

# Universal orderability of Legendrian isotopy classes

VLADIMIR CHERNOV AND STEFAN NEMIROVSKI

It is shown that non-negative Legendrian isotopy defines a partial order on the universal cover of the Legendrian isotopy class of the fibre of the spherical cotangent bundle of any manifold. This result is applied to Lorentz geometry in the spirit of the authors' earlier work on the Legendrian Low conjecture.

## 1. Introduction

### 1.1. Partial orders in contact geometry

Let  $(X, \ker \alpha)$  be a contact manifold with a co-oriented contact structure. A contact or Legendrian isotopy in  $X$  is called non-negative if individual points move in the direction of the co-orientation of the contact hyperplanes (formal definitions are given in §4.1).

Let  $\mathcal{C}$  denote either a connected component of the contactomorphism group of  $X$  or a Legendrian isotopy class in  $X$ . We write  $a \preceq b$  for two elements  $a, b \in \mathcal{C}$  if there is a non-negative isotopy connecting  $a$  to  $b$ . This partial relation has a natural lift to a partial relation  $\preceq$  on the universal cover  $\tilde{\mathcal{C}}$ , see §4.3 for details. It is clear that  $\preceq$  and  $\preceq$  are reflexive and transitive. Let us call  $\mathcal{C}$  *orderable* if  $\preceq$  is also antisymmetric (i.e. defines a partial order on  $\mathcal{C}$ ) and *universally orderable* if  $\preceq$  defines a partial order on  $\tilde{\mathcal{C}}$ .

The question of (universal) orderability for groups of contactomorphisms and Legendrian isotopy classes was apparently first raised by Eliashberg and Polterovich [14] and Bhupal [6]. There are now several papers treating various situations [13], [11], [8], [9], [27], [1], [29], [2], [7]. For reasons that will be explained in §1.2, we are particularly interested in the case, first considered by Colin, Ferrand, and Pushkar in [11], when  $\mathcal{C} = \text{Leg}(ST_x^*M)$  is

---

This work was partially supported by a grant from the Simons Foundation (#235674 to Vladimir Chernov). The second author was partially supported by SFB/TR 12 of the DFG and RFBR grant №13-01-12417-ofi-m.

the Legendrian isotopy class of the fibre of the spherical cotangent bundle  $ST^*M$  of a manifold  $M$ ,  $\dim M \geq 2$ . This class is orderable if the universal cover of  $M$  is non-compact by [9, Remark 8.2] or does not have the integral cohomology ring of a compact rank one symmetric space (CROSS) by [16, Theorem 1.2] and Proposition 4.7. On the other hand,  $\text{Leg}(ST_x^*M)$  is not orderable for every CROSS and, more generally, for any manifold  $M$  admitting a Riemannian  $Y_\ell^x$ -metric, see [9, Example 8.3]. The following special case of Theorem 4.10 shows that *universal* orderability holds for every  $M$ .

**Theorem 1.1.** *The Legendrian isotopy class of the fibre of  $ST^*M$  is universally orderable.*

This theorem is inspired by and generalises the result of Eliashberg, Kim, and Polterovich [13, Theorem 1.18] about the contactomorphism group of  $ST^*M$ . They proved (modulo an assumption removed in [9]) that the identity component  $\text{Cont}_0(ST^*M)$  is *universally* orderable for every closed manifold  $M$ . However,  $\text{Cont}_0(ST^*M)$  is not orderable for any manifold  $M$  admitting a Riemannian metric with periodic geodesic flow.

## 1.2. Causality and orderability

Let  $(\mathcal{X}, g)$  be a spacetime, that is, a time-oriented connected Lorentz manifold. Assume that  $g$  has signature  $(+, -, \dots, -)$  with at least two negative spacelike directions. A piecewise smooth curve in  $\mathcal{X}$  is called future-directed (f.d.) if its tangent vector at each point lies in the future hemicone defined by the time-orientation in the non-spacelike cone of the Lorentz metric.

The *causality relation*  $\leq$  on  $\mathcal{X}$  is defined by setting  $x \leq y$  if either  $x = y$  or there is a f.d. curve connecting  $x$  to  $y$ . This relation is always reflexive and transitive. If it is a partial order, the spacetime is said to be *causal*.

A causal spacetime is *globally hyperbolic* if all causal segments  $I_{x,y} = \{z \in \mathcal{X} \mid x \leq z \leq y\}$  are compact [5]. By another result of Bernal and Sánchez [4], a spacetime is globally hyperbolic if and only if it contains a smooth spacelike hypersurface  $M \subset \mathcal{X}$  such that every endless f.d. curve meets  $M$  exactly once. Such an  $M$  is called a *Cauchy surface* in  $\mathcal{X}$ .

Let  $\mathfrak{N}$  be the set of all f.d. non-parametrised null geodesics (i.e. light rays) in  $(\mathcal{X}, g)$ .  $\mathfrak{N}$  has a canonical structure of a contact manifold, see [21, §2] or [23, pp. 252–253]. The set of all null geodesics passing through a point  $x \in \mathcal{X}$  is a Legendrian sphere  $\mathfrak{S}_x \subset \mathfrak{N}$  called the *sky* (or the *celestial sphere*) of that point. The association  $x \mapsto \mathfrak{S}_x$  was one of the starting points of

Penrose's twistor theory, see e.g. [24]; its study in the context of contact geometry was initiated by Low [20].

There is a contactomorphism

$$\rho_M : \mathfrak{N} \xrightarrow{\cong} ST^*M$$

taking a null geodesic  $\gamma \in \mathfrak{N}$  to the equivalence class of the 1-form  $g(\dot{\gamma}, \cdot)$  on  $T_{\gamma \cap M}M$ , where  $\dot{\gamma}$  is a f.d. tangent vector to  $\gamma$  at  $\gamma \cap M$ . For every sky, its image  $\rho_M(\mathfrak{S}_x)$  in  $ST^*M$  is Legendrian isotopic to the fibre of  $ST^*M$ . Hence, we obtain a map

$$\mathfrak{s} : \mathcal{X} \longrightarrow \text{Leg}(ST_{\{\text{pt}\}}^*M)$$

from a globally hyperbolic spacetime  $\mathcal{X}$  to the Legendrian isotopy class of the fibre of the spherical cotangent bundle of its Cauchy surface  $M$ .

The following key observation is an immediate corollary of the proof of [8, Proposition 4.2] taking into account the opposite convention for the signature of the Lorentz metric.

**Proposition 1.2.**  $x \leq y \implies \mathfrak{s}(x) \preceq \mathfrak{s}(y)$ .

The converse implication does not hold in general. (For example, the map  $\mathfrak{s}$  may be non-injective, cf. [9, Example 10.5].) However, a useful sufficient condition for it to hold can be formulated purely in terms of  $ST^*M$ . This condition was an implicit underpinning of our work on the (Legendrian) Low conjecture in [8, 9].

