that the groupoid $\mathcal{U}(\mathfrak{M})$ acts on $\mathcal{P}(\mathfrak{M}_*)$ in the continuous way with respect to $\mathfrak{T}_{\mathfrak{M}_*}$ topology of $\mathcal{P}(\mathfrak{M}_*)$.

Summarizing the above considerations and applying the construction presented in the Appendix we have the following:

Theorem 3.3. *(i) The groupoids* $\mathcal{U}(\mathfrak{M}) *_l \mathfrak{M}$, $\mathcal{U}(\mathfrak{M}) *_r \mathfrak{M}$, $\mathcal{U}(\mathfrak{M}) * \mathcal{J}(\mathfrak{M})$, $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}^h$ *and* $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}^+$ *are topological groupoids with respect to the relative topology inherited from the product uniform topology of* $\mathcal{U}(\mathfrak{M}) \times \mathfrak{M}$ *.*

(ii) The groupoids $\mathcal{U}(\mathfrak{M}) *_{l_*} \mathfrak{M}_*$, $\mathcal{U}(\mathfrak{M}) *_{r_*} \mathfrak{M}_*$, $\mathcal{U}(\mathfrak{M}) * \mathcal{P}(\mathfrak{M}_*)$, $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^+$ *and* $\mathcal{U}(\mathfrak{M}) * \mathfrak{M}_*^h$ *are topological groupoids with respect to the relative topology inherited from the product uniform topology of* $\mathcal{U}(\mathfrak{M}) \times \mathfrak{M}_*$*.*

(iii) The groupoids listed above cover the groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$*.*

4. Banach-Lie groupoid structures of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

Now we show that the groupoids $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ have a canonically defined structures of complex and real Banach manifolds, respectively, which are consistent with their groupoid structures.

4.1. Banach manifold structure on $\mathcal{L}(\mathfrak{M})$

Let us begin from definition of a complex Banach manifold structure on the lattice $\mathcal{L}(\mathfrak{M})$ of projections of W^*-algebra $\mathfrak{M}$. For this reason for any

$p \in \mathcal{L}(\mathfrak{M})$ by $\Pi_p \subset \mathcal{L}(\mathfrak{M})$ we denote the subset of projections $q \in \mathcal{L}(\mathfrak{M})$ such that there exists the Banach splitting

$$\mathfrak{M} = q\mathfrak{M} \oplus (1-p)\mathfrak{M} \tag{4.1}$$

of $\mathfrak{M}$ on the right W^*-ideals. Note that from the condition (4.1) follows

$$q \wedge (1-p) = 0 \quad \text{and} \quad q \vee (1-p) = 1, \tag{4.2}$$

where "$\wedge$" and "$\vee$" are joint- and meet-operations on the projections in the lattice $\mathcal{L}(\mathfrak{M})$. Since for any pair of projections $e, f \in \mathcal{L}(\mathfrak{M})$ one has

$$(e \vee f) - e \sim f - (e \wedge f),$$

see [19], taking $e = 1-p$ and $f = q$ we obtain that $q \sim p$. So we have $\Pi_p \subset \mathcal{O}_p$ and thus $\Pi_{p'} \cap \Pi_p = \emptyset$ if $p' \not\sim p$. The inverse statement, i.e. that $p' \sim p$ implies $\Pi_{p'} \cap \Pi_p \neq \emptyset$ is not true in general case. For example for infinite W^*-algebra we can take $p \neq 1$ such that $p \sim 1$. Then $\Pi_1 = \{1\}$ and thus $\Pi_1 \cap \Pi_p = \emptyset$.

Using (4.1) we decompose

$$p = x - y \tag{4.3}$$

the projection p on two elements $x \in q\mathfrak{M}p$ and $y \in (1-p)\mathfrak{M}p$. In such a way we define the map $\varphi_p : \Pi_p \tilde{\to} (1-p)\mathfrak{M}p$ by

$$\varphi_p(q) := y. \tag{4.4}$$

Let us show that φ_p is a bijection of Π_p on the Banach space $(1-p)\mathfrak{M}p$. To this end for any $y \in (1-p)\mathfrak{M}p$ we define x by equality (4.3) and note that

$$p = px, \quad xp = x \quad \text{and} \quad x^2 = x. \tag{4.5}$$

Thus the left multiplication maps L_p and L_x on $\mathfrak{M}$ satisfy

$$L_p = L_p \circ L_x, \quad L_x = L_x \circ L_p \quad \text{and} \quad L_x \circ L_x = L_x \tag{4.6}$$

and

$$(1-x)\mathfrak{M} = Ker\ L_x = Ker\ L_p = (1-p)\mathfrak{M}. \tag{4.7}$$

From (4.6) one has $KerL_x = KerL_{xp} = Ker(L_x \circ L_p) \supset KerL_p = KerL_{px} = Ker(L_p \circ L_x) \supset KerL_x$ and thus $KerL_p = KerL_x$. For any $x^2 = x \in \mathfrak{M}$ if

$y \in KerL_x$ then $y = (1-x)y$. Thus one obtains that $KerL_x = (1-x)\mathfrak{M}$ and $KerL_p = (1-p)\mathfrak{M}$. The above proves (4.7).

From (4.7) we have

$$\mathfrak{M} = x\mathfrak{M} \oplus (1-p)\mathfrak{M}, \tag{4.8}$$

where $x\mathfrak{M}$ is right ideal of W^*-algebra generated by $x \in \mathfrak{M}$. Let us also note that

$$L_x : p\mathfrak{M} \tilde{\to} x\mathfrak{M} \quad \text{and} \quad L_p : x\mathfrak{M} \tilde{\to} p\mathfrak{M} \tag{4.9}$$

are mutually inverse isomorphisms of the corresponding right $\mathfrak{M}$-modules.

The left support $l(x)$ of $x \in \mathfrak{M}$ is the identity in W^*-subalgebra $x\mathfrak{M} \cap (x\mathfrak{M})^*$. Thus $l(x) \in x\mathfrak{M}$. This shows that $x\mathfrak{M} = l(x)\mathfrak{M}$ and

$$\mathfrak{M} = l(x)\mathfrak{M} \oplus (1-p)\mathfrak{M}, \tag{4.10}$$

i.e. $l(x) \in \Pi_p$. In such a way we prove that φ_p has the inverse defined by

$$\varphi_p^{-1}(y) := l(p+y). \tag{4.11}$$

Proposition 4.1. *If $x \in q\mathfrak{M}p$ is defined by the decomposition (4.3) then $x \in \mathcal{G}(\mathfrak{M})$ and $\boldsymbol{s}(x) = p$ and $\boldsymbol{t}(x) = q$. So one has section $\sigma_p : \Pi_p \to \boldsymbol{t}^{-1}(\Pi_p) \subset \mathcal{G}(\mathfrak{M})$ defined by*

$$\sigma_p(q) := x. \tag{4.12}$$

Proof. The above follows from (4.9) and from $x\mathfrak{M} = l(x)\mathfrak{M} = q\mathfrak{M}$. □

The following proposition describe the complex manifold structure on $\mathcal{L}(\mathfrak{M})$.

Proposition 4.2. *The atlas $(\Pi_p,\ \varphi_p)$, $p \in \mathcal{L}(\mathfrak{M})$, defines on $\mathcal{L}(\mathfrak{M})$ the structure of a complex Banach manifold of type $\mathfrak{G}$ in sense of [3], where $\mathfrak{G}$ is the set of Banach spaces $(1-p)\mathfrak{M}p$ indexed by elements $p \in \mathcal{L}(\mathfrak{M})$.*

Proof. Note that the domains Π_p, where $p \in \mathcal{L}(\mathfrak{M})$, of the maps $\varphi_p : \Pi_p \to (1-p)\mathfrak{M}p$ defined in (4.4) cover $\mathcal{L}(\mathfrak{M})$, i.e. $\bigcup_{p\in\mathcal{L}(\mathfrak{M})} \Pi_p = \mathcal{L}(\mathfrak{M})$.

