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Categorification of Clifford algebras and $\mathbf{U}_q(\mathfrak{sl}(1|1))$

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We construct families of differential graded algebras R_n and $R_n \boxtimes R_n$ for $n > 0$, and differential graded categories $DGP(R_n)$ generated by some distinguished projective R_n-modules. The category $DGP(R_n)$ gives an algebraic formulation of the *contact category* of a disk. The 0-th homology category $H^0(DGP(R_n))$ of $DGP(R_n)$ is a triangulated category and its Grothendieck group $K_0(R_n)$ is isomorphic to a Clifford algebra. We then categorify the multiplication on $K_0(R_n)$ to a functor $DGP(R_n \boxtimes R_n) \to DGP(R_n)$. We also construct a subcategory of $H^0(DGP(R_n))$ which categorifies an integral version of $\mathbf{U}_q(\mathfrak{sl}(1|1))$ as an algebra.

1. Introduction

1.1. Background

Categorification is a process in which we lift an integer to a vector space, a vector space to a category, and a linear map between vector spaces to a functor between categories. Two of the pioneering works are Khovanov homology defined by Khovanov [17] and knot Floer homology, defined independently by Ozsváth-Szabó [29] and Rasmussen [31], which categorify the Jones and Alexander polynomials, respectively. Khovanov homology and knot Floer homology are finer invariants of knots which take values in the homotopy category of chain complexes of graded vector spaces whose graded Euler characteristics agree with the polynomial invariants.

The Jones polynomial fits in the general framework of Reshetikhin-Turaev invariants [33] associated to the fundamental representation V_1 of the quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$. With an eye towards categorifying the Reshetikhin-Turaev invariants, Bernstein-Frenkel-Khovanov [2] formulated a program for categorifying representations of $\mathbf{U}_q(\mathfrak{sl}_2)$. The symmetric powers $V_1^{\otimes n}$ of $\mathbf{U}(\mathfrak{sl}_2)$ were categorified in [2] and extended to the graded case of $\mathbf{U}_q(\mathfrak{sl}_2)$ by Stroppel [37]. Other tensor product representations of $\mathbf{U}_q(\mathfrak{sl}_2)$ were categorified by Frenkel-Khovanov-Stroppel [6]. Chuang and Rouquier [4] categorified locally finite $\mathfrak{sl}_2$-representations. More generally, Rouquier [35] studied a 2-category associated with a Kac-Moody algebra and its 2-representation. For the quantum groups themselves, Lauda [24] gave a diagrammatic categorification of $\mathbf{U}_q(\mathfrak{sl}_2)$ and Khovanov-Lauda [20–22] extended it to the cases of $\mathbf{U}_q(\mathfrak{sl}_n)$. The program of categorifying Reshetikhin-Turaev invariants was brought to fruition by Webster [38, 39] using this diagrammatic approach.

For the Alexander polynomial, Kauffman-Saleur developed a representation-theoretic approach in the spirit of [23] via the quantum group $\mathbf{U}_q(\mathfrak{sl}(1|1))$ of the Lie superalgebra $\mathfrak{sl}(1|1)$. Rozansky-Saleur in [34] gave an associated quantum field theory description. Compared to the case of $\mathbf{U}_q(\mathfrak{sl}_2)$, the following question naturally appears in the context:

Question 1.1. Is there a categorical program for $\mathbf{U}_q(\mathfrak{sl}(1|1))$ and its fundamental representation which could recover knot Floer homology?

The first step in such a program is to categorify $\mathbf{U}_q(\mathfrak{sl}(1|1))$. Motivated by the strands algebra of Lipshitz-Ozsváth-Thurston [25], Khovanov [18] categorified the positive part of $\mathbf{U}_q(\mathfrak{gl}(1|2))$. Douglas-Manolescu [5] generalized the strands algebra associated to a surface to a differential 2-algebra associated to a circle. Recently, Sartori [36] announced a categorification of tensor products of the fundamental representation of $\mathbf{U}_q(\mathfrak{sl}(1|1))$ using the parabolic category $\mathcal{O}$. The category $\mathcal{O}$ was also used in a categorification of the Temperley-Lieb category by Stroppel [37] and a combinatorial approach to functorial quantum $\mathfrak{sl}_k$ knot invariants by Mazorchuk-Stroppel [28].

The goal of this paper is to present a categorification of the algebra structure of an integral version of $\mathbf{U}_q(\mathfrak{sl}(1|1))$. The motivation is from the contact category introduced by Honda [8], which presents an algebraic way to study contact topology in dimension 3. In particular, the distinguished basis in our categorification is given by geometric objects, called *dividing sets*, induced by contact structures. The connection between 3-dimensional contact topology and *Heegaard Floer homology* was established by Ozsváth and Szabó [30] in the closed case. Honda-Kazez-Matić generalized it to the case of a contact 3-manifold with *convex* boundary in [11] and formulated it in the framework of TQFT in [10]. The combinatorial properties of the contact categories were studied by Mathews in the case of disks [26] and annuli [27]. The connection to *bordered Heegaard Floer homology* defined in [25] is observed by Zarev [40]. Since the quantum $\mathfrak{sl}(1|1)$ knot invariant was categorified to Heegaard Floer homology, it is not too surprising that the contact topology can be used to categorify $\mathbf{U}_q(\mathfrak{sl}(1|1))$.

1.2. Results

We actually first give a categorification of a *Clifford algerba* Cl_n.

Definition 1.2. Define Cl_n as a unital $\mathbb{Z}[q^{\pm 1}]$-algebra with generators X_i's for $0 \leq i \leq n$, relations:

$$X_i^2 = 0;$$
$$X_i X_j = -X_j X_i \text{ if } |i-j| > 1;$$
$$X_i X_{i+1} + X_{i+1} X_i = q^{2i+1-n}.$$

It is easy to see that Cl_n is a Clifford algebra $Cl(V_n, Q_n)$ over $\mathbb{Z}[q^{\pm 1}]$, where V_n is a free $\mathbb{Z}[q^{\pm 1}]$-module spanned by $\{X_i \mid 0 \leq i \leq n\}$ and Q_n is a quadratic form on V_n given by:

$$Q_n\left(\sum_{i=0}^{n} a_i X_i\right) = \sum_{i=0}^{n-1} a_i a_{i+1} q^{2i+1-n},$$

for $a_i \in \mathbb{Z}[q^{\pm 1}]$. Note that the connection between $\mathbf{U}_q(\mathfrak{gl}(1|1))$ and Clifford algerbas was investigated by Reshetikhin-Stroppel-Webster [32] in clarifying the braid group action defined by R-matrices of $\mathbf{U}_q(\mathfrak{gl}(1|1))$.

Our first theorem is a categorification of the multiplication $\mathrm{m_n}$: $Cl_n \otimes_{\mathbb{Z}[q^{\pm 1}]} Cl_n \to Cl_n$ via triangulated categories:

Theorem 1.3. *For $n > 0$, there exist triangulated categories $\mathcal{CL}_{n,n}$ and $\mathcal{CL}_n$ whose Grothendieck groups are $Cl_n \otimes Cl_n$ and Cl_n, respectively. There* *exists an exact functor* $\mathcal{M}_n : \mathcal{CL}_{n,n} \to \mathcal{CL}_n$ *whose induced map on the Grothendieck groups* $K_0(\mathcal{M}_n) : K_0(\mathcal{CL}_{n,n}) \to K_0(\mathcal{CL}_n)$ *agrees with the multiplication* $\mathrm{m}_n : Cl_n \otimes Cl_n \to Cl_n$.

Remark 1.4. (1) The basis $\{X_i \mid 0 \le i \le n\}$ in our categorification is different from the basis $\{\psi_i, \psi_i^* \mid i \in \mathbb{Z}\}$ of the Clifford algebra in [14]. As a comparison, Khovanov used a different basis from $\{p_i, q_i \mid i \in \mathbb{Z}\}$ in the categorification of a *Heisenberg algebra* [19].
(2) Compared to the additive categorification of the Heisenberg algebra, the triangulated or differential graded categories are more useful for categorifying the Clifford algebra.

We view an integral version of $\mathbf{U}_q(\mathfrak{sl}(1|1))$ as a subalgebra of Cl_n as follows. The quantum group $\mathbf{U}_q(\mathfrak{sl}(1|1))$ is the unital associative $\mathbb{Q}(q)$-algebra with generators E, F, H, H^{-1} and relations:

$$\begin{aligned} HH^{-1} = H^{-1}H &= 1, \\ E^2 = F^2 &= 0, \\ HE = EH, HF &= FH, \\ EF + FE &= \frac{H - H^{-1}}{q - q^{-1}}. \end{aligned}$$

We consider two variants of $\mathbf{U}_q(\mathfrak{sl}(1|1))$: the *idempotent completion* $\mathbf{U}$ and the *integral form* $\mathbf{U}_n$ of $\mathbf{U}$. The idempotent completion $\mathbf{U}$ is obtained from $\mathbf{U}_q(\mathfrak{sl}(1|1))$ by replacing the unit by a collection of orthogonal idempotents 1_n for $n \in \mathbb{Z}$ such that

$$1_n 1_m = \delta_{n,m}, \;\; H1_n = 1_n H = q^n 1_n, \;\; 1_n E = E 1_n, \;\; 1_n F = F 1_n.$$

Definition 1.5. The integral form $\mathbf{U}_n$ is the unital associative $\mathbb{Z}[q^{\pm 1}]$-algebra with generators E, F and relations:

$$E^2 = F^2 = 0, \qquad EF + FE = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + \cdots + q^{1-n}.$$

Then $\mathbf{U}_n$ can be viewed as a subalgebra of Cl_n by setting

$$E = \sum_{\substack{0 \le i \le n \\ i \text{ even}}} X_i, \qquad F = \sum_{\substack{0 \le i \le n \\ i \text{ odd}}} X_i$$

We construct triangulated full subcategories $\mathcal{U}_n$ of $\mathcal{CL}_n$ and $\mathcal{U}_{n,n}$ of $\mathcal{CL}_{n,n}$ so that their Grothendieck groups are $\mathbf{U}_n$ and $\mathbf{U}_n \otimes \mathbf{U}_n$, respectively. Then the restriction of $\mathcal{M}_n$ in Theorem 1.3 to the subcategory $\mathcal{U}_{n,n}$ gives a categorification of the multiplication on $\mathbf{U}_n$.

Theorem 1.6. *For $n > 0$, there exist triangulated categories $\mathcal{U}_{n,n}$ and $\mathcal{U}_n$ whose Grothendieck groups are $\mathbf{U}_n \otimes \mathbf{U}_n$ and $\mathbf{U}_n$, respectively. There exists an exact functor $\mathcal{F}_n : \mathcal{U}_{n,n} \to \mathcal{U}_n$ whose induced map on the Grothendieck groups $K_0(\mathcal{F}_n) : K_0(\mathcal{U}_{n,n}) \to K_0(\mathcal{U}_n)$ agrees with the multiplication $\mathrm{f}_n : \mathbf{U}_n \otimes \mathbf{U}_n \to \mathbf{U}_n$.*

1.3. Motivation from contact topology in dimension 3

The contact category $\mathcal{C}(\Sigma, F)$ of (Σ, F) is an additive category associated to an oriented surface Σ and a finite subset F of $\partial\Sigma$. The objects of $\mathcal{C}(\Sigma, F)$ are formal direct sums of isotopy classes of *dividing sets* on Σ whose restrictions to $\partial\Sigma$ agree with F. A *dividing set* Γ on Σ is a properly embedded 1-manifold, possibly disconnected and possibly with boundary, which divides Σ into positive and negative regions. The *Euler number* $e(\Gamma)$ of a dividing set is the Euler characteristic of the positive region minus the Euler characteristic of the negative region. The morphism $\mathrm{Hom}_{\mathcal{C}(\Sigma,F)}(\Gamma_0, \Gamma_1)$ is an $\mathbb{F}_2$-vector space spanned by isotopy classes of *tight* contact structures on $\Sigma \times [0,1]$ with the dividing sets Γ_i on $\Sigma \times \{i\}$ for $i = 0, 1$. The composition is given by vertically stacking contact structures.

We give a very brief description of morphism spaces of contact categories. Since any tight contact structure preserves the Euler number of the dividing sets, we have $\mathrm{Hom}_{\mathcal{C}(\Sigma,F)}(\Gamma_0, \Gamma_1) = 0$ if $e(\Gamma_0) \neq e(\Gamma_1)$. It follows that $\mathcal{C}(\Sigma, F)$ is actually a disjoint union of its subcategories $\mathcal{C}(\Sigma, F; e)$ which are generated by dividing sets Γ with $e(\Gamma) = e$. Any dividing set with a contractible component is isomorphic to the zero object since there is no tight contact structure in a neighborhood of the dividing set by a criterion of Giroux [7]. As basic blocks of morphisms, *bypass attachments* introduced by Honda [9] locally change dividing sets as in Figure 1.

There is a refined version, called the *universal cover* $\tilde{\mathcal{C}}(\Sigma, F)$ of the contact category $\mathcal{C}(\Sigma, F)$ given as follows. Choose a dividing set $\Gamma_{0,e}$ as a base point for each subcategory $\mathcal{C}(\Sigma, F; e)$ of $\mathcal{C}(\Sigma, F)$. The basic objects of $\tilde{\mathcal{C}}(\Sigma, F)$ are pairs $(\Gamma, [\zeta])$, where Γ is an isotopy class of dividing sets on (Σ, F) with $e(\Gamma) = e$, and $[\zeta]$ is a homotopy class of a 2-plane field ζ on $\Sigma \times [0,1]$ which is contact near $\Sigma \times \{0, 1\}$ with the dividing sets $\Gamma_{0,e}$ on $\Sigma \times \{0\}$ and Γ on $\Sigma \times \{1\}$. The morphism set $\mathrm{Hom}_{\tilde{\mathcal{C}}(\Sigma,F)}((\Gamma_1, [\zeta_1]), (\Gamma_2, [\zeta_2]))$ is spanned by tight contact structures $\{\xi\}$ such that $[\zeta_2] = [\xi \cup \zeta_1]$, where $\xi \cup \zeta_1$ denotes a concatenation of the 2-plane fields ξ and ζ_1. In other words, the component

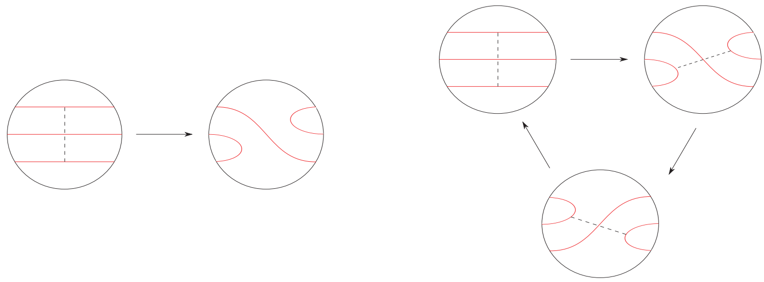

Figure 1: The picture on the left is a bypass attachment along the dashed arc; the one on the right is a distinguished triangle given by a triple of bypass attachments.

$[\zeta]$ gives a grading $\mathrm{gr}(\Sigma)$ on the objects of $\tilde{\mathcal{C}}(\Sigma, F)$ which takes values in homotopy classes of 2-plane fields. Equivalently, the grading $\mathrm{gr}(\Sigma)$ is given by a central extension by $\mathbb{Z}$ of the homology group $H_1(\Sigma)$, i.e., there is a short exact sequence: $0 \to \mathbb{Z} \to \mathrm{gr}(\Sigma) \to H_1(\Sigma) \to 0$. Note that a similar grading appears in bordered Heegaard Floer homology [25, Section 3.3]. The main feature of the universal cover $\tilde{\mathcal{C}}(\Sigma, F)$ is the existence of distinguished triangles given by a triple of bypass attachments as in Figure 1. The subgroup $\mathbb{Z}$ of the grading $\mathrm{gr}(\Sigma)$ is related to the shift functor in a triangulated category. In particular, Huang [12] showed that a triple of bypass attachments changes the $\mathbb{Z}$ component by 1.

This paper can be viewed as an algebraic formulation of the triangulated structure on the universal cover of the contact category of a disk, viewed as a rectangle. Let $\mathcal{C}_n$ be the universal cover of the contact category $\mathcal{C}(D_n, 2n+4)$ of a rectangle D_n with $n+2$ marked points on both the left and right sides of ∂D_n and no marked points on the top and bottom sides. The grading $\mathrm{gr}(D_n)$ is isomorphic to $\mathbb{Z}$ in this case. Although modifying a disk into a rectangle breaks the symmetry of the disk, it gives a monoidal structure on $\mathcal{C}_n$, i.e. a bifunctor $\rho_n : \mathcal{C}_n \times \mathcal{C}_n \to \mathcal{C}_n$. The monoidal structure ρ_n is given by horizontally stacking two dividing sets along their common boundaries for the objects and sideways stacking two contact structures for the morphisms as shown in Fig 2. Since the universal cover $\mathcal{C}_n$ is defined by choosing some base points $\Gamma_{0,e}$ for each Euler number e, the definition of ρ_n really depends on choices of stacking the base points. The algebraic formulation considered in this paper is motivated by a special choice of ρ_n. We won't discuss the detail about how ρ_n is defined over the base points. Although the motivation is from contact topology, our algebraic formulation can actually be constructed independently.