**Proposition 1.3.** *If  $\text{Leg}(ST_{\{\text{pt}\}}^*M)$  is orderable, then  $\mathfrak{s}(x) \preceq \mathfrak{s}(y) \implies x \leq y$ .*

*Proof.* If  $y \leq x$ , then  $\mathfrak{s}(y) \preceq \mathfrak{s}(x)$  by Proposition 1.2 and hence  $\mathfrak{s}(x) = \mathfrak{s}(y)$  by orderability. If  $x$  and  $y$  are not causally related, then the Legendrian links  $\mathfrak{s}(x) \sqcup \mathfrak{s}(y)$  and  $\mathfrak{s}(y) \sqcup \mathfrak{s}(x)$  are Legendrian isotopic (through links formed by skies of pairs of causally unrelated points) by [8, Lemma 4.3]. By the Legendrian isotopy extension theorem, there exists a  $\varphi \in \text{Cont}_0(ST^*M)$  such that  $\varphi(\mathfrak{s}(x) \sqcup \mathfrak{s}(y)) = \mathfrak{s}(y) \sqcup \mathfrak{s}(x)$ . It follows that  $\mathfrak{s}(y) \preceq \mathfrak{s}(x)$  and again  $\mathfrak{s}(x) = \mathfrak{s}(y)$  by orderability.

So we have to exclude the possibility that  $x \neq y$  but  $\mathfrak{s}(x) = \mathfrak{s}(y)$ . In this case,  $x$  and  $y$  are causally related by null geodesics. Assume that  $y \leq x$  (otherwise we are done). Moving from  $y$  a bit along any f.d. null geodesic, we obtain a point  $z$  with a different sky and such that  $y \leq z \leq x$ . Then  $\mathfrak{s}(y) \preceq$

$\mathfrak{s}(z) \preceq \mathfrak{s}(x)$  by Proposition 1.2 and hence  $\mathfrak{s}(z) = \mathfrak{s}(x) = \mathfrak{s}(y)$  by orderability, which contradicts the choice of  $z$ .  $\square$

It follows now from the orderability results cited above that the conclusion of Proposition 1.3 holds for a globally hyperbolic spacetime such that the universal cover of its Cauchy surface is either non-compact or does not have the integral cohomology ring of a CROSS. The remaining cases (e.g. the case when  $M$  is homotopy equivalent to a sphere) may be handled using the universal orderability result of the present paper.

Firstly, note that we may pass to a simply connected globally hyperbolic spacetime by considering the universal cover of  $\mathcal{X}$  with the pulled back Lorentz metric, see [10, Theorem 14]. If  $\mathcal{X}$  is simply connected, the map  $\mathfrak{s}$  admits a lift

$$\widetilde{\mathfrak{s}} : \mathcal{X} \longrightarrow \widetilde{\text{Leg}}(ST_{\{\text{pt}\}}^*M).$$

A careful inspection of the proofs of Propositions 1.2 and 1.3 shows that they remain true with  $\text{Leg}$ ,  $\mathfrak{s}$ , and  $\preceq$  replaced by  $\widetilde{\text{Leg}}$ ,  $\widetilde{\mathfrak{s}}$ , and  $\widetilde{\preceq}$ . Since  $\widetilde{\preceq}$  is always a partial order by Theorem 1.1, we obtain the following result.

**Theorem 1.4.** *Suppose that  $\mathcal{X}$  is a simply connected globally hyperbolic spacetime with Cauchy surface  $M$ . Then*

$$x \leq y \text{ in } \mathcal{X} \iff \widetilde{\mathfrak{s}}(x) \widetilde{\preceq} \widetilde{\mathfrak{s}}(y) \text{ in } \widetilde{\text{Leg}}(ST_{\{\text{pt}\}}^*M)$$

Thus, up to passing to a finite cover, the causality relation (and hence, by [22, Theorem 2], the conformal Lorentz structure) of a globally hyperbolic spacetime is always determined by the map  $x \mapsto \mathfrak{S}_x$  to the space of Legendrian spheres in its space of null geodesics.

### Acknowledgments

This paper owes very much to the seminal work of Eliashberg and Polterovich [14]. To a large extent, it implements their original strategy for proving [13, Theorem 1.18], cf. Remark 3.12. The authors are also very grateful to Yuli Rudyak for his valuable advice on the proof of Theorem 2.5.

## 2. A Legendrian non-displacement result

### 2.1. Generating hypersurfaces for Legendrian submanifolds in spherical cotangent bundles (after Eliashberg and Gromov [12, §4.2])

Let  $L \subset ST^*M$  be a Legendrian submanifold in the spherical cotangent bundle of a closed manifold  $M$ . Suppose that there exists a function  $f : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $N \geq 0$ , such that

- 1) 0 is not a critical value of  $f$ ;
- 2) the hypersurface  $\{f = 0\}$  is in general position with respect to the projection  $\pi_M : M \times \mathbb{R}^N \rightarrow M$ , that is to say, the subset

$$FT_f := \{x \in M \times \mathbb{R}^N \mid \{f = 0\} \text{ is tangent to } \{\pi_M(x)\} \times \mathbb{R}^N\}$$

is a submanifold cut out transversally by the equations  $f = 0$  and  $df|_{\mathbb{R}^N} = 0$ ;

- 3) the map

$$FT_f \ni x \longmapsto [\pi_{M*}df(x)] \in ST_{\pi_M(x)}^*M,$$

where  $\pi_{M*}df(x)$  is the unique 1-form at  $\pi_M(x)$  such that  $df(x)$  is its pull-back by  $\pi_M$ , defines a diffeomorphism  $FT_f \xrightarrow{\cong} L$ ;

- 4)  $f$  is equal to a non-degenerate quadratic form  $Q : \mathbb{R}^N \rightarrow \mathbb{R}$  outside of a compact subset in  $M \times \mathbb{R}^N$ .

Then  $\{f = 0\}$  is called a *quadratic at infinity generating hypersurface* for the Legendrian submanifold  $L \subset ST^*M$ .

**Example 2.1.** Let  $f : M \rightarrow \mathbb{R}$  be a smooth function such that 0 is not a critical value. Then the hypersurface  $\{f = 0\} \subset M (= M \times \mathbb{R}^0)$  generates the Legendrian submanifold

$$L_f = \{[df(x)] \in ST^*M \mid f(x) = 0\}.$$

Thus, in this case  $L$  is the Legendrian lift of the co-oriented generating hypersurface and this hypersurface is the wave front of  $L$ .

If  $Q' : \mathbb{R}^{N'} \rightarrow \mathbb{R}$  is another non-degenerate quadratic form and  $\varphi : \mathbb{R}^{N'} \rightarrow [0, 1]$  is a cut-off function with  $\|d\varphi\| \leq 1$  that is equal to 1 on a sufficiently

large ball, then the zero set of the function

$$\varphi \cdot (f - Q) + Q + Q' : M \times \mathbb{R}^{N+N'} \longrightarrow \mathbb{R}$$

is also a quadratic at infinity generating hypersurface for the same Legendrian submanifold  $L$ . This operation on generating hypersurfaces is called *stabilisation*.

