JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 14, Number 1, 269–295, 2016

Exact Lagrangian caps of Legendrian knots

Francesco Lin

We prove that any Legendrian knot in (S^3, ξ_{std}) bounds an exact Lagrangian surface in $\mathbb{R}^4 \setminus B^4$ after a sufficient number of stabilizations. In order to do this, we define Lagrangian projections, consisting of a knot projection along with some additional information, and construct a family of combinatorial moves which correspond to Lagrangian cobordisms between knots.

Introduction

Given a Legendrian knot K in the contact manifold (S^3, ξ_{std}), it is natural to ask whether or not it bounds an oriented Lagrangian surface L in its standard (convex) filling $(B^4, \omega_{\text{std}})$. In [1] it is shown that the existence of such a surface (which we call a *Lagrangian filling* of K) implies the identities

$$\begin{aligned} \text{tb}(K) &= -\chi(L) \\ \text{rot}(K) &= 0 \end{aligned}$$

for the Thurston-Bennequin number $\text{tb}(K)$ and the rotation number $\text{rot}(K)$ of the Legendrian knot. From the slice Thurston-Bennequin inequality ([12])

$$\text{tb}(K) + |\text{rot}(K)| \leq 2g_4(K) - 1,$$

where $g_4(K)$ is the 4-ball genus of K, it then follows that such a surface L realizes the 4-ball genus of K and K has maximal Thurston-Bennequin invariant within its topological class. Hence there are very restrictive limitations on the existence of such surfaces, even if we do not impose any exactness condition.

The same technique tells us much less about *Lagrangian caps* of K, i.e. Lagrangian surfaces L properly embedded in $\mathbb{R}^4 \setminus B^4$ bounding the given knot. This is because the constraints on the classical invariants in this case

are

$$\begin{aligned} \mathrm{tb}(K) &= \chi(L) \\ \mathrm{rot}(K) &= 0, \end{aligned}$$

and by applying the slice Thurston-Bennequin inequality again one obtains the much less restrictive conditions

$$\begin{aligned} g(L) &\geq 1 \\ \mathrm{tb}(K) &\leq -1. \end{aligned}$$

Before we state the main result of the present paper, we briefly recall some terminology. A Lagrangian L with Legendrian boundary K is *exact* if the pull-back of the primitive of the symplectic form λ_{std} on L is of the form df for some function $f : L \to \mathbb{R}$ with $f(K) = 0$. This is equivalent to say that $\lambda_{\mathrm{std}}|_L$ has integral zero along every closed curve on L and every arc with boundary on K. Moreover, recall that a *stabilization* of a Legendrian knot K is a Legendrian knot obtained by performing one of the local moves of Figure 1 in the front projection. The Legendrian isotopy type of the stabilized knot only depends on the effect of the stabilization on the rotation number (which is ± 1).

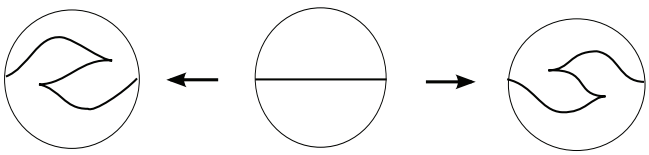

Figure 1: Stabilization of a Legendrian knot.

The following is the main result we will prove.

Theorem 1. *Given any Legendrian knot K in $(S^3, \xi_{\mathrm{std}})$, one can stabilize it a sufficient number of times so that the final result admits an oriented exact Lagrangian cap. More precisely, for every Legendrian knot K there is an $n(K)$ such that a knot K' obtained by stabilizing K more that $n(K)$ times and such that $\mathrm{rot}(K') = 0$ admits an exact Lagrangian cap.*

Remark 1. The higher dimensional analogue of the problem is governed by an h-principle, see [6]. In [2], the existence of such caps is used to show the

existence of *non orientable* closed embedded exact Lagrangians in $\mathbb{C}^2$ which are not uniruled and have infinite Gromov width. We have learned from [11] that our theorem implies the existence of exact Lagrangian surfaces in $(\mathbb{R}^4, \omega_{\text{std}})$ with open Whitney umbrella singularities, disproving a conjecture proposed by Givental in [9].

Here, and in the rest of the paper, we will work in the equivalent setting (for our problem) of $(\mathbb{R}^3, \xi_{\text{std}})$ where $\mathbb{R}^3$ has coordinates (x, y, z), $\xi_{\text{std}} = \ker(\alpha_{\text{std}})$ with $\alpha_{\text{std}} = dz + xdy$. With this convention the front projection is the projection on the yz-plane while the Lagrangian projection is the projection to the xy-plane, rotated clockwise by 90 degrees. We are interested in searching Lagrangian surfaces in the symplectization $(\mathbb{R}^{\geq 0}_t \times \mathbb{R}^3, d(e^t \alpha_{\text{std}}))$ where the first factor is

$$\mathbb{R}^{\geq 0}_t = \{t \in \mathbb{R} | t \geq 0\}.$$

On the other hand, it will be more convenient for us to work with surfaces in

$$\mathbb{R}^4_+ = \{(x', y', t', z') | t' \geq 0\} \subset (\mathbb{R}^4, \omega_{\text{std}})$$
$$\omega_{\text{std}} = dx' \wedge dy' + dt' \wedge dz',$$

with boundary condition on the hyperplane $\{t' = 0\}$. In this case, our preferred primitive for the symplectic form will be the one form

$$\lambda_{\text{std}} = x'dy' + (t' + 1)dz'.$$

These models are indeed equivalent for our purposes via the symplectomorphism

$$\Phi : \mathbb{R}^{\geq 0}_t \times \mathbb{R}^3 \to \mathbb{R}^4_+$$
$$(x, y, t, z) \mapsto (e^t x, y, e^t - 1, z),$$

which is a strict contactomorphism on the boundary and preserves the fixed primitives of the symplectic forms. We will always work with the latter, and drop the apostrophes in the notation.

In order to prove Theorem 1, we study the set of *Lagrangian diagrams* of a surface $L \subset \mathbb{R}^4_+$. A Lagrangian diagram is simply the Lagrangian projection of a generic slice on which the t-coordinate is constant together with the assignment to each component of the complement of the knot projection of

its volume with respect to the standard area form $dx \wedge dy$. We will introduce a set of combinatorial moves on such diagrams (up to some identifications) and prove that they can always be realized by *Lagrangian cobordisms*, i.e. properly embedded Lagrangians in $\mathbb{R}^3 \times [0,T]$ such that the restrictions to $\mathbb{R}^3 \times \{0\}$ and $\mathbb{R}^3 \times \{T\}$ are the given knots. These constructions are inspired by the techniques developed in [13] to show the existence of immersed exact Lagrangians in $\mathbb{C}^2$ with few self-intersections.

With this in hand, the strategy to prove Theorem 1 is the following. First, we prove that after a sufficient number of stabilizations each knot is exactly cobordant to the Legendrian unknot in Figure 2, which we denote by U_0. Then we will use the combinatorial moves to explicitly construct an exact Lagrangian cap of U_0. Also, we prove that if a knot K admits a cap (hence $\mathrm{rot}(K) = 0$) then also its double stabilization with rotation number zero admits a cap, justifying the "sufficiently many stabilization" statement in Theorem 1.