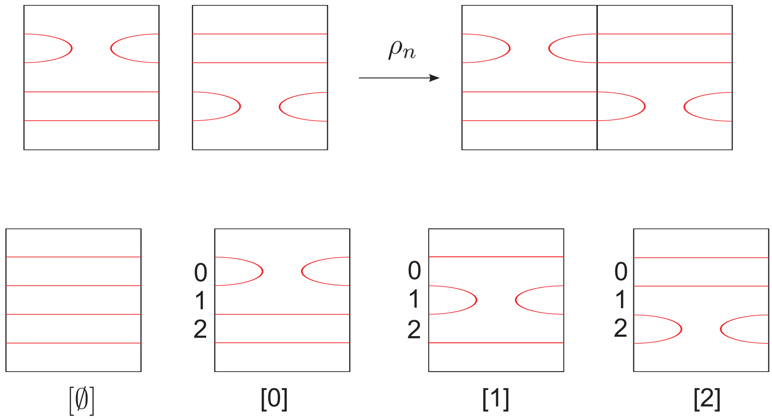

Figure 2: The top is a stacking of two dividing sets; the bottom is a distinguished collection of dividing sets: $[\emptyset], [0], [1], [2]$.

There is a collection of distinguished dividing sets $[\emptyset]$ and $[i]$'s for $0 \leq i \leq n$ as shown in Fig 2. Each distinguished object is actually a pair of a dividing set and its homotopy grading with respect to the base point. From now on we will focus on the dividing sets and ignore the homotopy grading for simplicity. Note that any nontrivial dividing set can be represented as a horizontal stacking of those dividing sets up to isotopy. Let $[i] \cdot [j]$ denote the horizontal stacking of dividing sets $[i]$ and $[j]$. Let $X_\emptyset$ and X_i's be classes of $[\emptyset]$ and $[i]$'s in the Grothendieck group of $\mathcal{C}_n$. Under a special choice of ρ_n they satisfy the following properties illustrated in Fig 3:

(1) $X_\emptyset$ is the unit since any dividing set is unchanged when stacking $[\emptyset]$ from both left and right.

(2) $X_i^2 = 0$ since the dividing set $[i] \cdot [i]$ contains a contractible loop.

(3) $X_i X_j = -X_j X_i$ for $|i-j| > 1$, since dividing sets $[i] \cdot [j]$ and $[j] \cdot [i]$ are in the same isotopy class as dividing sets, but their homotopy gradings differ by 1 from the special choice of ρ_n. Hence their classes differ by a minus sign in the Grothendieck group.

(4) $X_i X_{i+1} + X_{i+1} X_i = q^{2i+1-n} X_\emptyset$, since there exists a distinguished triangle: $[i] \cdot [i+1] \to [\emptyset] \to [i+1] \cdot [i] \xrightarrow{[1]} [i] \cdot [i+1]$. The morphism in $\mathrm{Hom}_{\mathcal{C}_n}([i+1] \cdot [i], [i] \cdot [i+1])$ is of cohomological degree 1 due to the choice of ρ_n. The exponent of q is related to the height of the location of the distinguished triangle.

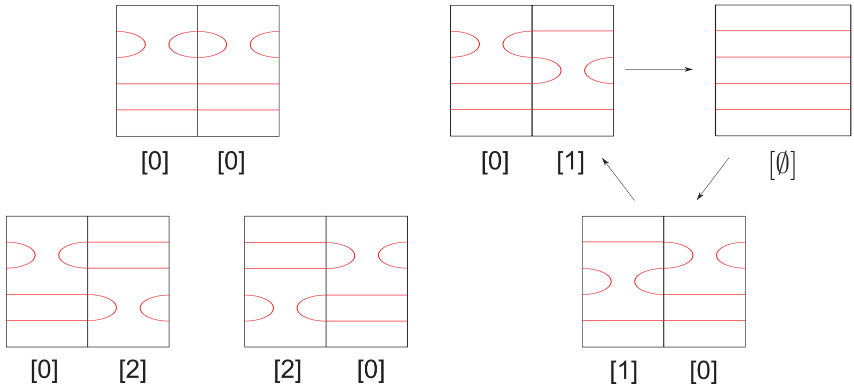

Figure 3: The top left picture represents $[0]\cdot[0]$; the bottom left compares $[0]\cdot[2]$ and $[2]\cdot[0]$; the right picture is a distinguished triangle: $[0]\cdot[1] \to [\emptyset] \to [1]\cdot[0]$.

The key ingredient in categorification is the existence of a canonical or distinguished basis. The morphism sets between objects in the basis are supposed to give relations on the level of Grothendieck group. In this perspective, the contact category $\mathcal{C}_n$ of a rectangle provides a distinguished basis {the dividing sets $[i]$'s} which lifts the generators X_i's of the Clifford algebra Cl_n. The multiplication on Cl_n is lifted to the monoidal structure on $\mathcal{C}_n$ and the relations in Cl_n are lifted to isomorphisms or distinguished triangles in $\mathcal{C}_n$. In a follow-up paper, we construct a categorical action of the Clifford algebra Cl_n which is motivated by stacking $\mathcal{C}_n$ with other contact categories. We conclude this section with the following question:

Question 1.7. What is the intrinsic connection between contact topology and the Clifford algebra?

1.4. The algebraic formulation

In this paper, we give an algebraic formulation of the contact categories $\mathcal{C}_n$ and the monoidal functor ρ_n in 3 steps[1]:

1) Define $\mathbb{F}_2$-algebras R_n motivated from considering morphisms in $\mathcal{C}_n$ between objects in certain distinguished basis. Then model $\mathcal{C}_n$ by DG categories $DGP(R_n)$ generated by some projective R_n-modules.

[1]In fact, the rest of the paper is just algebra which is motivated by the contact category.

2) Define multiplication m_n on the Grothendieck groups $K_0(R_n)$ of $DGP(R_n)$ and show that $K_0(R_n)$ are isomorphic to the Clifford algebras Cl_n.
3) Categorify $\mathrm{m}_n : K_0(R_n) \otimes K_0(R_n) \to K_0(R_n)$ to functors

$$\mathcal{M}_n : DGP(R_n \boxtimes R_n) \xrightarrow{T_n \otimes_{R_n \boxtimes R_n} -} DGP(R_n)$$

by tensoring with DG $(R_n, R_n \boxtimes R_n)$-bimodules T_n. Here, DG algebras $R_n \boxtimes R_n$ are variants of tensor products $R_n \otimes R_n$ by adding a nontrivial differential. The construction of T_n is motivated by the monoidal functors ρ_n on the topological side.

The rigorous definition of R_n is given in Definition 2.6 which does not need contact topology directly. We give more detail about how the algebra R_n is motivated from the contact category $\mathcal{C}_n$. It is easy to see from Definition 1.2 that Cl_n has a $\mathbb{Z}[q^{\pm 1}]$-basis:

$$\{X_{i_0} X_{i_1} \cdots X_{i_k} \mid n \geq i_0 > \cdots > i_k \geq 0\}.$$

This basis can be lifted to a collection of distinguished objects of $\mathcal{C}_n$:

$$\mathcal{B}_n = \{[i_0] \cdot [i_1] \cdots [i_k] \mid n \geq i_0 > \cdots > i_k \geq 0\}.$$

Here homotopy gradings of the distinguished objects are omitted. The key point is that there exists a choice of homotopy gradings such that $\mathrm{Hom}_{\mathcal{C}_n}(x, y)$ for any pair $x, y \in \mathcal{B}_n$ is concentrated at cohomological degree 0. Then the composition of morphisms in the contact category $\mathcal{C}_n$ defines an $\mathbb{F}_2$-algebra:

$$R_n = \bigoplus_{x, y \in \mathcal{B}_n} \mathrm{Hom}_{\mathcal{C}_n}(x, y).$$

More precisely, there is a correspondence from the topological model to the algebraic model:

$$\begin{aligned} \phi_n : \ \mathcal{C}_n &\to \mathbf{D}^b(R_n) \\ z &\mapsto \bigoplus_{x \in \mathcal{B}_n} \mathrm{Hom}(x, z), \end{aligned}$$

where $\mathbf{D}^b(R_n)$ is the bounded derived category of finitely generated left R_n-modules. In fact, the collection $\mathcal{B}_n$ of the distinguished objects generates the contact category $\mathcal{C}_n$ by taking iterated mapping cones. Associated R_n-modules $\phi_n(z)$ are good enough to distinguish objects z of $\mathcal{C}_n$. Roughly speaking ϕ_n can be viewed as an embedding of $\mathcal{C}_n$ into its triangulated envelope $\mathbf{D}^b(R_n)$. This method which describes a category by picking up a collection distinguished objects was used in Khovanov-Seidel's approach to certain *Fukaya categories* [23]. In order to describe homotopies between complexes of R_n-modules, we will work over DG categories $DGP(R_n)$ generated by distinguished projective R_n-modules corresponding to the basis $\mathcal{B}_n$ on the topological side.

We describe R_n in terms of a path algebra of a quiver Γ_n explicitly. The rigorous definition of Γ_n will be given in Definition 2.1. Here we only discuss the motivation from contact topology behind its definition. A key fact about the contact category $\mathcal{C}_n$ is that any morphism in R_n can be written as a product of elementary morphisms in R_n which are not decomposable. Therefore we construct the quiver Γ_n whose vertices are in one-to-one correspondence to the basis in $\mathcal{B}_n$, and whose arrows represent the elementary morphisms in R_n. More precisely, there exists an arrow

$$[i_0]\cdot[i_1]\cdots[i_k] \to [j_0]\cdot[j_1]\cdots[j_l]$$

if $l = k+2$ and $\{j_0, j_1, \ldots, j_l\} = \{i_0, i_1, \ldots, i_k\} \sqcup \{s+1, s\}$ for some s. Each arrow actually represents an elementary morphism which is a tight contact structure given by a single bypass attachment. For instance, there is an arrow $[\emptyset] \to [1]\cdot[0]$ in Γ_n as a part of the distinguished triangle $[0]\cdot[1] \to [\emptyset] \to [1]\cdot[0]$ in $\mathcal{C}_n$ as shown in Fig 3. We further impose a commutativity relation on the path algebra $\mathbb{F}_2\Gamma_n$ about squares in Γ_n. For instance, the following square commutes:

$$\begin{array}{ccc} [\emptyset] & \longrightarrow & [1][0] \\ \downarrow & & \downarrow \\ [3][2] & \longrightarrow & [3][2][1][0]. \end{array}$$

The relation comes from the fact in contact topology that two disjoint bypass attachments commute up to isotopy. Then we add an extra q-grading on the set of arrows and define a q-graded algebra R_n as a quotient of $\mathbb{F}_2\Gamma_n$ modulo the commutativity relation.

Let $P(\mathbf{x}) = R_n e(\mathbf{x})$ be a left projective R_n-module, where $e(\mathbf{x})$ is an idempotent of R_n for a vertex $\mathbf{x} \in \mathcal{B}_n$ of Γ_n. Let $DGP(R_n)$ be a DG category consisting of finitely iterated mapping cones of maps between distinguished projective R_n-modules in $\{P(\mathbf{x}) \mid \mathbf{x} \in \mathcal{B}_n\}$. As triangulated categories, the

0-th homology category $H^0(DGP(R_n))$ is equivalent to $\mathbf{K}^b(R_n)$, the homotopy category of bounded complexes of finitely generated projective q-graded R_n-modules. The Grothendieck group $K_0(R_n)$ of $H^0(DGP(R_n))$, is a free $\mathbb{Z}[q^{\pm1}]$-module over the vertex set $V(\Gamma_n) = \mathcal{B}_n$.

We then define a multiplication $\mathrm{m}_n : K_0(R_n) \otimes K_0(R_n) \to K_0(R_n)$ and show that $K_0(R_n)$ is isomorphic to the Clifford algebra Cl_n. Consider a tensor product $R_n \otimes R_n$ and its homotopy category $\mathbf{K}^b(R_n \otimes R_n)$ whose Grothendieck group is isomorphic to $K_0(R_n) \otimes K_0(R_n)$. To categorify m_n, we ideally need $(R_n, R_n \otimes R_n)$-bimodule to induce a functor $\mathbf{K}^b(R_n \otimes R_n) \to \mathbf{K}^b(R_n)$. But the construction of those bimodules requires more information on homotopies between complexes in $\mathbf{K}^b(R_n \otimes R_n)$. Thus, we deform $R_n \otimes R_n$ into $R_n \boxtimes R_n$ by adding a nontrivial differential and show that two algebras are quasi-isomorphic. Then consider a DG category $DGP(R_n \boxtimes R_n)$ whose 0-th homology category $H^0(DGP(R_n \boxtimes R_n))$ is equivalent to $\mathbf{K}^b(R_n \otimes R_n)$.

$$
\begin{array}{ccc}
DGP(R_n \boxtimes R_n) & \xrightarrow{\ \mathcal{M}_n\ } & DGP(R_n) \\
\Big\downarrow{\scriptstyle H^0} & & \Big\downarrow{\scriptstyle H^0} \\
H^0(DGP(R_n \boxtimes R_n)) & \xrightarrow{\ \mathcal{M}_n|_{H^0}\ } & H^0(DGP(R_n)) \\
\Big\downarrow{\scriptstyle K_0} & & \Big\downarrow{\scriptstyle K_0} \\
Cl_n \otimes Cl_n \cong K_0(R_n \otimes R_n) & \xrightarrow{\ \mathrm{m}_n\ } & K_0(R_n) \cong Cl_n.
\end{array}
$$

The key part of this paper is the construction of DG $(R_n, R_n \boxtimes R_n)$-bimodules T_n which is motivated by the monoidal functor ρ_n in the topological model. Then tensoring with T_n defines a functor $\mathcal{M}_n : DGP(R_n \boxtimes R_n) \to DGP(R_n)$. We show that $\mathcal{M}_n$ induces an exact functor $\mathcal{M}_n|_{H^0} : H^0(DGP(R_n \boxtimes R_n)) \to H^0(DGP(R_n))$. Let $\mathcal{CL}_{n,n}$ and $\mathcal{CL}_n$ denote $H^0(DGP(R_n \boxtimes R_n))$ and $H^0(DGP(R_n))$, respectively. Then $\mathcal{M}_n|_{H^0}$ categorifies the multiplication m_n on the Clifford algebra Cl_n shown in Theorem 1.3.

The organization of the paper. In Section 2 we construct the quivers $\Gamma_n, \Gamma_n \boxtimes \Gamma_n$ and the q-graded DG algebras $R_n, R_n \otimes R_n$ and $R_n \boxtimes R_n$. In Section 3 we define the multiplication on $K_0(R_n)$ and show that it is isomorphic to Cl_n. In Section 4 we give a categorification of the multiplication: $\mathcal{M}_n : DGP(R_n \boxtimes R_n) \to DGP(R_n)$ through the DG $(R_n, R_n \otimes R_n)$-bimodules T_n. In Sect. 5 we construct the subcategory $\mathcal{U}_n$ of $H^0(DGP(R_n))$.

It categorifies the integral version $\mathbf{U}_n$ of $\mathbf{U}_q(\mathfrak{sl}(1|1))$ in the sense that the restriction of $\mathcal{M}_n$ categorifies the multiplication on $\mathbf{U}_n$.

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2. The q-graded DG algebras R_n and $R_n \boxtimes R_n$

2.1. The q-graded DG algebra R_n

In this section, we define R_n as path algebras of quivers Γ_n, for $n > 0$. The definition of Γ_n is purely algebraic although it is motivated from the contact category $\mathcal{C}_n$ More precisely, the vertex set $V(\Gamma_n)$ is the basis $\mathcal{B}_n$ consisting of the distinguished objects of the contact category $\mathcal{C}_n$. The arrow set $A(\Gamma_n)$ is given by some morphisms in $\mathcal{C}_n$ between the objects in $\mathcal{B}_n$. Each object $\mathbf{x} \in \mathcal{B}_n$ corresponds to a projective R_n-module $P(\mathbf{x})$. We are interested in DG categories $DGP(R_n)$ generated by these projective R_n-modules $\{P(\mathbf{x}) \mid \mathbf{x} \in \mathcal{B}_n\}$.

2.1.1. The quiver Γ_n. We construct a family of quivers $\Gamma_n = (V(\Gamma_n), A(\Gamma_n))$, where $V(\Gamma_n)$ and $A(\Gamma_n)$ are vertex and arrow sets of Γ_n.

Definition 2.1 (Quiver $\Gamma_n = (V(\Gamma_n), A(\Gamma_n))$).

1) Let $V(\Gamma_n)$ be the set of decreasing sequences of integers bounded by n and 0, i.e., $V(\Gamma_n) = \{[\emptyset]\} \sqcup \{\mathbf{x} = [x_0, \ldots, x_i] \mid n \geq x_0 > \cdots > x_i \geq 0, x_k \in \mathbb{Z} \text{ for } 0 \leq k \leq i\}$.
2) Let $A(\Gamma_n)$ be the subset of $V(\Gamma_n) \times V(\Gamma_n)$, where $(\mathbf{x}, \mathbf{y}) \in A(\Gamma_n)$ for $\mathbf{x} = [x_0, x_1, \ldots, x_i]$, $\mathbf{y} = [y_0, y_1, \ldots, y_j]$ if $j = i + 2$ and $\{y_0, y_1, \ldots, y_j\} = \{x_0, x_1, \ldots, x_i\} \sqcup \{s+1, s\}$ as sets, for some s.

Notation 2.2. We write an arrow $(\mathbf{x} \xrightarrow{s} \mathbf{y})$ if $(\mathbf{x}, \mathbf{y}) \in A(\Gamma_n)$, where $\mathbf{y}$ is obtained from $\mathbf{x}$ by adding a pair of adjacent integers $\{s+1, s\}$. We write $s \in \mathbf{x} = [x_0, x_1, \ldots, x_i]$ if $s = x_k$ for some k and $s \notin \mathbf{x}$ otherwise.