**Theorem 2.2** (cf. [12, Theorem 4.2.1]). *Suppose that  $\{L_t\}_{t \in [0,1]}$  is a Legendrian isotopy in  $ST^*M$  such that  $L_0 = L_f$  for a function  $f : M \rightarrow \mathbb{R}$ . Then there exist an  $N \geq 0$  and a smooth family of quadratic at infinity generating hypersurfaces  $\{f_t = 0\} \subset M \times \mathbb{R}^N$  for  $L_t$  such that  $\{f_0 = 0\}$  is a stabilisation of  $\{f = 0\}$ .*

**Remark 2.3.** An inaccuracy in the proof of [12, Theorem 4.2.1] was pointed out and corrected by Pushkar' in two recent preprints [25] and [26]. A result similar to Theorem 2.2 may also be found in [15]. In a different context, generating hypersurfaces appeared in the classical text [3, §20.7].

**Example 2.4.** The spherical conormal bundle  $SN^*V \subset ST^*M$  of a submanifold  $V \subset M$  is a Legendrian submanifold. For instance, if  $V$  is a point  $v \in M$ , then  $SN^*V = ST_v^*M$  is the fibre of  $ST^*M$  at  $v$ . The co-geodesic flow of a Riemann metric on  $M$  defines a Legendrian isotopy of  $SN^*V$  to the Legendrian lift of the co-oriented boundary of a geodesic tube around  $V$ . Taking a function  $f : M \rightarrow \mathbb{R}$  such that  $\{f < 0\}$  is such a tube, we see that  $SN^*V$  is Legendrian isotopic to a submanifold of the form considered in Example 2.1. In particular, it has a quadratic at infinity generating hypersurface.

## 2.2. Legendrian non-displacement in the spherical cotangent bundle of a fibred manifold

Let  $\pi : P \rightarrow M$  be a submersion. For every subset  $U \subset ST^*M$ , its pull-back in  $ST^*P$  is the subset

$$\pi^*(U) := \{[\pi^*\xi] \in ST^*P \mid [\xi] \in U\}.$$

The pull-back of a Legendrian submanifold in  $ST^*M$  is a Legendrian submanifold of  $ST^*P$ . For instance, if  $f : M \rightarrow \mathbb{R}$  is a function such that 0 is not a critical value and  $L_f$  is the Legendrian submanifold in  $ST^*M$  generated

by  $\{f = 0\}$  as in Example 2.1, then

$$\pi^*(L_f) = L_{f \circ \pi} \subset ST^*P.$$

Similarly, the pull-back of the spherical conormal bundle of a submanifold  $V \subset M$  is the spherical conormal bundle of its pre-image  $\pi^{-1}(V) \subset P$ .

**Theorem 2.5.** *Let  $P$  and  $M$  be closed connected manifolds and  $\pi : P \rightarrow M$  a fibre bundle. Any Legendrian submanifold of  $ST^*P$  Legendrian isotopic to  $\pi^*(L_f)$  for some function  $f : M \rightarrow \mathbb{R}$  intersects  $\pi^*(ST^*M)$ .*

*Proof.* Let us assume that there is a Legendrian submanifold  $L \subset ST^*P - \pi^*(ST^*M)$  that is Legendrian isotopic to  $\pi^*(L_f)$ . The Legendrian submanifold  $\pi^*(L_f) = L_{f \circ \pi}$  has a (trivially) quadratic at infinity generating hypersurface  $\{f \circ \pi = 0\} \subset P$ . By Theorem 2.2, there exists a family of hypersurfaces  $H_t = \{f_t = 0\} \subset P \times \mathbb{R}^N$  such that

- 1)  $H_0 = \{f_0 = 0\}$  is a stabilisation of  $\{f \circ \pi = 0\} \subset P$ ;
- 2)  $H_1 = \{f_1 = 0\}$  generates  $L$ ;
- 3) outside of a compact subset of  $P \times \mathbb{R}^N$ , all hypersurfaces  $H_t$  coincide with the hypersurface  $P \times \{Q = 0\}$ , where  $Q$  is a non-degenerate quadratic form on  $\mathbb{R}^N$ .

Applying additional stabilisation, if necessary, we may assume that both inertia indices  $\varkappa_{\pm}(Q) \geq 2$ . This guarantees, in particular, that the hypersurfaces  $H_t$  are connected.

The assumption that  $L$  is disjoint from  $\pi^*(ST^*M)$  means that its generating hypersurface  $H_1$  is nowhere tangent to the fibres of the composite projection  $\pi_M : P \times \mathbb{R}^N \rightarrow P \xrightarrow{\pi} M$ . In other words, the restriction  $\pi_M|_{H_1}$  is a submersion. Since  $H_1$  is standard at infinity, it follows by Ehresmann's theorem that  $\pi_M|_{H_1} : H_1 \rightarrow M$  is a fibre bundle.

On the other hand, the fibre of the projection  $\pi_M|_{H_0} : H_0 \rightarrow M$  over a point  $x \in M$  is essentially a product of the form  $\pi^{-1}(x) \times \{Q = -f(x)\}$ . In particular, the fibres over the sets  $\{f > 0\}$  and  $\{f < 0\}$  are embedded in topologically different ways. (This statement will be made precise at the end of the proof.)

The main point of the following somewhat technical argument is to show that this behaviour of the fibres contradicts the fact that the hypersurfaces  $H_0$  and  $H_1$  are isotopic within the class of hypersurfaces satisfying condition (3).

Let us fix a parametrisation

$$\iota_t : H \longrightarrow H_t, \quad t \in [0, 1],$$

of the family  $\{H_t\}$  such that  $\iota_t$  is the same standard embedding outside of a compact subset in  $H$ . Let further

$$\pi_t := \pi_M|_{H_t} \circ \iota_t : H \longrightarrow M$$

be the induced family of projections to  $M$ .

Since  $\pi_1 : H \rightarrow M$  is a fibre bundle, we may apply the relative homotopy lifting property [19, Proposition 4.48] to the homotopy of maps

$$\pi_t : H \longrightarrow M, \quad t \in [0, 1].$$

It follows that there is a homotopy of (continuous) maps

$$\varphi_t : H \longrightarrow H, \quad t \in [0, 1],$$

such that  $\varphi_1 \equiv \text{id}_H$ ,  $\varphi_t = \text{id}_H$  outside of a compact set for all  $t$ , and

$$\pi_t = \pi_1 \circ \varphi_t, \quad t \in [0, 1].$$

Thus,  $\pi_0 = \pi_1 \circ \varphi_0$ , where  $\varphi_0 : H \rightarrow H$  is homotopic to the identity in the class of compactly supported self-maps of  $H$ .

