# Four-ball genus bounds and a refinement of the Ozsváth-Szabó tau invariant

JENNIFER HOM AND ZHONGTAO WU

Based on work of Rasmussen [Ras03], we construct a concordance invariant associated to the knot Floer complex, and exhibit examples in which this invariant gives arbitrarily better bounds on the 4-ball genus than the Ozsváth-Szabó $\tau$ invariant.

## 1. Introduction

The 4-*ball genus* of a knot $K \subset S^3$ is

$$g_4(K) = \min\{g(\Sigma) \mid \Sigma \text{ smoothly embedded in } B^4 \text{ with } \partial\Sigma = K\},$$

where $g(\Sigma)$ denotes the genus of the surface $\Sigma$. The 4-ball genus gives a lower bound on the unknotting number of a knot (that is, the minimal number of crossing changes needed to obtain the unknot). We say knots $K_1$ and $K_2$ are *concordant* if $g_4(K_1\# - K_2) = 0$, where $-K_2$ denotes the reverse of the mirror image of $K_2$.

In [OS03c], Ozsváth-Szabó defined a concordance invariant, $\tau$, that gives a lower bound for the 4-ball genus of a knot. This invariant is sharp on torus knots, giving a new proof of the Milnor conjecture, originally proved by Kronheimer-Mrowka using gauge theory [KM93]

The knot Floer homology package [OS04a, Ras03] associates to a knot $K$ a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex over the ring $\mathbb{F}[U, U^{-1}]$, where $\mathbb{F}$ denotes the field of two elements and $U$ is a formal variable. We denote this complex $CFK^\infty(K)$. The invariant $\tau$ depends only on a single $\mathbb{Z}$-filtration, and forgets the module structure. By studying the module structure together with the full $\mathbb{Z} \oplus \mathbb{Z}$-filtration, we obtain a concordance invariant, $\nu^+$, which gives a better bound on the 4-ball genus than $\tau$, in the sense that

(1.1) $$\tau(K) \leq \nu^+(K) \leq g_4(K).$$

Moreover, the gap between $\tau$ and $\nu^+$ can be made arbitrarily large.

**Theorem 1.** *For any positive integer p, there exists a knot K with $\tau(K) \geq$*
*0 and*

$$\tau(K) + p \leq \nu^+(K) = g_4(K).$$

**Remark 1.1.** The invariant $\nu^+$ is closely related to the sequence of local
$h$ invariants of Rasmussen [Ras03, Section 7], which Rasmussen uses to give
bounds on the 4-ball genus; indeed, $\nu^+$ corresponds to the first place in the
sequence where a zero appears.

In Proposition 3.7, we also show that the gap between $\nu^+$ and the knot
signature can be made arbitrarily large.

In the case of alternating knots (or, more generally, quasi-alternating
knots), the invariant $\nu^+$ is completely determined by the signature of the
knot.

**Theorem 2.** *Let $K \subset S^3$ be a quasi-alternating knot. Then,*

$$\nu^+(K) = \begin{cases} 0 & \text{if } \sigma(K) \geq 0, \\ -\frac{\sigma(K)}{2} & \text{if } \sigma(K) < 0. \end{cases}$$

We also have the following result when $K$ is strongly quasipositive. See
[Hed10] for background on strongly quasipositive knots.

**Proposition 3.** *If $K$ is strongly quasipositive, then*

$$\nu^+(K) = \tau(K) = g_4(K) = g(K).$$

*Proof.* [Hed10, Theorem 1.2] states that $\tau(K) = g_4(K) = g(K)$ if and only if
$K$ is strongly quasipositive. Since $\tau(K) \leq \nu^+(K) \leq g_4(K)$, the result follows.
□

**Organization.** In Section 2, we define the invariant $\nu^+$ and prove various
properties. In Section 3, we construct an infinite family of knots in order to
prove Theorem 1. Throughout, we work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

## 2. The invariant $\nu^+$

Heegaard Floer homology, introduced by Ozsváth and Szabó [OS04b], is an invariant for closed oriented $\mathrm{Spin}^c$ 3–manifolds $(Y, \mathfrak{s})$, taking the form of a collection of related homology groups: $\widehat{HF}(Y, \mathfrak{s})$, $HF^\pm(Y, \mathfrak{s})$, and $HF^\infty(Y, \mathfrak{s})$. There is a $U$–action on the Heegaard Floer homology groups $HF^\pm$ and $HF^\infty$. When $\mathfrak{s}$ is torsion, there is an absolute Maslov $\mathbb{Q}$–grading on the Heegaard Floer homology groups. The $U$–action decreases the grading by 2.

For a rational homology 3–sphere $Y$ with a $\mathrm{Spin}^c$ structure $\mathfrak{s}$, $HF^+(Y, \mathfrak{s})$ can be decomposed as the direct sum of two groups: the first group is the image of $HF^\infty(Y, \mathfrak{s}) \cong \mathbb{F}[U, U^{-1}]$ in $HF^+(Y, \mathfrak{s})$, which is isomorphic to $\mathcal{T}^+ = \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$, and its minimal absolute $\mathbb{Q}$–grading is an invariant of $(Y, \mathfrak{s})$, denoted by $d(Y, \mathfrak{s})$, the *correction term* [OS03a]; the second group is the quotient modulo the above image and is denoted by $HF_{\mathrm{red}}(Y, \mathfrak{s})$. Altogether, we have

$$HF^+(Y, \mathfrak{s}) = \mathcal{T}^+ \oplus HF_{\mathrm{red}}(Y, \mathfrak{s}).$$