The quiver Γ_n is decomposed into connected components according to a grading on $V(\Gamma_n)$:

Definition 2.3. The *Euler grading* $\mathrm{e} : V(\Gamma_n) \to \mathbb{Z}$ is defined as $\mathrm{e}(\mathbf{x}) = \sum_{k=0}^{i}(-1)^{x_k}$ for $\mathbf{x} = [x_0, x_1, \ldots, x_i]$ and $\mathrm{e}([\emptyset]) = 0$.

It is easy to see that $\mathbf{x}$ and $\mathbf{y}$ are in the same connected component of Γ_n if and only if they have the same Euler grading: $\mathrm{e}(\mathbf{x}) = \mathrm{e}(\mathbf{y})$. Therefore, $\Gamma_n = \sqcup_e \Gamma_{n,e}$, where $\Gamma_{n,e}$ is the connected component with Euler grading e.

Remark 2.4. The Euler grading e actually comes from the Euler number of a dividing set. Recall a dividing set divides the surface into positive and negative regions. Then the *Euler number* is the Euler characteristic of the positive region minus the Euler characteristic of the negative region.

Example 2.5 (Quiver Γ_2). The quiver Γ_2 has four components $\Gamma_{2,e}$ for $e = -1, 0, 1, 2$, where $\Gamma_{2,0}$ and $\Gamma_{2,1}$ are dual to each other.

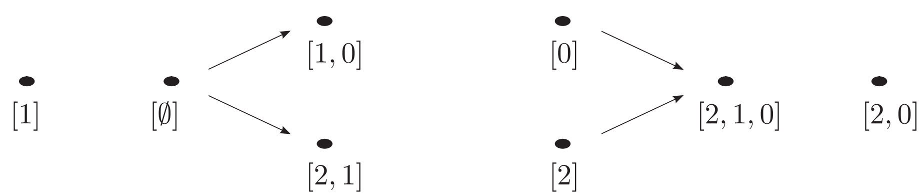


Figure 4: The quiver Γ_2.

2.1.2. The q-graded algebra R_n. In this subsection we define the q-graded algebra R_n as a quotient of the path algebra $\mathbb{F}_2\Gamma_n$ of the quiver Γ_n. $\mathbb{F}_2$ is fixed as the ground field throughout the paper.

Definition 2.6. R_n is an associative q-graded $\mathbb{F}_2$-algebra with a generator $r(\mathbf{x} \xrightarrow{s} \mathbf{y})$ for each arrow $(\mathbf{x} \xrightarrow{s} \mathbf{y})$ in Γ_n, idempotents $e(\mathbf{x})$ for each vertex $\mathbf{x}$ in Γ_n and relations:

$$e(\mathbf{x}) \cdot e(\mathbf{y}) = \delta_{\mathbf{x},\mathbf{y}} \cdot e(\mathbf{x}),$$
$$e(\mathbf{x}) \cdot r(\mathbf{x} \xrightarrow{s} \mathbf{y}) = r(\mathbf{x} \xrightarrow{s} \mathbf{y}) \cdot e(\mathbf{y}) = r(\mathbf{x} \xrightarrow{s} \mathbf{y}),$$
$$r(\mathbf{x} \xrightarrow{s} \mathbf{y}) \cdot r(\mathbf{y} \xrightarrow{t} \mathbf{w}) = r(\mathbf{x} \xrightarrow{t} \mathbf{z}) \cdot r(\mathbf{z} \xrightarrow{s} \mathbf{w}) \text{ for } |s - t| > 1.$$

The unit of R_n is the sum of idempotents: $\mathbf{1}_{R_n} = \sum_{\mathbf{x} \in V(\Gamma_n)} e(\mathbf{x})$. The q-grading deg on R_n is given on generators as: $\deg_{R_n}(e(\mathbf{x})) = 0, \deg_{R_n}(r(\mathbf{x} \xrightarrow{s} \mathbf{y})) = n - 1 - 2s$.

Remark 2.7. The last commutativity relation is about two paths from $\mathbf{x}$ to $\mathbf{w}$ if $\mathbf{w}$ is obtained from $\mathbf{x}$ by adding two disjoint pairs of adjacent integers $\{s+1, s\}$ and $\{t+1, t\}$, for $|s-t| > 1$.

We refer to the book [1] for an introduction to the representation theory of quivers. The nice property of the quiver Γ_n is that it has no oriented cycles. In particular, R_n is a finite dimensional algebra. Since $\{e(\mathbf{x}) \mid \mathbf{x} \in \Gamma_n\}$ is a complete set of primitive orthogonal idempotents in R_n, $\{P(\mathbf{x}) = R_n e(\mathbf{x}) \mid \mathbf{x} \in \Gamma_n\}$ forms a complete set of non-isomorphic indecomposable projective q-graded left R_n-modules, up to grading shifts. Let $A\{m\}$ denote A with its q-grading shifted by m, i.e., $A\{m\} = \{a \in A \mid \deg_{A\{m\}}(a) = \deg_A(a) - m\}$. Then any projective graded left R_n-module A is a direct sum of indecomposables $P(\mathbf{x})\{m\}$.

Consider $\mathbf{K}^b(R_n)$, the homotopy category of bounded cochain complexes of finitely generated projective graded modules over R_n with grading-preserving differentials. The chain maps are also grading-preserving. For any cochain complex $M = \{\cdots \to M^s \to M^{s+1} \to \cdots\} \in \mathbf{K}^b(R_n)$, let $M[p]$ be M with the cohomological grading shifted by p, i.e., $M[p]^s = M^{s+p}$. By a standard result in homological algebra, $\mathbf{K}^b(R_n)$ is a triangulated category.

Let $K_0(R_n)$ be the Grothendieck group of $\mathbf{K}^b(R_n)$. It is a $\mathbb{Z}[q^{\pm 1}]$-module generated by $[P]$ over all finitely generated projective graded R_n-modules P, subject to relations $[P\{1\}] = q[P]$, $[P[1]] = -[P]$ and $[P_2] = [P_1] + [P_3]$ for each short exact sequence $0 \to P_1 \to P_2 \to P_3 \to 0$. It is easy to see $K_0(R_n)$ is a free $\mathbb{Z}[q^{\pm 1}]$-module over the basis $\{[P(\mathbf{x})] \mid \mathbf{x} \in V(\Gamma_n)\}$. Let $\mathbb{Z}[q^{\pm 1}]\langle S \rangle$ denote the free $\mathbb{Z}[q^{\pm 1}]$-module generated by the set S, then

$$K_0(R_n) \cong \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma_n) \rangle \cong \mathbb{Z}[q^{\pm 1}]\langle \mathcal{B}_n \rangle.$$

Remark 2.8. The q-grading on R_n is only used to make $K_0(R_n)$ as a $\mathbb{Z}[q^{\pm 1}]$-module. The reader may ignore the q-grading in various modules at a first reading.

2.1.3. The DG category $DGP(R_n)$. We consider DG algebras and DG modules with an additional q-grading in a straightforward way. We refer to [3, Section 10] for an introduction to DG algebras and DG modules. Most of this subsection is standard homological algebra in the DG world. The goal is to define the DG category $DGP(R_n)$ consisting of iterated mapping cones of maps between the projective R_n-modules $\{P(\mathbf{x}) \mid \mathbf{x} \in V(\Gamma_n)\}$. The category $DGP(R_n)$ is our algebraic formulation of the contact category $\mathcal{C}_n$ on the topological side. Instead of studying chain maps up to homotopy in $\mathbf{K}^b(R_n)$, we want to keep track of various homotopies in the DG category $DGP(R_n)$.

The DG structure is crucial in the construction of functors between DG categories in Section 4.

Definition 2.9. A *q-graded DG algebra* (A, d) is a doubly graded $\mathbb{F}_2$-algebra $A = \bigoplus_{i,j} A^{i,j}$ with a unit $\mathbf{1}_A \in A^{0,0}$, where i is the cohomological grading and j is the q-grading. The differential d is an additive endomorphism of degree $(1, 0)$ such that for $a, b \in A$:

$$d^2 = 0,\ d(\mathbf{1}_A) = 0,$$
$$d(a \cdot b) = d(a) \cdot b + a \cdot d(b).$$

Definition 2.10. A *left q-graded DG module* (M, d_M) over a q-graded DG algebra (A, d) is a doubly graded unitary left A-module $M = \bigoplus_{i,j} M^{i,j}$, where i is the cohomological grading and j is the q-grading. The differential d_M is an additive endomorphism of degree $(1, 0)$ such that for $a \in A, m \in M$

$$d_M^2 = 0, \quad d_M(a \cdot m) = d(a) \cdot m + a \cdot d_M(m).$$

We view R_n as a q-graded DG algebra $(R_n, d = 0)$ which has trivial differential and is concentrated in cohomological grading 0. Let $DG(R_n)$ be the DG category of q-graded DG R_n-modules. We refer to [16] for an introduction to DG categories.

Definition 2.11 (DG category $DG(A)$ for a q-graded DG algebra A).

1) The objects of $DG(A)$ are left q-graded DG A-modules.
2) The space of morphisms

$$\big(\mathrm{Hom}_{DG(A)}(M, N), d\big) = \left(\bigoplus_i \mathrm{Hom}^i_{DG(A)}(M, N), \bigoplus_i d^i\right)$$

   is a cochain complex, where $\mathrm{Hom}^i_{DG(A)}(M, N)$ is the set of left A-module maps of degree $(i, 0)$ and

$$d^i : \mathrm{Hom}^i_{DG(A)}(M, N) \to \mathrm{Hom}^{i+1}_{DG(A)}(M, N)$$

   is given as $d(f) = d \circ f + f \circ d$.
3) A morphism f is *closed* if $d(f) = 0$; f is *exact* if $f = d(g)$ for some morphism g. Let $Z^i(\mathrm{Hom}_{DG(A)}(M, N))$ and $B^i(\mathrm{Hom}_{DG(A)}(M, N))$ denote

the subset of $\mathrm{Hom}^i_{DG(A)}(M,N)$ consisting of closed morphisms and exact morphisms respectively.

Remark 2.12. For $(R_n, d=0)$, a q-graded DG R_n-module is a cochain complex of q-graded R_n-modules. A closed morphism of degree 0 is a chain map.

Definition 2.13. The *0-th homology category* $H^0(DG(A))$ of the DG category $DG(A)$ has the same objects as $DG(A)$ and its morphisms are given by

$$\mathrm{Hom}_{H^0(DG(A))}(M,N) = Z^0(\mathrm{Hom}_{DG(A)}(M,N))/B^0(\mathrm{Hom}_{DG(A)}(M,N)).$$

Remark 2.14. The 0-th homology category $H^0(DG(A))$ is isomorphic to the homotopy category of q-graded DG A-modules.

Definition 2.15. A DG A-module P is called *projective* if the complex $\mathrm{Hom}_{DG(A)}(P,M)$ has zero cohomology when the cohomology $H(M)$ of $(M,d_M) \in Ob(DG(A))$ is zero.

Remark 2.16. The DG R_n-module $P(\mathbf{x}) = R_n e(\mathbf{x})$ is projective since it is a direct summand of R_n which is projective [3, Remark 10.12.2.3].

There are two automorphisms of $DG(A)$: [1] and $\{1\}$ with respect to the cohomological grading and the q-grading. There is another operation, called the *mapping cone*, which constructs a new object $C(f)$ from $f \in Z^0(\mathrm{Hom}_{DG(A)}(M,N))$

Definition 2.17 (Two shift functors and $C(f)$).

1) The shift functor $[1] : DG(A) \to DG(A)$ is an automorphism of $DG(A)$ such that,
$$(M[1])^{i,j} = M^{i+1,j}, \quad d_{M[1]} = d_M.$$
2) The shift functor $\{1\} : DG(A) \to DG(A)$ is an automorphism of $DG(A)$ such that,
$$(M\{1\})^{i,j} = M^{i,j+1}, \quad d_{M\{1\}} = d_M.$$
3) For $f \in Z^0(\mathrm{Hom}_{DG(A)}(M,N))$, $M,N \in Ob(DG(A))$, define the mapping cone $C(f) = N \oplus M[1]$ with the differential $d_{C(f)} = (d_N + f, d_M)$.

Definition 2.18. Let $DGP(R_n)$ be the smallest full subcategory of $DG(R_n)$ which contains the projective DG R_n-modules $\{P(\mathbf{x}) \mid \mathbf{x} \in V(\Gamma_n)\}$ and is closed under the two shift functors $[1], \{1\}$ and taking the mapping cones.

The objects of $DGP(R_n)$ are finitely iterated cones of closed morphisms between the projective modules $\{P(\mathbf{x}) \mid \mathbf{x} \in V(\Gamma_n)\}$ up to grading shifts. Since $\{P(\mathbf{x}) \mid \mathbf{x} \in \Gamma_n\}$ form a complete set of non-isomorphic indecomposable projective R_n-modules up to grading shifts, the 0-th homology category $H^0(DGP(R_n))$ is equivalent to $\mathbf{K}^b(R_n)$ as triangulated categories. Hence their Grothendieck groups are isomorphic:

$$K_0(H^0(DGP(R_n))) \cong K_0(\mathbf{K}^b(R_n)) = K_0(R_n).$$

2.2. The q-graded DG algebra $R_n \boxtimes R_n$

We will define the multiplication $\mathrm{m}_n : K_0(R_n) \otimes K_0(R_n) \to K_0(R_n)$ in Section 3. To categorify $K_0(R_n) \otimes K_0(R_n)$, we first consider the tensor product $R_n \otimes R_n$ and its homotopy category $\mathbf{K}^b(R_n \otimes R_n)$ whose Grothendieck group is isomorphic to $K_0(R_n) \otimes K_0(R_n)$. The algebra $R_n \otimes R_n$ can be described by a quiver $\Gamma_n \times \Gamma_n$. Then we construct a quiver $\Gamma_n \boxtimes \Gamma_n$ by adding more arrows on the product $\Gamma_n \times \Gamma_n$. The new arrows deform $R_n \otimes R_n$ into a nontrivial DG algebra $R_n \boxtimes R_n$. We show that two algebras are quasi-isomorphic, hence their homotopy categories are equivalent. Note that the nontrivial DG structure on $R_n \boxtimes R_n$ is only used to construct the DG $(R_n, R_n \boxtimes R_n)$-bimodule T_n in Section 4.

2.2.1. The q-graded algebra $R_n \otimes R_n$.

Definition 2.19. As an $\mathbb{F}_2$-algebra, $R_n \otimes_{\mathbb{F}_2} R_n$ is the tensor product of two R_n's over $\mathbb{F}_2$ with unit

$$\mathbf{1}_{R_n \otimes_{\mathbb{F}_2} R_n} = \sum_{\mathbf{x},\mathbf{y} \in V(\Gamma_n)} e(\mathbf{x}) \otimes e(\mathbf{y}).$$

The q-grading on generators is given as $\deg_{R_n \otimes_{\mathbb{F}_2} R_n}(a \otimes b) = \deg_{R_n}(a) + \deg_{R_n}(b)$.

For simplicity, we omit ground rings or fields in various tensor products. For instance, we write $R_n \otimes R_n$ for $R_n \otimes_{\mathbb{F}_2} R_n$, and $K_0(R_n) \otimes K_0(R_n)$ for $K_0(R_n) \otimes_{\mathbb{Z}[q^{\pm 1}]} K_0(R_n)$.

Since $\{e(\mathbf{x}) \otimes e(\mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in V(\Gamma_n)\}$ is a complete set of primitive orthogonal idempotents of $R_n \otimes R_n$, the modules $P'(\mathbf{x}, \mathbf{y}) = (R_n \otimes R_n)(e(\mathbf{x}) \otimes e(\mathbf{y})) = R_n e(\mathbf{x}) \otimes R_n e(\mathbf{y})$ form a complete set of non-isomorphic indecomposable projective graded left $R_n \otimes R_n$-modules, up to grading shifts. Consider $\mathbf{K}^b(R_n \otimes R_n)$, the homotopy category of bounded cochain complexes of finitely generated projective graded modules over $R_n \otimes R_n$ with grading-preserving differentials and chain maps. Let $K_0(R_n \otimes R_n)$ be the Grothendieck group of $\mathbf{K}^b(R_n \otimes R_n)$. It is easy to see that $K_0(R_n \otimes R_n)$ is a free $\mathbb{Z}[q^{\pm 1}]$-module over $\{[P'(\mathbf{x}, \mathbf{y})] \mid \mathbf{x}, \mathbf{y} \in V(\Gamma_n)\}$, i.e.,

$$\begin{aligned} K_0(R_n \otimes R_n) &= \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma_n) \times V(\Gamma_n)\rangle \\ &= \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma_n)\rangle \otimes \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma_n)\rangle. \end{aligned}$$

Hence we have $K_0(R_n \otimes R_n) = K_0(R_n) \otimes K_0(R_n)$.

2.2.2. The q-graded DG-algebra $R_n \boxtimes R_n$. We construct a family of quivers $\Gamma_n \boxtimes \Gamma_n$ viewed as a variant of the product $\Gamma_n \times \Gamma_n$ by adding more arrows.

Definition 2.20 (Quiver $\Gamma_n \boxtimes \Gamma_n = (V(\Gamma_n \boxtimes \Gamma_n), A(\Gamma_n \boxtimes \Gamma_n))$).