Suppose now that  $\beta \in H_c^*(H; \mathbb{Z}/2)$  is a cohomology class with compact support such that its restriction to a fibre  $\pi_0^{-1}(x)$  is non-zero. We claim that its restriction to every regular fibre of  $\pi_0$  must also be non-zero. Indeed, note first that

$$0 \neq \beta|_{\pi_0^{-1}(x)} = (\varphi_0^* \beta)|_{\pi_0^{-1}(x)} = (\varphi_0|_{\pi_0^{-1}(x)})^*(\beta|_{\pi_1^{-1}(x)}),$$

where the first equality holds because  $\varphi_0$  acts as the identity on the cohomology of  $H$  and the second because  $\varphi_0$  maps the fibre of  $\pi_0$  over  $x$  to the fibre of  $\pi_1$  over  $x$ . Since the fibres of the fibre bundle  $\pi_1$  are all isotopic, we conclude that

$$0 \neq \beta|_{\pi_1^{-1}(y)} \quad \text{for all } y \in M.$$

However, if  $y$  is a regular value of  $\pi_0$ , then the restriction

$$\varphi_0|_{\pi_0^{-1}(y)} : \pi_0^{-1}(y) \longrightarrow \pi_1^{-1}(y)$$

is a proper map of equidimensional manifolds that has  $\mathbb{Z}/2$ -degree 1. It follows from Poincaré duality and naturality of the  $\cup$ -product that this



map induces an injection on  $\mathbb{Z}/2$ -cohomology with compact support (cf. [28, Lemma 2.2]). Hence,

$$\beta|_{\pi_0^{-1}(y)} = (\varphi_0^*\beta)|_{\pi_0^{-1}(y)} = (\varphi_0|_{\pi_0^{-1}(y)})^*(\beta|_{\pi_1^{-1}(y)}) \neq 0,$$

as claimed.

Thus, to complete the proof of the theorem by contradiction, we need to exhibit a compactly supported cohomology class on  $H$  such that its restrictions to the regular fibres of  $\pi_0$  can be both non-zero and zero. Let  $V_+ \subset \mathbb{R}^N$  be a maximal linear subspace on which the form  $Q$  is positive definite. By construction,  $\dim V_+ = \varkappa_+ \geq 2$ . Consider the submanifold  $P \times V_+ \subset P \times \mathbb{R}^N$ . Its intersection with  $H_0$  is compact and defines (by duality) a compactly supported cohomology class on  $H_0$ . Let  $\beta$  be the pull-back of this class to  $H$ .

The restriction of  $\beta$  to a regular fibre  $\pi_0^{-1}(x)$  is then dual to the pre-image of the intersection of the fibre of  $\pi_M|_{H_0}$  over  $x$  with  $P \times V_+$ . If  $f(x) > 0$ , this intersection is empty. If  $f(x) < 0$ , it is the product  $\pi^{-1}(x) \times S$ , where the sphere  $S = V_+ \cap \{Q = -f(x)\}$  is the ‘waist’ of the quadric, and so the dual compactly supported cohomology class on  $\pi^{-1}(x) \times \{Q = -f(x)\}$  is non-zero. Hence,  $\beta$  has the required property.  $\square$

**Remark 2.6.** To explain the idea of the proof, let us apply it directly to the simplest case when  $N = 0$  and so  $H_t$  are (let us say, connected) hypersurfaces in  $P$  isotopic to  $H_0 = \pi^{-1}(\{f = 0\})$ . Choosing a parametrisation  $\iota_t : H \rightarrow H_t$ , we get a homotopy of maps  $\pi_t = \pi \circ \iota_t : H \rightarrow M$  such that  $\pi_1$  is a fibre bundle projection and  $\pi_0$  is not surjective (because its image is  $\{f = 0\} \subsetneq M$ ). Using the homotopy lifting property, we obtain a homotopy  $\varphi_t : H \rightarrow H$  such that  $\varphi_1 = \text{id}_H$  and  $\pi_0 = \pi_1 \circ \varphi_0$ . It follows from the latter equality that  $\varphi_0$  can not be surjective. This contradicts the fact that a map homotopic to the identity map of the *closed* manifold  $H$  has degree 1 and hence must be onto.

### 3. Transverse families of Legendrian submanifolds

#### 3.1. Families

A parametrised family of Legendrian submanifolds in a contact manifold  $(X, \ker \alpha)$  over a base  $B$  is a map

$$F : \underline{L} \longrightarrow X,$$

where  $\pi : \underline{L} \rightarrow B$  is a fibre bundle and the restriction

$$F|_{\pi^{-1}(b)} : \pi^{-1}(b) \longrightarrow L_b \subset X$$

is a Legendrian embedding for every  $b \in B$ . Two parametrised families  $F_1, F_2 : \underline{L} \rightarrow X$  are called equivalent if  $F_1 = F_2 \circ \Phi$  for a diffeomorphism  $\Phi : \underline{L} \rightarrow \underline{L}$  such that  $\pi \circ \Phi = \pi$ .

**Definition 3.1.** A family  $\mathcal{L} = \{L_b\}_{b \in B}$  of Legendrian submanifolds is an equivalence class of parametrised families.

A family of Legendrian submanifolds is called *constant* if  $L_b$  is the same Legendrian submanifold in  $X$  for all  $b \in B$ . Note that a constant family may have non-constant parametrisations.

**Definition 3.2.** A family of Legendrian submanifolds in  $(X, \ker \alpha)$  is called *transverse* if it has a parametrisation  $F : \underline{L} \rightarrow X$  such that the pull-back  $F^*\alpha$  of the contact form does not vanish anywhere on  $\underline{L}$ .

It is obvious that this property depends neither on the choice of a parametrisation of the family nor on the choice of a contact form defining the contact structure on  $X$ .

**Example 3.3 (Transverse families and positive isotopies).** The simplest example of a family of Legendrian submanifolds is a Legendrian isotopy  $\{L_t\}_{t \in [0,1]}$  that can be parametrised by the product  $L_0 \times [0,1]$ . This family is transverse if and only if the isotopy is either positive or negative, see §4.1.

**Example 3.4 (Transverse families and fibre bundles).** Let  $p : M \rightarrow B$  be a fibre bundle. Then  $\{SN^*p^{-1}(b)\}_{b \in B}$  is a transverse family of Legendrian submanifolds in  $ST^*M$  with base  $B$ . It is tautologically parametrised by  $\underline{L} = p^*(ST^*B)$  and the projection  $\pi : \underline{L} \rightarrow B$  is the composition

$$p^*(ST^*B) \subset ST^*M \longrightarrow M \xrightarrow{p} B$$

In particular, one can take  $p = \text{id}_M$  and obtain the family of all fibres of  $ST^*M$ .

### 3.2. Stabilisation of contact manifolds

Let  $(X, \ker \alpha)$  be a contact manifold and  $B$  an arbitrary manifold. The  $B$ -stabilisation of  $X$  is the contact manifold

$$X^B := (X \times T^*B, \ker(\alpha \oplus \lambda_{\text{can}})),$$

where  $\lambda_{\text{can}} = pdq$  is the canonical 1-form on  $T^*B$ . If  $\alpha' = e^f \alpha$  is another contact form defining the same (co-oriented) contact structure on  $X$ , then the map

$$(3.1) \quad X \times T^*B \ni (x, q, p) \longmapsto (x, q, e^{f(x)}p) \in X \times T^*B$$

defines a contactomorphism

$$(X \times T^*B, \ker(\alpha \oplus \lambda_{\text{can}})) \xrightarrow{\cong} (X \times T^*B, \ker(\alpha' \oplus \lambda_{\text{can}})).$$

Hence, the  $B$ -stabilisation of  $X$  is well-defined as a contact manifold.