We briefly recall the large $N$ surgery formula of [OS04a, Theorem 4.4]. We use the notation of [NW15]. Let $CFK^\infty(K)$ denote the knot Floer complex of $K$, which takes the form of a $\mathbb{Z} \oplus \mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{F}[U, U^{-1}]$. The $U$-action lowers each filtration by one. We will be particularly interested in the quotient complexes

$$A_k^+ = C\{\max\{i, j - k\} \geq 0\} \quad \text{and} \quad B^+ = C\{i \geq 0\}$$

where $i$ and $j$ refer to the two filtrations. The complex $B^+$ is isomorphic to $CF^+(S^3)$. There is a map

$$v_k^+ : A_k^+ \to B^+$$

defined by projection. One can also define a map

$$h_k^+ : A_k^+ \to B^+$$

defined by projection to $C\{j \geq k\}$, followed by shifting to $C\{j \geq 0\}$ via the $U$-action, and concluding with a chain homotopy equivalence between $C\{j \geq 0\}$ and $C\{i \geq 0\}$. These maps correspond to the maps induced on $HF^+$ by the two handle cobordism from $S_N^3(K)$ to $S^3$ [OS04a, Theorem 4.4].

Similarly, one can consider the subquotient complexes

$$\widehat{A}_k = C\{\max\{i, j - k\} = 0\} \quad \text{and} \quad \widehat{B} = C\{i = 0\} \cong \widehat{CF}(S^3)$$

and the maps

$$\widehat{v}_k : \widehat{A}_k \to \widehat{B} \quad \text{and} \quad \widehat{h}_k : \widehat{A}_k \to \widehat{B}.$$

The invariant $\tau$ is defined in [OS03c] to be

$$\tau(K) = \min\{k \in \mathbb{Z} \mid \iota_k \text{ induces a nontrivial map on homology}\},$$

where $\iota_k : C\{i = 0, j \le k\} \to \widehat{CF}(S^3)$ denotes inclusion. A slightly stronger concordance invariant, $\nu$, is defined in [OS11, Definition 9.1] to be

$$\nu(K) = \min\{k \in \mathbb{Z} \mid \widehat{v}_k : \widehat{A}_k \to \widehat{CF}(S^3)$$
$$\text{induces a nontrivial map in homology}\}.$$

The invariant $\nu(K)$ gives a lower bound for $g_4(K)$ and is equal to either $\tau(K)$ or $\tau(K) + 1$; in particular, in many cases $\nu$ gives a better 4-ball genus than $\tau$.

We can further refine these bounds by considering maps on $CF^+$ rather than $\widehat{CF}$.

**Definition 2.1.** Define $\nu^+(K)$ by

$$\nu^+(K) = \min\{k \in \mathbb{Z} \mid v_k^+ : A_k^+ \to CF^+(S^3), \ v_k^+(1) = 1\}.$$

Here, 1 denotes the lowest graded generator of the subgroup $\mathcal{T}^+$ in the homology of the complex, and we abuse our notations by identifying $A_k^+$ and $CF^+(S^3)$ with their homologies.

According to [NW15], the definition of $\nu^+(K)$ is equivalent to the smallest $k$ such that $V_k = 0$, where $V_k$ is the $U$-exponent of $v_k^+$ at sufficiently high gradings. We can define $H_k$ similarly in terms of $h_k^+$. By [NW14, Equation (13)] and [HLZ15, Lemma 2.5], the $V_k$'s and $H_k$'s satisfy

(2.1)                                    $$H_k = V_{-k}$$
(2.2)                                    $$H_k = V_k + k$$
(2.3)                              $$V_k - 1 \le V_{k+1} \le V_k$$

and are related to the correction terms in the surgery formula [NW15, Proposition 1.6]:

**Proposition 2.2.** *Suppose $p, q > 0$, and fix $0 \leq i \leq p - 1$. Then*

(2.4) $$d(S^3_{p/q}(K), i) = d(L(p, q), i) - 2 \max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}.$$

We have the following properties for $\nu^+$.

**Proposition 2.3.** *The invariant $\nu^+$ satisfies:*

1) *$\nu^+$ is a smooth concordance invariant.*

2) *$\nu^+(K) \geq 0$, and the equality holds if and only if $V_0 = 0$.*

3) *$\nu^+(K) \geq \nu(K) \geq \tau(K)$.*

*Proof.* To see 1, note that $V$'s are determined by the $d$-invariants of the surgered manifolds $S^3_n(K)$ [NW15, Proposition 1.6], and the $d$-invariants are concordance invariants. To see 2, note that $V_{-1} > H_{-1} = V_1 \geq 0$ by Equations (2.1) and (2.2). To see 3, chase the commutative diagram

$$\begin{array}{ccc} \widehat{A}_k & \xrightarrow{\ j_A\ } & A^+_k \\ {\scriptstyle \widehat{v}_k} \downarrow & & \downarrow {\scriptstyle v^+_k} \\ \widehat{B} & \xrightarrow{\ j_B\ } & B^+. \end{array}$$ $\qquad \square$

The $\nu^+$ invariant can be computed explicitly for quasi-alternating knots, a generalization of alternating knots introduced in [MO08]. In fact, Theorem 2 states that $\nu^+$ is completely determined by the signature of the knot, just as the $\tau$ invariant:

$$\nu^+(K) = \begin{cases} 0 & \text{if } \sigma(K) \geq 0, \\ -\frac{\sigma(K)}{2} & \text{if } \sigma(K) < 0. \end{cases}$$

*Proof of Theorem 2.* Let $K$ be quasi-alternating. By [OS03b, Corollary 1.5] and [MO08, Theorem 2], $d(S^3_1(K)) = 0$ when $\sigma(K) \geq 0$. This proves that $\nu^+(K) = 0$ when $\sigma(K) \geq 0$. On the other hand, the proof of Theorem 1.4 of [OS03b], together with [MO08, Theorem 2], implies that for any $s > 0$,

$$H_{\leq s + \frac{\sigma}{2} - 2}(A^+_s) \cong HF^+_{\leq s + \frac{\sigma}{2} - 2}(S^3).$$