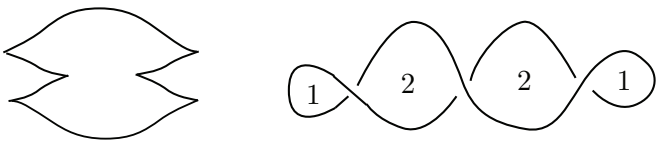

Figure 2: A front projection and a Lagrangian diagram of the knot U_0.

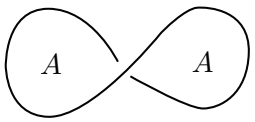

Figure 3: A Lagrangian diagram of U_1.

It is interesting to point out that while in the first part we can restrict ourselves to Lagrangians such that the generic slice where the t coordinate is constant is Legendrian, in order to construct a cap we necessarily have to rely on a wider class of Lagrangians. In fact the slices with t constant just before the cap closes are not Legendrian knots, but have a Lagrangian

diagram U_1 as in Figure 3. This is not the Lagrangian projection of a Legendrian knot because the crossing is the opposite of the one of the standard Legendrian unknot.

Plan of the paper. In Section 1 we show how to reduce the problem of constructing a cap to the single Legendrian knot U_0. In Section 2 we define Lagrangian diagrams and construct the combinatorial moves on them corresponding to Lagrangian cobordisms. In Section 3 we prove an exactness criterion for the cobordism in terms of the Lagrangian diagrams from which it arises. Finally in Section 4 we explicitly construct an exact Lagrangian cap for the knot U_0. In Section 5 we discuss some interesting problems that arise from the combinatorial methods investigated in this paper.

Acknowledgments. The author would like express his gratitude to Emmy Murphy for suggesting the problem and for all the very helpful and interesting discussions throughout all the time the present paper has been developed. He would also like to thank Kevin Sackel and the anonymous referee for the useful comments and corrections, the American Institute of Mathematics for organizing the conference *Contact topology in higher dimensions*, and Lisa Traynor in particular. The author was partially supported by NSF grant DMS-0805841.

1. Reduction to U_0

In this section we prove the following result.

Proposition 1. *Given any Legendrian knot K, there is a Legendrian knot K' obtained from K via stabilizations such that there exists an exact oriented Lagrangian cobordism from K' to the Legendrian unknot U_0 of Figure* 2.

We will work only using front projections and rely on the moves depicted in Figure 4, which appear for example in [3].

The moves R_1, R_2 and R_3 are the Legendrian Reidemeister moves. The fact that they can be realized by an exact Lagrangian cobordism follows from a result by Eliashberg and Gromov ([5]) stating that every Legendrian isotopy can be realized in this way. The handle attachment move H can be realized starting from Lagrangians in the first jet space $J^1\mathbb{R}^2$ by means of the following more general construction.

Any front projection (with coordinates (z, y_1, y_2)) with boundary in $y_2 = \{a, b\}$ such that:

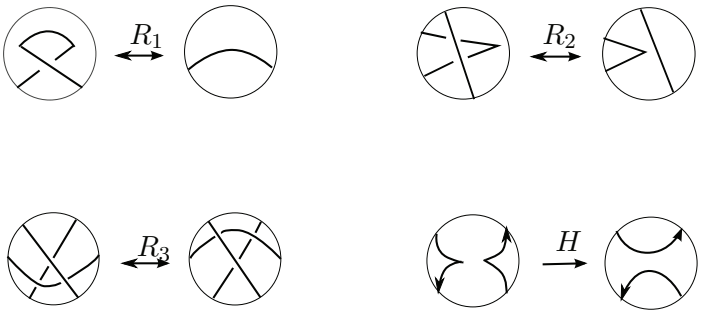

Figure 4: Moves on front projections.

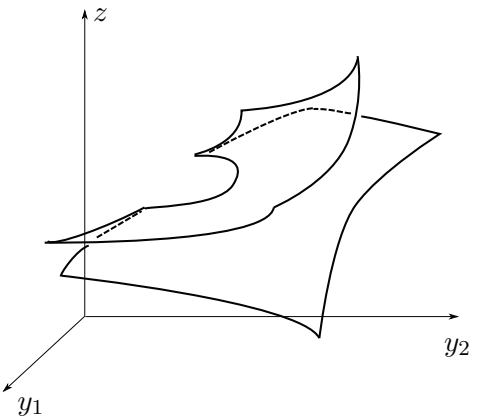

Figure 5: A front projection giving rise to the handle attachment move H.

- there are no points with the same y_i coordinates and parallel tangent planes at those points;
- the length of the Reeb chords in the slices with y_2 constant is increasing at the boundary;

corresponds to an exact embedded Lagrangian cobordism between the Legendrian knots arising as the slices $y_2 = \{a, b\}$. In particular, the front projection in Figure 5 corresponds to the move H.

Remark 2. Notice that in both constructions one does not have any control on the time interval in which the cobordism is defined. For example, in the jet space construction a non-trivial change of coordinates (depending on the front projection itself) is made in order to encode in a simple way the Legendrian boundary condition.

We are now ready to describe how to construct a Lagrangian cobordism to U_0.

Proof of Proposition 1. In our description we will add stabilizations along the way, but it will be clear from the construction that all of them can be performed at the beginning without affecting the final result. The proof is divided in various steps.

Step 1. We show that up to stabilizations, every Legendrian knot is cobordant to one such that every component is an unknot and in the front projection each component is contained in a disk disjoint from the other ones. To do so, we perform a sequence of local moves near each crossing. There are essentially two cases up to orientation reversing (see Figure 6). In the first case one adds two stabilizations, uses the move H and then R_1

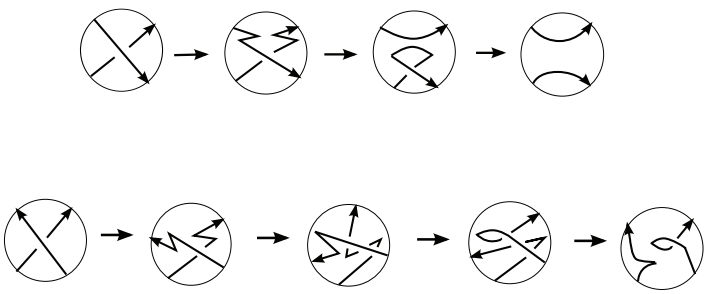

Figure 6: The two cases in Step 1.

to resolve the crossing. In the second case one adds two stabilizations and then performs an R_2 move, a handle attachment and another R_2 move to simplify the crossing. It is clear that after performing these moves to every crossing one obtains a Legendrian link as the claimed one.