1) $V(\Gamma_n \boxtimes \Gamma_n) = V(\Gamma_n) \times V(\Gamma_n)$.
2) Let $A(\Gamma_n \boxtimes \Gamma_n)$ be the subset of

$$V(\Gamma_n \boxtimes \Gamma_n) \times V(\Gamma_n \boxtimes \Gamma_n) = (V(\Gamma_n) \times V(\Gamma_n)) \times (V(\Gamma_n) \times V(\Gamma_n)),$$

where $(\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}') \in A(\Gamma_n \boxtimes \Gamma_n)$ if one of following holds:
a) $(\mathbf{x}, \mathbf{x}') \in A(\Gamma_n)$ and $\mathbf{y} = \mathbf{y}'$;
b) $(\mathbf{y}, \mathbf{y}') \in A(\Gamma_n)$ and $\mathbf{x} = \mathbf{x}'$;
c) $(\mathbf{x}, \mathbf{x}'), (\mathbf{y}, \mathbf{y}') \in A(\Gamma_n)$ and there exist some $s \in \{0, 1, \dots, n-1\}$ such that the corresponding arrows are $(\mathbf{x} \xrightarrow{s} \mathbf{x}')$ and $(\mathbf{y} \xrightarrow{s+1} \mathbf{y}')$

We denote the arrows for (a), (b) and (c) by $(\mathbf{x}, \mathbf{y} \xrightarrow{s,\emptyset} \mathbf{x}', \mathbf{y})$, $(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,s} \mathbf{x}, \mathbf{y}')$ and $(\mathbf{x}, \mathbf{y} \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}')$, respectively.

Remark 2.21. Arrows of types (a) and (b) are induced from those in each factor of $\Gamma_n \times \Gamma_n$. Arrows of new type (c) measure homotopies between arrows in certain squares as shown in the following example.

Example 2.22 (Quiver $\Gamma_2 \boxtimes \Gamma_2$). One connected component of $\Gamma_2 \boxtimes \Gamma_2$ is shown in Fig 5. The arrow $([\emptyset],[\emptyset]) \xrightarrow{0,1} ([1,0],[2,1])$ in red gives the homotopy between two paths:

$$([\emptyset],[\emptyset]) \xrightarrow{0,\emptyset} ([1,0],[\emptyset]) \xrightarrow{\emptyset,1} ([1,0],[2,1]) \quad \text{and}$$
$$([\emptyset],[\emptyset]) \xrightarrow{\emptyset,1} ([\emptyset],[2,1]) \xrightarrow{0,\emptyset} ([1,0],[2,1]).$$

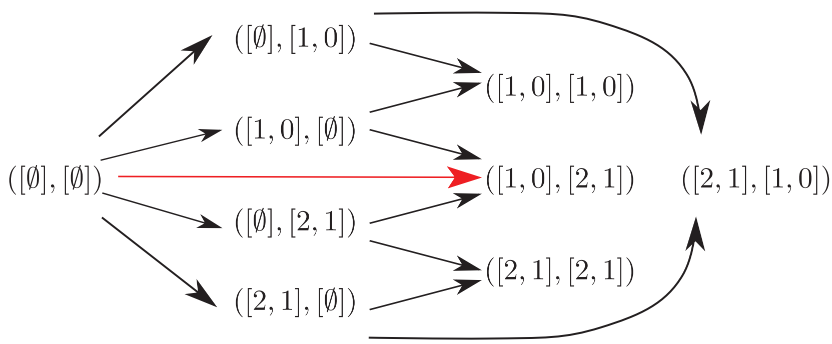


Figure 5: One component of the quiver $\Gamma_2 \boxtimes \Gamma_2$.

The algebra $R_n \boxtimes R_n$ is defined as a quotient of the path algebra $\mathbb{F}_2(\Gamma_n \boxtimes \Gamma_n)$ of the quiver $\Gamma_n \boxtimes \Gamma_n$ with a differential corresponding to the arrows of type (c).

Definition 2.23. $(R_n \boxtimes R_n, d)$ is an associative q-graded DG algebra with a differential d, a cohomological grading and a q-grading.

(A) $R_n \boxtimes R_n$ has idempotents $e(\mathbf{x},\mathbf{y})$ for each vertex $(\mathbf{x},\mathbf{y})$, generators $r(\mathbf{x},\mathbf{y} \xrightarrow{s,t} \mathbf{x}',\mathbf{y}')$ for each arrow $(\mathbf{x},\mathbf{y} \xrightarrow{s,t} \mathbf{x}',\mathbf{y}')$ in $\Gamma_n \boxtimes \Gamma_n$, here s or t maybe $\emptyset$. The relations consists of two types:

(1) relations from each factor of $\Gamma_n \boxtimes \Gamma_n$:

$$e(\mathbf{x},\mathbf{y}) \cdot e(\mathbf{x}',\mathbf{y}') = \delta_{\mathbf{x},\mathbf{x}'} \cdot \delta_{\mathbf{y},\mathbf{y}'} \cdot e(\mathbf{x},\mathbf{y});$$
$$e(\mathbf{x},\mathbf{y}) \cdot r(\mathbf{x},\mathbf{y} \xrightarrow{s,t} \mathbf{x}',\mathbf{y}') = r(\mathbf{x},\mathbf{y} \xrightarrow{s,t} \mathbf{x}',\mathbf{y}') \cdot e(\mathbf{x}',\mathbf{y}') = r(\mathbf{x},\mathbf{y} \xrightarrow{s,t} \mathbf{x},\mathbf{y}');$$
$$r(\mathbf{x},\mathbf{y} \xrightarrow{\emptyset,s} \mathbf{x},\mathbf{y}') \cdot r(\mathbf{x},\mathbf{y}' \xrightarrow{\emptyset,t} \mathbf{x},\mathbf{y}''') = r(\mathbf{x},\mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x},\mathbf{y}'') \cdot r(\mathbf{x},\mathbf{y}'' \xrightarrow{\emptyset,s} \mathbf{x},\mathbf{y}''');$$
$$r(\mathbf{x},\mathbf{y} \xrightarrow{s,\emptyset} \mathbf{x}',\mathbf{y}) \cdot r(\mathbf{x}',\mathbf{y} \xrightarrow{t,\emptyset} \mathbf{x}''',\mathbf{y}) = r(\mathbf{x},\mathbf{y} \xrightarrow{t,\emptyset} \mathbf{x}'',\mathbf{y}) \cdot r(\mathbf{x}'',\mathbf{y} \xrightarrow{s,\emptyset} \mathbf{x}''',\mathbf{y});$$

$$\begin{aligned} &r(\mathbf{x}, \mathbf{y} \xrightarrow{s,\emptyset} \mathbf{x}', \mathbf{y}) \cdot r(\mathbf{x}', \mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x}', \mathbf{y}') \\ = &r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x}, \mathbf{y}') \cdot r(\mathbf{x}, \mathbf{y}' \xrightarrow{s,\emptyset} \mathbf{x}', \mathbf{y}') \text{ if } t \neq s+1; \end{aligned}$$

(2) relations on the arrows of type (c) in $\Gamma_n \boxtimes \Gamma_n$:

$$\begin{aligned} &r(\mathbf{x}, \mathbf{y} \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}') \cdot r(\mathbf{x}', \mathbf{y}' \xrightarrow{\emptyset,t} \mathbf{x}', \mathbf{y}''') \\ = &r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x}, \mathbf{y}'') \cdot r(\mathbf{x}, \mathbf{y}'' \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}'''); \\ &r(\mathbf{x}, \mathbf{y} \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}') \cdot r(\mathbf{x}', \mathbf{y}' \xrightarrow{t,\emptyset} \mathbf{x}''', \mathbf{y}') \\ = &r(\mathbf{x}, \mathbf{y} \xrightarrow{t,\emptyset} \mathbf{x}'', \mathbf{y}) \cdot r(\mathbf{x}'', \mathbf{y} \xrightarrow{s,s+1} \mathbf{x}''', \mathbf{y}'); \\ &r(\mathbf{x}, \mathbf{y} \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}') \cdot r(\mathbf{x}', \mathbf{y}' \xrightarrow{t,t+1} \mathbf{x}''', \mathbf{y}''') \\ = &r(\mathbf{x}, \mathbf{y} \xrightarrow{t,t+1} \mathbf{x}'', \mathbf{y}'') \cdot r(\mathbf{x}'', \mathbf{y}'' \xrightarrow{s,s+1} \mathbf{x}''', \mathbf{y}'''). \end{aligned}$$

(B) The differential d is given on the generators as:

$$\begin{aligned} d(r(\mathbf{x}, \mathbf{y} \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}')) = &r(\mathbf{x}, \mathbf{y} \xrightarrow{s,\emptyset} \mathbf{x}', \mathbf{y}) \cdot r(\mathbf{x}', \mathbf{y} \xrightarrow{\emptyset,s+1} \mathbf{x}', \mathbf{y}') \\ &+ r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,s+1} \mathbf{x}, \mathbf{y}') \cdot r(\mathbf{x}, \mathbf{y}' \xrightarrow{s,\emptyset} \mathbf{x}', \mathbf{y}'), \end{aligned}$$

and $d(r) = 0$ otherwise. It is extended by $d(r_1 \cdot r_2) = d(r_1) \cdot r_2 + r_1 \cdot d(r_2)$ for generators r_1, r_2.

(C) The cohomological grading gr is given on generators as: $\mathrm{gr}(r(\mathbf{x}, \mathbf{y} \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}')) = -1$ and $\mathrm{gr}(r) = 0$ otherwise.

(D) The q-grading deg is given on generators as: $\deg(r(\mathbf{x}, \mathbf{y} \xrightarrow{s,t} \mathbf{x}', \mathbf{y}')) = (n-1-2s) + (n-1-2t)$ and $\deg(e(\mathbf{x}, \mathbf{y})) = 0$.

Remark 2.24. Relations of type (1) are analogous to those of $R_n \otimes R_n$ if we identify $e(\mathbf{x}, \mathbf{y}), r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,s} \mathbf{x}, \mathbf{y}'), r(\mathbf{x}, \mathbf{y} \xrightarrow{t,\emptyset} \mathbf{x}', \mathbf{y})$ with $e(\mathbf{x}) \otimes e(\mathbf{y}), e(\mathbf{x}) \otimes r(\mathbf{y} \xrightarrow{s} \mathbf{y}'), r(\mathbf{x} \xrightarrow{t} \mathbf{x}') \otimes e(\mathbf{y})$, respectively.

Lemma 2.25. *d is a differential on $R_n \boxtimes R_n$.*

Proof. It suffices to prove that d preserves the relations of type (2) in Definition 2.23 (A). We prove it for the first relation of type (2)

$$\begin{aligned} (*) \qquad &r(\mathbf{x}, \mathbf{y} \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}') \cdot r(\mathbf{x}', \mathbf{y}' \xrightarrow{\emptyset,t} \mathbf{x}', \mathbf{y}''') \\ = &r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x}, \mathbf{y}'') \cdot r(\mathbf{x}, \mathbf{y}'' \xrightarrow{s,s+1} \mathbf{x}', \mathbf{y}''') \end{aligned}$$

by checking an explicit example for $s = 0, t = 3$:

$$(\mathbf{x}, \mathbf{y}) = ([\emptyset], [\emptyset]), \quad (\mathbf{x}', \mathbf{y}') = ([1,0], [2,1]),$$
$$(\mathbf{x}, \mathbf{y}'') = ([\emptyset], [4,3]), \quad (\mathbf{x}', \mathbf{y}''') = ([1,0], [4,3,2,1]).$$

We rewrite the relation (*) as $\alpha\gamma_1 = \gamma_2\beta$ in terms of symbols in Fig 6. The differential of the arrow α (β) in red is the sum of two paths from its tail to its head. Two paths commute in each parallelogram without a diagonal arrow. Now we prove that the differential preserves $\alpha\gamma_1 = \gamma_2\beta$ by chasing the diagram:

$$\begin{aligned} d(\alpha\gamma_1) = d(\alpha)\gamma_1 &= \alpha_1\alpha_2\gamma_1 + \alpha_3\alpha_4\gamma_1 \\ &= \alpha_1\gamma_3\beta_2 + \alpha_3\gamma_4\beta_4 = \gamma_2\beta_1\beta_2 + \gamma_2\beta_3\beta_4 = d(\gamma_2\beta). \end{aligned}$$

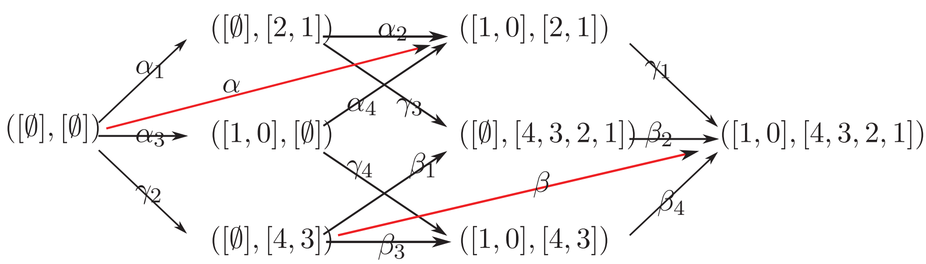


Figure 6:

The proof about all relations in general is similar and left to the reader. □

2.2.3. Relations between $R_n \boxtimes R_n$ and $R_n \otimes R_n$. We define DG categories $DGP(R_n \otimes R_n)$ and $DGP(R_n \boxtimes R_n)$. In order to compute the Grothendieck group of the 0-th homology category $H^0(DGP(R_n \boxtimes R_n))$, we show that $R_n \boxtimes R_n$ is quasi-isomorphic to $R_n \otimes R_n$ which has trivial differential. Then the homology categories $H^0(DGP(R_n \boxtimes R_n))$ and $H^0(DGP(R_n \otimes R_n))$ are equivalent. We will show that both Grothendieck groups are isomorphic to $K_0(R_n \otimes R_n)$.

Definition 2.26. A q-graded DG algebra A is *formal* if it is q-graded quasi-isomorphic to its cohomology $H(A)$.

Lemma 2.27. *The q-graded DG algebra $R_n \boxtimes R_n$ is formal and its cohomology $H(R_n \boxtimes R_n)$ is isomorphic to $R_n \otimes R_n$.*

Proof. It is easy to see that the cohomology $H(R_n \boxtimes R_n)$ is isomorphic to $R_n \otimes R_n$. We define a quasi-isomorphism $H : R_n \boxtimes R_n \to R_n \otimes R_n$ as follows:

$$\begin{aligned} e(\mathbf{x}, \mathbf{y}) &\mapsto e(\mathbf{x}) \otimes e(\mathbf{y}) \\ r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset, s} \mathbf{x}, \mathbf{y}') &\mapsto e(\mathbf{x}) \otimes r(\mathbf{y} \xrightarrow{s} \mathbf{y}') \\ r(\mathbf{x}, \mathbf{y} \xrightarrow{s, \emptyset} \mathbf{x}', \mathbf{y}) &\mapsto r(\mathbf{x} \xrightarrow{s} \mathbf{x}') \otimes e(\mathbf{y}) \\ r(\mathbf{x}, \mathbf{y} \xrightarrow{s, s+1} \mathbf{x}', \mathbf{y}') &\mapsto 0 \end{aligned}$$

□

Definition 2.28. (1) Let $DGP(R_n \boxtimes R_n)$ be the smallest full subcategory of $DG(R_n \boxtimes R_n)$ which contains the projective DG $R_n \boxtimes R_n$-modules $\{P(\mathbf{x}, \mathbf{y}) = (R_n \boxtimes R_n)e(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in V(\Gamma_n)\}$ and is closed under the two shift functors $[1], \{1\}$ and taking the mapping cones.

(2) Let $DGP(R_n \otimes R_n)$ be the smallest full subcategory of $DG(R_n \otimes R_n)$ which contains the projective DG $R_n \otimes R_n$-modules $\{P'(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in V(\Gamma_n)\}$ and is closed under the two shift functors $[1], \{1\}$ and taking the mapping cones.

We give the connection between $H^0(DGP(R_n \boxtimes R_n))$ and $H^0(DGP(R_n \otimes R_n))$ as follows. Let $Z^0(DG(A))$ be an abelian category with the same objects as $DG(A)$, whose morphisms are

$$\mathrm{Hom}_{Z^0(DG(A))}(M, N) = Z^0(\mathrm{Hom}_{DG(A)}(M, N)).$$

Consider the homotopy category $KDG(A)$ and derived category $DDG(A)$ of $Z^0(DG(A))$. They are triangulated categories. Let $KPDG(A)$ be the full subcategory of $KDG(A)$ consisting of projective q-graded DG A-modules. The localization functor induces an equivalence: $KPDG(A) \to DDG(A)$ [3, Corollary 10.12.2.9].

For any quasi-isomorphism $F : A \to B$ of DG algebras, the derived induction functor $ind = B \bigotimes^L {}_A - : DDG(A) \to DDG(B)$ is an equivalence of categories [3, Theorem 10.12.5.1]. The induced functor ind on $KPDG(A)$ is the induction functor $B \otimes_A -$. It maps any projective DG A-module P to a projective DG B-module $B \otimes_A P$ since

$$\mathrm{Hom}_{DG(B)}(B \otimes_A P, N) \cong \mathrm{Hom}_{DG(A)}(P, Res(N))$$

has zero cohomology for any DG B-module N with $H(N) = 0$, where $Res(N)$ is the restriction of N as a A-module. Hence we have an equivalence $B \otimes_A - :$

$KPDG(A) \to KPDG(B)$. In particular, $ind_n : KPDG(R_n \boxtimes R_n) \to KPDG(R_n \otimes R_n)$ is an equivalence since $R_n \boxtimes R_n$ is quasi-isomorphic to $R_n \otimes R_n$. We have the following equivalence for their subcategories:

Lemma 2.29. *The* 0*-th homology category* $H^0(DGP(R_n \boxtimes R_n))$ *is equivalent to the* 0*-th homology category* $H^0(DGP(R_n \otimes R_n))$.