In particular, if we let $s = -\sigma/2$ when $\sigma(K) < 0$, then

$$H_{\leq -2}(A^+_s) \cong HF^+_{\leq -2}(S^3) \cong 0.$$

Here, the gradings of the homology of both sides are inherited from the grading on $CFK^\infty(K)$. Thus, the element $1 \in \mathcal{T}^+ \subset H_*(A_s^+)$ has grading $-2V_s$. In light of the vanishing of the homology group $H_{\leq -2}(A_s^+)$, we must have $V_s = 0$. So

$$\nu^+(K) \leq s = -\sigma(K)/2$$

from the definition. We also know that

$$\nu^+(K) \geq \tau(K) = -\sigma(K)/2$$

for a quasi-alternating knot $K$. Hence, $\nu^+(K) = -\sigma(K)/2$. □

Next, we show that $\nu^+$ also give a lower bound for the four-ball genus of a knot.

**Proposition 2.4.** $\nu^+(K) \leq g_4(K)$

*Proof.* This follows from [Ras03, Corollary 7.4]. The function $h_k(K)$ in [Ras03] is the same as $V_k$ in [NW15]. □

**Remark 2.5.** [Ras03, Corollary 7.4] states that $g_4(K) \geq V_k + k$ for all $k \leq g_4(K)$, so one might wonder if other $V_k$'s can give stronger 4-ball genus bounds. However, since $V_k - 1 \leq V_{k+1} \leq V_k$, it follows that $\nu^+$ is the best 4-ball genus bound obtainable from the sequence of $V_k$'s.

## 3. Four-ball genus bound

In this section, we exhibit some examples of knots whose $\nu^+$ invariant is arbitrarily better than the corresponding $\tau$ invariant. Hence, the $\nu^+$ invariant indeed gives us significantly improved four-ball genus bound for some particular knots. We will show that for any integer $n \geq 2$, there exists a knot $K$ with $\tau(K) \geq 0$ and

$$\tau(K) + n = \nu^+(K) = g_4(K).$$

Let $K_{p,q}$ denote the $(p, q)$-cable of $K$, where $p$ denotes the longitudinal winding. Without loss of generality, we will assume throughout that $p > 0$. Let $T_{p,q}$ denote the $(p, q)$-torus knot (that is, the $(p, q)$-cable of the unknot), and $T_{p,q;m,n}$ the $(m, n)$-cable of $T_{p,q}$. We begin with a single example of a knot for which $\nu^+$ gives a better 4-ball genus bound than $\tau$.

**Proposition 3.1.** *Let $K$ be the knot $T_{2,9}\# - T_{2,3;2,5}$. We have*

$$\tau(K) = 0, \qquad \nu(K) = 1, \quad \text{and} \quad \nu^+(K) = 2.$$

*Proof.* The torus knot $T_{2,9}$ is an $L$-space knot, as is $T_{2,3;2,5}$ [Hed09, Theorem 1.10], so their knot Floer complexes are completely determined by their Alexander polynomials [OS05, Theorem 1.2] (cf. [Hom14b, Remark 6.6]). We have that

$$\Delta_{T_{2,9}}(t) = t^8 - t^7 + t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$$

and

$$\begin{aligned}\Delta_{T_{2,3;2,5}}(t) &= \Delta_{T_{2,3}}(t^2) \cdot \Delta_{T_{2,5}}(t) \\ &= t^8 - t^7 + t^4 - t + 1.\end{aligned}$$

Furthermore, we have that $CFK^\infty(-K) \cong CFK^\infty(K)^*$ [OS04a, Section 3.5], where $CFK^\infty(K)^*$ denotes the dual of $CFK^\infty(K)$. Thus, $CFK^\infty(-T_{2,3;2,5})$ is generated over $\mathbb{F}[U, U^{-1}]$ by

$$[y_0, 0, -4], \quad [y_1, -1, -4], \quad [y_1, -1, -1], \quad [y_3, -3, -1], \quad [y_4, -4, 0],$$

where we write $[y, i, j]$ to denote that the generator $y$ has filtration level $(i, j)$. The differential is given by

$$\begin{aligned}\partial y_0 &= y_1 \\ \partial y_2 &= y_1 + y_3 \\ \partial y_4 &= y_3.\end{aligned}$$

The complex $CFK^\infty(T_{2,9})$ is generated by

$$[x_0, 0, 4], \quad [x_1, 1, 4], \quad [x_2, 1, 3], \quad [x_3, 2, 3], \quad [x_4, 2, 2],$$
$$[x_5, 3, 2], \quad [x_6, 3, 1], \quad [x_7, 4, 1], \quad [x_8, 4, 0].$$

The differential is given by

$$\begin{aligned}\partial x_1 &= x_0 + x_2 \\ \partial x_3 &= x_2 + x_4 \\ \partial x_5 &= x_4 + x_6 \\ \partial x_7 &= x_6 + x_8.\end{aligned}$$

The complexes $CFK^\infty(-T_{2,3;2,5})$ and $CFK^\infty(T_{2,9})$ are depicted in Figures 1 and 2, respectively. (More precisely, $CFK^\infty$ consists of the complexes pictured tensored with $\mathbb{F}[U, U^{-1}]$, where $U$ lowers $i$ and $j$ each by 1.) In particular, we see that $\tau(-T_{2,3;2,5}) = -4$ since $y_0$ generates the vertical homology, and that $\tau(T_{2,9}) = 4$ since $x_0$ generates the vertical homology. Since $\tau$ is additive under connected sum, it follows that