Step 2. We show that we can reduce to a standard unknot $K(n, m)$ as in Figure 7. Here m is number of descending cusps pointing to the right,

and n is the number of ascending cusps pointing to the left (for example, the knot in the figure is $K(1,2)$). In fact, it is a classical consequence (see

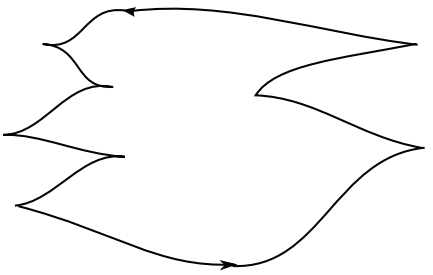

Figure 7: A standard form for Legendrian unknots.

for example [8]) of the Thurston-Bennequin inequality and the Eliashberg-Fraser Theorem ([4]) that every Legendrian unknot is Legendrian isotopic to some $K(n,m)$ (here in order to construct this isotopy one might need to use some Legendrian R_3 moves). As in our front projection each component is cointained in a disk disjoint from the other ones, we can put every component of our link in such a form. Using then an H move we can connect sum them along the cusps as in Figure 8 (notice that the orientations match) in order to obtain a single unknot. The final result can also be isotoped to the standard model.

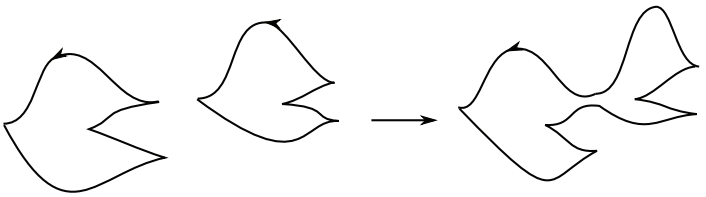

Figure 8: Connected sum of $K(m,n)$ knots along cusps.

Step 3. We construct a cobordism to U_0. After adding some stabilizations, we can suppose our knot is a $K(n,n)$ with $n \geq 1$. Our goal is to

construct a cobordism to $U_0 = K(1,1)$, and the proof will follow by induction if we prove that there is a cobordism from $K(n,n)$ to $K(n-1,n-1)$. The procedure is shown in Figure 9. We can perform an H move on the top pair of cusps pointing inward, obtaining a $K(0,0)$ and a $K(n-1,n-1)$. Then, we can connect sum these two knots along the cusps, and obtain a $K(n-1,n-1)$. □

The same idea of Step 3 (and in particular the sequence of moves depicted in Figure 9) can be exploited to show the following easy lemma.

Lemma 1. *Given a Legendrian knot K_0, there is an exact Lagrangian cobordism from its double stabilization with the same rotation number to itself.*

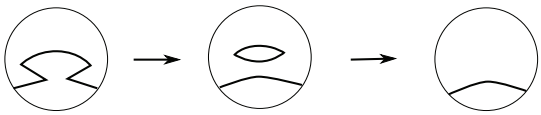

Figure 9: A cobordism from the double stabilization.

2. The combinatorial moves

As observed in the introduction, in order to construct caps we necessarily have to leave the Legendrian world and allow more general Lagrangian cobordisms in which the slices are not necessarily Legendrian knots. In this section we show how to construct these, and we first start defining the class of combinatorial objects we will be dealing with.

Definition 1. A *Lagrangian diagram* D is a regular projection of a smooth link $K \subset \mathbb{R}^3$ to the xy-plane, and the assignment to each domain of the complement of the projection of its area (with respect to the standard area form).

For example, Figure 2 (on the right) and Figure 3 are Lagrangian diagrams of an unknot. It is important to note that the diagram D has to be the projection of some link $K \subset \mathbb{R}^3$. In order to develop a combinatorial model, we introduce an identification between Lagrangian diagrams as follows.

Definition 2. Given a Lagrangian diagram D, define $\Delta(D)$ to be the set of positive real numbers arising as the collection consisting of all areas appearing in the diagram and all absolute values of differences between distinct areas. We say that another Lagrangian diagram D' is *commensurate* to D if the following conditions hold:

- the two projections are isotopic *as knot diagrams in* $\mathbb{R}^2$;
- the difference between the two areas assigned to a same domain under the bijection induced by the isotopy is strictly smaller than any element in $\Delta(D) \cup \Delta(D')$.

It is important to remark that being commensurate is *not* an equivalence relation on Lagrangian diagrams.

Notation. We can extend Definition 1 allowing for a domain to be assigned the number 0, which means that its area is much smaller than every non-zero area and every absolute value of the difference of two distinct areas.

For example, the Lagrangian diagram of Figure 2 is commensurate to the one on the left in Figure 10, but not to the one on the right. For the diagram on the left $\Delta = \{1, 1.5, 2.5\}$ while for the diagram on the right $\Delta = \{1, 2\}$.

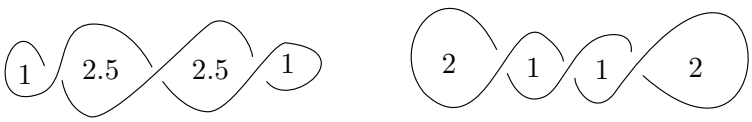

Figure 10: Non equivalent Lagrangian diagrams.

We now state the main result of the section describing some combinatorial moves on Lagrangian diagrams that correspond to Lagrangian cobordisms between the respective knots.

Proposition 2. *Any sequence of local moves as in Figure* 11 *between Lagrangian diagrams* D_0 *and* D_1 *with* $A, \delta > 0$ *can be realized by an orientable Lagrangian cobordism in the following sense. Suppose we have a link* K_0 *with Lagrangian diagram* D_0*. Then there exists a link* K_1 *with Lagrangian diagram commensurate to* D_1 *and an orientable Lagrangian* L *properly embedded in* $\mathbb{R}^3 \times [0, \varepsilon] \subset (\mathbb{R}^4_+, \omega_{\text{std}})$ *for some* $\varepsilon > 0$ *such that* $L \cap \{t = 0\}$ *coincides with* $K_0 \subset \mathbb{R}^3$ *and* $L \cap \{t = \varepsilon\}$ *coincides with* $K_1 \subset \mathbb{R}^3$*.*

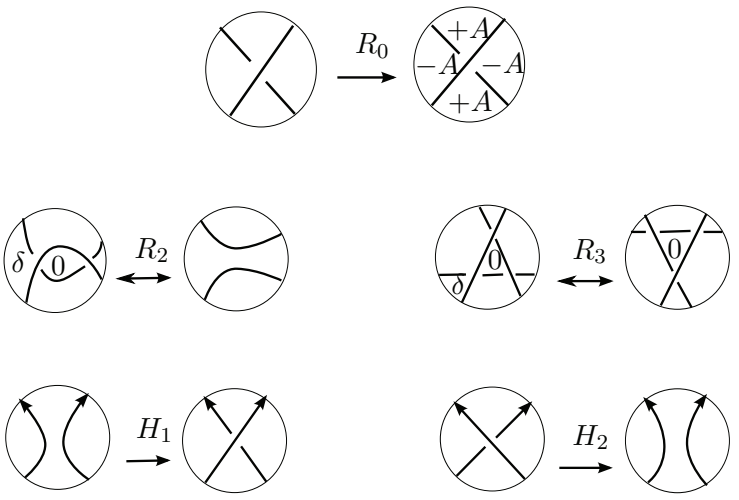

Figure 11: Combinatorial moves on Lagrangian diagrams.