Now we find the explicit formulae for the transitions maps

$$\varphi_p \circ \varphi_{p'}^{-1} : \varphi_{p'}(\Pi_p \cap \Pi_{p'}) \to \varphi_p(\Pi_p \cap \Pi_{p'}) \tag{4.13}$$

in the case when $\Pi_p \cap \Pi_{p'} \neq \emptyset$. For this reason let us take for $q \in \Pi_p \cap \Pi_{p'}$ the following splittings

$$\begin{aligned} \mathfrak{M} &= q\mathfrak{M} \oplus (1-p)\mathfrak{M} = p\mathfrak{M} \oplus (1-p)\mathfrak{M} \\ \mathfrak{M} &= q\mathfrak{M} \oplus (1-p')\mathfrak{M} = p'\mathfrak{M} \oplus (1-p')\mathfrak{M}. \end{aligned} \tag{4.14}$$

The splittings (4.14) lead to the corresponding decompositions of p and p'

$$\begin{aligned} p &= x - y & \qquad p &= a + b \\ p' &= x' - y' & \qquad 1 - p &= c + d \end{aligned} \tag{4.15}$$

where $x \in q\mathfrak{M}p$, $y \in (1-p)\mathfrak{M}p$, $x' \in q\mathfrak{M}p'$, $y' \in (1-p')\mathfrak{M}p'$, $a \in p'\mathfrak{M}p$, $b \in (1-p')\mathfrak{M}p$, $c \in p'\mathfrak{M}(1-p)$ and $d \in (1-p')\mathfrak{M}(1-p)$. Combining equations from (4.15) we obtain

$$q = \iota(x') + y'\iota(x') \tag{4.16}$$

$$q = (a + cy)\iota(x) + (b + dy)\iota(x). \tag{4.17}$$

Comparing (4.16) and (4.17) we find that

$$\iota(x') = (a + cy)\iota(x) \tag{4.18}$$

$$y'\iota(x') = (b + dy)\iota(x). \tag{4.19}$$

After substitution (4.18) into (4.19) and noting that $\mathbf{t}(a + cy) \leqslant p'$ we get

$$y' = (\varphi_{p'} \circ \varphi_p^{-1})(y) = (b + dy)\iota(a + cy). \tag{4.20}$$

All operations involved in the right-hand-side of equality (4.20) are smooth. □

Remark 4.3. If projections p' and p are equivalent, i.e. if there exists $x \in \mathcal{G}(\mathfrak{M})$ such that $p = \mathbf{s}(x)$ and $p' = \mathbf{t}(x)$, then the Banach spaces $(1-p)\mathfrak{M}p$ and $(1-p')\mathfrak{M}p'$ are isomorphic.

See also [1] for the investigation of infinite-dimensional Grassmannians as homogeneous spaces of the Banach-Lie group $U(\mathfrak{M})$. Note, that when $\mathfrak{M}$ is a finite W^*-algebra then the orbits of the inner action of the groupoid $\mathcal{U}(\mathfrak{M})$ and the orbits of the inner action of unitary group $U(\mathfrak{M})$ on the lattice $\mathcal{L}(\mathfrak{M})$ coincide.

4.2. Banach-Lie groupoid structure of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

Now let us introduce a structure of a Banach smooth manifold on $\mathcal{G}(\mathfrak{M})$.

For this purpose taking $p, \tilde{p} \in \mathcal{L}(\mathfrak{M})$ we define the covering of $\mathcal{G}(\mathfrak{M})$ by subsets:

$$\Omega_{\tilde{p}p} := \mathbf{t}^{-1}(\Pi_{\tilde{p}}) \cap \mathbf{s}^{-1}(\Pi_p). \tag{4.21}$$

Let us note here that $\Omega_{\tilde{p}p} \neq \emptyset$ if and only if $\tilde{p} \sim p$. Note also that the set Ω_{pp} is a subgroupoid of $\mathcal{G}(\mathfrak{M})$. If $\Omega_{\tilde{p}p} \neq \emptyset$ then one has the one-to-one map

$$\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p \tag{4.22}$$

of $\Omega_{\tilde{p}p}$ on an open subset of the direct sum of the Banach subspaces of the W^*-algebra $\mathfrak{M}$. This map we define by

$$\psi_{\tilde{p}p}(x) := \left(\varphi_{\tilde{p}}(\mathbf{t}(x)), \iota(\sigma_{\tilde{p}}(\mathbf{t}(x)))x\sigma_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x))\right), \tag{4.23}$$

where $\sigma_p(q) \in q\mathfrak{M}p$ and $\varphi_p(q) \in (1-p)\mathfrak{M}p$ are obtained from the decomposition

$$p = \sigma_p(q) - \varphi_p(q) \tag{4.24}$$

of p with respect to (4.1). Recall that $\sigma_p : \Pi_p \to \mathbf{t}^{-1}(\Pi_p)$ is a section defined in (4.12).

Proposition 4.4. *The maps*

$$\left(\Omega_{\tilde{p}p}, \psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p\right), \tag{4.25}$$

where $(p, \tilde{p}) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ are pairs of equivalent projections, form a smooth atlas on the groupoid $\mathcal{G}(\mathfrak{M})$ in sense of [3].

Proof. The map $\psi_{\tilde{p}p}^{-1} : \psi_{\tilde{p}p}(\Omega_{\tilde{p}p}) \to \Omega_{\tilde{p}p}$ inverse to (4.23) looks as follows

$$\psi_{\tilde{p}p}^{-1}(\tilde{y}, z, y) := \sigma_{\tilde{p}}(\tilde{q})z\iota(\sigma_p(q)) = (\tilde{p}+\tilde{y})z\iota(p+y) \tag{4.26}$$

where $\tilde{q} = l(\tilde{p}+\tilde{y})$ and $q = l(p+y)$ are left supports of $\tilde{p}+\tilde{y}$ and $p+y$, respectively. The transition maps

$$\psi_{\tilde{p}'p'} \circ \psi_{\tilde{p}p}^{-1} : \psi_{\tilde{p}p}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p}) \to \psi_{\tilde{p}'p'}(\Omega_{\tilde{p}'p'} \cap \Omega_{\tilde{p}p})$$

for $(\tilde{y}, z, y) \in \psi_{\tilde{p}p}(\Omega_{\tilde{p'}p'} \cap \Omega_{\tilde{p}p})$ are given by

$$(\psi_{\tilde{p'}p'} \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) := (\tilde{y'}, z', y'), \tag{4.27}$$

where

$$\tilde{y'} = (\varphi_{\tilde{p'}} \circ \varphi_{\tilde{p}}^{-1})(\tilde{y}) = (\tilde{b} + \tilde{d}\tilde{y})\iota(\tilde{a} + \tilde{c}\tilde{y}) \tag{4.28}$$

$$y' = (\varphi_{p'} \circ \varphi_{p}^{-1})(y) = (b + dy)\iota(a + cy) \tag{4.29}$$

and

$$z' = \iota(\tilde{p'} + \tilde{y'})(\tilde{p} + \tilde{y})z\iota(p + y)(p' + y'). \tag{4.30}$$

We note that all maps in (4.28), (4.29), (4.30) are smooth. □

The smooth (analytic) Banach manifold structure on $\mathcal{G}(\mathfrak{M})$ has the type $\mathfrak{G}$, where $\mathfrak{G}$ is the set of Banach spaces $(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$ indexed by the pair of equivalent elements of $\mathcal{L}(\mathfrak{M})$.

Theorem 4.5. *The groupoid $\mathcal{G}(\mathfrak{M})$ is a Banach-Lie groupoid over the base $\mathcal{L}(\mathfrak{M})$ with respect to the smooth (analytic) Banach manifold structure of type $\mathfrak{G}$ defined by the atlas (4.25).*

Proof. We show that all groupoid maps and the groupoid product are smooth (analytic) with respect to the considered Banach manifold structure.

(i) For the source and target map we have

$$(\varphi_p \circ \mathbf{s} \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) = y, \tag{4.31}$$

$$(\varphi_{\tilde{p}} \circ \mathbf{t} \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) = \tilde{y}. \tag{4.32}$$

We assumed in (4.31) and (4.32) that $(\tilde{y}, z, y) \in \psi_{\tilde{p}p}(\Omega_{\tilde{p}p})$, $\mathbf{s}(\psi_{\tilde{p}p}^{-1}(\tilde{y}, z, y)) \in \Pi_p$ and $\mathbf{t}(\psi_{\tilde{p}p}^{-1}(\tilde{y}, z, y)) \in \Pi_{\tilde{p}}$ respectively. We conclude from (4.31) and (4.32) that $\varphi_p \circ \mathbf{s} \circ \psi_{\tilde{p}p}^{-1}$ and $\varphi_{\tilde{p}} \circ \mathbf{t} \circ \psi_{\tilde{p}p}^{-1}$ are smooth (analytic) submersions.

(ii) For the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ we have

$$(\psi_{\tilde{p}p} \circ \varepsilon \circ \varphi_p^{-1})(y) = \left((\varphi_{\tilde{p}} \circ \varphi_p^{-1})(y), \iota(\sigma_{\tilde{p}}(\varphi_p^{-1}(y)))\sigma_p(\varphi_p^{-1}(y)), y\right), \tag{4.33}$$

where $y \in \varphi_p(\Pi_p)$. Since $\sigma_{\tilde{p}} : \Pi_{\tilde{p}} \to \mathbf{t}^{-1}(\Pi_{\tilde{p}})$ and $\sigma_p : \Pi_p \to \mathbf{t}^{-1}(\Pi_p)$ are smooth (analytic) sections we obtain that $\psi_{\tilde{p}p} \circ \varepsilon \circ \varphi_p^{-1}$ is smooth (analytic) map too.

(iii) The inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ takes $\Omega_{\tilde{p}p}$ onto $\Omega_{p\tilde{p}}$ and we have

$$(\psi_{p\tilde{p}} \circ \iota \circ \psi_{\tilde{p}p}^{-1})(\tilde{y}, z, y) = (y, \iota(z), \tilde{y}). \tag{4.34}$$

Thus ι is a complex smooth (analytic) map.