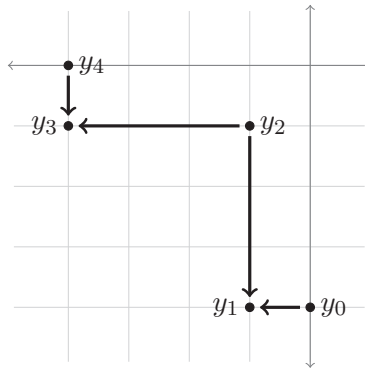$$\tau(-T_{2,3;2,5}\#T_{2,9}) = 0,$$
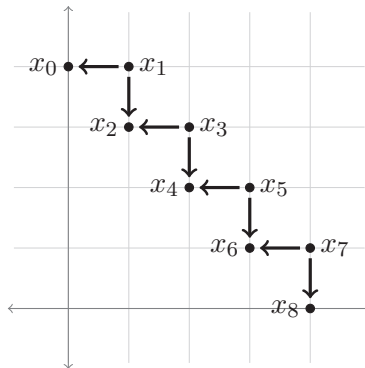
as desired.



Figure 1: $CFK^\infty(-T_{2,3;2,5})$



Figure 2: $CFK^\infty(T_{2,9})$

The knot Floer complex satisfies a Künneth formula [OS04a, Theorem 7.1]:

$$CFK^\infty(K_1 \# K_2) \cong CFK^\infty(K_1) \otimes_{\mathbb{F}[U,U^{-1}]} CFK^\infty(K_2).$$

In particular, we may compute $CFK^\infty(T_{2,9} \# - T_{2,3;2,5})$ as the tensor product of $CFK^\infty(T_{2,9})$ and $CFK^\infty(-T_{2,3;2,5})$ , where

$$[x, i, j] \otimes [y, k, \ell] = [xy, i + k, j + \ell].$$

The generators, filtration levels, and differentials in the tensor product are listed below.

$$\partial[x_0 y_0, 0, 0] = x_0 y_1$$
$$\partial[x_1 y_0, 1, 0] = x_1 y_1 + x_0 y_0 + x_2 y_0$$
$$\partial[x_2 y_0, 1, -1] = x_2 y_1$$
$$\partial[x_3 y_0, 2, -1] = x_3 y_1 + x_2 y_0 + x_4 y_0$$
$$\partial[x_4 y_0, 2, -2] = x_4 y_1$$
$$\partial[x_5 y_0, 3, -2] = x_5 y_1 + x_4 y_0 + x_6 y_0$$
$$\partial[x_6 y_0, 3, -3] = x_6 y_1$$
$$\partial[x_7 y_0, 4, -3] = x_7 y_1 + x_6 y_0 + x_8 y_0$$
$$\partial[x_8 y_0, 4, -4] = x_8 y_1$$
$$\partial[x_0 y_1, -1, 0] = 0$$
$$\partial[x_1 y_1, 0, 0] = x_0 y_1 + x_2 y_1$$
$$\partial[x_2 y_1, 0, -1] = 0$$
$$\partial[x_3 y_1, 1, -1] = x_2 y_1 + x_4 y_1$$
$$\partial[x_4 y_1, 1, -2] = 0$$
$$\partial[x_5 y_1, 2, -2] = x_4 y_1 + x_6 y_1$$
$$\partial[x_6 y_1, 2, -3] = 0$$
$$\partial[x_7 y_1, 3, -3] = x_6 y_1 + x_8 y_1$$
$$\partial[x_8 y_1, 3, -4] = 0$$
$$\partial[x_0 y_2, -1, 3] = x_0 y_1 + x_0 y_3$$
$$\partial[x_1 y_2, 0, 3] = x_1 y_1 + x_1 y_3 + x_0 y_2 + x_2 y_2$$
$$\partial[x_2 y_2, 0, 2] = x_2 y_1 + x_2 y_3$$
$$\partial[x_3 y_2, 1, 2] = x_3 y_1 + x_3 y_3 + x_2 y_2 + x_4 y_2$$
$$\partial[x_4 y_2, 1, 1] = x_4 y_1 + x_4 y_3$$

$$\partial[x_5 y_2, 2, 1] = x_5 y_1 + x_5 y_3 + x_4 y_2 + x_6 y_2$$
$$\partial[x_6 y_2, 2, 0] = x_6 y_1 + x_6 y_3$$
$$\partial[x_7 y_2, 3, 0] = x_7 y_1 + x_7 y_3 + x_6 y_2 + x_8 y_2$$
$$\partial[x_8 y_2, 3, -1] = x_8 y_1 + x_8 y_3$$
$$\partial[x_0 y_3, -4, 3] = 0$$
$$\partial[x_1 y_3, -3, 3] = x_0 y_3 + x_2 y_3$$
$$\partial[x_2 y_3, -3, 2] = 0$$
$$\partial[x_3 y_3, -2, 2] = x_2 y_3 + x_4 y_3$$
$$\partial[x_4 y_3, -2, 1] = 0$$
$$\partial[x_5 y_3, -1, 1] = x_4 y_3 + x_6 y_3$$
$$\partial[x_6 y_3, -1, 0] = 0$$
$$\partial[x_7 y_3, 0, 0] = x_6 y_3 + x_8 y_3$$
$$\partial[x_8 y_3, 0, -1] = 0$$
$$\partial[x_0 y_4, -4, 4] = x_0 y_3$$
$$\partial[x_1 y_4, -3, 4] = x_1 y_3 + x_0 y_4 + x_2 y_4$$
$$\partial[x_2 y_4, -3, 3] = x_2 y_3$$
$$\partial[x_3 y_4, -2, 3] = x_3 y_3 + x_2 y_4 + x_4 y_4$$
$$\partial[x_4 y_4, -2, 2] = x_4 y_3$$
$$\partial[x_5 y_4, -1, 2] = x_5 y_3 + x_4 y_4 + x_6 y_4$$
$$\partial[x_6 y_4, -1, 1] = x_6 y_3$$
$$\partial[x_7 y_4, 0, 1] = x_7 y_3 + x_6 y_4 + x_8 y_4$$
$$\partial[x_8 y_4, 0, 0] = x_8 y_3$$