We spell out the meaning of Figure 11 in more detail. If a domain is not assigned any number then its area remains unchanged when the move is performed. On the other hand, it will be clear from the construction that the actual areas of the domains in the projection do change, but this change can be made arbitrarily small (hence yielding a knot with a Lagrangian diagram commensurate to the one we want to obtain).

The move R_0 (defined for $A > 0$) is allowed only if the areas of the domains with label $-A$ are bigger than A. The $\delta > 0$ in the diagrams for R_2 and R_3 indicates the necessity of the indicated domain to have area strictly bigger that the one labeled with 0 (and the move can be performed in both cases also in the case that the symmetric domain is labeled with a δ). The moves R_2 and R_3 work with any choices of the crossings and are reversible (as the double arrows suggest). Conversely, the remaining moves can be performed only with the crossings shown in Figure 11, and are not reversible. Notice that in the moves H_1 and H_2 we deal with oriented diagrams. This is done in order to construct orientable cobordisms.

The rest of the section is dedicated to the construction of the Lagrangian cobordisms corresponding to the combinatorial moves described in Proposition 2. We first construct a Lagrangian cobordism L_0 defined for some small

$\varepsilon > 0$ which is topologically a product

$$L_0 \cong K \times [0, \varepsilon] \subset \mathbb{R}^3 \times [0, \varepsilon],$$

such that the Lagrangian diagram of $K \times \{\varepsilon\}$ is commensurate to the one of $K \times \{0\} = K_0$. W then modify such a cobordism in a small neighborhood of the crossing involved in the local move. The whole construction is based on the following key lemma which appears already in [13] and describes a way to construct Lagrangian cobordisms starting from a family of immersed loops in the plane.

Lemma 2. *Suppose we are given a smooth map*

$$\varphi(t, \vartheta) = (x(t, \vartheta), y(t, \vartheta)) : [0, T] \times S^1 \to \mathbb{R}^2$$

such that $\varphi(t, -) : S^1 \to \mathbb{R}^2$ is an immersion with total signed area fixed with respect to t. Then there exists $z : [0, T] \times S^1 \to \mathbb{R}$ such that the map

$$\begin{gathered} \tilde{\varphi} : [0, T] \times S^1 \to \mathbb{R}^4_+ \\ (t, \vartheta) \mapsto (x(t, \vartheta), y(t, \vartheta), t, z(t, \vartheta)) \end{gathered}$$

is a Lagrangian immersion. Furthermore, the map $\tilde{\varphi}$ is an embedding if and only if whenever $\varphi(t, \vartheta) = \varphi(t, \vartheta')$, $z(t, \vartheta) \neq z(t, \vartheta')$.

Proof. By imposing $\mathrm{Im}\tilde{\varphi}$ to be isotropic we obtain

$$0 = \tilde{\varphi}^* \omega(\partial/\partial t, \partial/\partial \vartheta) = \frac{\partial x}{\partial t}\frac{\partial y}{\partial \vartheta} - \frac{\partial x}{\partial \vartheta}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial \vartheta}.$$

Setting

$$J(t, \vartheta) = -\frac{\partial x}{\partial t}\frac{\partial y}{\partial \vartheta} + \frac{\partial x}{\partial \vartheta}\frac{\partial y}{\partial t},$$

if for every t one has

$$\int_{S^1} J(s, \vartheta) d\vartheta = 0$$

then one can define a map $\tilde{\varphi}$ with isotropic image by setting

$$z(t, \vartheta) = z(t, \vartheta_0) + \int_{\vartheta_0}^{\vartheta} J(t, \vartheta) d\vartheta$$

where $\vartheta_0 \in S^1$ is a basepoint and $z(-, \vartheta_0)$ is any fixed function. On the other hand if the total signed area of the curves is C then by Stokes' theorem we

have

$$0 = \frac{d}{dt}C = \frac{d}{dt}\int_{S^1}\varphi_t^*(xdy) = \int_{S^1}\mathcal{L}_{\partial/\partial t}\varphi_t^*(xdy) =$$
$$= \int_{S^1} d(\varphi_t^* i_{\partial/\partial t} xdy) + \int_{S^1} i_{\partial/\partial t} dx \wedge dy = \int_{S^1} \frac{\partial x}{\partial t}\frac{\partial y}{\partial \vartheta} - \frac{\partial x}{\partial \vartheta}\frac{\partial y}{\partial t} d\vartheta.$$

From the expression of $d\tilde{\varphi}$ we see that if each curve φ_t is immersed then $\tilde{\varphi}$ will be an immersion. Finally, the embeddedness criterion is straightforward. □

Proof of Proposition 2. As previously discussed, we first construct a Lagrangian cobordism L_0 between the given knot K_0 and a new knot with Lagrangian diagram commensurate to D_0. This cobordism will be defined for some small $\varepsilon > 0$ and will be topologically a product. The (parametrized) knot K_0 comes with the function

$$J_0(\vartheta) = dz/d\vartheta.$$

One can find a family of immersions $\varphi : [0,1] \times S^1 \to \mathbb{R}^2$ with constant signed area such that

- the restriction $\varphi(0,-)$ is the parametrization of K_0;
- for the function J defined in Lemma 2, $J(0,-) = J_0$.

Notice that the function J can be interpreted as the scalar product between $\partial\varphi/\partial\vartheta$ and the $\pi/2$ clockwise rotation of $\partial\varphi/\partial s$ so determines only the component of $\partial\varphi/\partial t$ normal to the curve, hence one can actually arrange the total change of area to be zero.

After restricting to a smaller time interval $[0,\varepsilon]$, such a map induces via the construction of Lemma 2 an embedded Lagrangian cobordism with the property that all the slices with s constant have Lagrangian diagram commensurate to D_0. In order to prove Proposition 2, we modify this family of immersions φ in a neighborhood of the crossing where each local move is performed as follows.

Move R_0. This move can be achieved using a family of immersions ψ as in Figure 12.

We spell out the meaning of the picture. It refers to a neighborhood of the crossing where the areas of the domains on which the move decreases the area is bigger than A. In agreement with our convention, we schematically represent the map φ we start with as the constant diagram on the left of the picture. We then modify such a map to the new family $\psi = \{\psi_t\}$ shown in

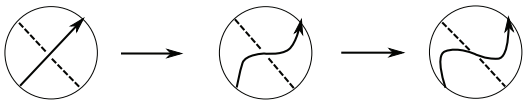

Figure 12: Constructing the move R_0.

the picture. Such a family agrees with φ_t near the boundary of the neighborhood for $t \in [0, \varepsilon]$ and for $t \geq 0$ small in the whole neighborhood. It is clear that such a family ψ_t can be chosen so that the total area remains constant. Furthermore, we can arrange that the family ψ still defines an embedded Lagrangian cobordism. This is because our family ψ can be chosen so that it respects the previous requirements and in addition the integral of its J function along the solid arc before the crossing is bigger for every t than the corresponding integral for φ. This implies that at the crossing the z coordinate of the new induced knot will be bigger than the starting one, hence in particular that the cobordism is embedded. Here we use in a fundamental way that the crossing is the one depicted in the picture and not the opposite one. Finally, it is clear that the family can also be chosen so that the area of the domains changes by A. Notice that the specular construction of the similar move with $A < 0$ will in general give rise to a Lagrangian with a self-intersection.