(iv) Let us take $x_1 \in \Omega_{\tilde{p}_1 p_1}$ and $x_2 \in \Omega_{\tilde{p}_2 p_2}$ such that $\mathbf{s}(x_1) = \mathbf{t}(x_2) \in \Pi_{\tilde{p}_2} \cap \Pi_{p_1}$. Assuming $\psi_{\tilde{p}_1 p_1}(x_1) = (\tilde{y_1}, z_1, y_1)$ and $\psi_{\tilde{p}_2 p_2}(x_2) = (\tilde{y_2}, z_2, y_2)$ we obtain that

$$\begin{aligned} &\psi_{\tilde{p}_1 p_2}(\psi_{\tilde{p}_1 p_1}^{-1}(\tilde{y_1}, z_1, y_1)\psi_{\tilde{p}_2 p_2}^{-1}(\tilde{y_2}, z_2, y_2)) \\ &= (\tilde{y_1}, z_1 \iota(\sigma_{p_1}(\varphi_{p_1}^{-1}(y_1)))(\sigma_{\tilde{p}_2}(\varphi_{\tilde{p}_2}^{-1}(\tilde{y_2})))z_2, y_2). \end{aligned} \tag{4.35}$$

To summarize, we conclude that $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a Banach-Lie groupoid. □

4.3. Banach-Lie groupoid structure of $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

In order to investigate the structure of real Banach manifold on $\mathcal{U}(\mathfrak{M})$ we recall that one can define $\mathcal{U}(\mathfrak{M})$ as the set of the fixed points of the groupoid automorphism $J : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ defined in (2.10). Let us note that J is a bijection of the domain $\Omega_{\tilde{p}p}$ of the chart (4.22) defined in (4.23). Recall also that $\mathcal{G}(\mathfrak{M})_p^{\tilde{p}}$ is an open subset of the Banach subspace $\tilde{p}\mathfrak{M}p$. Since J is an involution and $J(x) = x$ for $x \in \mathcal{U}(\mathfrak{M})$ one has

$$(DJ(x))^2 = \mathbf{1} \tag{4.36}$$

for the Fréchet derivative $DJ(x) : T_x\mathcal{G}(\mathfrak{M}) \to T_x\mathcal{G}(\mathfrak{M})$ of the map J at the element $x \in \mathcal{U}(\mathfrak{M})$. Thus we obtain a Banach splitting

$$T_x\mathcal{G}(\mathfrak{M}) = T_x^+\mathcal{G}(\mathfrak{M}) \oplus T_x^-\mathcal{G}(\mathfrak{M}) \tag{4.37}$$

of the tangent space $T_x\mathcal{G}(\mathfrak{M})$ defined by the projections

$$P_\pm(x) := \frac{1}{2}\left(\mathbf{1} \pm DJ(x)\right). \tag{4.38}$$

In the next, instead of $\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$ we will use a new chart defined by

$$\theta_{\tilde{p}p}(x) := ((\varphi_{\tilde{p}}(\mathbf{t}(x)), (u_{\tilde{p}}(\mathbf{t}(x)))^* x u_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x))) = (\tilde{y}, v, y), \tag{4.39}$$

where

$$u_p(\mathbf{s}(x)) := \sigma_p(\mathbf{s}(x))|\sigma_p(\mathbf{s}(x))|^{-1} \tag{4.40}$$

$$u_{\tilde{p}}(\mathbf{t}(x)) := \sigma_{\tilde{p}}(\mathbf{t}(x))|\sigma_{\tilde{p}}(\mathbf{t}(x))|^{-1} \tag{4.41}$$

are partial isometries defined by the polar decompositions of $\sigma_p(\mathbf{s}(x))$ and $\sigma_{\tilde{p}}(\mathbf{t}(x))$, respectively. The coordinates $(\tilde{y}, v, y)$ defined in (4.39) passing through the set

$$((1-\tilde{p})\mathfrak{M}\tilde{p}) \times \mathcal{G}(\mathfrak{M})_p^{\tilde{p}} \times ((1-p)\mathfrak{M}p)$$

and $x \in \mathcal{U}(\mathfrak{M}) \cap \Omega_{\tilde{p}p}$ if and only if $v \in \mathcal{U}(\mathfrak{M})_p^{\tilde{p}}$. Using the chart (4.39) we find

$$\begin{aligned} &D\left(\theta_{\tilde{p}p} \circ J \circ \theta_{\tilde{p}p}^{-1}\right)(\tilde{y}, v, y)(\triangle\, \tilde{y}, \triangle\, v, \triangle\, y) \\ &= (\triangle\, \tilde{y}, -\iota(v^*)(\triangle\, v)^*\iota(v^*), \triangle\, y) \end{aligned} \tag{4.42}$$

for $(\triangle\, \tilde{y}, \triangle\, v, \triangle\, y) \in (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p \cong T_x\mathcal{G}(\mathfrak{M})$. After these preliminary remarks let us formulate

Theorem 4.6. *(i) The groupoid $\mathcal{U}(\mathfrak{M})$ of partial isometries has a structure of a real Banach manifold of type $\mathfrak{G}$, where the family $\mathfrak{G}$ consist of the real Banach spaces*

$$(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus i\tilde{p}\mathfrak{M}^h p \oplus (1-p)\mathfrak{M}p$$

parameterized by the pairs $(\tilde{p}, p) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ of equivalent projections.

(ii) The groupoid $\mathcal{U}(\mathfrak{M})$ is a closed real Banach Lie subgroupoid of $\mathcal{G}(\mathfrak{M})$ when $\mathcal{G}(\mathfrak{M})$ is endowed with the real Banach Lie groupoid structure underlaying its complex Banach Lie groupoid structure.

Proof. In order to show that $\mathcal{U}(\mathfrak{M})$ is a real Banach submanifold of $\mathcal{G}(\mathfrak{M})$ we define for each $x \in \Omega_{\tilde{p}p} \cap \mathcal{U}(\mathfrak{M})$ the subset $\Omega^x_{\tilde{p}p} \subset \Omega_{\tilde{p}p}$ consisting of such elements $x' \in \Omega_{\tilde{p}p}$ for which $u \in \mathcal{G}(\mathfrak{M})_p^{\tilde{p}}$ defined by $\theta_{\tilde{p}p}(x') = (\tilde{y}', u, y')$ satisfies the inequality

$$\|v - u\| < 1 \tag{4.43}$$

where the partial isometry v is the second coordinate of x defined by (4.39).

From the inequality (4.43) we find that $v^*u \in G(p\mathfrak{M}p)$ satisfies

$$\|p - v^*u\| < 1 \tag{4.44}$$

So, one can define the map $\theta^x_{\tilde{p}p} : \Omega^x_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$ as follows

$$\theta^x_{\tilde{p}p}(x') := (\tilde{y}', log(v^*u), y'). \tag{4.45}$$

If $x' \in \Omega^x_{\tilde{p}p} \cap \mathcal{U}(\mathfrak{M})$ then $v^*u \in U(p\mathfrak{M}p)$. Thus $\log(v^*u) \in ip\mathfrak{M}^h p$, which implies that the image of the map

$$\theta^x_{\tilde{p}p} : \Omega^x_{\tilde{p}p} \cap \mathcal{U}(\mathfrak{M}) \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus i\tilde{p}\mathfrak{M}^h p \oplus (1-p)\mathfrak{M}p \tag{4.46}$$

is contained in the first component of the Banach splitting

$$\begin{aligned} &(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p \\ &= \Big((1-\tilde{p})\mathfrak{M}\tilde{p} \oplus i\tilde{p}\mathfrak{M}^h p \oplus (1-p)\mathfrak{M}p\Big) \oplus \Big(\{0\} \oplus \tilde{p}\mathfrak{M}^h p \oplus \{0\}\Big) \end{aligned} \tag{4.47}$$

of the real Banach space underling of the complex Banach space $(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$. Hence we have the atlas $(\Omega^x_{\tilde{p}p} \cap \mathcal{U}(\mathfrak{M}), \theta^x_{\tilde{p}p})$ on $\mathcal{U}(\mathfrak{M})$ parametrized by $(\tilde{p}, p) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ and $x \in \mathcal{U}(\mathfrak{M})$. This atlas defines on $\mathcal{U}(\mathfrak{M})$ a structure of a real Banach manifold which is consistent with the groupoids structure of $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. The consistence of (4.46) with the splitting (4.47) means that $\mathcal{U}(\mathfrak{M})$ is a submanifold of $\mathcal{G}(\mathfrak{M})$. For the definition of Banach submanifold see Section 2 of Chapter 2 in [10]. Summarizing we see that $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a Banach Lie subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. □

Since the group $G(p\mathfrak{M}p)$ is a complexification of the unitary group $U(p\mathfrak{M}p)$ we conclude from Theorem 4.6 that the groupoid $\mathcal{G}(\mathfrak{M})$ can be considered in some sense as a complexification of $\mathcal{U}(\mathfrak{M})$.

4.4. Short exact sequence of Banach-Lie groupoids

In Subsection 2.3 we have defined the inner subgroupoid $\mathcal{J}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M})$ of partially invertible elements of a W^*-algebra $\mathfrak{M}$. Now we show that $\mathcal{J}(\mathfrak{M})$ is a Banach-Lie subgroupoid of $\mathcal{G}(\mathfrak{M})$.