Proof. Notice that $ind_n(P(\mathbf{x},\mathbf{y})) = P'(\mathbf{x},\mathbf{y})$ and $H^0(DGP(R_n \boxtimes R_n))$ is a full subcategory of $KPDG(R_n \boxtimes R_n)$. We have a restriction of ind_n: $H^0(DGP(R_n \boxtimes R_n)) \to H^0(DGP(R_n \otimes R_n))$. It is fully faithful since $ind_n : KPDG(R_n \boxtimes R_n) \to KPDG(R_n \otimes R_n)$ is an equivalence.

Any object N in $H^0(DGP(R_n \otimes R_n))$ is a finitely iterated cone of maps between $P'(\mathbf{x},\mathbf{y})$'s, hence it is isomorphic to an object $ind_n(M)$ for some M in $H^0(DGP(R_n \boxtimes R_n))$. Therefore, the restriction of $ind_n : H^0(DGP(R_n \boxtimes R_n)) \to H^0(DGP(R_n \otimes R_n))$ induces an equivalence of categories. □

Since $\{P'(\mathbf{x},\mathbf{y}) \mid \mathbf{x},\mathbf{y} \in \Gamma_n\}$ form a complete set of non-isomorphic indecomposable projective $R_n \otimes R_n$-modules up to grading shifts, the 0-th homology category $H^0(DGP(R_n \otimes R_n))$ is equivalent to $\mathbf{K}^b(R_n \otimes R_n)$.

Corollary 2.30. *There are isomorphisms of the Grothendieck groups:*

$$\begin{aligned} K_0(H^0(DGP(R_n \boxtimes R_n))) &\cong K_0(H^0(DGP(R_n \otimes R_n))) \\ &\cong K_0(\mathbf{K}^b(R_n \otimes R_n)) = K_0(R_n) \otimes K_0(R_n). \end{aligned}$$

3. The multiplication on $K_0(R_n)$

The goal of this section is to define the multiplication $\mathrm{m}_n : K_0(R_n) \otimes K_0(R_n) \to K_0(R_n)$ and show that $K_0(R_n)(\cong \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma)\rangle)$ is isomorphic to the Clifford algebra Cl_n. We fix some $n > 0$ throughout this and next sections and omit the subscript n.

We actually define a *higher multiplication* M as a $\mathbb{Z}[q^{\pm 1}, h^{\pm 1}]$-bilinear map

$$\mathrm{M} : \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle \times \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle \to \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle.$$

The multiplication m is a specialization of M to $h = -1$. The specialization is the shadow of a decategorification passing from complexes to their Euler charateristic, where the variable h corresponds to the cohomological grading in $DGP(R)$. The higher multiplication M will be our main tool to construct the $(R, R \boxtimes R)$-bimodule T in Section 4.

Given any pair of decreasing sequences $\mathbf{x} = [x_0, x_1, \ldots, x_i]$ and $\mathbf{y} = [y_0, y_1, \ldots, y_j] \in V(\Gamma)$, their concatenation $\mathbf{x} \cdot \mathbf{y} = [x_0, x_1, \ldots, x_i, y_0, y_1, \ldots, y_j]$ may not be decreasing. The definition of $\mathrm{M}(\mathbf{x}, \mathbf{y})$ gives several rules to represent a non-decreasing sequence as a linear combination of decreasing ones. Since any decreasing sequences $\mathbf{x}$ corresponds to the projective module $P(\mathbf{x}) \in DGP(R)$, the multiplication $\mathrm{M}(\mathbf{x}, \mathbf{y})$ gives a projective resolution of the module corresponding to the concatenation $\mathbf{x} \cdot \mathbf{y}$.

Definition 3.1 (Higher multiplication M**, special cases).** Define $\mathrm{M}(\mathbf{x},\mathbf{y})$, where $\mathbf{x}, \mathbf{y}$ are sequences of length at most 1 in $V(\Gamma)$:

1) $\mathrm{M}([a], [b]) = [a, b]$ if $a > b$ and $\mathrm{M}([\emptyset], [a]) = \mathrm{M}([a], [\emptyset]) = [a]$.
2) $\mathrm{M}([a], [a]) = 0$.[2]
3) $\mathrm{M}([a], [b]) = h^{(-1)^{a+b+1}}[b, a]$ if $a < b - 1$.
4) $\mathrm{M}([a], [b]) = q^{2a+1-n}[\emptyset] + h[b, a]$ if $a = b - 1$.

Remark 3.2. (1) The specialization of M to $h = -1$ for the special cases agrees with the defining relations of the Clifford algebra Cl_n in Definition 1.2 by identifying $[a] \in V(\Gamma)$ with $X_a \in Cl_n$.

(2) The exponent of q in M corresponds to the q-grading in $R : \deg(r(\mathbf{x} \xrightarrow{s} \mathbf{y})) = n - 1 - 2s$.

In the rest of this section we generalize $\mathrm{M}(\mathbf{x}, \mathbf{y})$ in Definition 3.1 to any $\mathbf{x}, \mathbf{y} \in V(\Gamma_n)$ in 3 steps:

Step 1: Use Definition 3.1 (3) to exchange smaller numbers in $\mathbf{x}$ with larger numbers in $\mathbf{y}$ if their differences are greater than 1.

Step 2: Use Definition 3.1 (4) to express an adjacent increasing pair as a sum of the empty sequence and the decreasing pair.

Step 3: Use Definition 3.1 (1), (2) and the distribution law to get the multiplication in general.

Example 3.3 (Explanation of 3 steps). Let $\mathbf{x} = [1, 0], \mathbf{y} = [2, 1]$ and $\mathbf{x} \cdot \mathbf{y}$ denote $\mathrm{M}(\mathbf{x}, \mathbf{y})$ in this example. We explain 3 steps in computing $[1, 0] \cdot [2, 1]$ whose concatenation is $[1, 0, 2, 1]$:

[2]Note that $0 \in \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle$ is different from both sequences $[0], [\emptyset] \in V(\Gamma)$.

Step 1: The difference between $2 \in \mathbf{y}$ and $0 \in \mathbf{x}$ is greater than 1, so we apply Definition 3.1 (3) to move $2 \in \mathbf{y}$ in front of $0 \in \mathbf{x}$ in the concatenation:

$$[1, 0, 2, 1] = h^{-1}[1, 2, 0, 1].$$

Step 2: Apply Definition 3.1 (4) to adjacent increasing pairs $[1, 2]$ and $[0, 1]$:

$$h^{-1}[1, 2, 0, 1] = h^{-1}[1, 2] \cdot [0, 1] = h^{-1}(q^{3-n}[\emptyset] + h[2, 1]) \cdot (q^{1-n}[\emptyset] + h[1, 0]).$$

Step 3: Apply the distribution law and Definition 3.1 (1), (2):

$$\begin{aligned} &h^{-1}(q^{3-n}[\emptyset] + h[2, 1]) \cdot (q^{1-n}[\emptyset] + h[1, 0]) \\ &= h^{-1}q^{4-2n}[\emptyset] + q^{1-n}[2, 1] + q^{3-n}[1, 0] + h[2, 1] \cdot [1, 0] \\ &= h^{-1}q^{4-2n}[\emptyset] + q^{1-n}[2, 1] + q^{3-n}[1, 0]. \end{aligned}$$

The definition of M in general is given as follows. We first want to keep track of the exponent of h in Step 1. The cohomological shifting number $\mu(\mathbf{x}, \mathbf{y})$ counts the shift in h as in Definition 3.1 (3) when exchanging numbers with difference greater than 1.

Definition 3.4 (Cohomological shifting μ). Define $\mu : V(\Gamma) \times V(\Gamma) \to \mathbb{Z}$ by

$$\mu(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{i} \sum_{l=0}^{j} \mu(x_k, y_l)$$

for $\mathbf{x} = [x_0, \dots, x_i]$ and $\mathbf{y} = [y_0, \dots, y_j]$, where

$$\mu(x_k, y_l) = \begin{cases} (-1)^{x_k + y_l + 1} & \text{if } x_k < y_l - 1; \\ 0 & \text{otherwise.} \end{cases}$$

In Step 2, we want to find all adjacent increasing pairs for $\mathbf{x}$ and $\mathbf{y}$ and then apply Definition 3.1 (4). Let $p(\mathbf{x}, \mathbf{y})$ be the number of adjacent increasing pairs $\{s, s+1 \mid s \in \mathbf{x}, s+1 \in \mathbf{y}\}$ for $\mathbf{x}, \mathbf{y} \in V(\Gamma)$. Since all these pairs are distinct, they can be ordered as $\{s_1, s_1+1\}, \dots, \{s_{p(\mathbf{x},\mathbf{y})}, s_{p(\mathbf{x},\mathbf{y})}+1\}$ such that $s_1 > \cdots > s_{p(\mathbf{x},\mathbf{y})}$, where $s_i(\mathbf{x}, \mathbf{y})$ is the smaller number in the i-th pair for $\mathbf{x}, \mathbf{y}$. We write s_i for $s_i(\mathbf{x}, \mathbf{y})$ when there is no confusion. The multiplication $\mathrm{M}([s], [s+1])$ is defined in Definition 3.1 (4).

Definition 3.5. Define $\beta : \{0, 1, \dots, n-1\} \to \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle$ by

$$\beta(s) = q^{2s+1-n}[\emptyset] + h[s+1, s].$$

Let $\alpha'_i(\mathbf{x},\mathbf{y})$ be the i-th non-increasing sequence consisting of

$$\{x_k \in \mathbf{x} \mid s_{i+1}+1 \leq x_k < s_i\} \text{ and } \{y_l \in \mathbf{y} \mid s_{i+1}+1 < y_l \leq s_i\}$$

for $0 \leq i \leq p(\mathbf{x},\mathbf{y})$. Here we assume $s_0 = +\infty$, and $s_{p(\mathbf{x},\mathbf{y})+1} = -\infty$. Note that α'_i could be $[\emptyset]$. We now define the following since we want to set α'_i to zero if it has repetitions:

Definition 3.6 (i-th sequence α_i). Define

$$\alpha_i : V(\Gamma) \times V(\Gamma) \to \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle$$

by

$$\alpha_i(\mathbf{x},\mathbf{y}) = D(\alpha'_i(\mathbf{x},\mathbf{y})),$$

where D : {non-increasing sequences of integers bounded by n and $0\} \to \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle$ is given as

$$D(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if the sequence } \mathbf{x} \text{ is decreasing;} \\ 0 & \text{otherwise.} \end{cases}$$

We finally consider Step 3 and generalize Definition 3.1 (1) and (2) to a gluing map G_k for k elements in $V(\Gamma)$.

Definition 3.7. Let $G_k : (\mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle)^{\times k} \to \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\langle V(\Gamma)\rangle$ be a $\mathbb{Z}[q^{\pm 1}, h^{\pm 1}]$-multilinear map defined over the basis $V(\Gamma)^{\times k}$ as follows:

$$G_k(\mathbf{x^1},\mathbf{x^2},\ldots,\mathbf{x^k}) = \begin{cases} [x^1_0,\ldots,x^1_{i_1},x^2_0,\ldots,x^2_{i_2},\ldots,x^k_0,\ldots,x^k_{i_k}] & \text{if } x^j_{i_j} > x^{j+1}_0 \\ & \text{for } 1 \leq j \leq k-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{x^1} = [x^1_0,\ldots,x^1_{i_1}], \mathbf{x^2} = [x^2_0,\ldots,x^2_{i_2}],\ldots,\mathbf{x^k} = [x^k_0,\ldots,x^k_{i_k}]$. Here we assume $x^j_{i_j} > x^{j+1}_0$ is always true when $\mathbf{x^j}$ or $\mathbf{x^{j+1}}$ is $[\emptyset]$.

Remark 3.8. The gluing map G_2 is associative: $G_2(G_2(\mathbf{x},\mathbf{y}),\mathbf{z}) = G_2(\mathbf{x}, G_2(\mathbf{y},\mathbf{z}))$, and G_k is a composition of G_{k-1} and G_2: $G_k(\mathbf{x^1},\mathbf{x^2},\ldots,\mathbf{x^k}) = G_{k-1}(G_2(\mathbf{x^1},\mathbf{x^2}),\ldots,\mathbf{x^k})$. The gluing map G_k is used in the distribution law in Step 3.

We are now in a position to define the higher multiplication M in general.

Definition 3.9 (Higher multiplication M**).** Define M over the basis $\{(\mathbf{x},\mathbf{y}) \mid \mathbf{x},\mathbf{y} \in V(\Gamma)\}$ by

$$\mathrm{M}(\mathbf{x},\mathbf{y}) = h^{\mu(\mathbf{x},\mathbf{y})} G_{2p(\mathbf{x},\mathbf{y})+1}(\alpha_0(\mathbf{x},\mathbf{y}), \beta(s_1(\mathbf{x},\mathbf{y})), \ldots, \beta(s_{p(\mathbf{x},\mathbf{y})}(\mathbf{x},\mathbf{y})), \alpha_{p(\mathbf{x},\mathbf{y})}(\mathbf{x},\mathbf{y})).$$

Remark 3.10. (1) The definition of M above reduces to Definition 3.1 in special cases.

(2) The multiplication $\mathrm{M}(\mathbf{x},\mathbf{y})$ agrees with the gluing map $G_2(\mathbf{x},\mathbf{y})$ if $G_2(\mathbf{x},\mathbf{y}) \neq 0$.

Definition 3.11 (Multiplication m**).** The multiplication

$$\mathrm{m} : \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma)\rangle \times \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma)\rangle \to \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma)\rangle$$

is defined as the specialization of M to $h = -1$.

The specialization of M to $h = -1$ is exactly the multiplication on Cl_n since they agree on the generators as shown in Remark 3.2. In other words, we compute the multiplication on Cl_n in terms of the $\mathbb{Z}[q^{\pm 1}]$-basis $V(\Gamma_n)$ when specializing M to $h = -1$.

Proposition 3.12. *There is an isomorphism of* $\mathbb{Z}[q^{\pm 1}]$*-algebras:*

$$\begin{aligned} \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma_n)\rangle &\to Cl_n \\ [a] &\mapsto X_a. \end{aligned}$$

4. Categorification of the multiplication on $K_0(R)$

We fix some n for the rest of the paper and denote R_n by R for simplicity. We construct a DG $(R, R \boxtimes R)$-bimodule T in Section 4.1. Tensoring with T defines a functor $\mathcal{M} : DGP(R \boxtimes R) \to DGP(R)$ and its induced functor $\mathcal{M}|_{H^0} : H^0(DGP(R \boxtimes R)) \to H^0(DGP(R))$ on the homology categories. We show that $\mathcal{M}|_{H^0}$ categorifies the multiplication m on $K_0(R)$.

4.1. The DG $(R, R \boxtimes R)$-bimodule T

We construct a DG $(R, R \boxtimes R)$-bimodule T in 3 steps:

1) define the left R-module $T = \bigoplus_{\mathbf{x},\mathbf{y} \in V(\Gamma)} T(\mathbf{x},\mathbf{y})$ in Section 4.1.1;

2) define the differential $d = \sum_{\mathbf{x},\mathbf{y} \in V(\Gamma)} d(\mathbf{x},\mathbf{y})$ on left R-submodules $T(\mathbf{x},\mathbf{y})$ in Section 4.1.2;
3) define the right DG $(R \boxtimes R)$-module structure on T in Sections 4.1.3 and 4.1.4.

Since the construction is rather technical, we explain the main points in the following example.

Example 4.1. We use the higher multiplication $\mathrm{M}(\mathbf{x},\mathbf{y})$ in Definition 3.9 to define $T(\mathbf{x},\mathbf{y}) \in DGP(R)$, for $(\mathbf{x},\mathbf{y}) \in \{([\emptyset],[\emptyset]),\ ([1,0],[\emptyset]),\ ([\emptyset],[2,1]),\ ([1,0],[2,1])\}$. Roughly speaking, $T(\mathbf{x},\mathbf{y})$ is a projective DG R-module which lifts $\mathrm{M}(\mathbf{x},\mathbf{y})$. For instance, we define

$$T([\emptyset],[\emptyset]) = P([\emptyset])$$

since $\mathrm{M}([\emptyset],[\emptyset]) = [\emptyset]$. Similarly, we define

$$T([1,0],[\emptyset]) = P([1,0]), \quad T([\emptyset],[2,1]) = P([2,1]).$$

To define $T([1,0],[2,1])$, recall from Example 3.3 that

$$\mathrm{M}([1,0],[2,1]) = h^{-1}([\emptyset] + h[2,1]) \cdot ([\emptyset] + h[1,0])$$

by ignoring the q-grading for simplicity. Note that h corresponds to the cohomological grading in $DGP(R)$. Then one factor $([\emptyset] + h[2,1])$ is lifted to a complex $P([\emptyset]) \to P([2,1])$, whose differential is given by the right multiplication with $r([\emptyset] \to [2,1]) \in R$. Define the DG R-module $T([1,0],[2,1])$ as a tensor product of two complexes up to a total grading shift:

$$(P([\emptyset]) \to P([2,1])) \otimes (P([\emptyset]) \to P([1,0])) = P([\emptyset]) \to P([2,1]) \oplus P([1,0]).$$

In general, we define $T(\mathbf{x},\mathbf{y}) \in DGP(R)$ for all $\mathbf{x},\mathbf{y} \in V(\Gamma)$ and set the left DG R-module $T = \bigoplus_{\mathbf{x},\mathbf{y} \in V(\Gamma)} T(\mathbf{x},\mathbf{y})$.

