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Relative quasimorphisms and stably unbounded norms on the group of symplectomorphisms of the Euclidean spaces

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In the paper where Burago-Ivanov-Polterovich defined the notion of conjugation-invariant norms on groups, they asked whether there exists a group with stably bounded commutator length admitting stably unbounded norms. We show that the kernel of the Calabi homomorphism of the group of symplectomorphisms of the even-dimensional Euclidean space with compact support is such a group. To prove its stable unboundedness, we consider quasimorphisms relative to a conjugation-invariant norm.

1. Introduction

Burago, Ivanov and Polterovich defined the notion of conjugation-invariant norms on groups in [BIP] and they gave a number of its applications.

Definition 1.1 ([BIP]). Let G be a group. A function $\nu\colon G \to \mathbb{R}$ is a conjugation-invariant norm on G if ν satisfies the following axioms:

(1) $\nu(1) = 0$;

(2) $\nu(f) = \nu(f^{-1})$ for every $f \in G$;

(3) $\nu(fg) \leq \nu(f) + \nu(g)$ for every $f, g \in G$;

(4) $\nu(f) = \nu(gfg^{-1})$ for every $f, g \in G$;

(5) $\nu(f) > 0$ for every $f \neq 1 \in G$.

A conjugation-invariant norm ν is said to be a stably unbounded norm if there exists $f \in G$ such that $\lim_{n\to\infty} \frac{\nu(f^n)}{n} > 0$.

The most important conjugation-invariant norm is the commutator length $\mathrm{cl}\colon G \to \mathbb{Z}$ on a perfect group. $\mathrm{cl}(h)$ is defined by

$$\mathrm{cl}(h) = \min\{k \mid \exists f_1, \ldots, f_k, g_1, \ldots, g_k \in G;\ h = [f_1, g_1], \ldots, [f_k, g_k]\}.$$

They asked the following question.

Problem 1.2 ([BIP] Open Problem). *Does there exist a perfect group G which satisfies the following conditions:*

(1) *The commutator length of G is stably bounded;*

(2) *G admits a stably unbounded conjugation-invariant norm?*

In this paper, we give an affirmative answer by showing that the kernel of the Calabi homomorphism $\mathrm{Cal}\colon \mathrm{Symp}_0^c(\mathbb{R}^{2n}, \omega_0) \to \mathbb{R}$ admits stably unbounded conjugation-invariant norms. $\mathrm{Symp}_0^c(\mathbb{R}^{2n}, \omega_0)$ denotes the identity component of the group of symplectomorphisms of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ with compact support. Let $\mathrm{Cal}\colon \mathrm{Symp}_0^c(\mathbb{R}^{2n}, \omega_0) \to \mathbb{R}$ be the Calabi homomorphism on $\mathrm{Symp}_0^c(\mathbb{R}^{2n}, \omega_0)$ [B].

Our main result is as follows.

Theorem 1.3. Ker(Cal) *admits a stably unbounded conjugation-invariant norm.*

Ker(Cal) is known to be simple [B] and thus perfect. The commutator length of Ker(Cal) is known to be stably bounded ([BIP] Example 1.23, [K] Theorem 4.2). Thus Theorem 1.3 gives an affirmative answer to Problem 1.2.

To show Theorem 1.3, we define relative quasimorphisms in Section 2. There we show that stable nontriviality of relative quasimorphisms implies unboundedness of norm controlled commutator length. This norm is a fragmentation norm.

A relationship between estimation of the conjugation-ivariant norm and the volume of the open subset used in the definition of fragmentation norm also appears in the paper of Le Roux ([LR]).

1.1. Acknowledgment

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2. Relative quasimorphisms

For $p, q \in \mathbb{R}_{>0} \cup \{\infty\}$ define the (ν, p, q)-commutator subgroup $[G, G]_{\nu,p,q}$ of G with a conjugation-invariant norm ν to be the subgroup generated by commutators $[f, g]$ such that $\nu(f) \leq p, \nu(g) \leq q$.