**Example 3.5.** Suppose that  $X = ST^*M$  is the spherical cotangent bundle of a manifold  $M$  with its canonical contact structure. Then the  $B$ -stabilisation of  $X$  is naturally contactomorphic to the open subset

$$ST^*(M \times B) - \pi_B^*(ST^*B)$$

of the spherical cotangent bundle of the product  $M \times B$ , where  $\pi_B : M \times B \rightarrow B$  is the projection. Indeed, let  $\alpha$  be any contact form on  $ST^*M$ . There exists a unique fibrewise starshaped embedding  $\iota : ST^*M \hookrightarrow T^*M$  such that  $\iota^* \lambda_{\text{can}} = \alpha$ . The map

$$ST^*M \times T^*B \ni (\xi, \eta) \longmapsto [\iota(\xi) \oplus \eta] \in S(T^*M \times T^*B) = ST^*(M \times B)$$

defines a contactomorphism

$$(ST^*M \times T^*B, \ker(\alpha \oplus \lambda_{\text{can}})) \xrightarrow{\cong} ST^*(M \times B) - \pi_B^*(ST^*B).$$

### 3.3. Legendrian suspension

To each Legendrian family  $\mathcal{L}$  in  $X$  with base  $B$ , we can associate a Legendrian submanifold  $\tilde{\mathcal{L}}$  in the  $B$ -stabilisation of  $X$  called the *Legendrian suspension* of  $\mathcal{L}$ . Indeed, let  $F : \underline{L} \rightarrow X$  be a parametrisation of the family.

The pull-back  $F^*\alpha \in \Lambda^1(\underline{L})$  vanishes on the tangent spaces to the fibres of  $\pi : \underline{L} \rightarrow B$  and therefore there is a unique fibrewise map

$$\tilde{\alpha} : \underline{L} \longrightarrow T^*B$$

such that  $\tilde{\alpha}^*\lambda_{\text{can}} = F^*\alpha$ . Then  $\tilde{\mathcal{L}}$  is the image of the Legendrian embedding

$$(F, -\tilde{\alpha}) : \underline{L} \longrightarrow X \times T^*B.$$

Note that the Legendrian submanifold  $\tilde{\mathcal{L}}$  does not depend on the choice of the parametrisation of the family. Furthermore, if  $\alpha' = e^f\alpha$ , then the associated Legendrian submanifold  $\tilde{\mathcal{L}}'$  in  $(X \times T^*B, \ker(\alpha' \oplus \lambda_{\text{can}}))$  is the image of  $\tilde{\mathcal{L}}$  under the contactomorphism (3.1).

**Example 3.6.** The suspension of a constant family whose image is a Legendrian submanifold  $L \subset X$  is the product submanifold  $L \times N^*B \subset X \times T^*B = X^B$ , where  $N^*B$  is fancy notation for the zero section in  $T^*B$  (the zero section is the conormal bundle of  $B$  considered as a submanifold of itself).

**Example 3.7.** It is an immediate corollary of the definitions that a family of Legendrian submanifolds in  $X$  is transverse if and only if its Legendrian suspension does not intersect the subset

$$X \times N^*B \subset X \times T^*B = X^B,$$

where again  $N^*B$  is the zero section of  $T^*B$ .

**Example 3.8 (Suspension in  $ST^*M$ ).** In the case when  $X = ST^*M$ , it is natural to combine Examples 3.6 and 3.7 with the contactomorphism

$$(ST^*M)^B \xrightarrow{\cong} ST^*(M \times B) - \pi_B^*(ST^*B)$$

from Example 3.5:

(i) The suspension of a constant family with image  $L \subset ST^*M$  is identified with the pull-back  $\pi_M^*(L) \subset ST^*(M \times B)$ , where  $\pi_M$  is the projection on  $M$ .

(ii) The image in  $ST^*(M \times B)$  of the suspension of a transverse family in  $ST^*M$  is contained in the open set

$$ST^*(M \times B) - \pi_B^*(ST^*B) - \pi_M^*(ST^*M),$$

where  $\pi_B$  and  $\pi_M$  are the projections on  $B$  and  $M$ , respectively.

**Remark 3.9.** Eliashberg and Polterovich [14, §2.2] introduced contact stabilisation and Legendrian suspension in the case when  $B$  is the circle and for families of Legendrian submanifolds parametrised by product bundles  $L \times S^1$ , i.e. for Legendrian loops with trivial monodromy. Our Examples 3.6 and 3.7 are straightforward generalisations of the discussion of constant and positive Legendrian loops there.

### 3.4. Non-contractibility of transverse families

Two families of Legendrian submanifolds in  $(X, \ker \alpha)$  over the same base  $B$  are homotopic if they are restrictions to  $B \times \{0\}$  and  $B \times \{1\}$  of a family of Legendrian submanifolds over  $B \times [0, 1]$ . A family is called *contractible* if it is homotopic to a constant family.

**Theorem 3.10.** *Let  $M$  be a closed manifold. Suppose that an isotopy class of Legendrian submanifolds of  $ST^*M$  contains  $L_f$  for a function  $f : M \rightarrow \mathbb{R}$ . Then there is no contractible transverse family of Legendrian submanifolds over a closed base  $B$  in that class.*

*Proof.* The Legendrian suspensions of homotopic families over  $B$  are Legendrian isotopic. Using Example 3.5, we identify the  $B$ -stabilisation of  $ST^*M$  with  $ST^*(M \times B) - \pi_B^*(ST^*B)$ . The suspension of a constant family with image  $L_f$  is the pull-back  $\pi_M^*(L_f) \subset ST^*(M \times B)$ , whereas the suspension of a transverse family lies in  $ST^*(M \times B) - \pi_M^*(ST^*M)$ , see Example 3.8. Such Legendrian submanifolds cannot be Legendrian isotopic by Theorem 2.5 applied to the product bundle  $\pi_M : M \times B \rightarrow M$ . Hence, a transverse family in the Legendrian isotopy class of  $L_f$  can not be contractible.  $\square$

As a first and typical example, let us apply this theorem to the Legendrian isotopy class of the fibre of  $ST^*M$ . Since a positive Legendrian loop is a transverse family over the circle, we obtain the following result:

**Corollary 3.11.** *There are no contractible positive Legendrian loops in the Legendrian isotopy class of the fibre of  $ST^*M$ .*

This corollary can only be meaningful for a closed manifold  $M$  with finite fundamental group such that the integral cohomology ring of its universal cover is generated by a single element, because otherwise there are no positive loops in that Legendrian isotopy class whatsoever by [9, Corollary 8.1] and [16, Theorem 1.2].

**Remark 3.12 (A shortcut to a partial order on  $\widetilde{\text{Cont}}_0(ST^*M)$ ).** The nonexistence of contractible positive Legendrian loops in some Legendrian isotopy class on a contact manifold  $X$  implies the nonexistence of contractible positive loops of contactomorphisms in  $\text{Cont}_0(X)$ . Hence, combining Corollary 3.11 with [14, Criterion 1.2.C], we obtain a proof of the universal orderability of  $\text{Cont}_0(ST^*M)$  for any closed manifold  $M$  that is independent of [13], [9], and [2].