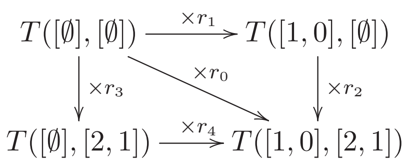
The right multiplication with $r(\mathbf{x},\mathbf{y} \xrightarrow{s,t} \mathbf{x}',\mathbf{y}')$ on T will be defined as a map:

$$\times r(\mathbf{x},\mathbf{y} \xrightarrow{s,t} \mathbf{x}',\mathbf{y}') : T(\mathbf{x},\mathbf{y}) \to T(\mathbf{x}',\mathbf{y}'),$$

which is compatible with the left R-module structure on T. In particular, the map should be a morphism in $\mathrm{Hom}_{DGP(R)}(T(\mathbf{x},\mathbf{y}), T(\mathbf{x}',\mathbf{y}'))$. Then a part of the bimodule T can be described by the following diagram whose vertices are $T(\mathbf{x},\mathbf{y})$'s. The arrows represent the right multiplication on T with the following generators in $R \boxtimes R$:

$$r_1 = r([\emptyset],[\emptyset] \xrightarrow{0,\emptyset} [1,0],[\emptyset]), \quad r_2 = r([1,0],[\emptyset] \xrightarrow{\emptyset,1} [1,0],[2,1]),$$
$$r_3 = r([\emptyset],[\emptyset] \xrightarrow{\emptyset,1} [\emptyset],[2,1]), \quad r_4 = r([\emptyset],[2,1] \xrightarrow{0,\emptyset} [1,0],[2,1]),$$
$$r_0 = r([\emptyset],[\emptyset] \xrightarrow{0,1} [1,0],[2,1]).$$

They satisfy $d(r_0) = r_1 r_2 + r_3 r_4$ as shown in Example 5.



Then we use the definition of $T(\mathbf{x},\mathbf{y})$'s above to get a corresponding diagram in $DGP(R)$, where the dashed arrows in the bottom right corner of Fig 7 denote the differential in $T([1,0],[2,1])$.

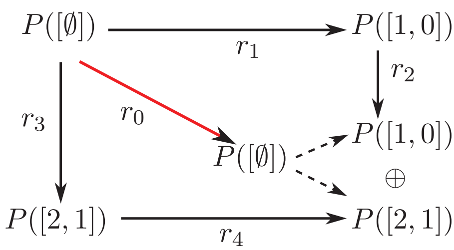


Figure 7: The construction of $(R, R \boxtimes R)$-bimodule T.

We define the remaining 5 arrows as morphisms in $DGP(R)$ in the following. Each arrow from $P(\mathbf{z})$ to $P(\mathbf{w})$ is given by the right multiplication with a generator $r(\mathbf{z} \to \mathbf{w}) \in R$. These morphisms in $DGP(R)$ define the right DG $R \boxtimes R$-module structure on T. For instance, the right multiplication with r_1 on T is a map:

$$\times r_1 : T([\emptyset],[\emptyset]) \to T([1,0],[\emptyset])$$

which is defined as the morphism from $P([\emptyset]) = T([\emptyset],[\emptyset])$ to $P([1,0]) = T([1,0],[\emptyset])$ on the top arrow in Fig 7. Similarly, we define the right multiplication with other generators. The point is that the right multiplication with r_1r_2 and r_3r_4 do not agree. They actually differ by the differential of the right multiplication with r_0 by chasing the diagram in Fig 7:

$$d(m \times r_0) = m \times (r_1r_2 + r_3r_4) = m \times d(r_0),$$

where $m \in P([\emptyset])$ in the top left corner of Fig 7. This gives T the right DG $R \boxtimes R$-module structure.

4.1.1. *T* as a left *R*-module. We use the higher multiplication $\mathrm{M}(\mathbf{x}, \mathbf{y})$ to construct $T(\mathbf{x}, \mathbf{y})$, for $\mathbf{x}, \mathbf{y} \in V(\Gamma)$. Recall from Definition 3.9 that:

$$\mathrm{M}(\mathbf{x}, \mathbf{y}) = h^{\mu(\mathbf{x},\mathbf{y})} G_{2p(\mathbf{x},\mathbf{y})+1}(\alpha_0(\mathbf{x}, \mathbf{y}), \beta(s_1(\mathbf{x}, \mathbf{y})), \ldots, \\ \beta(s_{p(\mathbf{x},\mathbf{y})}(\mathbf{x}, \mathbf{y})), \alpha_{p(\mathbf{x},\mathbf{y})}(\mathbf{x}, \mathbf{y}))$$

is a Laurent polynomial of h. Let $\mathrm{M}^k : V(\Gamma) \times V(\Gamma) \to \mathbb{Z}[q^{\pm 1}]\langle V(\Gamma)\rangle$ be the coefficient of h^k in M:

$$\mathrm{M}(\mathbf{x}, \mathbf{y}) = \sum_{k=-\infty}^{+\infty} \mathrm{M}^k(\mathbf{x}, \mathbf{y}) h^k = \sum_{k=\mu(\mathbf{x},\mathbf{y})}^{\mu(\mathbf{x},\mathbf{y})+p(\mathbf{x},\mathbf{y})} \mathrm{M}^k(\mathbf{x}, \mathbf{y}) h^k.$$

We want to expand $\mathrm{M}^k(\mathbf{x}, \mathbf{y})$ further in terms of q. We will omit $\mathbf{x}, \mathbf{y}$ in $\mu(\mathbf{x}, \mathbf{y}), p(\mathbf{x}, \mathbf{y}), s_i(\mathbf{x}, \mathbf{y})$ when $\mathbf{x}, \mathbf{y}$ are understood.

Let $\mathcal{I}_{k-\mu}(\mathbf{x}, \mathbf{y}) = \{A \subset \{1, 2, \ldots, p\} \mid |A| = k - \mu\}$ be the collection of all $(k - \mu)$-element subsets of $\{1, 2, \ldots, p\}$ for $\mu \leq k \leq \mu + p$. Let $\eta : \mathcal{I}_{k-\mu}(\mathbf{x}, \mathbf{y}) \to \mathbb{Z}$ be the overall shift in q:

$$\eta(A) = \sum_{i \notin A} (2s_i + 1 - n).$$

Let $\beta_A : \{s_1, \ldots, s_p\} \to V(\Gamma)$ be a choice of components of β in Definition 3.5 depending on an index set $A \in \mathcal{I}_{k-\mu}(\mathbf{x}, \mathbf{y})$:

$$\beta_A(s_i) = \begin{cases} [s_i + 1, s_i] & \text{if } i \in A; \\ [\emptyset] & \text{otherwise.} \end{cases}$$

Definition 4.2 (Expansion of M^k). For any given $\mathbf{x}, \mathbf{y} \in V(\Gamma)$ and $A \in \mathcal{I}_{k-\mu}(\mathbf{x}, \mathbf{y})$, let

$$\mathrm{M}^k_A(\mathbf{x}, \mathbf{y}) = G_{2p+1}(\alpha_0, \beta_A(s_1), \ldots, \beta_A(s_p), \alpha_p) \in V(\Gamma) \sqcup \{0\}$$

be the coefficient of $q^{\eta(A)}$ in $\mathrm{M}^k(\mathbf{x},\mathbf{y})$:

$$\mathrm{M}^k(\mathbf{x},\mathbf{y}) = \sum_{A\in\mathcal{I}_{k-\mu}} \mathrm{M}^k_A(\mathbf{x},\mathbf{y})q^{\eta(A)}.$$

Recall that $P(\mathbf{x}) = R\cdot e(\mathbf{x})$ and $P(\mathbf{x})\{n\}$ is $P(\mathbf{x})$ with the q-grading shifted by n. Let $P(0)=0$ denote the trivial R-module. We now define $T=\bigoplus_{\mathbf{x},\mathbf{y}\in V(\Gamma)} T(\mathbf{x},\mathbf{y})$ as a left R-module.

Definition 4.3. Define $T(\mathbf{x},\mathbf{y})=\bigoplus_k T^k(\mathbf{x},\mathbf{y})$ as left projective R-modules, where

$$T^k(\mathbf{x},\mathbf{y}) = \bigoplus_{A\in\mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y})} P(\mathrm{M}^k_A(\mathbf{x},\mathbf{y}))\{\eta(A)\}.$$

Remark 4.4. The class of $T^k(\mathbf{x},\mathbf{y})$ in the Grothendieck group $K_0(R)$ is $\mathrm{M}^k(\mathbf{x},\mathbf{y})$.

4.1.2. *T* as a left DG *R*-module. We define the differential

$$d = \sum_{\mathbf{x},\mathbf{y}\in V(\Gamma)} d(\mathbf{x},\mathbf{y}) \quad \text{on } T,$$

where $d(\mathbf{x},\mathbf{y})$ are differentials on the left R-submodules $T(\mathbf{x},\mathbf{y})$ of T. More precisely,

$$d(\mathbf{x},\mathbf{y}) = \sum_k d^k(\mathbf{x},\mathbf{y}) = \sum_k \sum_{\substack{A\in\mathcal{I}_{k-\mu}\\ B\in\mathcal{I}_{k+1-\mu}}} d^B_A(\mathbf{x},\mathbf{y}),$$

where $d^k(\mathbf{x},\mathbf{y}) : T^k(\mathbf{x},\mathbf{y}) \to T^{k+1}(\mathbf{x},\mathbf{y})$ is defined on each summand by

$$d^B_A(\mathbf{x},\mathbf{y}) : P(\mathrm{M}^k_A(\mathbf{x},\mathbf{y}))\{\eta(A)\} \to P(\mathrm{M}^{k+1}_B(\mathbf{x},\mathbf{y}))\{\eta(B)\}.$$

Given $\mathbf{x},\mathbf{y}\in V(\Gamma)$, if $A\in\mathcal{I}_{k-\mu}$ is a subset of $B\in\mathcal{I}_{k+1-\mu}$ and

$$\begin{aligned}\mathrm{M}^k_A &= G_{2p+1}(\alpha_0,\beta_A(s_1),\dots,\beta_A(s_p),\alpha_p)\\ \mathrm{M}^{k+1}_B &= G_{2p+1}(\alpha_0,\beta_B(s_1),\dots,\beta_B(s_p),\alpha_p)),\end{aligned}$$

are both nonzero, then they only differ by a pair of adjacent numbers $\{s_{i(A,B)}+1, s_{i(A,B)}\}$ at $\beta_A(s_{i(A,B)})$ and $\beta_B(s_{i(A,B)})$. Here we write $i(A,B)$ for the unique element in $B-A$. Hence there exists a generator $r(\mathrm{M}^k_A \xrightarrow{s_{i(A,B)}} \mathrm{M}^{k+1}_B)\in R$.

Definition 4.5. Given $\mathbf{x}, \mathbf{y} \in V(\Gamma)$, $A \in \mathcal{I}_{k-\mu}, B \in \mathcal{I}_{k+1-\mu}$, define

$$d_A^B(\mathbf{x}, \mathbf{y}) : P(\mathrm{M}_A^k)\{\eta(A)\} \to P(\mathrm{M}_B^{k+1})\{\eta(B)\}$$

as a left R-module map given by multiplying $r(\mathrm{M}_A^k \xrightarrow{s_{i(A,B)}} \mathrm{M}_B^{k+1})$ from the right if $B = A \sqcup \{i(A,B)\}$ and $\mathrm{M}_A^k, \mathrm{M}_B^{k+1} \in V(\Gamma)$. Otherwise define $d_A^B(\mathbf{x}, \mathbf{y}) = 0$.

Remark 4.6. The map d_A^B preserves the q-grading due to the q-grading shifting $\{\eta(A)\}, \{\eta(B)\}$ on the modules.

Lemma 4.7. *d is a differential, i.e., $d^{k+1}(\mathbf{x}, \mathbf{y}) \circ d^k(\mathbf{x}, \mathbf{y}) = 0$ for any $\mathbf{x}, \mathbf{y} \in V(\Gamma)$.*

Proof. It suffices to prove that

$$(**) \qquad d^{k+1}(\mathbf{x}, \mathbf{y}) \circ d^k(\mathbf{x}, \mathbf{y})\big|_{P(\mathrm{M}_A^k)\{\eta(A)\}} = \sum_{\substack{B \in \mathcal{I}_{k+1-\mu} \\ C \in \mathcal{I}_{k+2-\mu}}} d_B^C \circ d_A^B = 0$$

for any $A \in \mathcal{I}_{k-\mu}(\mathbf{x}, \mathbf{y})$. By definition d_A^B, d_B^C are both nonzero if and only if

$$B = A \sqcup \{i(A,B)\}, \qquad C = B \sqcup \{i(B,C)\},$$

for some $i(A,B), i(B,C)$ and $\mathrm{M}_A^k, \mathrm{M}_B^{k+1}, \mathrm{M}_C^{k+2} \in V(\Gamma)$. Then there exists another index set $B' = A \sqcup \{i(B,C)\}$ such that $C = B' \sqcup \{i(A,B)\}$ and $\mathrm{M}_{B'}^{k+1} \in V(\Gamma)$. Moreover, $d_A^{B'}, d_{B'}^C$ are both nonzero by definition. Then the map

$$d_B^C \circ d_A^B : P(\mathrm{M}_A^k)\{\eta(A)\} \to P(\mathrm{M}_C^{k+2})\{\eta(C)\}$$

is the right multiplication by

$$r(\mathrm{M}_A^k \xrightarrow{s_{i(A,B)}} \mathrm{M}_B^{k+1}) \cdot r(\mathrm{M}_B^{k+1} \xrightarrow{s_{i(B,C)}} \mathrm{M}_C^{k+2})$$

and the map $d_{B'}^C \circ d_A^{B'}$ is the right multiplication by

$$r(\mathrm{M}_A^k \xrightarrow{s_{i(B,C)}} \mathrm{M}_{B'}^{k+1}) \cdot r(\mathrm{M}_{B'}^{k+1} \xrightarrow{s_{i(A,B)}} \mathrm{M}_C^{k+2}).$$

We have $d_B^C \circ d_A^B + d_{B'}^C \circ d_A^{B'} = 0$ since

$$\begin{aligned} &r(\mathrm{M}_A^k \xrightarrow{s_{i(A,B)}} \mathrm{M}_B^{k+1}) \cdot r(\mathrm{M}_B^{k+1} \xrightarrow{s_{i(B,C)}} \mathrm{M}_C^{k+2}) \\ = &r(\mathrm{M}_A^k \xrightarrow{s_{i(B,C)}} \mathrm{M}_{B'}^{k+1}) \cdot r(\mathrm{M}_{B'}^{k+1} \xrightarrow{s_{i(A,B)}} \mathrm{M}_C^{k+2}). \end{aligned}$$

This implies Equation (**). □

4.1.3. The right $R \boxtimes R$-multiplication.. In this subsection we define the right multiplication with the generators $e(\mathbf{x}, \mathbf{y})$ and $r(\mathbf{x}, \mathbf{y} \xrightarrow{s,t} \mathbf{x}', \mathbf{y}')$ of $R \boxtimes R$. Let $m \times r$ denote the right multiplication for $m \in T, r \in R \boxtimes R$ and $m \cdot a$ denote the multiplication in R for $m \in P(\mathbf{x}) \subset R, a \in R$. Note that $m \cdot a$ is not the left R multiplication on T. In general, the right multiplication with $r(\mathbf{x}, \mathbf{y} \xrightarrow{s,t} \mathbf{x}', \mathbf{y}')$ on T is a map:

$$\times r(\mathbf{x}, \mathbf{y} \xrightarrow{s,t} \mathbf{x}', \mathbf{y}') : T(\mathbf{x}, \mathbf{y}) \to T(\mathbf{x}', \mathbf{y}'),$$

which is given by a morphism in $\mathrm{Hom}_{DGP(R)}(T(\mathbf{x}, \mathbf{y}), T(\mathbf{x}', \mathbf{y}'))$. The definition of $m \times r$ is rather involved and occupies the next several pages.

(1) Let $r = e(\mathbf{x}, \mathbf{y})$. Then define $m \times e(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x},\mathbf{x}'}\delta_{\mathbf{y},\mathbf{y}'}m$ for $m \in T(\mathbf{x}', \mathbf{y}')$.

(2) Let $r = r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x}, \mathbf{y}')$. Let us abbreviate $\mu = \mu(\mathbf{x}, \mathbf{y})$, $\mu' = \mu(\mathbf{x}, \mathbf{y}')$, $p = p(\mathbf{x}, \mathbf{y})$, $p' = p(\mathbf{x}, \mathbf{y}')$, $s_i = s_i(\mathbf{x}, \mathbf{y})$ and $s_i' = s_i(\mathbf{x}, \mathbf{y}')$. Since $\mathbf{y}' = \mathbf{y} \sqcup \{t+1, t\}$ and in particular $t+1 \notin \mathbf{y}$, we have $t \notin \{s_1, \dots, s_p\}$. Let $a(t) \in \{1, \dots, p\}$ be the number such that $s_{a(t)} > t > s_{a(t)+1}$.