Definition 2.1. Let G be a group. Let ν be a conjugation-invariant norm on G and $p, q \in \mathbb{R}_{>0} \cup \{\infty\}$. We define the (ν, p, q)-*commutator length* $\mathrm{cl}_{\nu,p,q} : [G, G]_{\nu,p,q} \to \mathbb{R}$ by

$$\mathrm{cl}_{\nu,p,q} = \min\{k \mid \exists f_1, \dots, f_k, g_1, \dots, g_k; \\ \nu(f_i) \leq p, \nu(g_j) \leq q \, (i, j = 1, \dots, k);\, h = [f_1, g_1] \cdots [f_k, g_k]\}.$$

Proposition 2.2. $\mathrm{cl}_{\nu,p,q}$ *is a conjugation-invariant norm on* $[G, G]_{\nu,p,q}$.

Proof. By the definition of $[G, G]_{\nu,p,q}$, we have $\mathrm{cl}_{\nu,p,q}(f) < \infty$ for every $f \in [G, G]_{\nu,p,q}$.

Since

$$[f, g]^{-1} = gfg^{-1}f^{-1} = g(fg^{-1}f^{-1}g)g^{-1} = g[f, g^{-1}]g^{-1},$$

$\mathrm{cl}_{\nu,p,q}$ satisfies the axiom (2) of Definition 1.1. Other axioms are satisfied obviously. □

A similar idea to (ν, p, q)-commutator length appears in the paper of Tsuboi ([T] Lemma 3.1).

To estimate the (ν, p, q)-commutator length, we consider relative quasimorphisms.

Definition 2.3. Let G be a group with a conjugation-invariant norm ν. A function $\mu\colon G \to \mathbb{R}$ is called a *quasimorphism relative to the norm* ν if there exists a positive number C such that for every f and $g \in G$

$$|\mu(fg) - \mu(f) - \mu(g)| < C \min\{\nu(f), \nu(g)\}.$$

The idea of relative quasimorphisms already appeared in [EP06], [FOOO], [MVZ], etc.

Proposition 2.4. *For a quasimorphism μ relative to ν and $p, q \in \mathbb{R}_{>0}$, if there exists $h_0 \in [G,G]_{\nu,p,q}$ such that $\lim_{k\to\infty} \frac{\mu(h_0^k)}{k} > 0$, the conjugation-invariant norm $\mathrm{cl}_{\nu,p,q}$ on $[G,G]_{\nu,p,q}$ is stably unbounded.*

Proof. For f and $g \in G$ that satisfy $\nu(f) \leq p$ and $\nu(g) \leq q$, respectively, we have

$$\begin{aligned}
&|\mu([f,g])| = |\mu(fgf^{-1}g^{-1}))| \\
&\leq |\mu(f) + \mu(gf^{-1}g^{-1})| + C\nu(f) \leq \cdots \\
&\leq |\mu(f) + \mu(g) + \mu(f^{-1}) + \mu(g^{-1})| + C\nu(f) + C\nu(g) + C\nu(f^{-1}) \\
&\leq |\mu(f) + \mu(f^{-1})| + |\mu(g) + \mu(g^{-1})| + C\nu(f) + C\nu(g) + C\nu(f^{-1}) \\
&\leq (|\mu(1)| + C\nu(f)) + (|\mu(1)| + C\nu(g)) + C\nu(f) + C\nu(g) + C\nu(f^{-1}) \\
&\leq (3p + 2q)C + 2|\mu(1)|,
\end{aligned}$$

and

$$\begin{aligned}
&\nu([f,g]) = \nu(fgf^{-1}g^{-1})) \\
&\leq \nu(f) + \nu(gf^{-1}g^{-1}) \\
&\leq 2p.
\end{aligned}$$

Thus for any $h = [f_1, g_1] \cdots [f_k, g_k] \in [G,G]_{\nu,p,q}$,

$$\begin{aligned}
\mu(h) &\leq \sum_{i=1}^{k} |\mu([f_i, g_i])| + \sum_{j=1}^{k-1} \nu([f_j, g_j]) \\
&\leq ((3p + 2q)C + 2|\mu(1)|)k + 2p(k-1) \\
&= ((3p + 2q)C + 2|\mu(1)| + 2p)k - 2p,
\end{aligned}$$

and hence

$$\mu(h) \leq ((3p + 2q)C + 2|\mu(1)| + 2p)\mathrm{cl}_{\nu,p,q}(h) - 2p.$$

By dividing $\mu(h^k)$ by k and passing to the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{\mu(h^k)}{k} \leq ((3p+2q)C + 2|\mu(1)| + 2p) \lim_{k\to\infty} \frac{\mathrm{cl}_{\nu,p,q}(h^k)}{k}.$$