Move R_2. The construction is achieved by the family of immersions depicted in Figure 13 (here the orientations are introduced only to describe the family). In particular, the part of the arcs before the first crossing is pulled to the left, while a neighborhood of the arcs connecting the two crossing points are pulled down (for the lower arc) and up (for the upper arc). Such a family is chosen as before so that the total area remains constant and coincides with φ_t near the boundary of the neighborhood for $t \in [0, \varepsilon]$ and for $t \geq 0$ small in the whole neighborhood. We are assured that such a family exists because the area enclosed by the crossing is much smaller that the area of the domain towards which we are pulling the arcs. The embeddedness of the induced cobordism can be arranged exactly as in the construction of move the R_0. The inverse move is constructed using an analogous family.

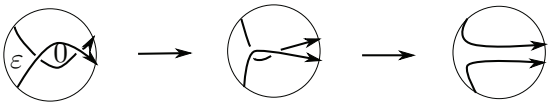

Figure 13: Constructing the move R_2.

Move R_3. The construction of this move is totally analogous to the previous ones.

Move $H1$. The handle attachment moves cannot be achieved by the same construction, as the topology of the cobordism changes. In order to construct them, we can rely on the local model shown in Figure 14 provided by a neighborhood of the Lagrangian submanifold

$$\{(x, y, t, z) \mid t = z^2 - y^2, x = 2yz\} \subset \mathbb{R}^4$$

around the origin. Alternatively, one can also rely on the handle attaching moves constructed in Section 1 using front projections and jet spaces. As

Figure 14: The local model for a handle attachment.

before, we describe how the local move is performed in a small neighborhood of the crossing affected by the local move. Suppose we start with a family of immersions $\{\varphi_t\}$ defined for a time $\varepsilon > 0$. After restricting to smaller arcs, we can obtain a local model for the handle attachment $\tilde{L}$ that can be realized in a time $\varepsilon/2$ (we just need to restrict the Lagrangian we start with to a smaller time interval). Pick a neighborhood $\mathcal{U}$ of the crossing involved in the local move. Then, in time $\varepsilon/2$ we can construct an isotopy of the arcs inside this neighborhood so that the induced arcs in $\mathbb{R}^3$ agree with the lower boundary of our local attaching model $\tilde{L}$ in a smaller neighborhood $\mathcal{U}'$. Clearly such a family can be chosen so that it preserves the area and

coincides with $\{\varphi_t\}$ near the boundary of $\mathcal{U}$. We just then attach $\tilde{L}$ inside $\mathcal{U}'$ and modify the family of immersions for $t \in [\varepsilon/2, \varepsilon]$ in $\mathcal{U} \setminus \mathcal{U}'$ so that together they define a smooth embedded Lagrangian for $t \in [0, \varepsilon]$.

Move H_2. The move H_2 can be obtained as a combination of the previous moves as in Figure 15. In particular, one performs a H_1 move in a small neighborhood above the crossing such that the resulting pair of crossing encloses a very small area and then an R_2 move.

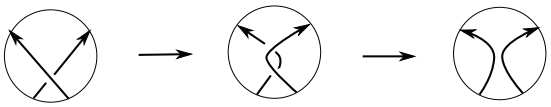

Figure 15: Constructing the move H_2.

Finally, it is clear that given a sequence of moves each Lagrangian cobordism can be constructed so that the total result is smooth. This concludes the proof. □

We conclude this section by constructing caps for the knots with Lagrangian diagram U_1 and fillings for the knots with the mirror Lagrangian diagram (i.e. the Lagrangian diagram with the opposite crossing, which is also the Lagrangian projection of the standard Legendrian unknot).

Lemma 3. *Any knot with Lagrangian diagram commensurate to U_1 admits a Lagrangian cap which is topologically a disk. Similarly, any knot with Lagrangian diagram commensurate to the mirror of U_1 admits a Lagrangian filling which is topologically a disk.*

Proof. In order to construct a cap we rely on a suitable translation of the Lagrangian open disk parametrized in polar coordinates by

$$\begin{aligned} \psi : \mathbb{R}^2 &\to \mathbb{R}^4 \\ (r, \vartheta) &\mapsto \left(r\cos\vartheta, r\sin 2\vartheta, -2r, r(\sin\vartheta - \frac{1}{3}\sin^3\vartheta) \right). \end{aligned}$$

Following the proof of Proposition 2, we can construct a Lagrangian cobordism (which is topologically a cylinder) from the given knot to (a suitable translation) of one of the curves $\psi(r_0, -)$, and then cap the latter using the

image via ψ of the disk of radius r_0. The construction of the filling is analogous, as one can use the map obtained from ψ by changing sign to the last two coordinates. □

3. An exactness criterion

So far we have constructed embedded Lagrangians which are not necessarily exact, a condition which is well known to be very restrictive. For instance it is easy to construct non exact closed embedded Lagrangians in $(\mathbb{R}^4, \omega_{\text{std}})$, while a celebrated result by Gromov ([10]) tells us that there cannot be any exact closed embedded Lagrangian. In this section we discuss a criterion on sequences of combinatorial moves that assures that there exists an *exact* Lagrangian cobordism corresponding to them. We first introduce a definition.

Definition 3. Given a Lagrangian cobordism $L \subset \mathbb{R}^4_+$, we define its *negative boundary* to be $K_0 = L \cap \{t = 0\}$. If the negative boundary K_0 of L is Legendrian we say that L is *exact relative to its negative boundary* if $\lambda_{\text{std}}|_L$ is of the form df for a function $f : L \to \mathbb{R}$ that vanishes on K_0.

Clearly, L is exact relative to its negative boundary if and only if the integral of λ is zero on every closed curve and on every embedded arc with both endpoints on the negative boundary K_0. Furthermore, in the case L is a cap of K_0 then L is exact in the sense of the Introduction. The primitive

$$\lambda_{\text{std}} = xdy + (t+1)dz$$

is particularly nice because its integral along a closed curve with constant t coordinate is exactly the area enclosed by its projection to the xy-plane. For example, the sequence of moves in Figure 16 corresponds to a Lagrangian torus which is manifestly non exact, and is in fact a standard Clifford torus. Here the dashed arrow on the left indicates the standard filling constructed in Lemma 3, and is followed by an H_2 move, an H_1 move and finally one uses Lemma 3 again to construct the cap.

Before stating our exactness criterion we introduce a simple definition.

Definition 4. Given a projection of a link K, we say that two components of the link K_1, K_2 are *vertically split* if the image of the projection of K can be covered with two disjoint disks Δ_1, Δ_2 with $K_i \subset \Delta_i$ for $i = 1, 2$.