The right multiplication

$$\begin{aligned}&\times r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x}, \mathbf{y}') :\\ &\bigoplus_{A \in \mathcal{I}_{k-\mu}} P(\mathrm{M}^k_A(\mathbf{x}, \mathbf{y}))\{\eta(A)\} \to \bigoplus_{B \in \mathcal{I}_{k-\mu'}} P(\mathrm{M}^k_B(\mathbf{x}, \mathbf{y}'))\{\eta(B)\}\end{aligned}$$

is defined on a case-by-case basis as follows:

(2A) Suppose $t-1 \notin \mathbf{x}$ and $t \notin \mathbf{x}$. We have $\mu' = \mu, p' = p$. Decompose $\alpha_{a(t)} = G_2(\alpha_{a(t)+}, \alpha_{a(t)-})$, where $\alpha_{a(t)+}$ is the subsequence of $\alpha_{a(t)}$ consisting of numbers greater than t and $\alpha_{a(t)-}$ is the complementary sequence of $\alpha_{a(t)+}$ in $\alpha_{a(t)}$. Then we have

$$\begin{aligned}\mathrm{M}(\mathbf{x}, \mathbf{y}) &= h^{\mu} G_{2p+1}(\alpha_0, \dots, \alpha_{a(t)}, \dots, \alpha_p)\\ &= h^{\mu} G_{2p+2}(\alpha_0, \dots, \alpha_{a(t)+}, \alpha_{a(t)-}, \dots, \alpha_p),\\ \mathrm{M}(\mathbf{x}, \mathbf{y}') &= h^{\mu} G_{2p+3}(\alpha_0, \dots, \alpha_{a(t)+}, [t+1, t], \alpha_{a(t)-}, \dots, \alpha_p).\end{aligned}$$

Define a function $f : \mathcal{I}_{k-\mu}(\mathbf{x}, \mathbf{y}) \to \mathcal{I}_{k-\mu'}(\mathbf{x}, \mathbf{y}')$ as the identity. Then two non-increasing sequences $\mathrm{M}^k_A(\mathbf{x}, \mathbf{y})$ and $\mathrm{M}^k_{f(A)}(\mathbf{x}, \mathbf{y}')$ differ by the pair $\{t+1, t\}$. If both sequences are decreasing, then there exists a generator $r(\mathrm{M}^k_A(\mathbf{x}, \mathbf{y}) \xrightarrow{t} \mathrm{M}^k_{f(A)}(\mathbf{x}, \mathbf{y}')) \in R$; otherwise we still write $r(\mathrm{M}^k_A(\mathbf{x}, \mathbf{y}) \xrightarrow{t} \mathrm{M}^k_{f(A)}(\mathbf{x}, \mathbf{y}'))$ to denote $0 \in R$.

For $m \in P(\mathrm{M}^k_A(\mathbf{x},\mathbf{y}))\{\eta(A)\}$, we define the right multiplication

$$\begin{aligned} &m \times r(\mathbf{x},\mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x},\mathbf{y}') \\ &= m \cdot r(\mathrm{M}^k_A(\mathbf{x},\mathbf{y}) \xrightarrow{t} \mathrm{M}^k_{f(A)}(\mathbf{x},\mathbf{y}')) \in P(\mathrm{M}^k_{f(A)}(\mathbf{x},\mathbf{y}'))\{\eta(f(A))\}. \end{aligned}$$

(2B) Suppose $t-1 \notin \mathbf{x}$ and $t \in \mathbf{x}$. We have $\mu' = \mu, p' = p+1$. Consider a decomposition similar to that in (2A): $\alpha_{a(t)} = G_2(\alpha_{a(t)+}, \alpha_{a(t)-})$, here $t \in \alpha_{a(t)-}$. In this case, t is in some adjacent increasing pair for $\mathbf{x},\mathbf{y}'$ since $t \in \mathbf{x}$ and $t+1 \in \mathbf{y}'$. In particular, $t = s'_{a(t)+1} \in \{s'_1, \dots, s'_{p'}\}$. Then we have

$$\begin{aligned} \mathrm{M}(\mathbf{x},\mathbf{y}) &= h^{\mu} G_{2p+2}(\alpha_0, \dots, \beta(s_{a(t)}), \alpha_{a(t)+}, \alpha_{a(t)-}, \dots, \alpha_p), \\ \mathrm{M}(\mathbf{x},\mathbf{y}') &= h^{\mu} G_{2p+3}(\alpha_0, \dots, \beta(s'_{a(t)}), \alpha_{a(t)+}, \beta(t), \alpha_{a(t)-}, \dots, \alpha_p) \\ &= h^{\mu} G_{2p+3}(\alpha_0, \dots, \beta(s'_{a(t)}), \alpha_{a(t)+}, \beta(s'_{a(t)+1}), \alpha_{a(t)-}, \dots, \alpha_p). \end{aligned}$$

Define $f : \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y}) \to \mathcal{I}_{k-\mu'}(\mathbf{x},\mathbf{y}')$ for $A \in \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y})$ by

$$f(A) = \{a \mid a \in A, a \leq a(t)\} \sqcup \{a+1 \mid a \in A, a > a(t)\}.$$

We have $\beta_{f(A)}(s'_{a(t)+1}) = [\emptyset]$ since $a(t)+1 \notin f(A)$. Hence $\mathrm{M}^k_A(\mathbf{x},\mathbf{y}) = \mathrm{M}^k_{f(A)}(\mathbf{x},\mathbf{y}')$.

For $m \in P(\mathrm{M}^k_A(\mathbf{x},\mathbf{y}))\{\eta(A)\}$, we define the right multiplication

$$m \times r(\mathbf{x},\mathbf{y} \xrightarrow{\emptyset,t} \mathbf{x},\mathbf{y}') = m \cdot e(\mathrm{M}^k_{f(A)}(\mathbf{x},\mathbf{y}')) \in P(\mathrm{M}^k_{f(A)}(\mathbf{x},\mathbf{y}'))\{\eta(f(A))\}.$$

(2C) Suppose $t-1 \in \mathbf{x}$ and $t \notin \mathbf{x}$. We have $\mu' = \mu + \mu(t-1,t+1) = \mu - 1$ and $p' = p+1$. Consider a decomposition

$$\alpha_{a(t)} = G_3(\alpha_{a(t)+}, [t-1], \alpha_{a(t)-}),$$

where $\alpha_{a(t)+}$ is the subsequence of $\alpha_{a(t)}$ consisting of numbers greater than t and $\alpha_{a(t)-}$ is the subsequence of $\alpha_{a(t)}$ consisting of numbers less than $t-1$. The number $t-1$ is in some adjacent increasing pair for $\mathbf{x},\mathbf{y}'$ since $t-1 \in \mathbf{x}$ and $t \in \mathbf{y}'$. In particular, $t-1 = s'_{a(t)+1}$. Then we have

$$\begin{aligned} \mathrm{M}(\mathbf{x},\mathbf{y}) &= h^{\mu} G_{2p+3}(\alpha_0, \dots, \beta(s_{a(t)}), \alpha_{a(t)+}, [t-1], \alpha_{a(t)-}, \dots, \alpha_p), \\ \mathrm{M}(\mathbf{x},\mathbf{y}') &= h^{\mu'} G_{2p+4}(\alpha_0, \dots, \beta(s'_{a(t)}), \alpha_{a(t)+}, [t+1], \beta(t-1), \alpha_{a(t)-}, \dots, \alpha_p) \\ &= h^{\mu'} G_{2p+4}(\alpha_0, \dots, \beta(s'_{a(t)}), \alpha_{a(t)+}, [t+1], \beta(s'_{a(t)+1}), \alpha_{a(t)-}, \dots, \alpha_p). \end{aligned}$$

Define $f : \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y}) \to \mathcal{I}_{k-\mu'}(\mathbf{x},\mathbf{y}')$ for $A \in \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y})$ by

$$f(A) = \{a \mid a \in A, a \leq a(t)\} \sqcup \{a(t)+1\} \sqcup \{a+1 \mid a \in A, a > a(t)\}.$$

We have $\beta_{f(A)}(s'_{a(t)+1}) = [t, t-1]$ since $a(t)+1 \in f(A)$. Then there exists a generator $r(\mathrm{M}^k_A(\mathbf{x},\mathbf{y}) \xrightarrow{t} \mathrm{M}^k_{f(A)}(\mathbf{x},\mathbf{y}'))$ if $\mathrm{M}^k_A(\mathbf{x},\mathbf{y})$ and $\mathrm{M}^k_{f(A)}(\mathbf{x},\mathbf{y}')$ are both nonzero. The definition of the right multiplication is the same as that in (2A).

(2D) Suppose $t-1, t \in \mathbf{x}$. We have $\mu' = \mu - 1$ and $p' = p+2$. Consider a decomposition

$$\alpha_{a(t)} = G_4(\alpha_{a(t)+}, [t], [t-1], \alpha_{a(t)-}),$$

where $\alpha_{a(t)+}$ is the subsequence of $\alpha_{a(t)}$ consisting of numbers greater than t and $\alpha_{a(t)-}$ is the subsequence of $\alpha_{a(t)}$ consisting of numbers less than $t-1$. The numbers t and $t-1$ are in some adjacent increasing pairs for $\mathbf{x},\mathbf{y}'$ since $t-1, t \in \mathbf{x}$ and $t, t+1 \in \mathbf{y}'$. In particular, we have $t = s'_{a(t)+1}$ and $t-1 = s'_{a(t)+2}$. Then we have

$$\begin{aligned}
\mathrm{M}(\mathbf{x},\mathbf{y}) &= h^{\mu} G_{2p+4}(\alpha_0, \ldots, \beta(s_{a(t)}), \alpha_{a(t)+}, [t], [t-1], \alpha_{a(t)-}, \ldots, \alpha_p);\\
\mathrm{M}(\mathbf{x},\mathbf{y}') &= h^{\mu'} G_{2p+5}(\alpha_0, \ldots, \beta(s'_{a(t)}), \alpha_{a(t)+}, \beta(t), [\emptyset], \beta(t-1),\\
&\qquad \alpha_{a(t)-}, \ldots, \alpha_p)\\
&= h^{\mu'} G_{2p+5}(\alpha_0, \ldots, \beta(s'_{a(t)}), \alpha_{a(t)+}, \beta(s'_{a(t)+1}), [\emptyset], \beta(s'_{a(t)+2}),\\
&\qquad \alpha_{a(t)-}, \ldots, \alpha_p).
\end{aligned}$$

Define $f : \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y}) \to \mathcal{I}_{k-\mu'}(\mathbf{x},\mathbf{y}')$ for $A \in \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y})$ by

$$f(A) = \{a \mid a \in A, a \leq a(t)\} \sqcup \{a(t)+2\} \sqcup \{a+2 \mid a \in A, a > a(t)\}.$$

We have $\beta_{f(A)}(s'_{a(t)+1}) = [\emptyset], \beta_{f(A)}(s'_{a(t)+2}) = [t, t-1]$ since $a(t)+1 \notin f(A)$, $a(t)+2 \in f(A)$. Hence $\mathrm{M}^k_A(\mathbf{x},\mathbf{y}) = \mathrm{M}^k_{f(A)}(\mathbf{x},\mathbf{y}')$. The definition of the right multiplication is the same as that in (2B).

(3) Let $r = r(\mathbf{x},\mathbf{y} \xrightarrow{t,\emptyset} \mathbf{x}',\mathbf{y})$. Definitions of the right multiplication are similar to those in (2) and break into 4 cases, depending on whether $t+2 \in X$ and whether $t+1 \in X$.

(4) Let $r = r(\mathbf{x},\mathbf{y} \xrightarrow{t,t+1} \mathbf{x}',\mathbf{y}')$. Let us abbreviate $\mu' = \mu(\mathbf{x}',\mathbf{y}'), p' = p(\mathbf{x}',\mathbf{y}')$ and $s'_i = s_i(\mathbf{x}',\mathbf{y}')$. We have $\mu' = \mu - 1, p' = p+2$ and a decomposition $\alpha_{a(t)} = G_2(\alpha_{a(t)+}, \alpha_{a(t)-})$ which is similar to that in (2A). The numbers $t+1$

and t are in some increasing adjacent pairs for $\mathbf{x}', \mathbf{y}'$ since $t, t+1 \in \mathbf{x}$ and $t+1, t+2 \in \mathbf{y}'$. In particular, we have $t+1 = s'_{a(t)+1}$, $t = s'_{a(t)+2}$ and

$$\begin{aligned}\mathrm{M}(\mathbf{x},\mathbf{y}) &= h^{\mu} G_{2p+2}(\alpha_0, \dots, \beta(s_{a(t)}), \alpha_{a(t)+}, \alpha_{a(t)-}, \dots, \alpha_p);\\ \mathrm{M}(\mathbf{x}',\mathbf{y}') &= h^{\mu'} G_{2p+5}(\alpha_0, \dots, \beta(s'_{a(t)}), \alpha_{a(t)+}, \beta(t+1), [\emptyset], \beta(t),\\ &\qquad \alpha_{a(t)-}, \dots, \alpha_p)\\ &= h^{\mu'} G_{2p+5}(\alpha_0, \dots, \beta(s'_{a(t)}), \alpha_{a(t)+}, \beta(s'_{a(t)+1}), [\emptyset], \beta(s'_{a(t)+2}),\\ &\qquad \alpha_{a(t)-}, \dots, \alpha_p).\end{aligned}$$

Define $f : \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y}) \to \mathcal{I}_{k-1-\mu'}(\mathbf{x}',\mathbf{y}')$ for $A \in \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y})$ by

$$f(A) = \{a \mid a \in A, a \leq a(t)\} \sqcup \{a+2 \mid a \in A, a > a(t)\}.$$

We have $\beta_{f(A)}(s'_{a(t)+1}) = \beta_{f(A)}(s'_{a(t)+2}) = [\emptyset]$ since $a(t)+1, a(t)+2 \notin f(A)$. Hence $\mathrm{M}^k_A(\mathbf{x},\mathbf{y}) = \mathrm{M}^{k-1}_{f(A)}(\mathbf{x}',\mathbf{y}')$.

For $m \in P(\mathrm{M}^k_A(\mathbf{x},\mathbf{y}))\{\eta(A)\}$, we define the right multiplication

$$m \times r(\mathbf{x},\mathbf{y} \xrightarrow{t,t+1} \mathbf{x}',\mathbf{y}') = m \cdot e(\mathrm{M}^{k-1}_{f(A)}(\mathbf{x}',\mathbf{y}')) \in P(\mathrm{M}^{k-1}_{f(A)}(\mathbf{x}',\mathbf{y}'))\{\eta(f(A))\}.$$

This concludes the definition of the right $R \boxtimes R$-multiplication.

Remark 4.8. The definition above is compatible with the q-grading on T.

4.1.4. *T* as a right DG ($R \boxtimes R$)-module. We need to show that the above definition gives T a right DG $R \boxtimes R$-module structure. More precisely, we need to verify that

1) $(m \times r_1) \times r_2 = (m \times r'_1) \times r'_2$, if $r_1 \cdot r_2 = r'_1 \cdot r'_2$ for $m \in T$ and generators $r_1, r_2, r'_1, r'_2 \in R \boxtimes R$.
2) $d(m \times r) = d(m) \times r + m \times d(r)$, for $m \in T$ and $r \in R \boxtimes R$.

A direct verification of the second equation for $r = r([\emptyset],[\emptyset] \xrightarrow{0,1} [1,0],[2,1])$ is given in Example 4.1. We prove it for general cases in the following lemma and leave the verification of the first equation for the reader.

Lemma 4.9. *The differential satisfies the Leibniz rule with respect to the right multiplication:*

$$d(m \times r) = d(m) \times r + m \times d(r),$$

for $m \in P(\mathrm{M}^k_{A_0}(\mathbf{x},\mathbf{y}))\{\eta(A_0)\}$, $A_0 \in \mathcal{I}_{k-\mu}(\mathbf{x},\mathbf{y})$ *and* $r = r(\mathbf{x},\mathbf{y} \xrightarrow{t,t+1} \mathbf{x}',\mathbf{y}')$.