In order to exhibit the Banach submanifold structure of $\mathcal{J}(\mathfrak{M})$ we define the charts

$$\kappa_p : \Omega_p \to p\mathfrak{M}p \oplus (1-p)\mathfrak{M}p,$$

parametrized by $p \in \mathcal{L}(\mathfrak{M})$, where

$$\Omega_p := \mathcal{J}(\mathfrak{M}) \cap \Omega_{pp} = \bigcup_{q \in \Pi_p} \mathbf{t}^{-1}(q) \cap \mathbf{s}^{-1}(q)$$

and

$$\kappa_p(x) := \left(\iota(\sigma_p(\mathbf{t}(x)))x\sigma_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x))\right). \tag{4.48}$$

The transition maps $\kappa_{p'} \circ \kappa_p^{-1} : \kappa_p(\Omega_{p'} \cap \Omega_p) \to \kappa_{p'}(\Omega_{p'} \cap \Omega_p)$ for (4.48) are

$$\begin{aligned} (z', y') &= \left(\kappa_{p'} \circ \kappa_p^{-1}\right)(z, y) \\ &= \big(\iota(p' + y')(p + y)z\iota(p + y)(p' + y'), (b + dy)\iota(a + cy)\big). \end{aligned} \tag{4.49}$$

Note here that for $x \in \mathcal{J}(\mathfrak{M})$ one has $\mathbf{s}(x) = \mathbf{t}(x)$. So, it follows from (4.23) that $x \in \Omega_p = \mathcal{J}(\mathfrak{M}) \cap \Omega_{pp}$ if and only if $\tilde{y} = y$. Thus, using Theorem 4.5 we see that the groupoid $\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a Banach-Lie subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Let us take the subset

$$\bigcup_{\tilde{p} \sim p} (\Pi_{\tilde{p}} \times \Pi_p) \subset \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M}) \tag{4.50}$$

of the pair groupoid $\mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. Since $\bigcup_{\tilde{p} \sim p} (\Pi_{\tilde{p}} \times \Pi_p)$ is an open subset of $\mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ closed with respect to the groupoid operations one can consider $\bigcup_{\tilde{p} \sim p} (\Pi_{\tilde{p}} \times \Pi_p) \rightrightarrows \mathcal{L}(\mathfrak{M})$ as a Banach-Lie subgroupoid of the pair groupoid $\mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Proposition 4.7. *One has the following short exact sequence of Banach-Lie groupoids*

$$\begin{array}{ccccc} \mathcal{J}(\mathfrak{M}) & \longrightarrow & \mathcal{G}(\mathfrak{M}) & \longrightarrow & \bigcup_{\tilde{p} \sim p} (\Pi_{\tilde{p}} \times \Pi_p) \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}(\mathfrak{M}) & \xrightarrow{id} & \mathcal{L}(\mathfrak{M}) & \xrightarrow{id} & \mathcal{L}(\mathfrak{M}), \end{array} \tag{4.51}$$

i.e. the quotient groupoid $\mathcal{G}(\mathfrak{M})/\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ *is isomorphic with the groupoid* $\bigcup_{\tilde{p}\sim p}(\Pi_{\tilde{p}} \times \Pi_p) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Proof. According to Chapter I, §2 of [11] the quotient groupoid $\mathcal{G}(\mathfrak{M})/\mathcal{J}(\mathfrak{M})$ consists of the classes $[x]$ of the equivalence relation on $\mathcal{G}(\mathfrak{M})$ defined as follows: elements $x_1, x \in \mathcal{G}(\mathfrak{M})$ are equivalent if and only if there exists $g \in G(p\mathfrak{M}p)$, where $p = \mathbf{t}(x_1)$ such that $gx_1 = x$.

One easily checks that all groupoid operations of $\mathcal{G}(\mathfrak{M})/\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ are inherited from those of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. We observe also that $(\mathcal{G}(\mathfrak{M})/\mathcal{J}(\mathfrak{M}))^q_p$ is a one element set if and only if $p \sim q$ and it is the empty set in the opposite case. Thus we can identify $\mathcal{G}(\mathfrak{M})/\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ with $\bigcup_{\tilde{p}\sim p}(\Pi_{\tilde{p}} \times \Pi_p) \rightrightarrows \mathcal{L}(\mathfrak{M})$. □

5. Groupoids and Banach Lie-Poisson structure of $\mathfrak{M}_*$

In [13] it was shown that the predual space $\mathfrak{M}_*$ of a W^*-algebra $\mathfrak{M}$ has canonically defined Lie-Poisson structure. This follows from $ad^*(\mathfrak{M})$-invariance of Banach subspace $\mathfrak{M}_* \subset \mathfrak{M}^*$, where $ad_x(y) := xy - yx$. One defines the Lie-Poisson bracket of $f, g \in C^\infty(\mathfrak{M}_*, \mathbb{C})$ as follows

$$\{f, g\}(\omega) := \langle \omega, [Df(\omega), Dg(\omega)]\rangle \tag{5.1}$$

for $\omega \in \mathfrak{M}_*$. Note that Fréchet derivatives $Df(\omega), Dg(\omega)$ belong to $\mathfrak{M}$ which allows to take the commutator of them. The predual space $\mathfrak{M}_*$ as well as the Lie-Poisson bracket (5.1) is invariant with respect to the Ad^*-action of the Banach group $G(\mathfrak{M})$.

The pairing between $\mathfrak{M}^h_*$ and the real Banach-Lie algebra $i\mathfrak{M}^h$ of anti-hermitian elements of $\mathfrak{M}$ is defined by

$$\mathfrak{M}^h_* \times i\mathfrak{M}^h \ \ni \ (\omega, x) \ \mapsto \ i\langle \omega, x\rangle \ \in \ \mathbb{R}.$$

Hence, multiplying the right hand side of definition (5.1) by $i = \sqrt{-1}$ one obtains the Lie-Poisson bracket for real valued functions $f, g \in C^\infty(\mathfrak{M}^h_*, \mathbb{R})$ defined on the hermitian part $\mathfrak{M}^h_*$ of $\mathfrak{M}_*$.

As in the complex case the Banach Lie-Poisson structure of $\mathfrak{M}^h_*$ is $Ad^*(U(\mathfrak{M}))$-invariant. If the isotropy subgroup $U(\mathfrak{M})_\rho := \{g \in U(\mathfrak{M}) : Ad^*_g\rho = \rho\}$ is a Banach-Lie subgroup of $U(\mathfrak{M})$, then the connected components of the coadjoint orbits $Ad^*_{U(\mathfrak{M})}\rho$ of $\rho \in \mathfrak{M}^h_*$ are in general weakly symplectic leaves of the real Banach-Lie-Poisson space $\mathfrak{M}^h_*$, see Theorem 7.3 and Theorem 7.4 in [13].

5.1. Tangent $TG(\mathfrak{M}) \rightrightarrows \mathfrak{M}$ and precotangent $T_*G(\mathfrak{M}) \rightrightarrows \mathfrak{M}_*$ groupoids of $\mathfrak{M}$

Now let us apply the definitions of the groupoids structures on the tangent bundle TG and cotangent bundle T^*G of a Lie group G, e.g. see [11], to the case of Banach-Lie group $G(\mathfrak{M})$. We will do this with some modification. Namely in our considerations we replace the cotangent bundle $T^*G(\mathfrak{M})$ by the pre-cotangent bundle $T_*G(\mathfrak{M})$ of $G(\mathfrak{M})$. Note that in the finite dimensional case the bundles T^*G and T_*G are canonically isomorphic. In our case the cotangent bundle $T^*G(\mathfrak{M})$, contrary to the pre-cotangent bundle $T_*G(\mathfrak{M})$ does not have the symplectic structure related to the Banach Lie-Poisson structure of $\mathfrak{M}_*$ defined by (5.1).

The groupoid structure on $TG(\mathfrak{M})$ is defined as follows. The base of $TG(\mathfrak{M})$ is the tangent space $T_eG(\mathfrak{M})$ at the identity element $e \in G(\mathfrak{M})$. The source map $\mathbf{s} : TG(\mathfrak{M}) \to T_eG(\mathfrak{M})$ and the target map $\mathbf{t} : TG(\mathfrak{M}) \to T_eG(\mathfrak{M})$ are defined as follows

$$\begin{aligned} \mathbf{s}(a) &:= DL_{\pi(a)^{-1}}(\pi(a))a, \\ \mathbf{t}(a) &:= DR_{\pi(a)^{-1}}(\pi(a))a, \end{aligned} \tag{5.2}$$

where $a \in TG(\mathfrak{M})$ and $\pi : TG(\mathfrak{M}) \to G(\mathfrak{M})$ is the canonical projection on the base. The identity section $\varepsilon : T_eG(\mathfrak{M}) \to TG(\mathfrak{M})$ is done by the inclusion of the fibre $T_eG(\mathfrak{M}) \subset TG(\mathfrak{M})$. The involution $\iota : TG(\mathfrak{M}) \to TG(\mathfrak{M})$ one defines by

$$\iota(a) := DL_{\pi(a)^{-1}}(e) \circ DR_{\pi(a)^{-1}}(\pi(a))a. \tag{5.3}$$

Finally the groupoid product is defined by

$$ab := DL_{\pi(a)}(\pi(b))b \tag{5.4}$$

if and only if $(a, b) \in TG(\mathfrak{M})^{(2)}$, i.e. $\mathbf{s}(a) = \mathbf{t}(b)$.