In particular,

$$0 < ((3p+2q)C + 2|\mu(1)| + 2p)^{-1} \lim_{k\to\infty} \frac{\mu(h_0^k)}{k} \leq \lim_{k\to\infty} \frac{\mathrm{cl}_{\nu,p,q}(h_0^k)}{k}.$$

□

3. Construction of relative quasimorphisms

Fix a non-empty open set U of $\mathbb{R}^{2n}$, we denote by $\mathrm{Symp}^c_{0,U}(\mathbb{R}^{2n}, \omega_0)$ the set of elements of $\mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$ with support in U. By the fragmentation lemma [B], any element ϕ of $\mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$ can be written as a product of elements of the form $\psi\theta\psi^{-1}$ with $\theta \in \mathrm{Symp}^c_{0,U}(\mathbb{R}^{2n}, \omega_0)$, $\psi \in \mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$.

Let $\nu_U(\phi)$ be the minimal number of factors in such a product. This is called *the fragmentation norm relative to* U [BIP]. ν_U is a conjugation-invariant norm.

We are going to show that the Maslov quasimorphism $\tilde{\beta}$ on $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ induces a quasimorphism $\mathcal{B}$ relative to ν_U.

For $g \in \mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$, choose an isotopy $(g^t)_{t\in[0,1]}$ in $\mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$ between $g^0 = 1$ and $g^1 = g$. For each point x in $\mathbb{R}^{2n}$, the differential is given as a $2n \times 2n$ matrix $A(x, g^t)$ belonging to the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. Thus the path $(A(x, g^t))_{t\in[0,1]}$ on $\mathrm{Sp}(2n, \mathbb{R})$ represents an element of the universal covering $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$. Let $\tilde{\beta}\colon \widetilde{\mathrm{Sp}}(2n, \mathbb{R}) \to \mathbb{R}$ be the Maslov quasimorphism [BG], [C], [EP09]. Berge and Ghys showed that $\tilde{\beta}((A(x, g^t))_{t\in[0,1]})$ does not depend on the choice of an isotopy $(g_t)_{t\in[0,1]}$ [BG], and we denote $\tilde{\beta}([(A(x, g^t))_{t\in[0,1]}])$ by $\beta(g, x)$.

Lemma 3.1 ([BG]). *There exists a constant $C > 0$ such that for every $x \in \mathbb{R}^{2n}$ and $f, g \in \mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$*

$$|\beta(fg, x) - \beta(f, g(x)) - \beta(g, x)| < C.$$

Proof. For the representatives $\{f^t\}_{t\in[0,1]}$ and $\{g^t\}_{t\in[0,1]}$ of f and $g \in \mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$. $fg \in \mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$ is represented by the concatenation of g^t and $f^t g$. Since $\tilde{\beta}$ is a quasimorphism on $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$, the lemma follows. □

We define the function $\mathcal{B}\colon \mathrm{Symp}_0^c(\mathbb{R}^{2n},\omega_0)\to\mathbb{R}$ by

$$\mathcal{B}(g)=\int_{x\in\mathbb{R}^{2n}}\beta(g,x)\omega_0^n.$$

Note that since $\beta(g,x)=0$ for $x\notin\bigcup_{t\in[0,1]}(g^t)^{-1}(\mathrm{supp}(g^t))$, $\mathcal{B}(g)<\infty$ for every $g\in\mathrm{Symp}_0^c(\mathbb{R}^{2n},\omega_0)$.

Berge and Ghys showed that $\mathcal{B}$ is stably non-trivial.

Theorem 3.2 ([BG] Corollaire 4.5). *Define the function* $\bar{\mathcal{B}}\colon \mathrm{Symp}_0^c(\mathbb{R}^{2n},\omega_0)\to\mathbb{R}$ *by*

$$\bar{\mathcal{B}}(g)=\lim_{k\to\infty}\frac{\mathcal{B}(g^k)}{k}.$$

Then there exists $g_0\in\mathrm{Ker}(\mathrm{Cal})$ *such that* $\bar{\mathcal{B}}(g_0)>0$.

The following proposition completes the proof of Theorem 1.3.

Proposition 3.3. *Let* U *be a non-trivial bounded open subset of* $\mathbb{R}^{2n}$ *and* ν_U *be the fragmentation norm relative to* U. $\mathcal{B}$ *is a quasimorphism relative to* ν_U.