## 4. Non-negative Legendrian isotopies and partial orders

*All Legendrian submanifolds are henceforth assumed closed and connected.*

### 4.1. Non-negative Legendrian and contact isotopies

A Legendrian isotopy  $\{L_t\}_{t \in [0,1]}$  in a contact manifold  $(X, \ker \alpha)$  is called non-negative if some (and hence every) parametrisation  $\iota_t : L_0 \rightarrow L_t$  satisfies

$$(4.1) \quad \alpha \left( \frac{d}{dt} \iota_t(x) \right) \geq 0$$

for all  $t \in [0, 1]$  and  $x \in L_0$ . Similarly, an isotopy of contactomorphisms  $\{\varphi_t\}_{t \in [0,1]}$  is called non-negative if its contact Hamiltonian

$$(4.2) \quad H(\varphi_t(x), t) := \alpha \left( \frac{d}{dt} \varphi_t(x) \right) \geq 0$$

for all  $t \in [0, 1]$  and  $x \in X$ . If the inequalities in (4.1) and (4.2) are strict, the isotopies are called positive.

**Proposition 4.1.** *If  $\{L_t\}_{t \in [0,1]}$  is a non-negative Legendrian isotopy, then there exists a compactly supported non-negative contact isotopy  $\{\varphi_t\}_{t \in [0,1]}$  such that  $\varphi_0 = \text{id}_X$  and  $\varphi_t(L_0) = L_{\chi(t)}$  for a non-decreasing function  $\chi : [0, 1] \rightarrow [0, 1]$ .*

**Remark 4.2.** The Legendrian isotopy extension theorem asserts that for every parametrisation  $\iota_t : L_0 \rightarrow L_t$  of an arbitrary Legendrian isotopy, there is a contact isotopy  $\varphi_t$  such that  $\varphi_0 = \text{id}$  and  $\varphi_t|_{L_0} = \iota_t$ , see e.g. [17, Theorem 2.6.2]. However, it is not true that there exists a non-negative contact extension for every parametrisation of a non-negative Legendrian isotopy. For instance, a non-negative contact isotopy can only extend the constant parametrisation of a constant isotopy, cf. [18, Proof of Lemma 4.12(i)].

*Proof of Proposition 4.1.* By the Legendrian version of the Darboux–Weinstein theorem, a Legendrian submanifold  $L$  has a neighbourhood  $U$  contactomorphic to a neighbourhood  $U'$  of the zero section in the 1-jet bundle  $\mathcal{J}^1(L) = \mathbb{R} \oplus T^*L$  with its canonical contact structure  $\ker(du - \lambda_{\text{can}})$ . Any Legendrian submanifold sufficiently close to  $L$  in  $C^1$ -topology is then represented as the graph of the 1-jet of a function  $f : L \rightarrow \mathbb{R}$ . Non-negativity of a Legendrian isotopy of such graphs means simply that, for the corresponding functions  $f_t$  on  $L$ , the value  $f_t(q)$  is a non-decreasing function of  $t$  for every  $q \in L$ . In this situation, it is easy to produce the required contact extension supported in the neighbourhood  $U$ . Indeed, suppose that  $f_t : L \rightarrow \mathbb{R}$  are functions such that  $f_0 \equiv 0$  and  $\dot{f}_t \geq 0$ . The contact isotopy of  $\mathcal{J}^1(L)$  given by

$$(4.3) \quad \xi \longmapsto \xi + j^1 f_t(q) \quad \text{for } \xi \in \mathcal{J}_q^1(L), q \in L,$$

is non-negative and extends the Legendrian isotopy of the graphs. A compactly supported contact isotopy of  $U \cong U' \subset \mathcal{J}^1(L)$  with the same properties is obtained by multiplying the contact Hamiltonian of the isotopy (4.3) by a non-negative cut-off function  $\psi : U' \rightarrow \mathbb{R}$  that is equal to 1 on a slightly smaller neighbourhood of the zero section.

Now let us choose a subdivision  $0 = t_0 < t_1 < \dots < t_{k-1} < t_k = 1$  of the segment  $[0, 1]$  such that  $L_t$ ,  $t \in [t_j, t_{j+1}]$ , are sufficiently close to  $L_{t_j}$  in the sense of the preceding paragraph. Then for each  $j = 0, \dots, k-1$  we obtain a compactly supported non-negative contact isotopy  $\{\varphi_{j,t}\}_{t \in [t_j, t_{j+1}]}$  such that  $\varphi_{j,t_j} = \text{id}_X$  and  $\varphi_{j,t}(L_{t_j}) = L_t$ . The required isotopy  $\{\varphi_t\}_{t \in [0,1]}$  is a smoothed concatenation of the isotopies  $\{\varphi_{j,t}\}$ . Namely, let  $\chi : [0, 1] \rightarrow [0, 1]$  be any non-decreasing smooth function such that

- 1)  $\chi(t_j) = t_j$  for all  $j = 0, \dots, k$ ;
- 2)  $\{t \mid \chi'(t) = 0\} = \{t \mid \chi^{(n)}(t) = 0 \text{ for all } n \in \mathbb{N}\} = \{t_1, \dots, t_{k-1}\}$ .

Set  $\tilde{\varphi}_{j,t} := \varphi_{j,\chi(t)}$  and define

$$\varphi_t := \tilde{\varphi}_{j,t} \circ \tilde{\varphi}_{j-1,t_j} \circ \dots \circ \tilde{\varphi}_{0,t_1}$$

for  $t \in [t_j, t_{j+1}]$ . □

**Remark 4.3.** Note for future use that the derivative of  $\chi$  does not vanish at  $t_0 = 0$ .

## 4.2. Non-negative and positive Legendrian loops

A Legendrian loop is a family of Legendrian submanifolds over the circle. It is often convenient to view loops as Legendrian isotopies  $\{L_t\}_{t \in [0,1]}$  such that  $L_0 = L_1$  and the gluing at the endpoints is smooth. If the circle is oriented, non-negative and positive Legendrian loops are defined in the obvious way.

**Lemma 4.4.** *If a non-negative Legendrian loop  $\{L_\theta\}_{\theta \in S^1}$  is positive on some non-empty interval  $I \subset S^1$ , then it can be  $C^\infty$ -approximated by a positive Legendrian loop.*

*Proof.* Let  $[\theta_1, \theta_2] \subset I$  be an interval that is small enough so that  $L_\theta$ ,  $\theta \in [\theta_1, \theta_2]$ , can be represented as graphs of 1-jets of functions  $f_\theta$  in  $\mathcal{J}^1(L_{\theta_2})$ . The positivity of the loop on  $I$  implies that this family of functions is pointwise *strictly* increasing. In particular, the submanifolds  $L_{\theta_1}$  and  $L_{\theta_2}$  are disjoint.

The restriction of our loop to  $S^1 - (\theta_1, \theta_2)$  is a non-negative Legendrian isotopy with *disjoint* ends. By [9, Lemma 2.2] it can be  $C^\infty$ -approximated by a positive Legendrian isotopy  $\{\tilde{L}_\theta\}_{\theta \in S^1 - (\theta_1, \theta_2)}$  with the same ends.