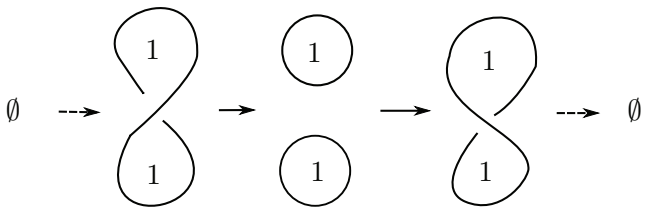

Figure 16: A non exact torus in $\mathbb{R}^4$.

Proposition 3. *Suppose we are given a Legendrian link K_0 with Lagrangian diagram D_0 and a sequence of Lagrangian diagrams related by combinatorial moves starting at D_0 such that*

- *at each stage every component of the link has total signed area zero;*
- *if a handle attachment move which merges two components is performed, then these components are vertically split for the projection to the xy-plane.*

Then the sequence of combinatorial moves can be realized by a Lagrangian cobordism which is exact relative to its negative boundary K_0.

The conditions on the sequence of diagrams imposed by this proposition are rather strict, but they are general enough to be applied to our case. On the other hand, there are many subtleties that come in play when one tries to generalize this result. For example, the sequence of moves in Figure 17 corresponds to a non exact Lagrangian torus in $\mathbb{R}^4$ such that each component of any slice encloses area zero. More in detail, after the usual filling provided by Lemma 3 one performs an R_2 move, an R_0 move, an H_1 move, an R_2 move, an H_2 move, an R_0 move, an R_2 move, an R_0 move, an R_2 move and finally one uses the cap from Lemma 3. It is easy to check that all these moves are compatible with the orientation of the starting unknot.

We will prove that a sequence of Lagrangian diagrams related by a sequence of combinatorial moves satisfying the conditions of Proposition 3 can be realized by a Lagrangian cobordism such that the integrals of the primitive λ_{std} on some fixed curves can be made arbitrary. In particular

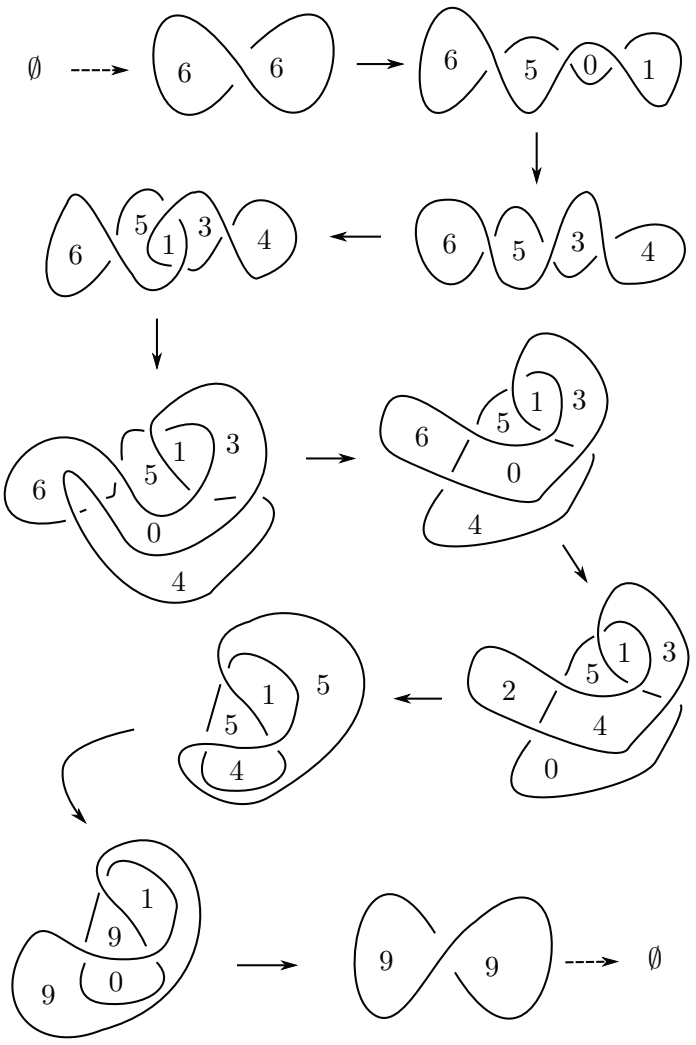

Figure 17: A non exact torus with all enclosed areas zero.

for different choices of these integrals we obtain non-Hamiltonian isotopic Lagrangians.

Proof of Proposition 3. In order to prove the exactness of a Lagrangian cobordism L relative to its negative boundary we just need to check that the integral of λ_{std} along a well-chosen collection of closed curves and arcs is

zero. A possible choice is displayed in Figure 18. In particular the collection consists of the following:

- for every handle attachment move that creates a new component of the link, a closed curve α_i on which the t coordinate is constant;
- for every handle attachment move that merges two components of the link, a curve β_i which is either closed or properly embedded with boundary in the negative boundary K_0.

We just need to check the condition on this set of curves because it generates the homology relative to the negative boundary K_0 and λ_{std} vanishes when restricted to K_0 as this is a Legendrian link by hypothesis.

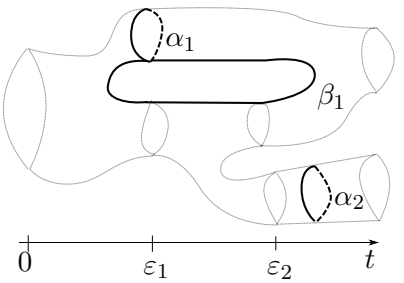

Figure 18: The set of curves on which that we need to check the integral condition.

The integral condition on the curves α_i is exactly the fact that at each stage every component of the link has total area zero in the hypothesis of the proposition. It is easy to see that our Lagrangian can be constructed so that the integral along each of these curves is exactly zero. Hence we just have to show that it is possible to construct a Lagrangian such that the integral along each of the curves β_i is zero.

Suppose that we are given a handle attachment cobordism which merges two components of the link, and let β_i be the corresponding curve. We show how to modify the construction of such a cobordism exploiting the freedom of choice of the function $z(s, \vartheta_0)$ in Lemma 2. This will change the handle attachment, but will preserve the Lagrangian projections. Looking again at the construction of the move H_1 in the proof of Proposition 2, before the local model for the handle attachment is glued we have a Lagrangian cobordism (the tube on the left in Figure 19) induced by a family of immersions

$$\varphi : [0, \varepsilon/2] \times S^1 \to \mathbb{R}^2.$$

We can suppose without loss of generality that $\tilde{\varphi}(t, \vartheta_0)$ gives a parametrization of the solid subarc $\gamma \subset \beta_i$ shown in Figure 19 for $t \in [0, \varepsilon/2]$. Any choice of the height function $z(t, \vartheta_0)$ such that its value agrees with the one we started with near the endpoints will give rise to a cobordism that agrees with the original one outside this small tube. Because of the vertical split hypothesis in the statement of the proposition, by modifying accordingly also the z functions of the other components of the link which project to the same disk in the xy-plane we can obtain an embedded cobordism. Furthermore, one has

$$\int_\gamma \lambda_{\text{std}} = \int_0^{\varepsilon/2} \left[x(t) y'(t) dt + (t+1) z'(t, \vartheta_0) \right] dt$$

which can be given any desired value for an appropriate choice of the function $z(t, \vartheta_0)$. Hence we can arrange that the new Lagrangian cobordism is such that the total integral along β_i is any value we want, and in particular it can be made exactly zero. □

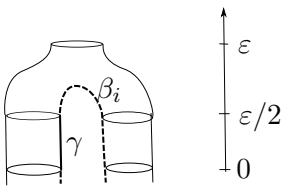

Figure 19: Perturbing the family of immersions for $s \in [0, \varepsilon/2]$.