Proof. Take $f,g\in\mathrm{Symp}_0^c(\mathbb{R}^{2n},\omega_0)$. Since g preserves ω_0,

$$\begin{aligned}
&|\mathcal{B}(fg)-\mathcal{B}(f)-\mathcal{B}(g)|\\
&=\left|\int_{\mathbb{R}^{2n}}\beta(fg,x)\omega_0^n-\int_{\mathbb{R}^{2n}}\beta(f,x)\omega_0^n-\int_{\mathbb{R}^{2n}}\beta(g,x)\omega_0^n\right|\\
&=\left|\int_{\mathbb{R}^{2n}}(\beta(fg,x)-\beta(f,g(x))-\beta(g,x))\omega_0^n\right|.
\end{aligned}$$

Let f^t and g^t be isotopies on $\mathrm{Symp}_0^c(\mathbb{R}^{2n},\omega_0)$ such that $f^0=\mathrm{id}$, $f^1=f$ and $g^0=\mathrm{id}$, $g^1=g$. Let $(f\sharp g)^t$ be the concatenation of g^t and f^t.

If $x\notin\bigcup_{t\in[0,1]}(\mathrm{supp}(g^t))$, then $(A(x,(f\sharp g)^t))_{t\in[0,1]}\simeq(A(x,f^t)_{t\in[0,1]})$ and $A(x,g^t)=\mathrm{id}$.

If $x\notin\bigcup_{t\in[0,1]}g^{-1}(\mathrm{supp}(f^t))$, then $(A(x,(f\sharp g)^t))_{t\in[0,1]}\simeq(A(x,g^t)_{t\in[0,1]})$ and $A(g(x),f^t)=\mathrm{id}$.

Thus for every $x \notin \bigcup_{t\in[0,1]}(\mathrm{supp}(g^t)) \cap \bigcup_{t\in[0,1]} g^{-1}(\mathrm{supp}(f^t))$,

$$\begin{aligned}&\beta(fg,x) - \beta(f,g(x)) - \beta(g,x)\\ &= \beta(((A(x,(f\sharp g)^t))_{t\in[0,1]})) - \beta(((A(x,f^t))_{t\in[0,1]})) - \beta(((A(x,g^t))_{t\in[0,1]}))\\ &= 0.\end{aligned}$$

By the definition of the fragmentation norm, we can choose $\{f^t\}$ and $\{g^t\}$ satisfying

$$\mathrm{Vol}\left(\bigcup_{t\in[0,1]}(\mathrm{supp}(g^t)), \omega_0^n\right) \le \nu_U(g)\cdot \mathrm{Vol}(U,\omega_0^n), \quad \text{and}$$
$$\mathrm{Vol}\left(\bigcup_{t\in[0,1]} g^{-1}(\mathrm{supp}(f^t)), \omega_0^n\right) \le \nu_U(f)\cdot \mathrm{Vol}(U,\omega_0^n).$$

Thus by using Lemma 3.1 we obtain the following inquality.

$$\begin{aligned}&\left|\int_{\mathbb{R}^{2n}}(\beta(fg,x) - \beta(f,g(x)) - \beta(g,x))\omega_0^n\right|\\ &\le C\,\mathrm{Vol}(U,\omega_0^n)\,\min\{\nu_U(f),\nu_U(g)\}.\end{aligned}$$

□

Proof of Theorem 1.3. Take $p \in \mathbb{R}_{\ge 1}$ and $q \in \mathbb{R}_{>0}$, a non-trivial bounded open subset U of $\mathbb{R}^{2n}$. Denote ν_U by ν. Since $p \ge 1$, $[\mathrm{Ker(Cal)}, \mathrm{Ker(Cal)}]_{\nu,p,q}$ is a non-trivial subgroup of Ker(Cal). Since Ker(Cal) is simple [B], $[\mathrm{Ker(Cal)}, \mathrm{Ker(Cal)}]_{\nu,p,q} = \mathrm{Ker(Cal)}$. Thus $\mathrm{cl}_{\nu,p,q}$ is defined as a conjugation-invariant norm on Ker(Cal).

$\mathcal{B}$ on Proposition 3.3 is a quasimorphism relative to ν. On the other hand, Theorem 3.2 implies that there exists an element in Ker(Cal) for which $\bar{\mathcal{B}}$ is non zero. Thus Proposition 2.4 implies that $\mathrm{cl}_{\nu,p,q}$ is stably unbounded. □

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