It remains to interpolate between  $f_{\theta_1} < 0$  and  $f_{\theta_2} \equiv 0$  by a new strictly increasing family of functions on  $L_{\theta_2}$  so that the union of the corresponding Legendrian isotopy over  $[\theta_1, \theta_2]$  with  $\{\tilde{L}_\theta\}_{\theta \in S^1 - (\theta_1, \theta_2)}$  is a smooth positive loop.  $\square$

**Proposition 4.5.** *If a Legendrian isotopy class contains a non-constant non-negative (contractible) Legendrian loop, then it contains a positive (contractible) Legendrian loop.*

*Proof.* A non-constant non-negative Legendrian loop in  $(X, \ker \alpha)$  defines a non-negative Legendrian isotopy  $\iota_t : L_0 \rightarrow L_t$ ,  $t \in [0, 1]$ , such that  $\iota_1(L_0) = L_0$  and

$$\alpha \left( \frac{d}{dt} \iota_t \Big|_{t=0} (x_0) \right) > 0$$

for some  $x_0 \in L_0$ . By Proposition 4.1 and Remark 4.3, there exists a compactly supported non-negative contact isotopy  $\{\varphi_t\}_{t \in [0,1]}$  such that  $\varphi_0 \equiv \text{id}_X$ ,  $\varphi_t(L_0) = L_{\chi(t)}$ , and

$$\alpha \left( \frac{d}{dt} \varphi_t \Big|_{t=0} (x_0) \right) = \chi'(0) \alpha \left( \frac{d}{dt} \iota_t \Big|_{t=0} (x_0) \right) > 0.$$

Let  $U \subseteq L_0$  be a neighbourhood of  $x_0$  on which this strict inequality continues to hold.

Since  $L_0$  is compact and connected, there exist contactomorphisms  $\Psi_j \in \text{Cont}_0(X, \ker \alpha)$ ,  $j = 0, \dots, k$ , such that



- 1)  $\Psi_0 = \text{id}_X$ ;
- 2)  $\Psi_j(L_0) = L_0$  for all  $j$ ;
- 3)  $\Psi_j$  is isotopic to  $\text{id}_X$  within the class of contactomorphisms preserving  $L_0$ ;
- 4)  $\bigcup_{j=0}^k \Psi_j(U) = L_0$ .

To construct  $\Psi_j$ , note first that there exist diffeomorphisms of  $L_0$  isotopic to  $\text{id}_{L_0}$  and having property (4). These diffeomorphisms extend to contactomorphisms of  $X$  with the required properties by the Legendrian isotopy extension theorem.

Let  $\psi_j := (\Psi_j)^{-1} \circ \Psi_{j-1}$  for  $j = 1, \dots, k$  so that  $(\Psi_j)^{-1} = \psi_j \circ \dots \circ \psi_1$  for each  $j = 1, \dots, k$ . Consider the contact isotopy

$$(4.4) \quad \tilde{\varphi}_t := \varphi_t \circ \psi_k \circ \varphi_t \circ \dots \circ \psi_1 \circ \varphi_t, \quad t \in [0, 1].$$

By property (4) of  $\Psi_j$  and the choice of  $U$ , we have

$$\alpha\left(\frac{d}{dt}\tilde{\varphi}_t|_{t=0}(x)\right) > 0$$

for all  $x \in L_0$ . Hence, the Legendrian isotopy

$$\tilde{L}_t := \tilde{\varphi}_t(L_0)$$

is positive on some interval  $[0, \varepsilon)$ ,  $\varepsilon > 0$ . Since  $\tilde{L}_0 = \tilde{L}_1$  by property (2), smoothing  $\{\tilde{L}_t\}$  at 0 and 1 gives us a non-negative Legendrian loop that is positive on a slightly smaller interval  $(\varepsilon', \varepsilon)$ ,  $\varepsilon > \varepsilon' > 0$ . By Lemma 4.4, this loop can be approximated by an everywhere positive loop.

Finally, it follows from formula (4.4) and property (3) of  $\Psi_j$  that the obtained positive loop is homotopic to the  $(k + 1)$ -fold iteration of the original Legendrian loop. In particular, starting with a contractible non-constant non-negative Legendrian loop, we get a contractible positive one.  $\square$

**Remark 4.6.** The proof of the proposition is a modification of the second step in the proof of [14, Proposition 2.1.B].

### 4.3. Partial orders on Legendrian isotopy classes and their universal coverings

Let  $L$  be a Legendrian submanifold in a contact manifold  $(X, \ker \alpha)$ . Denote by  $\text{Leg}(L)$  the Legendrian isotopy class of  $L$ , i.e. the space of all Legendrian submanifolds Legendrian isotopic to  $L$  with  $C^\infty$ -topology, and let  $\Pi : \widetilde{\text{Leg}}(L) \rightarrow \text{Leg}(L)$  be the universal covering of this space.

For  $L_1, L_2 \in \text{Leg}(L)$ , write  $L_1 \preceq L_2$  if there is a non-negative Legendrian isotopy connecting  $L_1$  to  $L_2$ . This partial relation admits a natural lift to  $\widetilde{\text{Leg}}(L)$ . For  $\ell_1, \ell_2 \in \widetilde{\text{Leg}}(L)$ , write  $\ell_1 \preceq \ell_2$  if there exists a path  $\gamma \subset \widetilde{\text{Leg}}(L)$  connecting  $\ell_1$  to  $\ell_2$  such that  $\Pi(\gamma)$  is a non-negative Legendrian isotopy.

It is clear that  $\preceq$  and  $\preceq$  are reflexive and transitive. The Legendrian isotopy class  $\text{Leg}(L)$  is called *orderable* if  $\preceq$  is a partial order on it, i.e. if  $\preceq$  is also antisymmetric:

$$L_1 \preceq L_2 \text{ and } L_2 \preceq L_1 \implies L_1 = L_2.$$

$\text{Leg}(L)$  is called *universally orderable* if  $\preceq$  is a partial order on  $\widetilde{\text{Leg}}(L)$ .

**Proposition 4.7.** *A Legendrian isotopy class is orderable if and only if it does not contain a positive Legendrian loop and universally orderable if and only if it does not contain a contractible positive Legendrian loop.*

*Proof.* The ‘only if’ part is obvious from the definitions. It is equally obvious that a class is (universally) orderable if it does not contain a (contractible) non-constant non-negative Legendrian loop. The result follows now from Proposition 4.5.  $\square$

**Remark 4.8.** Proposition 4.7 is a Legendrian version of [14, Criterion 1.2.C].

The following orderability result was obtained in [9] in the case when  $V$  is a point and is a consequence of [18, Corollary 4.14] and Proposition 4.1 in the general case.