As a base for groupoid structure of $T_*G(\mathfrak{M})$ we take the pre-cotangent space $T_{*e}G(\mathfrak{M})$ at $e \in G(\mathfrak{M})$. The identity section $\varepsilon_* : T_{*e}G(\mathfrak{M}) \to T_*G(\mathfrak{M})$ we define as an inclusion $T_{*e}G(\mathfrak{M}) \subset T_*G(\mathfrak{M})$.

Let us take $\xi \in T_*G(\mathfrak{M})$ and let $\pi(\xi) \in G(\mathfrak{M})$ be the projection of ξ on the base. Then one defines the source and target maps as follows:

$$\begin{aligned} \mathbf{s}_*(\xi) &:= (DL_{\pi(\xi)}(e))^*\xi, \\ \mathbf{t}_*(\xi) &:= (DR_{(\pi(\xi))}(e))^*\xi. \end{aligned} \tag{5.5}$$

The inversion $\iota_* : \; T_*G(\mathfrak{M}) \;\to\; T_*G(\mathfrak{M})$ is defined by

$$\iota_*(\xi) := (DL_{\pi(\xi)}(\pi(\xi))^{-1})^* \circ (DR_{(\pi(\xi))}(e))^*\xi. \tag{5.6}$$

The product of elements $\xi, \eta \in T_*G(\mathfrak{M})$ such that $\mathbf{s}(\xi) = \mathbf{t}(\eta)$ is given by

$$\xi\eta := (DL_{(\pi(\xi))^{-1}}(\pi(\xi)\pi(\eta)))^*\eta. \tag{5.7}$$

The precotangent bundle $T_*G(\mathfrak{M})$ is a weak symplectic complex Banach manifold with the weak symplectic form defined in the following way

$$\begin{aligned} &\Omega_L(g,\rho)((a,\xi),(b,\eta)) \\ &\quad = \langle \eta, DL_{g^{-1}}(g)a\rangle - \langle \xi, DL_{g^{-1}}(g)b\rangle - \langle \rho, [DL_{g^{-1}}(g)a, DL_{g^{-1}}(g)b]\rangle, \end{aligned} \tag{5.8}$$

where $g \in G(\mathfrak{M}),\; a, b \in T_gG(\mathfrak{M}),\; \rho, \xi, \eta \in T_{*e}G(\mathfrak{M})$, see [13]. Thus defined weak symplectic structure is consistent with the groupoid structure of $T_*G(\mathfrak{M})$ in the sense of [6], [9], [21], [22]. Hence one can consider $T_*G(\mathfrak{M})$ as a weak symplectic groupoid.

The definition of action groupoid structure on the product $G \times M$, where G is a group acting on a set M, can be found in Appendix D. From this general definition one gets action groupoid structures on $G(\mathfrak{M}) \times \mathfrak{M}$ and $G(\mathfrak{M}) \times \mathfrak{M}_*$ defined by adjoint $Ad : G(\mathfrak{M}) \to Aut\ \mathfrak{M}$ and co-adjoint $Ad^* : G(\mathfrak{M}) \to Aut\ \mathfrak{M}_*$ representation of Banach-Lie group $G(\mathfrak{M})$:

$$Ad_g x = gxg^{-1} \tag{5.9}$$

$$\langle Ad^*_g \omega, x\rangle := \langle \omega, Ad_{g^{-1}} x\rangle, \tag{5.10}$$

where $x \in \mathfrak{M}$ and $\omega \in \mathfrak{M}_*$ respectively.

The vector bundles trivializations $\phi : \; TG(\mathfrak{M}) \to G(\mathfrak{M}) \times \mathfrak{M}$ and $\phi_* : T_*G(\mathfrak{M}) \to G(\mathfrak{M}) \times \mathfrak{M}_*$ defined by

$$\phi(a) := (\pi(a), DL_{\pi(a)^{-1}}(\pi(a))a) \tag{5.11}$$

$$\phi_*(\xi) := (\pi(\xi), (DL_{\pi(\xi)}(e))^*\xi) \tag{5.12}$$

give the canonical groupoid isomorphisms $\phi : TG(\mathfrak{M}) \to G(\mathfrak{M}) \times \mathfrak{M}$ and $\phi_* : T_*G(\mathfrak{M}) \to G(\mathfrak{M}) \times \mathfrak{M}_*$. To this end we use the identifications $T_eG(\mathfrak{M}) \cong \mathfrak{M}$ and $T_{*e}G(\mathfrak{M}) \cong \mathfrak{M}_*$.

Let us define the injective immersions of the groupoids $\Lambda : TG(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M}) * \mathfrak{M}$ and $\Lambda_* : T_*G(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M}) * \mathfrak{M}_*$ by:

$$\Lambda(a) := \big(\pi(a)\; l(DL_{\pi(a)^{-1}}(\pi(a))a), DL_{\pi(a)^{-1}}(\pi(a))a\big) \tag{5.13}$$

$$\Lambda_*(\xi) := \big(\pi(\xi)\; l((DL_{\pi(\xi)}(e))^*\xi), DL_{\pi(\xi)}(e)^*\xi\big) \tag{5.14}$$

respectively.

Proposition 5.1. *One has the following groupoid monomorphisms*

$$\begin{array}{ccc} TG(\mathfrak{M}) & \xrightarrow{\ \Lambda\ } & \mathcal{G}(\mathfrak{M}) * \mathfrak{M} \\ s \Big\downarrow\Big\downarrow t & & \tilde{s} \Big\downarrow\Big\downarrow \tilde{t} \\ \mathfrak{M} & \xrightarrow{\ id\ } & \mathfrak{M} \end{array} \tag{5.15}$$

and

$$\begin{array}{ccc} T_*G(\mathfrak{M}) & \xrightarrow{\ \Lambda_*\ } & \mathcal{G}(\mathfrak{M}) * \mathfrak{M}_* \\ s \Big\downarrow\Big\downarrow t & & \tilde{s} \Big\downarrow\Big\downarrow \tilde{t} \\ \mathfrak{M}_* & \xrightarrow{\ id\ } & \mathfrak{M}_* \end{array} \tag{5.16}$$

of the groupoids $TG(\mathfrak{M}) \rightrightarrows \mathfrak{M}$ and $T_*G(\mathfrak{M}) \rightrightarrows \mathfrak{M}_*$ into the action groupoids $\mathcal{G}(\mathfrak{M}) * \mathfrak{M} \rightrightarrows \mathfrak{M}$ and $\mathcal{G}(\mathfrak{M}) * \mathfrak{M}_* \rightrightarrows \mathfrak{M}_*$, where Λ and Λ_* are defined in (5.13) and (5.14)

Proof. In order to see that $\Lambda : TG(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M}) * \mathfrak{M}$ commutes with source and target maps we note that

$$\begin{aligned} (\tilde{\mathbf{s}} \circ \Lambda)(a) &= \tilde{\mathbf{s}}\left(\pi(a)\; l(DL_{\pi(a)^{-1}}(\pi(a))a), DL_{\pi(a)^{-1}}(\pi(a))a\right) \\ &= DL_{\pi(a)^{-1}}(\pi(a))a = (id \circ \mathbf{s})(a), \\ (\tilde{\mathbf{t}} \circ \Lambda)(a) &= \tilde{\mathbf{t}}\left(\pi(a)\; l(DL_{\pi(a)^{-1}}(\pi(a))a), DL_{\pi(a)^{-1}}(\pi(a))a\right) \\ &= Ad_{\pi(a)\; l(DL_{\pi(a)^{-1}}(\pi(a))a)} DL_{\pi(a)^{-1}}(\pi(a))a \\ &= Ad_{\pi(a)} DL_{\pi(a)^{-1}}(\pi(a))a = (id \circ \mathbf{t})(a). \end{aligned}$$

Since $l(DL_{\pi(a)^{-1}}(\pi(a))a) = Ad_{\pi(a)}l(DL_{\pi(b)^{-1}}(\pi(b))b)$ the following shows that Λ preserves also the groupoid product

$$\begin{aligned}
&\Lambda(a)\Lambda(b)\\
&= \big(\pi(a)\ l(DL_{\pi(a)^{-1}}(\pi(a))a), DL_{\pi(a)^{-1}}(\pi(a))a\big)\\
&\quad \big(\pi(b)\ l(DL_{\pi(b)^{-1}}(\pi(b))b), DL_{\pi(b)^{-1}}(\pi(b))b\big)\\
&= \big(\pi(a)\ l(DL_{\pi(a)^{-1}}(\pi(a))a)(\pi(b))\ l(DL_{\pi(b)^{-1}}(\pi(b))b), DL_{\pi(b)^{-1}}(\pi(b))b\big)\\
&= \Lambda(DL_{\pi(a)}(\pi(b))b) = \Lambda(ab).
\end{aligned}$$

The proof for (5.16) can be done in the similar way. □

5.2. Tangent $T\mathcal{G}(\mathfrak{M}) \rightrightarrows T\mathcal{L}(\mathfrak{M})$ and precotangent $T_*\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{A}_*\mathcal{G}(\mathfrak{M})$ prolongations of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

Now instead of the complex Banach-Lie group $G(\mathfrak{M})$ let us consider the groupoid of partially invertible elements $\mathcal{G}(\mathfrak{M})$. In this case we come to the following statements.