4. An exact Lagrangian cap for U_0

In this section we complete the proof of Theorem 1 by constructing an exact Lagrangian cap for the knot U_0 using the set of combinatorial moves we have previously developed. Before describing the construction, we introduce some *curl moves* which will be useful in the construction. This is because there is not an analogue of the R_1 move for Lagrangian diagrams, so we cannot generally get rid of the curls. On the other hand, these moves allow us to move these curls around the diagram.

Lemma 4. *Each of the moves in Figure* 20 *can be realized by a Lagrangian cobordism. For C_1, the two domains on the right need to have area bigger*

than A, while for C_2 the two upper domains need to have area bigger than A.

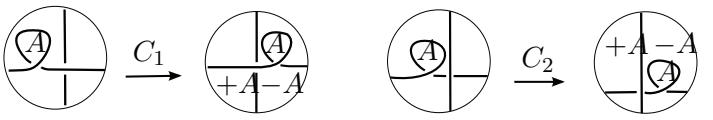

Figure 20: The curl moves.

Similarly there are analogous inverse moves, and there are also versions in which the curl has the opposite crossing.

Proof. We just show how to construct the move C_1, as the other is analogous. This is explained in Figure 21. In particular, the moves are the following:

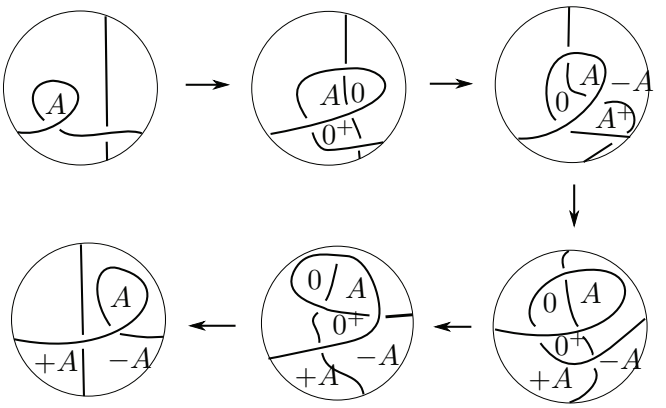

Figure 21: Construction of the move C_1.

- A R_2 move such that the new created domains have small area (here 0^+ indicates a small number bigger that 0);
- An R_0 move (here we need the upper right domain to have area bigger than A, and A^+ indicates number slightly bigger that A);
- An R_0 move crossing (here we need the lower right domain to have area bigger than A);

- An R_3 move;
- An R_2 move. □

Our strategy is to construct an exact Lagrangian cobordism to a union of a component U_1 as in Figure 3 together with two standard Legendrian unknots (whose Lagrangian diagram is isotopic to the mirror of U_1). The first can be easily capped using Lemma 3. To get rid of the other components, we will use the following *unknot trick* before constructing the final cap.

Lemma 5. *Provided $A > B$, the move of Figure 22 can be realized by a Lagrangian cobordism.*

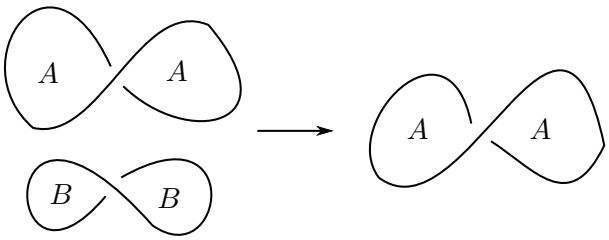

Figure 22: The unknot trick. Again, the upper left diagram does not correspond to a Legendrian knot while the lower left one does.

Proof. The proof is contained in Figure 23. The first move is an H_1 move (it is straightforward to see that this can be done for any choice of the orientations). Then one simply performs an R_0 and an R_2 move as suggested in the figure. □

Finally, we are ready to give a proof of Theorem 1.

Proof of Theorem 1. We show how to construct an exact cap for the knot U_0. First, one performs the moves displayed in Figure 24, which are in order an R_0, an R_2, an R_0, a C_1, a C_2 and an H_2. One then focuses on the right component of the result, and use the moves indicated in Figure 25, which are two R_0 moves, two C_2 moves and an H_2 move. At this point, one can use the unknot trick to get rid of all the Legendrian unknot components and then cap the remaining one using Lemma 3. We hence obtain an oriented Lagrangian cap of the knot U_0, and we can perform the construction so that the result is exact by the criterion provided by Proposition 3. This is because

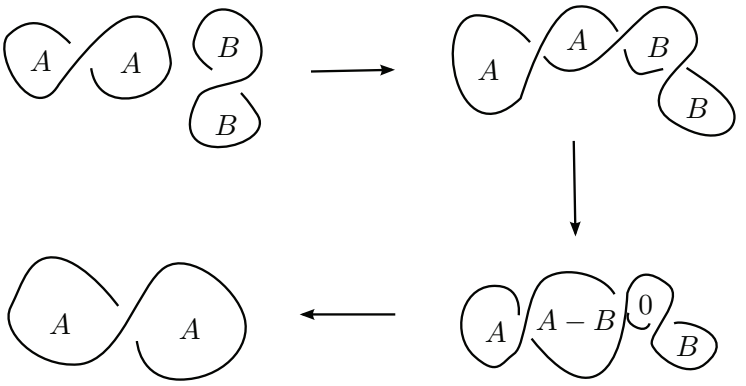

Figure 23: Proof of the unknot trick.

no handle attachment moves are used in the construction of the curl moves, and all the handle attachments (including those used for the unknot trick) are performed on vertically split components. Finally, the theorem follows from the existence of this cap for U_0 and Proposition 1 and Lemma 1 in Section 1. □

5. Further problems

The combinatorial moves used in order to prove Theorem 1 might be adapted to other problems related to Legendrian knots. On the other hand, even if it is easy to construct a Lagrangian projection of a given Legendrian knot form a front projection (see for example the algorithm in [7]), we are not aware of any easy way to determine a Lagrangian diagram, i.e. a way to assign areas to the domains. This is a big limitation for the techniques developed in the present paper, especially when one tries to work out some practical examples.