We perform the following change of basis on $CFK^\infty(T_{2,9} \# - T_{2,3;2,5})$. In the linear combinations below, we have ordered the terms so that the first basis element has the greatest filtration and thus determines the filtration level of the linear combination.

$$z_0 = x_0 y_0$$
$$z_1 = x_0 y_1$$
$$z_2 = x_0 y_2 + x_1 y_3 + x_3 y_3 + x_4 y_4$$
$$z_3 = x_1 y_2$$
$$z_4 = x_2 y_2 + x_3 y_3 + x_1 y_1 + x_4 y_4$$
$$z_5 = x_3 y_2 + x_5 y_4 + x_1 y_0$$

$$z_6 = x_4y_2 + x_5y_3 + x_3y_1 + x_6y_4 + x_2y_0$$
$$z_7 = x_5y_2 + x_7y_4 + x_3y_0$$
$$z_8 = x_6y_2 + x_7y_3 + x_5y_1 + x_4y_0$$
$$z_9 = x_7y_2$$
$$z_{10} = x_8y_2 + x_7y_1 + x_4y_0 + x_5y_1$$
$$z_{11} = x_8y_3$$
$$z_{12} = x_8y_4$$

$$w_0^i = x_{2i+1}y_4 \qquad\qquad i = 0,1,2,3$$
$$w_1^i = x_{2i}y_4 \qquad\qquad i = 0,1,2,3$$
$$w_2^i = x_{2i}y_3 \qquad\qquad i = 0,1,2,3$$
$$w_3^i = x_{2i+1}y_3 + x_{2i+2}y_4 \qquad\qquad i = 0,1,2,3$$
$$w_0^{i+4} = x_{2i+1}y_0 \qquad\qquad i = 0,1,2,3$$
$$w_1^{i+4} = x_{2i+1}y_1 + x_{2i}y_0 \qquad\qquad i = 0,1,2,3$$
$$w_2^{i+4} = x_{2i+2}y_1 \qquad\qquad i = 0,1,2,3$$
$$w_3^{i+4} = x_{2i+2}y_0 \qquad\qquad i = 0,1,2,3.$$

See Figure 3.

Notice that the basis elements $\{z_i\}_{i=0}^{12}$ generate a direct summand $C$ of $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$. See Figure 4. Since the total homology of this summand is non-zero, this summand determines both $\nu$ and $\nu^+$. We write $\widehat{A}_s$ and $A_s^+$ to refer to the associated subquotient complexes of $C$.

The vertical homology of $C$ is generated by $z_0$. The generator $z_0$ in $C\{i = 0\}$ is not the image of any cycle in $\widehat{A}_0$. On the other hand, $z_0$ is non-zero in $H_*(\widehat{A}_1)$. Hence $\nu(T_{2,9}\# - T_{2,3;2,5}) = 1$.

The cycle $z_6$ generates $H_*(C)$. Moreover, the cycle $Uz_6$ is non-zero in $H_*(A_1^+)$; see Figure 5. The cycle $Uz_6$ is a boundary in $A_2^+$ as in Figure 6, while the cycle $z_6$ is non-zero in $H_*(A_2^+)$. It follows that $\nu^+(T_{2,9}\# - T_{2,3;2,5}) = 2$, as desired. $\qquad\qquad\square$

**Corollary 3.2.** *Let* $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. *Then*

$$\tau(K) = 0, \quad \nu(K) = 1, \quad \text{and} \quad \nu^+(K) = 2.$$

*Proof.* By [HKL16, Theorem B.1],

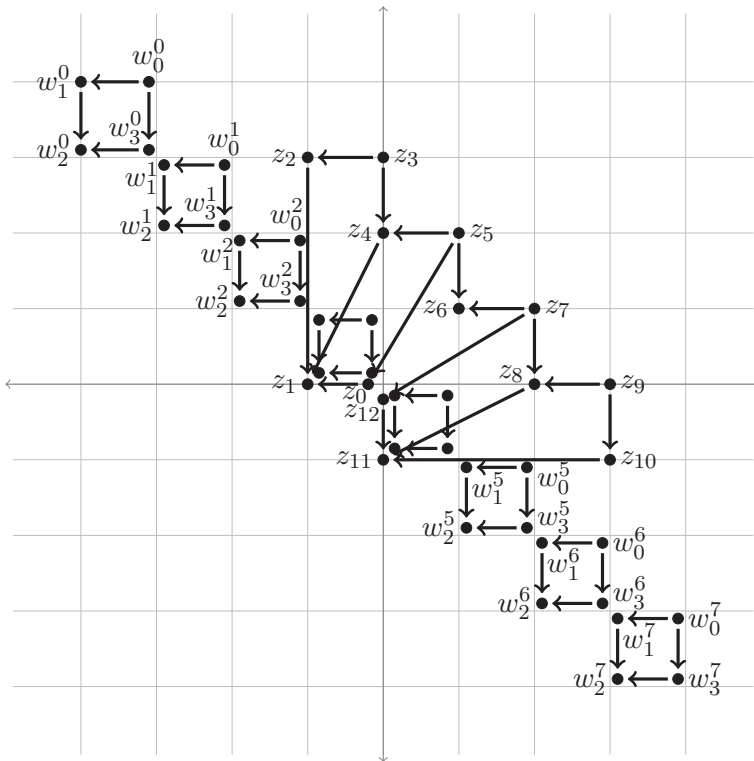$$CFK^\infty(T_{2,5}\#2T_{2,3}) \cong CFK^\infty(T_{2,9}) \oplus A,$$