Since $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a Banach Lie groupoid one can define its tangent prolongation $T\mathcal{G}(\mathfrak{M}) \rightrightarrows T\mathcal{L}(\mathfrak{M})$ which is a Banach Lie $\mathcal{VB}$-groupoid, (see e.g. Definition 11.2.1 in [12]), i.e. one has

$$\begin{array}{ccc}
T\mathcal{G}(\mathfrak{M}) & \xrightarrow{\tilde{q}} & \mathcal{G}(\mathfrak{M})\\
D\mathbf{s}\ \Big\downarrow\Big\downarrow\ D\mathbf{t} & & \mathbf{s}\ \Big\downarrow\Big\downarrow\ \mathbf{t}\\
T\mathcal{L}(\mathfrak{M}) & \xrightarrow{q} & \mathcal{L}(\mathfrak{M})
\end{array} \tag{5.17}$$

where the vector bundle projections q and $\tilde{q}$ on the bases define the groupoid morphism, the tangent maps $D\mathbf{s}, D\mathbf{t}, D\iota, D\varepsilon$ are vector bundle morphisms.

Let us note also that the map

$$(\tilde{q}, D\mathbf{s}) : T\mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M}) \times T\mathcal{L}(\mathfrak{M})$$

of tangent groupoid $T\mathcal{G}(\mathfrak{M})$ on $\mathcal{G}(\mathfrak{M}) \times_{\mathcal{L}(\mathfrak{M})} T\mathcal{L}(\mathfrak{M}) := \{(x, v) \in \mathcal{G}(\mathfrak{M}) \times T\mathcal{L}(\mathfrak{M});\ \mathbf{s}(x) = q(v)\}$ is a surjective submersion.

Since $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a Banach Lie groupoid we can define its Banach Lie algebroid $\mathcal{A}\mathcal{G}(\mathfrak{M})$ which is a vector bundle $\varrho : \mathcal{A}\mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ over the lattice $\mathcal{L}(\mathfrak{M})$. It follows from the general theory of $\mathcal{VB}$-groupoids that the bundle $\varrho : \mathcal{A}\mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ is the core of the tangent prolongation groupoid $T\mathcal{G}(\mathfrak{M}) \rightrightarrows T\mathcal{L}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. For definitions of the Lie algebroid of a Lie groupoid $G \rightrightarrows M$ and the core of $\mathcal{VB}$-groupoid see for example Definition 3.5.8 in [12] or Appendix F, and Subsection 11.2 of [12], respectively.

The algebroid $\mathcal{A}\mathcal{G}(\mathfrak{M})$ and its predual $\mathcal{A}_*\mathcal{G}(\mathfrak{M})$ are most crucial for the Poisson aspect of the investigated theory. Namely, extending the considerations from the finite dimensional case, see e.g. [12], [21], [22], to the Banach-Lie context we obtain the Banach-Lie $\mathcal{VB}$-groupoid

$$\begin{array}{ccc} T_*\mathcal{G}(\mathfrak{M}) & \xrightarrow{\ \tilde{q}_*\ } & \mathcal{G}(\mathfrak{M}) \\ \tilde{\mathbf{s}} \Big\downarrow\Big\downarrow \tilde{\mathbf{t}} & & \mathbf{s} \Big\downarrow\Big\downarrow \mathbf{t} \\ \mathcal{A}_*\mathcal{G}(\mathfrak{M}) & \xrightarrow{\ q_*\ } & \mathcal{L}(\mathfrak{M}) \end{array} \tag{5.18}$$

precotangent to the one presented in (5.17), where q_* and $\tilde{q}_*$ are the projections on the base. One defines the source $\tilde{\mathbf{s}}$ and target $\tilde{\mathbf{t}}$ maps in (5.18) as follows. Let $\phi \in T_{*x}\mathcal{G}(\mathfrak{M})$ and $x \in \mathcal{A}\mathcal{G}(\mathfrak{M})$ such that $\varrho(x) = p \in \mathcal{L}(\mathfrak{M})$, then

$$\langle \tilde{\mathbf{s}}(\phi), x\rangle := \langle \phi, DL_g(\varepsilon(p))(x - D\varepsilon(p)D\mathbf{t}(\varepsilon(p))x)\rangle, \tag{5.19}$$
$$\langle \tilde{\mathbf{t}}(\phi), x\rangle := \langle \phi, DR_g(\varepsilon(p))x\rangle. \tag{5.20}$$

The product $\phi \bullet \psi$ of $\phi \in T_{*x}\mathcal{G}(\mathfrak{M})$ and $\psi \in T_{*y}\mathcal{G}(\mathfrak{M})$, where $\tilde{\mathbf{s}}(\phi) = \tilde{\mathbf{t}}(\psi) \in \mathcal{A}_{*p}\mathcal{G}(\mathfrak{M})$ and $\mathbf{s}(x) = \mathbf{t}(y) = p \in \mathcal{L}(\mathfrak{M})$, one defines by

$$\langle \phi \bullet \psi, \xi \cdot \eta\rangle = \langle \phi, \xi\rangle + \langle \psi, \eta\rangle, \tag{5.21}$$

where $\xi \in T_x\mathcal{G}(\mathfrak{M})$, $\eta \in T_y\mathcal{G}(\mathfrak{M})$ satisfy $D\mathbf{s}(\xi) = D\mathbf{t}(\eta)$ and $\xi \cdot \eta \in T_{xy}\mathcal{G}(\mathfrak{M})$ is the product of ξ and η in the tangent groupoid $T\mathcal{G}(\mathfrak{M})$. The above definitions we obtain as a direct generalization of those accepted in the finite dimensional case, e.g. Subsection 11.5 of [12].

The groupoid $T_*\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{A}_*\mathcal{G}(\mathfrak{M})$ is a weak symplectic Banach-Lie realization of the Banach-Poisson bundle $\mathcal{A}_*\mathcal{G}(\mathfrak{M})$, which Poisson structure is determined by the algebroid structure of $\mathcal{AG}(\mathfrak{M})$. We note here that diagram (5.18) is the groupoid version of the diagram

$$\begin{array}{ccc} T_*G(\mathfrak{M}) & \xrightarrow{\tilde{q}_*} & G(\mathfrak{M}) \\ \tilde{\mathbf{s}} \downarrow\downarrow \tilde{\mathbf{t}} & & \mathbf{s} \downarrow\downarrow \mathbf{t} \\ \mathfrak{M}_* & \xrightarrow{q_*} & \{\mathbf{1}\} \end{array} \tag{5.22}$$

valid for the group $G(\mathfrak{M})$.

The proofs of these statements are the direct generalizations of the proofs for the finite dimensional case (see Theorem 11.5.18 in [12]) to the context of the Banach-Lie groupoids theory.

Finally let us mention that all objects considered above belong to the category of complex analytic Banach manifolds. They have their real analytic counterparts if we replace the group $G(\mathfrak{M})$ and the groupoid $\mathcal{G}(\mathfrak{M})$ by $U(\mathfrak{M})$ and $\mathcal{U}(\mathfrak{M})$ respectively, and $\mathfrak{M}$ ($\mathfrak{M}_*$) by $i\mathfrak{M}^h$ ($\mathfrak{M}^h_*$).

A detailed investigation of Banach -Lie Poisson geometry related to W^*-algebras needs a longer treatment in a separate paper, which is currently in preparation.

5.3. The Atiyah sequence of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

The Atiyah sequence is a short exact sequence of Lie algebroids naturally related to a principal bundle, e.g. see Section 3 of [12].

In the context of the paper we define the Atiyah sequence as the short exact sequence of Banach-Lie algebroids which are algebroids of the groupoids of (4.51). Let us describe these algebroids explicitly.

The inner groupoid $\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a totally intransitive Banach-Lie groupoid such that $\mathbf{s}^{-1}(p) = \mathbf{t}^{-1}(p) = G(p\mathfrak{M}p)$ for $p \in \mathcal{L}(\mathfrak{M})$. Thus one identifies $\mathcal{AJ}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ with the bundle $\mathcal{A}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ defined in (2.30). Since $\mathcal{A}(\mathfrak{M}) \to \mathcal{J}(\mathfrak{M})$ is a bundle of the associative Banach algebras (W^*-algebras) it could be considered as a bundle of Banach-Lie algebras, i.e. it is an intransitive Banach-Lie algebroid over $\mathcal{L}(\mathfrak{M})$.

Recall that sections of the Lie algebroid $\mathcal{A}G$ of a Lie groupoid $G \rightrightarrows M$ (e.g. see Section 3 of [12]) are identified with the right invariant vector fields on G tangent to the $\mathbf{s}$-fibres. Such vector fields are defined in the unique way by their values at the identity elements of G. So, by definition the algebroid $\mathcal{AG}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ is

$$\mathcal{AG}(\mathfrak{M}) := \bigsqcup_{p\in\mathcal{L}(\mathfrak{M})} T^{\mathbf{s}}_{\varepsilon(p)}\mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M}), \tag{5.23}$$

where $T^{\mathbf{s}}_{\varepsilon(p)}\mathcal{G}(\mathfrak{M})$ is the Banach space tangent to $\mathbf{s}^{-1}(p)$ at $\varepsilon(p) \in \mathcal{G}(\mathfrak{M})$.