Another interesting task is to find a good generalization of the exactness criterion provided by Proposition 3, and in particular one that fits in our combinatorial model. This might be related to very subtle issues in contact and symplectic geometry, and in fact it is not improbable that one can find a non exact torus such that all the areas enclosed are zero and the handle attachment that merges two components is performed on two unlinked (but

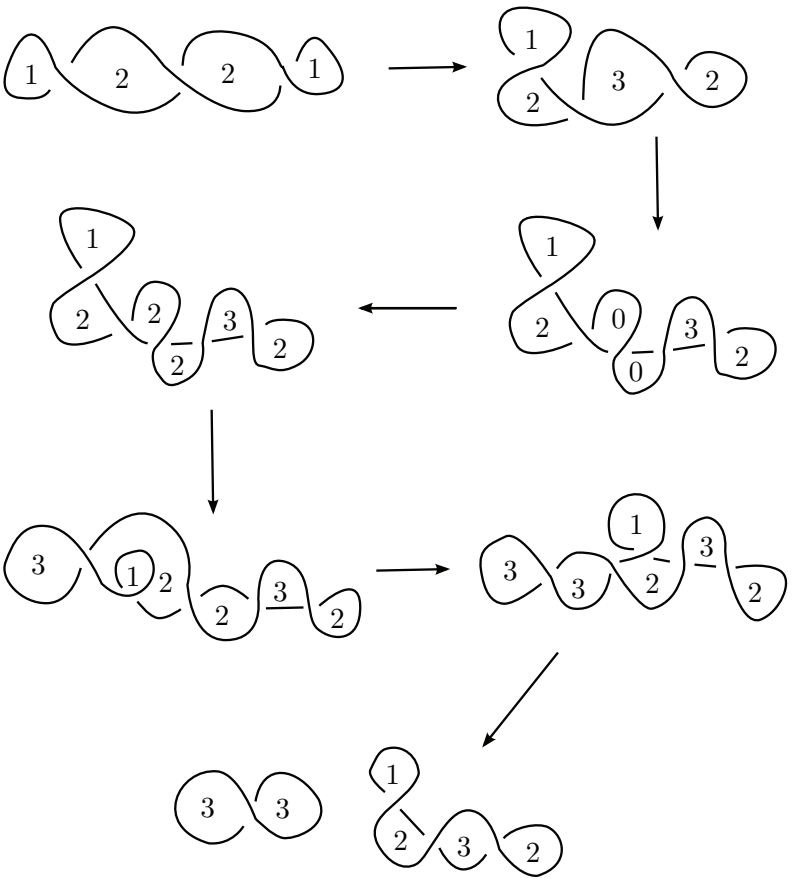

Figure 24: Constructing a cap for U_0 - part 1.

not vertically split) components. This would make the strict requirements in Proposition 3 look not that excessive.

Finally, there are two more questions that naturally arise from our treatment.

Question 1. *Is there any Legendrian knot with an exact Lagrangian cap of genus* 1*?*

While every cap we have constructed in the paper has genus at least 2, this case is not ruled out by the slice Thurston-Bennequin inequality (while the genus 0 case is).

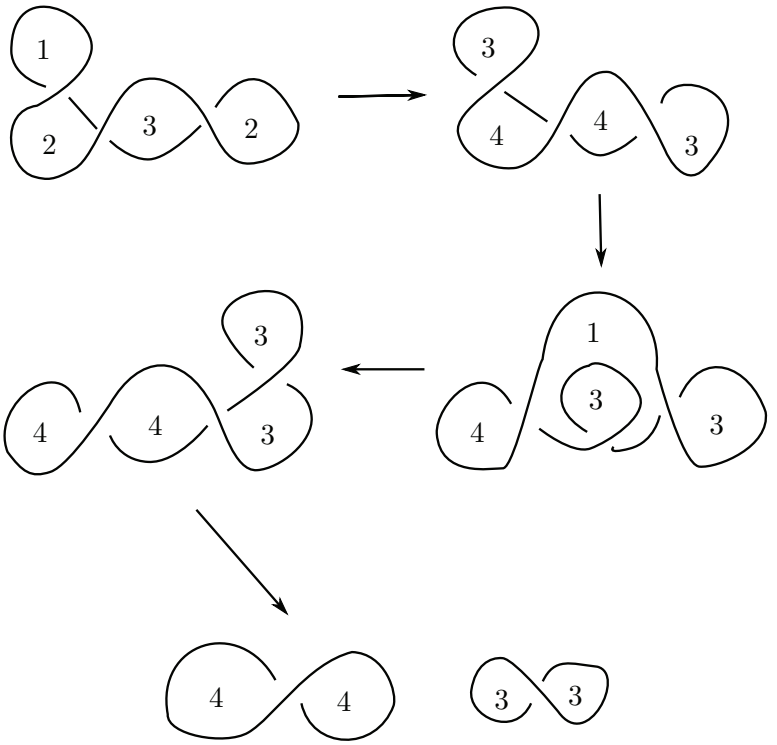

Figure 25: Constructing a cap for U_0 - part 2.

Question 2. *Is there any not destabilizable Legendrian knot which admits an exact Lagrangian cap?*

This is a very interesting question, as it is a fundamental assumption for Theorem 1 that we can add as many stabilizations as desired.

References

[1] B. Chantraine, *Lagrangian concordance of Legendrian knots*, Algebr. Geom. Topol., **10**(1):63–85, 2010.

[2] G Dimitroglou Rizell, *Exact lagrangian caps and non-uniruled lagrangian submanifolds*, Arkiv för Matematik, pages 1–28, 2014.

[3] E. Ekholm, Ko Honda, and T Kálmán, *Legendrian knots and exact lagrangian cobordisms*, arXiv:math/1212.1519, 2012.

[4] Y. Eliashberg and M. Fraser, *Classification of topologically trivial Legendrian knots*, In: Geometry, topology, and dynamics (Montreal, PQ, 1995), volume 15 of *CRM Proc. Lecture Notes*, pages 17–51. Amer. Math. Soc., Providence, RI, 1998.

[5] Y. Eliashberg and M. Gromov, *Lagrangian intersection theory: finite-dimensional approach*, In: Geometry of differential equations, volume 186 of *Amer. Math. Soc. Transl. Ser. 2*, pages 27–118. Amer. Math. Soc., Providence, RI, 1998.

[6] Y. Eliashberg and E. Murphy, *Lagrangian caps*, Geometric and Functional Analysis, **23**(5):1483–1514, 2013.

[7] J. Etnyre, *Legendrian and transversal knots*, pages 105–185, 2005.

[8] H. Geiges, *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2008.

[9] A. B. Givental′, *Lagrangian imbeddings of surfaces and the open Whitney umbrella*, Funktsional. Anal. i Prilozhen., **20**(3):35–41, 96, 1986.

[10] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math., **82**(2):307–347, 1985.

[11] S. Nemirovski and K. Siegel, *Rationally convex domains and singular Lagrangian surfaces in* $\mathbb{C}^2$, arXiv:math/1410.4652, 2014.

[12] L. Rudolph, *The slice genus and the Thurston-Bennequin invariant of a knot*, Proc. Amer. Math. Soc., **125**(10):3049–3050, 1997.

[13] D. Sauvaget, *Curiosités lagrangiennes en dimension 4*, Ann. Inst. Fourier (Grenoble), **54**(6):1997–2020 (2005), 2004.

Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA 02139-4307, USA
E-mail address: linf@math.mit.edu

Received January 11, 2014