Figure 3: $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$ after a change of basis

where $A$ is acyclic (i.e., its total homology vanishes). Since acyclic summands
do not affect $\tau$, $\nu$, and $\nu^+$, the result follows. $\qquad\qquad\square$

**Lemma 3.3.** Let $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then $g_4(K) = 2$.

*Proof.* When $p, q > 0$, the genus of $T_{p,q}$ is equal to $\frac{(p-1)(q-1)}{2}$. We can con-
struct a genus 4 Seifert surface $F$ for $-T_{2,3;2,5} = (-T_{2,3})_{-2,5}$ by taking two
parallel copies of the genus one Seifert surface for $-T_{2,3}$ and connecting them
with 5 half-twisted bands. The knot $-T_{2,3}\#T_{-2,5}$ sits on $F$. To see this, con-
sider one copy of the Seifert surface for $-T_{2,3}$ together with the half-twisted
bands and a small neighborhood of a segment connecting the ends of the
bands.

Take the boundary sum of $F$ with the genus two Seifert surface for $T_{2,5}$
and with two copies of the genus one Seifert surface for $T_{2,3}$ to obtain a
surface $F'$. The surface $F'$ is a genus 8 Seifert surface for $K$. The genus
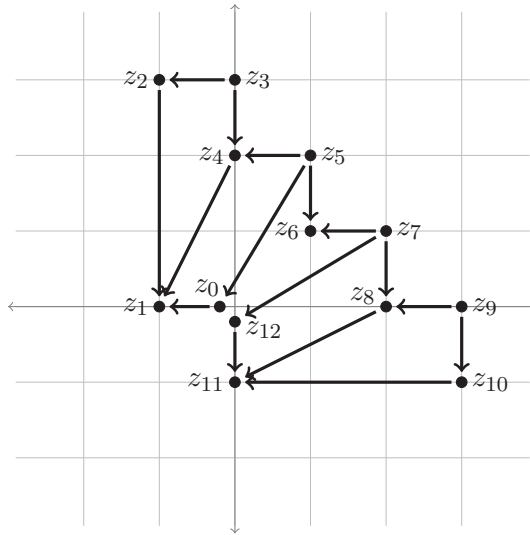
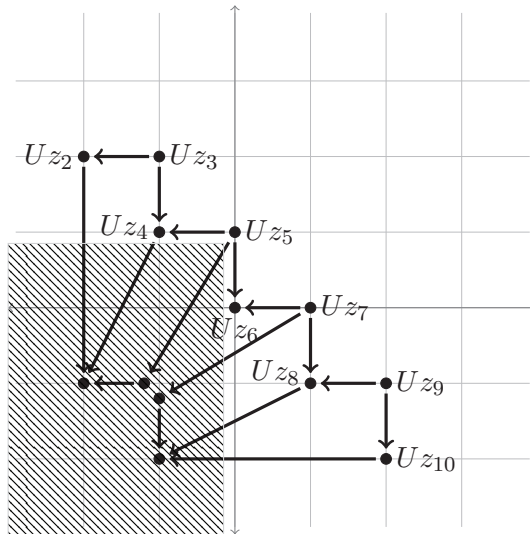Figure 4: The relevant summand of $CFK^\infty(T_{2,9}\# - T_{2,3;2,5})$



Figure 5: The generators $\{Uz_i\}$ in $A_1^+$

6 slice knot $J = -T_{2,3}\#T_{-2,5}\#T_{2,3}\#T_{2,5}$ sits on this surface. Performing
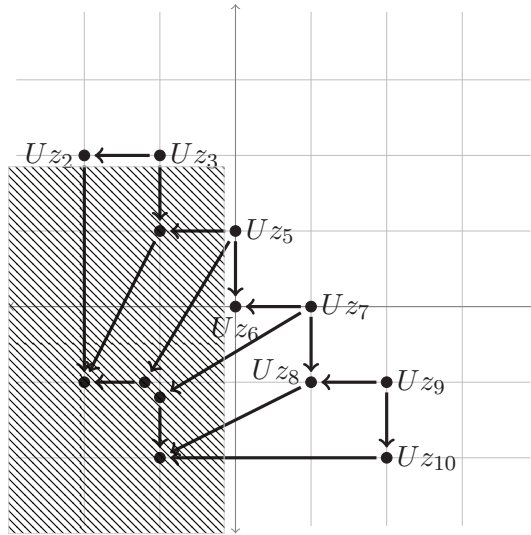
Figure 6: The generators $\{Uz_i\}$ in $A_2^+$

surgery along $J$ on $F'$ in $B^4$ yields a genus two slice surface for $K$. Since $\nu^+(K) = 2$ and $\nu^+(K) \leq g_4(K)$, it follows that $g_4(K) = 2$.                    $\square$

In order to prove the main theorem, we will consider certain cables of the knot $K = T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. We first compute $\tau$ of these cables.