Proposition 5.2. *For $p \in \mathcal{L}(\mathfrak{M})$ one has the following isomorphism of Banach spaces*

$$T^{\mathbf{s}}_{\varepsilon(p)}\mathcal{G}(\mathfrak{M}) \cong \mathfrak{M}p.$$

Proof. Let us take $]-\epsilon,\epsilon[\ni t \mapsto x(t) \in \mathbf{s}^{-1}(p) = \mathcal{G}(\mathfrak{M}) \cap \mathfrak{M}p$ such that $x(0) = p$. Thus from $\frac{d}{dt}x(t)|_{t=0} = \frac{d}{dt}(x(t)p)|_{t=0} = \frac{d}{dt}x(t)|_{t=0}p$ one has $\frac{d}{dt}x(t)|_{t=0} \in \mathfrak{M}p$. The above shows that $T^{\mathbf{s}}_{\varepsilon(p)}\mathcal{G}(\mathfrak{M}) \subset \mathfrak{M}p$. Let us show that $\mathfrak{M}p \subset T^{\mathbf{s}}_{\varepsilon(p)}\mathcal{G}(\mathfrak{M})$. For this reason, for $0 \neq x \in (1-p)\mathfrak{M}p$ we define $x(t) := p + tx$, where $t \in \mathbb{R}$. If $|t| < \frac{1}{\|x\|}$ then

$$x^*(t)x(t) = (p+tx)^*(p+tx) = p + t^2x^*x \in G(p\mathfrak{M}p).$$

Thus $x(t) \in \mathcal{G}(\mathfrak{M})_p \subset \mathcal{G}(\mathfrak{M})$ if $|t| < \frac{1}{\|x\|}$. Since $x(0) = p$ and $\frac{d}{dt}x(t)|_{t=0} = x$ we find that $x \in T^{\mathbf{s}}_{\varepsilon(p)}\mathcal{G}(\mathfrak{M})$. If $x \in p\mathfrak{M}p$ then $x(t) = p + tx \in G(p\mathfrak{M}p)$ for $|t| < \frac{1}{\|x\|}$ and thus $x \in T_{\varepsilon(p)}\mathcal{J}(\mathfrak{M}) \subset T^{\mathbf{s}}_{\varepsilon(p)}\mathcal{G}(\mathfrak{M})$. From the above and from the Banach splitting $\mathfrak{M}p = p\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$ we obtain $\mathfrak{M}p \subset T^{\mathbf{s}}_{\varepsilon(p)}\mathcal{G}(\mathfrak{M})$. □

We conclude from Proposition 5.2 the following

Proposition 5.3. *The algebroid $\mathcal{AG}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is isomorphic, as a vector bundle, to the bundle $\mathcal{M}_L(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of the left $\mathfrak{M}$-modules over $\mathcal{L}(\mathfrak{M})$. Hence $\Gamma^\infty(\mathcal{M}_L(\mathfrak{M}))$ inherits the Lie algebra structure of $\Gamma^\infty(\mathcal{AG}(\mathfrak{M}))$.*

Since $\bigcup_{\tilde{p}\sim p}(\Pi_{\tilde{p}} \times \Pi_p)$ is an open subset of $\mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ containing the diagonal the Banach Lie algebroid of $\bigcup_{\tilde{p}\sim p}(\Pi_{\tilde{p}} \times \Pi_p) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is the tangent bundle $T\mathcal{L}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of the lattice $\mathcal{L}(\mathfrak{M})$.

Now, since there is a functorial correspondence between the category of the Banach-Lie groupoids and the category of the Banach-Lie algebroids we conclude the following

Proposition 5.4. *One has the exact sequence of Banach-Lie algebroids*

$$\mathcal{A}(\mathfrak{M}) \xrightarrow{\iota} \mathcal{M}_L(\mathfrak{M}) \xrightarrow{\mathbf{a}} T\mathcal{L}(\mathfrak{M}) \tag{5.24}$$

over $\mathcal{L}(\mathfrak{M})$*, where* ι *and* $\mathbf{a}$ *are the inclusion monomorphism and anchor map, respectively.*

Let us note that the algebroid $\mathcal{AG}(\mathfrak{M}) = \mathcal{M}_L(\mathfrak{M})$ splits into transitive Banach-Lie algebroids over each orbit $\mathcal{O}_p \subset \mathcal{L}(\mathfrak{M})$, $p \in \mathcal{L}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. There are the Atiyah algebroids of the $G(p\mathfrak{M}p)$-principal bundles $\mathbf{t} : \mathbf{s}^{-1}(p) \to \mathcal{O}_p$ defined in (2.15). The above considerations justify us to calling (5.24) the Atiyah sequence of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Appendix A. The lattice of projections

The set of orthogonal projections $\mathcal{L}(\mathfrak{M})$ of W^*-algebra $\mathfrak{M}$ has canonically defined structure of complete orthomodular lattice, see Chapter III in [20], and points (i-vi) below. Let us shortly describe this structure. For details we refer also to [17, 19].

The order $p \leqslant q$, the meet $p \wedge q$ and the joint $p \vee q$ are operations in $\mathcal{L}(\mathfrak{M})$ defined by

(i)

$$p \leqslant q \;\Leftrightarrow\; pq = p;$$

(ii)

$$p \wedge q := \lim_{n\to\infty} (pq)^n,$$

the limit in (ii) is taken in the sense of $s(\mathfrak{M}, \mathfrak{M}_*)$-topology;

(iii)

$$p \vee q := (p^\perp \wedge q^\perp)^\perp,$$

where the orthocomplementation $\perp : \mathcal{L}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ is defined as

$$p^\perp := 1 - p.$$

The lattice $\mathcal{L}(\mathfrak{M})$ is also a complete lattice, i.e. for any set of projections $\{p_\alpha\}_{\alpha\in I}$ one has

(iv)

$$\bigvee_{\alpha\in I} p_\alpha \in \mathcal{L}(\mathfrak{M}),$$

(v)

$$\bigwedge_{\alpha \in I} p_\alpha \in \mathcal{L}(\mathfrak{M}).$$

We can easily see that

$$p \wedge p^\perp = 0, \quad p \vee p^\perp = 1, \quad p^{\perp\perp} = p, \quad p \leqslant q \ \Rightarrow\ q^\perp \leqslant p^\perp,$$

and

(vi)

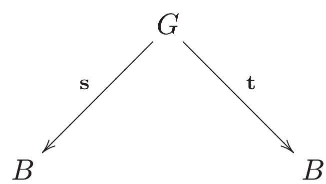
$$p \leqslant q \Rightarrow\ q = p \vee (q \wedge p^\perp).$$

The last property is called the orthomodular property.

Appendix B. Groupoids

Let us recall that a **groupoid with the base set** B (set of objects) is a set G such that:

i) there is a pair of maps



called **source** and **target** map respectively;

ii) for set of composable pairs

$$G^{(2)} := \{(g,h) \in G \times G; \quad \mathbf{s}(g) = \mathbf{t}(h)\}$$

one has a **product map** $m : G^{(2)} \to G$, denoted by

$$m(g,h) =: gh \tag{B.1}$$

such that
(a) $\mathbf{s}(gh) = \mathbf{s}(h)$, $\mathbf{t}(gh) = \mathbf{t}(g)$,
(b) associativity: $k(gh) = (kg)h$;

iii) there is an injection $\varepsilon : B \to G$ called the **identity section**, such that

$$\varepsilon(\mathbf{t}(g))g = g = g\varepsilon(\mathbf{s}(g));$$

iv) there exists an **inversion** $\iota : G \to G$ denoted by

$$\iota(g) =: g^{-1}, \tag{B.2}$$

such that

$$\iota(g)g = \varepsilon(\mathbf{s}(g)), \qquad g\iota(g) = \varepsilon(\mathbf{t}(g))$$

for all $g \in G$.

A groupoid G gives rise to a hierarchy of sets
$G^{(0)} := \varepsilon(B) \simeq B$
$G^{(1)} := G$
$G^{(2)} := \{(g,h) \in G \times G;\ \mathbf{s}(g) = \mathbf{t}(h)\}$
$\vdots$
$G^{(k)} := \{(g_1, g_2, ..., g_k) \in G \times G \times \cdots \times G;\ \mathbf{t}(g_i) = \mathbf{s}(g_{i-1}),\ i = 2, 3, \ldots, k\}$

In the paper we will consider the topological (differentiable) groupoids. Because of this let us recall that the groupoid G is called a topological (differentiable) groupoid if G and B have the topologies (differential manifold structure) such that:

i) the product map (B.1) and the involution (B.2) are continuous (differentiable);

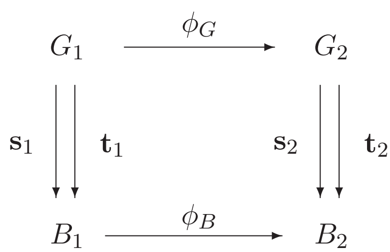
ii) the injection $\varepsilon : B \to G$ is an embedding (differentiable embedding).

¿From $\varepsilon \circ \mathbf{s}(g) = gg^{-1}$ and $\varepsilon \circ \mathbf{t}(g) = g^{-1}g$ it follows that source map and target map are continuous (differentiable). By definition the topology of $G^{(k)}$, for $k = 0, 1, 2, ...$, is inherited from G. In case of differentiable groupoid one assumes that the source and target maps are submersions.