Proof. Consider the index maps in the definition of right multiplication with $r(\mathbf{x}, \mathbf{y} \xrightarrow{t,t+1} \mathbf{x}', \mathbf{y}')$:

$$
\begin{aligned}
&f : \mathcal{I}_{k-\mu}(\mathbf{x}, \mathbf{y}) \to \mathcal{I}_{k-1-\mu'}(\mathbf{x}', \mathbf{y}'),\\
&f(A) = \{a \mid a \in A, a \leq a(t)\}\\
&\qquad \sqcup \{a+2 \mid a \in A, a > a(t)\}, \quad \text{for } A \in \mathcal{I}_{k-\mu}(\mathbf{x}, \mathbf{y});\\
&g : \mathcal{I}_{k+1-\mu}(\mathbf{x}, \mathbf{y}) \to \mathcal{I}_{k-\mu'}(\mathbf{x}', \mathbf{y}'),\\
&g(B) = \{b \mid b \in B, b \leq a(t)\}\\
&\qquad \sqcup \{b+2 \mid b \in B, b > a(t)\}, \quad \text{for } B \in \mathcal{I}_{k+1-\mu}(\mathbf{x}, \mathbf{y}).
\end{aligned}
$$

Let $\overline{\mathcal{B}'} = \{B' \in \mathcal{I}_{k+1-\mu}(\mathbf{x}, \mathbf{y}) \mid B' \supset A_0\}$, $\overline{\mathcal{B}} = \{B \mid B = g(B') \text{ for some } B' \in \overline{\mathcal{B}'}\}$ and

$$
\begin{aligned}
&\mathcal{B} = \{B \in \mathcal{I}_{k-\mu'}(\mathbf{x}', \mathbf{y}') \mid B \supset f(A_0)\} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \overline{\mathcal{B}},\\
&\text{where} \quad \mathcal{B}_1 = \{B = f(A_0) \sqcup \{a(t)+1\}\}, \quad \mathcal{B}_2 = \{B = f(A_0) \sqcup \{a(t)+2\}\}.
\end{aligned}
$$

Then for $m \in P(\mathrm{M}^k_{A_0}(\mathbf{x}, \mathbf{y}))\{\eta(A_0)\}$, we have

$$
\begin{aligned}
& && d\left(m \times r(\mathbf{x}, \mathbf{y} \xrightarrow{t,t+1} \mathbf{x}', \mathbf{y}')\right)\\
&\text{(right multiplication in Section 4.1.3 (4))} && = d(m \cdot e(\mathrm{M}^{k-1}_{f(A_0)}(\mathbf{x}', \mathbf{y}')))\\
&\text{(differential in Section 4.1.2)} && = \textstyle\sum_{B \in \mathcal{B}} d^B_{f(A_0)}(m)\\
&\text{(decomposition of } \mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \overline{\mathcal{B}}) && = \textstyle\sum_{B \in \overline{\mathcal{B}}} m \cdot r(\mathrm{M}^{k-1}_{f(A_0)}(\mathbf{x}', \mathbf{y}') \xrightarrow{s_{i(f(A_0),B)}} \mathrm{M}^k_B(\mathbf{x}', \mathbf{y}'))\\
& && \quad + m \cdot r(\mathrm{M}^{k-1}_{f(A_0)}(\mathbf{x}', \mathbf{y}') \xrightarrow{s_{a(t)+1}} \mathrm{M}^k_{B_1}(\mathbf{x}', \mathbf{y}'))\\
& && \quad + m \cdot r(\mathrm{M}^{k-1}_{f(A_0)}(\mathbf{x}', \mathbf{y}') \xrightarrow{s_{a(t)+2}} \mathrm{M}^k_{B_2}(\mathbf{x}', \mathbf{y}'))\\
&(\mathrm{M}^{k-1}_{f(A_0)}(\mathbf{x}', \mathbf{y}') = \mathrm{M}^k_{A_0}(\mathbf{x}, \mathbf{y})) && = \textstyle\sum_{B' \in \overline{\mathcal{B}'}} m \cdot r(\mathrm{M}^k_{A_0}(\mathbf{x}, \mathbf{y}) \xrightarrow{s_{i(A_0,B')}} \mathrm{M}^{k+1}_{B'}(\mathbf{x}, \mathbf{y}))\\
&(\mathrm{M}^k_{g(B')}(\mathbf{x}', \mathbf{y}') = \mathrm{M}^{k+1}_{B'}(\mathbf{x}, \mathbf{y})) && \quad + m \cdot r(\mathrm{M}^{k-1}_{f(A_0)}(\mathbf{x}', \mathbf{y}') \xrightarrow{t+1} \mathrm{M}^k_{B_1}(\mathbf{x}', \mathbf{y}'))\\
&(s_{a(t)+1} = t+1, s_{a(t)+2} = t) && \quad + m \cdot r(\mathrm{M}^{k-1}_{f(A_0)}(\mathbf{x}', \mathbf{y}') \xrightarrow{t} \mathrm{M}^k_{B_2}(\mathbf{x}', \mathbf{y}'))\\
&\text{(right multiplication in Section 4.1.3 (2- 4))} && = d(m) \times r(\mathbf{x}, \mathbf{y} \xrightarrow{t,t+1} \mathbf{x}', \mathbf{y}')\\
& && \quad + m \times r(\mathbf{x}, \mathbf{y} \xrightarrow{\emptyset,t+1} \mathbf{x}, \mathbf{y}') \times r(\mathbf{x}, \mathbf{y}' \xrightarrow{t,\emptyset} \mathbf{x}, \mathbf{y}')\\
& && \quad + m \times r(\mathbf{x}, \mathbf{y} \xrightarrow{t,\emptyset} \mathbf{x}', \mathbf{y}) \times r(\mathbf{x}', \mathbf{y} \xrightarrow{\emptyset,t+1} \mathbf{x}', \mathbf{y}')\\
& && = d(m) \times r(\mathbf{x}, \mathbf{y} \xrightarrow{t,t+1} \mathbf{x}', \mathbf{y}')\\
& && \quad + m \times d(r(\mathbf{x}, \mathbf{y} \xrightarrow{t,t+1} \mathbf{x}', \mathbf{y}')).
\end{aligned}
$$

□

It is easy to see that the left R-module and the right $R \boxtimes R$-module structures on T are compatible:

$$a \cdot (m \times r) = (a \cdot m) \times r,$$

for $a \in R, r \in R \boxtimes R$ and $m \in T$. Here $a \cdot m$ denotes the left R multiplication on T. We finally have the DG $(R, R \boxtimes R)$-bimodule T.

4.2. The functor $\mathcal{M} : DGP(R \boxtimes R) \to DGP(R)$

We show that tensoring with T over $R \boxtimes R$ maps the projective DG $R \boxtimes R$-module $P(\mathbf{x}, \mathbf{y}) = (R \boxtimes R)e(\mathbf{x}, \mathbf{y})$ to a projective DG R-module in $DGP(R)$.

Lemma 4.10. *The tensor product $T \otimes_{R \boxtimes R} P(\mathbf{x}, \mathbf{y})$ is the DG R-module*

$$T(\mathbf{x}, \mathbf{y}) = \left(\bigoplus_k T^k(\mathbf{x}, \mathbf{y}), \sum_k d^k(\mathbf{x}, \mathbf{y}) \right)$$

in $DGP(R)$ for any $\mathbf{x}, \mathbf{y} \in V(\Gamma)$.

Proof. Since $T = \bigoplus_{\mathbf{x}', \mathbf{y}' \in V(\Gamma)} T(\mathbf{x}', \mathbf{y}')$ as left DG R-modules, it follows that $T \otimes P(\mathbf{x}, \mathbf{y})$ is the quotient of $\bigoplus_{\mathbf{x}', \mathbf{y}' \in V(\Gamma)} (T(\mathbf{x}', \mathbf{y}') \times P(\mathbf{x}, \mathbf{y}))$ by the relation

$$\{(m \times r, e(\mathbf{x}, \mathbf{y})) = (m,\ r \cdot e(\mathbf{x}, \mathbf{y})) \mid m \in T(\mathbf{x}', \mathbf{y}'), r \in R \boxtimes R\}.$$

Since $T(\mathbf{x}', \mathbf{y}') \times P(\mathbf{x}, \mathbf{y})$ is spanned by $\{(m,\ r \cdot e(\mathbf{x}, \mathbf{y})) \mid m \in T(\mathbf{x}', \mathbf{y}'),\ r \cdot e(\mathbf{x}, \mathbf{y}) \neq 0\}$, the tensor product $T \otimes P(\mathbf{x}, \mathbf{y})$ is isomorphic to

$$\{(m \times r, e(\mathbf{x}, \mathbf{y})) \mid m \in T(\mathbf{x}', \mathbf{y}'), r \cdot e(\mathbf{x}, \mathbf{y}) \neq 0\} \cong T(\mathbf{x}, \mathbf{y}). \qquad \square$$

Since $DGP(R \boxtimes R)$ is generated by the $P(\mathbf{x}, \mathbf{y})$'s, we obtain the functor

$$\mathcal{M} : DGP(R \boxtimes R) \xrightarrow{T \otimes_{R \boxtimes R} -} DGP(R).$$

The following lemma implies that we have an induced functor on their homology categories

$$\mathcal{M}|_{H^0} : H^0(DGP(R \boxtimes R)) \to H^0(DGP(R)).$$

Lemma 4.11. *The functor $\mathcal{M}$ preserves closed and exact morphisms.*

Proof. For any $g \in \mathcal{H}om_{DGP(R\boxtimes R)}(N, N')$, we have

$$\mathcal{M}(g) = id_T \otimes g \in \mathcal{H}om_{DGP(R)}(T \otimes N, T \otimes N').$$

It suffices to prove $d(id_T \otimes g) = id_T \otimes d(g)$. For any $t \in T, n \in N$,

$$\begin{aligned}(d(id_T \otimes g))(t \otimes n) &= d \circ (id_T \otimes g)(t \otimes n) + (id_T \otimes g) \circ d(t \otimes n) \\ &= d(t \otimes g(n)) + (id_T \otimes g)(d(t) \otimes n + t \otimes d(n)) \\ &= d(t) \otimes g(n) + t \otimes d(g(n)) + d(t) \otimes g(n) + t \otimes g(d(n)) \\ &= t \otimes d(g(n)) + t \otimes g(d(n)) \\ &= (id_T \otimes d(g))(t \otimes n).\end{aligned}$$

□

Note that $\mathcal{M}|_{H^0}$ is an exact functor since $\mathcal{M}$ also preserves mapping cones. Then $\mathcal{M}|_{H^0}$ induces a $\mathbb{Z}[q^{\pm 1}]$-linear map $K_0(\mathcal{M}|_{H^0}) : K_0(R \otimes R) \to K_0(R)$ under the isomorphisms

$$K_0(H^0(DGP(R \boxtimes R))) \cong K_0(R) \otimes K_0(R), \quad K_0(H^0(DGP(R))) \cong K_0(R).$$

Proof of Theorem 1.3. The isomorphism $K_0(R_n) \cong Cl_n$ was proved in Proposition 3.12. In order to prove that $\mathcal{M}_n|_{H^0} : H^0(DGP(R_n \boxtimes R_n)) \to H^0(DGP(R_n))$ categorifies the multiplication m_n, we compute $K_0(\mathcal{M}|_{H^0})$ in terms of the basis $\{[P(\mathbf{x}, \mathbf{y})]\}$ of $K_0(R) \otimes K_0(R)$. By Remark 4.4 and Lemma 4.10 we have

$$\begin{aligned}K_0(\mathcal{M}|_{H^0})(\mathbf{x}, \mathbf{y}) &= K_0(\mathcal{M}|_{H^0})([P(\mathbf{x}, \mathbf{y})]) \\ &= [T \otimes P(\mathbf{x}, \mathbf{y})] = [T(\mathbf{x}, \mathbf{y})] \\ &= \sum_k [T^k(\mathbf{x}, \mathbf{y})] h^k|_{h=-1} = \sum_k \mathrm{M}^k(\mathbf{x}, \mathbf{y}) h^k|_{h=-1} \\ &= \mathrm{m}(\mathbf{x}, \mathbf{y}).\end{aligned}$$

Hence, we finish the proof of Theorem 1.3. □

5. A categorification of U_n via a subcategory of $H^0(DGP(R_n))$

5.1. U_n as a subalgebra of $K_0(R_n)$

We include $\mathbf{U}_n$ into $K_0(R_n)$ as a subalgebra for $n > 0$.

Lemma 5.1. *There is an inclusion of* $\mathbb{Z}[q^{\pm 1}]$*-algebras:*

$$\begin{aligned}\imath_n:\ \mathbf{U}_n &\to K_0(R_n)\\ 1 &\mapsto 1\\ E &\mapsto \sum_{\substack{0\leq i\leq n\\ i \text{ even}}} X_i\\ F &\mapsto \sum_{\substack{0\leq i\leq n\\ i \text{ odd}}} X_i\end{aligned}$$

Proof. It suffices to show that $\imath_n$ maps the relations of $\mathbf{U}_n$ in Definition 1.5 to the relations of $K_0(R_n)$ in Proposition 3.12:

$$\begin{aligned}(\imath_n(E))^2 &= \left(\sum_{i \text{ even}} X_i\right)^2\\ &= \sum_{i \text{ even}} X_i^2 + \sum_{i<j \text{ even}} (X_iX_j + X_jX_i) = 0 = \imath_n(E^2).\end{aligned}$$

Similarly $(\imath_n(F))^2 = 0 = \imath_n(F^2)$. We also have

$$\begin{aligned}\imath_n(E)\imath_n(F) + \imath_n(F)\imath_n(E) &= \sum_{\substack{i \text{ even}\\ j \text{ odd}}} X_iX_j + \sum_{\substack{i \text{ even}\\ j \text{ odd}}} X_jX_i\\ &= \sum_{i=0}^{n-1}(X_iX_{i+1} + X_{i+1}X_i) + \sum_{i<j-1}(X_iX_j + X_jX_i)\\ &= \sum_{i=0}^{n-1} q^{2i+1-n} + 0\\ &= \imath_n(EF + FE).\end{aligned}$$

□

Similarly, we have an inclusion $\imath_{n,n} = \imath_n \otimes \imath_n : \mathbf{U}_n \otimes \mathbf{U}_n \to K_0(R_n \otimes R_n)$.

Hence $\mathbf{U}_n$ and $\mathbf{U}_n \otimes \mathbf{U}_n$ can be viewed as subalgebras of $K_0(R_n)$ and $K_0(R_n \otimes R_n)$, respectively. The restriction of $\mathrm{m}_n : K_0(R_n \otimes R_n) \to K_0(R_n)$ to $\mathrm{f}_n : \mathbf{U}_n \otimes \mathbf{U}_n \to \mathbf{U}_n$ gives the algebra structure on $\mathbf{U}_n$. We will lift subalgebras to subcategories in the next section.

5.2. A subcategory of $H^0(DGP(R_n))$ categorifying $\mathbf{U}_n$

Since $K_0(R_n)$ is isomorphic to the Grothendieck group of $H^0(DGP(R_n))$, we can formally construct $\mathcal{U}_n$ as a triangulated full subcategory of

$H^0(DGP(R_n))$ whose Grothendieck group is the subalgebra $\mathbf{U}_n$. We define a bifunctor

$$\begin{aligned}\chi_n : H^0(DGP(R_n)) \times H^0(DGP(R_n)) &\to H^0(DGP(R_n \otimes R_n)) \\ &\to H^0(DGP(R_n \boxtimes R_n)),\end{aligned}$$

where the first map is given by tensoring two DG R_n-modules over $\mathbb{F}_2$ and the second map is an inverse of the equivalence in Lemma 2.29 which maps $P'(\mathbf{x}, \mathbf{y})$ to $P(\mathbf{x}, \mathbf{y})$ for any pair $\mathbf{x}, \mathbf{y} \in \Gamma_n$. Let $\rho_n = \mathcal{M}_n|_{H^0} \circ \chi_n$:

$$\begin{aligned}H^0(DGP(R_n)) \times H^0(DGP(R_n)) &\to H^0(DGP(R_n \boxtimes R_n)) \\ &\to H^0(DGP(R_n)).\end{aligned}$$

Notice that $\rho_n(M, P([\emptyset]))) = \rho_n(P([\emptyset]), M) = M$, for any $M \in H^0(DGP(R_n))$.

To define $\mathcal{U}_n$, we first lift 1 to $P([\emptyset])$, q to $P([\emptyset])\{1\}$, and q^{-1} to $P([\emptyset])\{-1\}$. Letters E and F are lifted to

$$\mathcal{E} = \bigoplus_{i \text{ even}} P([i]) \in H^0(DGP(R_n)),$$
$$\mathcal{F} = \bigoplus_{i \text{ odd}} P([i]) \in H^0(DGP(R_n)).$$

Then for the multiplication A_1A_2 of $A_1, A_2 \in \{1, q, q^{-1}, E, F\}$, we lift it to

$$\mathcal{A}_1\mathcal{A}_2 = \rho_n(\mathcal{A}_1, \mathcal{A}_2),$$

where $\mathcal{A}_i$ is the lifting of A_i defined above for $i = 1, 2$. For multiplication of 3 letters $A_1, A_2, A_3 \in \{1, q, q^{-1}, E, F\}$, we have different lifting of multiplication for different orders. For instance, we lift $(A_1A_2)A_3$ to

$$\rho_n(\rho_n(\mathcal{A}_1, \mathcal{A}_2), \mathcal{A}_3),$$

and $A_1(A_2A_3)$ to

$$\rho_n(\mathcal{A}_1, \rho_n(\mathcal{A}_2, \mathcal{A}_3)).$$

For multiplication of more letters, the definition of lifting is similar.

Then we define $\mathcal{U}_n$ as the smallest triangulated full subcategory of $H^0(DGP(R_n))$ containing the lifting of multiplication of all finitely many letters in $\{1, q, q^{-1}, E, F\}$ for all possible orders.

Remark 5.2. An equation in $\mathbf{U}_n$ may not be lifted to an isomorphism in $\mathcal{U}_n$. For example, the equation $E^2 = 0 \in \mathbf{U}_n$ is lifted to

$$\mathcal{E}\mathcal{E} = \bigoplus_{i,j \text{ even}} \mathcal{M}_n(P([i],[j])) \in H^0(DGP(R_n)).$$

As a cochain complex, $\mathcal{E}\mathcal{E} = (\mathcal{E}\mathcal{E})^{-1} \oplus (\mathcal{E}\mathcal{E})^0$ has zero differential, where

$$(\mathcal{E}\mathcal{E})^{-1} = (\mathcal{E}\mathcal{E})^0 = \bigoplus_{\substack{i,j \text{ even} \\ i>j}} P([i,j]).$$

It is not isomorphic to $0 \in H^0(DGP(R_n))$.

Next we define $\mathcal{U}_{n,n}$ as the smallest triangulated full subcategory of $H^0(DGP(R_n \boxtimes R_n))$ containing $\{\chi_n(\mathcal{X},\mathcal{Y}) \mid \mathcal{X},\mathcal{Y} \in \mathcal{U}_n\}$.

Proof of Theorem 1.6. Since $\mathbf{U}_n$ is generated by q, q^{-1}, E and F as an algebra and $\mathcal{U}_n$ is the smallest triangulated full subcategory containing the lifting of multiplication of q, q^{-1}, E and F, it follows that $K_0(\mathcal{U}_n) = \mathbf{U}_n$. Similarly, we have $K_0(\mathcal{U}_{n,n}) = \mathbf{U}_n \otimes \mathbf{U}_n$.

Since the exact functor $\mathcal{M}_n|_{H^0} : H^0(DGP(R_n \boxtimes R_n)) \to H^0(DGP(R_n))$ maps $\mathcal{U}_{n,n}$ into $\mathcal{U}_n$, let $\mathcal{F}_n : \mathcal{U}_{n,n} \to \mathcal{U}_n$ be the restriction. Then $K_0(\mathcal{F}_n) : K_0(\mathcal{U}_{n,n}) \to K_0(\mathcal{U}_n)$ agrees with the multiplication $\mathrm{f}_n : \mathbf{U}_n \otimes \mathbf{U}_n \to \mathbf{U}_n$. Hence we proved Theorem 1.6. □

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