**Lemma 3.4.**  *Let $K$ be the knot $T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then*

$$\tau(K_{p,3p-1}) = \frac{3p(p-1)}{2}.$$

*Proof.* Recall from [Hom14a, Definition 3.4] that the invariant $\varepsilon(K)$ is defined to be $-1$ if $\tau(K) < \nu(K)$. The equality then follows from [Hom14a, Theorem 1], which states that if $\varepsilon(K) = -1$, then

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q+1)}{2}.$$                    $\square$

**Proposition 3.5.**  *Let $K$ be the knot $T_{2,5}\#2T_{2,3}\# - T_{2,3;2,5}$. Then*

$$\nu^+(K_{p,3p-1}) = g_4(K_{p,3p-1}) = \frac{p(3p-1)}{2} + 1.$$

*Proof.* Let $p, q > 0$. For an arbitrary knot $J$ and its cable $J_{p,q}$, there is a reducible surgery

$$S^3_{pq}(J_{p,q}) \cong S^3_{q/p}(J) \# L(p, q).$$

We apply the surgery formula (2.4) for the above knot surgery when $J$ is the unknot. Note that $\max\{V_i, H_{i-pq}\} = V_i$ when $0 \le i \le \frac{pq}{2}$ since $V_i = H_{-i}$ and $H_{i-1} \le H_i$. Thus, we have

$$(3.1) \qquad d(L(pq, 1), i) - 2V_i(T_{p,q}) = d(L(q, p), \phi_1(i)) + d(L(p, q), \phi_2(i))$$

for all $0 \le i \le \frac{pq}{2}$.

Here, we identify the Spin$^c$ structure of a rational homology sphere by an integer $i$ as in [NW15], and $\phi_1(i)$ and $\phi_2(i)$ are the projection of the Spin$^c$ structure to the two factors of the reducible manifold. In particular, we can identify $\phi_1(i)$ with some integers between 0 and $q - 1$ and $\phi_2(i)$ with some integers between 0 and $p - 1$. The maps $\phi_1$ and $\phi_2$ are independent of the knot $J$, and in principle, can be determined from an explicit geometric description of the reducible surgery (cf [Hed09]). For the purpose of our argument below, we do not need it.

Similarly, apply (2.4) for an arbitrary knot $J$. We have

$$d(L(pq, 1), i) - 2V_i(J_{p,q})$$
$$= d(L(q, p), \phi_1(i)) - 2 \max \left\{ V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(J), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(J) \right\}$$
$$+ d(L(p, q), \phi_2(i)).$$

for all $i \le \frac{pq}{2}$.

Compared with Equation (3.1) and using the fact $V_i(T_{p,q}) \ge 0$, we deduce that for all $i \le \frac{pq}{2}$,

$$V_i(J_{p,q}) = V_i(T_{p,q}) + \max \left\{ V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(J), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(J) \right\}$$
$$\ge \max \left\{ V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(J), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(J) \right\}$$

From now on, let us specialize to the case when $K = T_{2,5} \# 2T_{2,3} \# - T_{2,3;2,5}$ and $q = 3p - 1$. We claim that

$$\max \left\{ V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K) \right\} > 0.$$

To see this, note that $V_0(K), V_1(K) > 0$ as $\nu^+(K) = 2$. When $0 \le \phi_1(i) < 2p$, $V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K) > 0$. Otherwise, $2p \le \phi_1(i) < q = 3p - 1$, and then $H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K) > 0$ since $H_{-k} = V_k$ and $V_0(K), V_1(K) > 0$.

Hence, $V_i(K_{p,q}) > 0$ for all $i \leq \frac{pq}{2}$. This implies that

$$\nu^+(K_{p,3p-1}) \geq \frac{p(3p-1)}{2} + 1.$$

On the other hand,

$$g_4(K_{p,q}) \leq pg_4(K) + \frac{(p-1)(q-1)}{2},$$

since one can construct a slice surface for $K_{p,q}$ from $p$ parallel copies of a slice surface for $K$ together with $(p-1)q$ half-twisted bands. By Lemma 3.3, $g_4(K) = 2$, so when $q = 3p - 1$, the right-hand side of the above inequality is $\frac{p(3p-1)}{2} + 1$. Hence

$$\frac{p(3p-1)}{2} + 1 \leq \nu^+(K_{p,3p-1}) \leq g_4(K_{p,3p-1}) \leq \frac{p(3p-1)}{2} + 1,$$

so $\nu^+(K_{p,3p-1}) = g_4(K_{p,3p-1}) = \frac{p(3p-1)}{2} + 1$.  $\square$

Note that $\nu^+(K_{p,3p-1}) - \tau(K_{p,3p-1}) = p + 1$ for $K = T_{2,5} \# 2T_{2,3} \# - T_{2,3;2,5}$. This proves Theorem 1.

A similar argument shows that $\nu^+$ gives a sharp four-ball genus bound for certain other cable knots as well.

**Proposition 3.6.** *Let $K$ be a knot with $\nu^+(K) = g_4(K) = n$, then*

$$\nu^+(K_{p,(2n-1)p-1}) = g_4(K_{p,(2n-1)p-1}) = \frac{p((2n-1)p-1)}{2} + 1.$$

*Proof.* Let $q = (2n-1)p - 1$. In the proof of Proposition 3.5, we showed

$$V_i(K_{p,q}) \geq \max\left\{ V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K) \right\}$$

for all $0 \leq i \leq \frac{pq}{2}$. We claim that

$$\max\left\{ V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K), H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K) \right\} > 0.$$

To see this, note that $V_i(K) > 0$ for all $i < n$. When $0 \leq \phi_1(i) < np$, $V_{\lfloor \frac{\phi_1(i)}{p} \rfloor}(K) > 0$. Otherwise, $np \leq \phi_1(i) < q = (2n-1)p - 1$, and then

$H_{\lfloor \frac{\phi_1(i)-q}{p} \rfloor}(K) > 0$. Hence, $V_i(K_{p,q}) > 0$ for all $i \leq \frac{pq}{2}$. This implies that

$$\nu^+(K_{p,q}) \geq \frac{pq}{2} + 1 = \frac{p((2n-1)p-1)}{2} + 1.$$

On the other hand,

$$
\begin{aligned}
g_4(K_{p,q}) &\leq pg_4(K) + \frac{(p-1)(q-1)}{2} \\
&= pn + \frac{(p-1)((2n-1)p-2)}{2} \\
&= \frac{p((2n-1)p-1)}{2} + 1.
\end{aligned}
$$