Appendix C. Groupoid morphisms

A **morphism** ϕ of two groupoids G_1 and G_2 over bases B_1 and B_2 can be depicted by the following commutative diagram

(C.1)



By definition one also has

$$\phi_B \circ \mathbf{s}_1 = \mathbf{s}_2 \circ \phi_G \quad \text{and} \quad \phi_B \circ \mathbf{t}_1 = \mathbf{t}_2 \circ \phi_G$$

and

$$\phi_G(g)\phi_G(h) = \phi_G(gh)$$

for $(g,h) \in G_1^{(2)}$. If $\phi_G : G_1 \hookrightarrow G_2$ and $\phi_B : B_1 \hookrightarrow B_2$ are inclusion maps one says that G_1 is a **subgroupoid** of G_2. The subgroupoid $G_1 \subset G_2$ is a **wide subgroupoid** of G_2 if $\phi_B(\mathbf{s}_1(G_1)) = \phi_B(\mathbf{t}_1(G_1)) = B_2$.

An example of a groupoid morphism is given by

$$\begin{array}{ccc} G & \xrightarrow{(\mathbf{s},\mathbf{t})} & B \times B \\ \mathbf{s} \downarrow\downarrow \mathbf{t} & & pr_1 \downarrow\downarrow pr_2 \\ B & \xrightarrow{id} & B \end{array} \tag{C.2}$$

where $B \times B$ is the pair groupoid, i.e. $\mathbf{s} := pr_1$, $\mathbf{t} := pr_2$, $\iota(x,y) := (y,x)$, $\varepsilon(x) = (x,x)$ and $m((y,z),(x,y)) = (x,z)$.

Appendix D. Action groupoids

If a group G acts on a set M

$$G \times M \ni (g,m) \mapsto g \cdot m \in M$$

one can define on the set $G \times M$ a groupoid structure, which is called the **action groupoid** structure. For this case one defines

i) source and target maps $\mathbf{s},\ \mathbf{t} : G \times M \to M$ as

$$\mathbf{s}(g,m) := m \in M \quad \text{and} \quad \mathbf{t}(g,m) := g \cdot m\ ; \tag{D.1}$$

ii) the groupoid product

$$(g,m)(h,n) := (gh,n) \tag{D.2}$$

on the set of composable pairs

$$(G \times M)^{(2)} := \{((g,m),(h,n)) \in (G \times M) \times (G \times M) : \quad m = h \cdot n\}$$

iii) the identity section $\varepsilon : M \to G \times M$ by

$$\varepsilon(m) = (e, m); \tag{D.3}$$

iv) the involution $\iota : G \times M \to G \times M$ by

$$\iota(g, m) = (g^{-1}, g \cdot m). \tag{D.4}$$

Appendix E. Groupoid actions

We recall the definition of a **left action of a groupoid** G **on a set** M. One assumes for this reason that there exists a map (moment map)

$$\mu : M \to B \tag{E.1}$$

and one defines the space

$$G *_l M := \{(g, r) \in G \times M : \quad \mathbf{s}(g) = \mu(r)\}. \tag{E.2}$$

Then a left action of groupoid G on M is defined as a map $G *_l M \ni (g, r) \mapsto g \cdot r \in M$ with properties:

$$\begin{aligned} &(gh) \cdot r = g \cdot (h \cdot r) \\ &\mu(g \cdot r) = \mathbf{t}(g) \\ &\varepsilon(\mu(r)) \cdot r = r. \end{aligned} \tag{E.3}$$

For the **right action of** G **on** M instead of (E.3) we have

$$\begin{aligned} &r \cdot (gh) = (r \cdot h) \cdot g \\ &\mu(r \cdot g) = \mathbf{s}(g) \\ &r \cdot \varepsilon(\mu(r)) = r, \end{aligned} \tag{E.4}$$

where $(g, r) \in G *_r M := \{(g, r) \in G \times M : \quad \mathbf{t}(g) = \mu(r)\}$.

As an example let us take the **canonical left action** of G on its base B. In this case $M := B$, $\mu := id$ and

$$G *_l B = \{(g, x) : \quad x = \mathbf{s}(g)\} \tag{E.5}$$

The action map is defined by

$$G *_l B \ \ni \ (g, x) \ \mapsto \ g \cdot x := \mathbf{t}(g). \tag{E.6}$$

The defining properties (E.3) follow from the corresponding properties of the maps $\mathbf{s}, \mathbf{t}$, ε and the product map (B.1).

One can generalize the notion of the action groupoid defined in Appendix D replacing the group G by a groupoid.

Definition E.1. The set $\tilde{G} := G *_l M$ has a groupoid structure $G *_l M \rightrightarrows M$ over M defined as follows:

i) source map and target map are given by $\tilde{\mathbf{s}}(g, r) := r \in M$ and $\tilde{\mathbf{t}}(g, r) := g \cdot r \in M$;

ii) the set of composable pairs

$$\tilde{G}^{(2)} := \{((g, r), (h, n)) \in \tilde{G} \times \tilde{G};\ \ \mathbf{t}(h) = \mathbf{s}(g)\}$$

and the product map $\tilde{m} : \tilde{G}^{(2)} \to \tilde{G}$ is defined as

$$\tilde{m}((g, r), (h, n)) = (gh, n); \tag{E.7}$$

iii) the identity section $\tilde{\varepsilon} : M \to \tilde{G}$ is defined by

$$\tilde{\varepsilon}(r) = (\varepsilon(\mu(r)), r); \tag{E.8}$$

iv) the involution $\tilde{\iota} : \tilde{G} \to \tilde{G}$ is defined by

$$\tilde{\iota}(g, r) = (\iota(g), g \cdot r). \tag{E.9}$$

Similarly as in the group case the groupoid $G *_l M \rightrightarrows M$ is called the **action groupoid**.

In the case when G is a topological groupoid and M is a topological space we obtain on $\tilde{G}$ a structure of a topological groupoid if the moment map μ and the action G on M are continuous. The topological structure of $\tilde{G} \subset G \times M$ is inherited from product topology of $G \times M$.

One calls the morphism depicted in (C.1) a **covering morphism** if for each $x \in B_1$ the restriction $\phi_G : \mathbf{s}_1^{-1}(x) \to \mathbf{s}_2^{-1}(\phi_B(x))$ of ϕ_G to the $\mathbf{s}$-level of x is bijection.

The diagram

$$\begin{array}{ccc} G *_l M & \xrightarrow{\ pr_1\ } & G \\ \tilde{\mathbf{s}} \downarrow\downarrow \tilde{\mathbf{t}} & & \mathbf{s} \downarrow\downarrow \mathbf{t} \\ M & \xrightarrow{\ \mu\ } & B \end{array} \tag{E.10}$$

where

$$\phi_G(g,r) := pr_1(g,r) = g \quad \text{and} \quad \phi_B(r) := \mu(r),$$

gives an example of the covering morphism of groupoids.

Appendix F. Algebroids

A **Lie algebroid** on manifold M, see e.g. Definition 3.3.1 in [12], is a vector bundle (A, ϱ, M) with a vector bundle map $\mathbf{a} : A \to TM$ over M (anchor map) and a bracket $[\ ,\ ] : \Gamma A \times \Gamma A \to \Gamma A$ which is

i) $\mathbb{R}$-bilinear, alternating, satisfies the Jacobi identity,

ii) $[X, uY] = u[X, Y] + \mathbf{a}(X)(u)Y$

iii) $\mathbf{a}([X, Y]) = [\mathbf{a}(X), \mathbf{a}(Y)]$

for $X, Y \in \Gamma A,\ u \in C^\infty(M)$.

A **Lie algebroid of a Lie groupoid** $G \rightrightarrows M$ is the vector bundle

$$\mathcal{A}G := \bigcup_{m\in M} T_{\varepsilon(m)}(\mathbf{s}^{-1}(m))$$

with the anchor map $\mathbf{a} : \mathcal{A}G \to TM$ defined by

$$\mathbf{a} := D\mathbf{t}|_{\varepsilon(M)}.$$

Any section of $\mathcal{A}G$ one can consider as the restriction $\mathcal{X}|_{\varepsilon(M)}$ to $\varepsilon(M)$ of the vector field $\mathcal{X} \in \Gamma^\infty TG$ tangent to the $\mathbf{s}$-fibres and invariant with respect to the right translations $R_g : G_{\mathbf{t}(g)} \to G_{\mathbf{s}(g)}$. So, the Lie bracket of $\mathcal{X}|_{\varepsilon(M)}, \mathcal{Y}|_{\varepsilon(M)} \in \Gamma^\infty \mathcal{A}G$ one defines as follows

$$[\mathcal{X}|_{\varepsilon(M)}, \mathcal{Y}|_{\varepsilon(M)}] := [\mathcal{X}, \mathcal{Y}]|_{\varepsilon(M)}.$$

The bundle $\mathcal{A}G$ with the above Lie bracket and $\mathbf{a} = D\mathbf{t}|_{\mathcal{A}G}$ is a Lie algebroid which is called the Lie algebroid of the Lie groupoid $G \rightrightarrows M$.

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