So $\nu^+(K_{p,(2n-1)p-1}) = g_4(K_{p,(2n-1)p-1}) = \frac{p((2n-1)p-1)}{2} + 1$. □

We conclude by showing that the knot signature cannot detect the four-ball genus of the knots used in Theorem 1. Recall that

$$\frac{1}{2}|\sigma(K)| \leq g_4(K).$$

**Proposition 3.7.** *Let $K = T_{2,5} \# 2T_{2,3} \# - T_{2,3;2,5}$. Then for $p > 0$,*

$$\frac{1}{2}|\sigma(K_{p,3p-1})| + 2p - 2 \leq g_4(K_{p,3p-1}).$$

*Proof.* We have that $\sigma(T_{2,q}) = 1 - q$. By [Shi71, Theorem 9],

$$
\sigma(K_{p,q}) = \begin{cases} \sigma(T_{p,q}) & \text{if } p \text{ is even} \\ \sigma(K) + \sigma(T_{p,q}) & \text{if } p \text{ is odd.} \end{cases}
$$

Thus, $\sigma(T_{2,3;2,5}) = -4$ and since signature is additive under connected sum,

$$
\begin{aligned}
\sigma(T_{2,5} \# 2T_{2,3} \# - T_{2,3;2,5}) &= -4 + 2(-2) - (-4) \\
&= -4.
\end{aligned}
$$

We showed in Lemma 3.3 that $g_4(K) = 2$, so for $K$, the signature is indeed strong enough to detect the four-ball genus. However, we will now show that it is not strong enough to detect the four-ball genus of $K_{p,3p-1}$. We have that

$$
\begin{aligned}
|\sigma(K_{p,3p-1})| &\leq |\sigma(K)| + |\sigma(T_{p,3p-1})| \\
&\leq 4 + (p-1)(3p-2) = 3p^2 - 5p + 6,
\end{aligned}
$$

where the second inequality follows from the fact that when $p, q > 0$,

$$|\sigma(T_{p,q})| \leq 2g_4(T_{p,q}) = (p-1)(q-1).$$

On the other hand,

$$2g_4(K_{p,3p-1}) = 3p^2 - p + 2,$$

so

$$|\sigma(K_{p,3p-1})| + 4p - 4 \leq 2g_4(K_{p,3p-1}). \qquad \square$$

Recall from Proposition 3.5 that $g_4(K_{p,3p-1}) = \nu^+(K_{p,3p-1})$. A consequence of Proposition 3.7 is that the gap between $\frac{1}{2}\sigma$ and $\nu^+$ can be made arbitrarily large.

## References

[Hed09] Matthew Hedden, *On knot Floer homology and cabling II*, Int. Math. Res. Not. IMRN (2009), no. 12, 2248–2274.

[Hed10] Matthew Hedden, *Notions of positivity and the Ozsváth-Szabó concordance invariant*, J. Knot Theory Ramifications, **19** (2010), no. 5, 617–629.

[HKL16] Matthew Hedden, Se-Goo Kim, and Charles Livingston, *Topologically slice knots of smooth concordance order two*, J. Differential Geom., **102** (2016), no. 3, 353–393.

[HLZ15] Jennifer Hom, Tye Lidman, and Nicholas Zufelt, *Reducible surgeries and Heegaard Floer homology*, Math. Res. Lett., **22** (2015), no. 3, 763–788.

[Hom14a] Jennifer Hom, *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, J. Topol., **7** (2014), no. 2, 287–326.

[Hom14b] Jennifer Hom, *The knot Floer complex and the smooth concordance group*, Comment. Math. Helv., **89** (2014), no. 3, 537–570.

[KM93] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces. I*, Topology, **32** (1993), no. 4, 773–826. `1241873 (94k:57048)`

[MO08] Ciprian Manolescu and Peter Ozsváth, *On the Khovanov and knot Floer homologies of quasi-alternating links*, Proceedings of Gökova Geometry-Topology Conference 2007, Gökova Geometry/Topology Conference (GGT), Gökova, 2008, pp. 60–81.

[NW15] Yi Ni and Zhongtao Wu, *Cosmetic surgeries on knots in $S^3$*, J. Reine Angew. Math., **706** (2015), 1–17.

[NW14] Y. Ni and Z. Wu, *Heegaard Floer correction terms and rational genus bounds*, Adv. Math., **267** (2014), 360–380.

[OS03a] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math., **173** (2003), no. 2, 179–261.

[OS03b] Peter Ozsváth and Zoltán Szabó, *Heegaard Floer homology and alternating knots*, Geom. Topol., **7** (2003), 225–254.

[OS03c] Peter Ozsváth and Zoltán Szabó, *Knot Floer homology and the four-ball genus*, Geom. Topol., **7** (2003), 615–639.

[OS04a] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and knot invariants*, Adv. Math., **186** (2004), no. 1, 58–116.

[OS04b] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2), **159** (2004), no. 3, 1027–1158.

[OS05] Peter Ozsváth and Zoltán Szabó, *On knot Floer homology and lens space surgeries*, Topology, **44** (2005), no. 6, 1281–1300.

[OS11] Peter S. Ozsváth and Zoltán Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol., **11** (2011), no. 1, 1–68.

[Ras03] Jacob Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, Harvard University, 2003.

[Shi71] Yaichi Shinohara, *On the signature of knots and links*, Trans. Amer. Math. Soc., **156** (1971), 273–285.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY
NEW YORK, NY 10027, USA
*E-mail address*: hom@math.columbia.edu

DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG
SHATIN, HONG KONG
*E-mail address*: ztwu@math.cuhk.edu.hk