**Theorem 4.9.** *Let  $M$  be a manifold with non-compact universal covering and  $V \subset M$  a simply connected closed submanifold of codimension  $\geq 2$ . Then the Legendrian isotopy class  $\text{Leg}(SN^*V)$  of the spherical conormal bundle of  $V$  is orderable.*

However,  $\text{Leg}(SN^*V)$  is not orderable in general. For instance, the condition that the universal covering of  $M$  is non-compact is necessary if  $\dim M \leq$

3, see [9, Example 8.3]. The assumption that  $V$  is simply connected can not be removed either, as shown by the example of  $V = S^1 \times \{\text{pt}\} \subset S^1 \times S^2 = M$ . Nevertheless, orderability can be restored by passing to the universal covering of the Legendrian isotopy class.

**Theorem 4.10.** *The Legendrian isotopy class  $\text{Leg}(SN^*V)$  of the spherical conormal bundle of a connected closed submanifold  $V \subset M$  of codimension  $\geq 2$  is **universally** orderable.*

*Proof.* Suppose that  $\text{Leg}(SN^*V)$  is not universally orderable. Then it contains a contractible positive Legendrian loop by Proposition 4.7. Since this loop and its homotopy to a constant loop are compact, we may assume that  $M$  is a closed manifold. Recall now that  $SN^*V$  is Legendrian isotopic to  $L_f$  for a suitable function  $f : M \rightarrow \mathbb{R}$ , see Example 2.4. Hence, the existence of a contractible positive Legendrian loop (i.e. of a contractible transverse family over the circle) in  $\text{Leg}(SN^*V)$  contradicts Theorem 3.10.  $\square$

**Remark 4.11.** The argument in the proof of Theorem 4.10 shows that  $\text{Leg}(L_f)$  is universally orderable for every function  $f : M \rightarrow \mathbb{R}$  on a closed manifold such that 0 is not a critical value and the hypersurface  $\{f = 0\}$  is connected.

## References

- [1] P. Albers and U. Frauenfelder, *A variational approach to Givental's nonlinear Maslov index*, *Geom. Funct. Anal.*, **22** (2012), 1033–1050.
- [2] P. Albers and W. J. Merry, *Orderability, contact non-squeezing, and Rabinowitz Floer homology*, Preprint, [arXiv:1302.6576](https://arxiv.org/abs/1302.6576).
- [3] V. I. Arnol'd, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts*, *Monographs in Mathematics* **82**, Birkhäuser Boston, Inc., Boston, MA, 1985.
- [4] A. Bernal and M. Sánchez, *On smooth Cauchy hypersurfaces and Geroch's splitting theorem*, *Comm. Math. Phys.*, **243** (2003), 461–470.
- [5] A. Bernal and M. Sánchez, *Globally hyperbolic spacetimes can be defined as "causal" instead of "strongly causal"*, *Class. Quant. Grav.*, **24** (2007), 745–750.

- [6] M. Bhupal, *A partial order on the group of contactomorphisms of  $\mathbb{R}^{2n+1}$  via generating functions*, Turkish J. Math., **25** (2001), 125–135.
- [7] M. S. Borman and F. Zapolsky, *Quasimorphisms on contactomorphism groups and contact rigidity*, Geom. Topol., **19** (2015), 365–411.
- [8] V. Chernov and S. Nemirovski, *Legendrian links, causality, and the Low conjecture*, Geom. Funct. Anal., **19** (2010), 1320–1333.
- [9] V. Chernov and S. Nemirovski, *Non-negative Legendrian isotopy in  $ST^*M$* , Geom. Topol., **14** (2010), 611–626.
- [10] V. Chernov (Tchernov) and Yu. Rudyak, *Linking and causality in globally hyperbolic space-times*, Comm. Math. Phys., **279** (2008), 309–354.
- [11] V. Colin, E. Ferrand, and P. Pushkar, *Positive isotopies of Legendrian submanifolds and applications*, Preprint, arXiv:1004.5263 and <http://people.math.jussieu.fr/~ferrand/publi/PIL.pdf>.
- [12] Ya. Eliashberg and M. Gromov, *Lagrangian intersection theory: finite-dimensional approach*, Geometry of differential equations, 27–118, Amer. Math. Soc. Transl. Ser. 2, **186**, AMS, Providence, RI, 1998.
- [13] Y. Eliashberg, S. S. Kim, and L. Polterovich, *Geometry of contact transformations and domains: orderability versus squeezing*, Geom. Topol., **10** (2006), 1635–1747; *Erratum*, Geom. Topol., **13** (2009), 1175–1176.
- [14] Y. Eliashberg and L. Polterovich, *Partially ordered groups and geometry of contact transformations*, Geom. Funct. Anal., **10** (2000), 1448–1476.
- [15] E. Ferrand, *On a theorem of Chekanov*, Symplectic singularities and geometry of gauge fields (Warsaw, 1995), 39–48, Banach Center Publ., **39**, Polish Acad. Sci., Warsaw, 1997.
- [16] U. Frauenfelder, C. Labrousse, and F. Schlenk, *Slow volume growth for Reeb flows on spherizations and contact Bott–Samelson theorems*, J. Topol. Anal., **7** (2015), 407–451.
- [17] H. Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008.
- [18] S. Guillermou, M. Kashiwara, and P. Schapira, *Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems*, Duke Math. J., **161** (2012), 201–245.

- [19] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
- [20] R. J. Low, *Causal relations and spaces of null geodesics*, DPhil Thesis, Oxford University (1988).
- [21] R. J. Low, *The space of null geodesics*, Proceedings of the Third World Congress of Nonlinear Analysts, Part 5 (Catania, 2000). *Nonlinear Anal.*, **47** (2001), 3005–3017.
- [22] D. B. Malament, *The class of continuous timelike curves determines the topology of spacetime*, *J. Mathematical Phys.*, **18** (1977), 1399–1404.
- [23] J. Natário, and P. Tod, *Linking, Legendrian linking and causality*, *Proc. London Math. Soc.* (3), **88** (2004), 251–272.
- [24] R. Penrose, *The complex geometry of the natural world*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 189–194, Acad. Sci. Fennica, Helsinki, 1980.
- [25] P. E. Pushkar, *On Hamiltonian and contact isotopy liftings*, Preprint, [arXiv:1602.07948](https://arxiv.org/abs/1602.07948).
- [26] P. E. Pushkar, *Chekanov-type theorem for spherized cotangent bundles*, Preprint, [arXiv:1602.08743](https://arxiv.org/abs/1602.08743).
- [27] S. Sandon, *Equivariant homology for generating functions and orderability of lens spaces*, *J. Symplectic Geom.*, **9** (2011), 123–146.
- [28] C. T. C. Wall, *Surgery on compact manifolds*, London Mathematical Society Monographs, No. 1. Academic Press, London–New York, 1970.
- [29] F. Zapolsky, *Geometry of contactomorphism groups, contact rigidity, and contact dynamics in jet spaces*, *Int. Math. Res. Notices* (2013), no. 20, 4687–4711.

DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE  
6188 KEMENY HALL, HANOVER, NH 03755-3551, USA  
*E-mail address:* `Vladimir.Chernov@dartmouth.edu`

STEKLOV MATHEMATICAL INSTITUTE  
GUBKINA 8, 119991 MOSCOW, RUSSIA  
& MATHEMATISCHES INSTITUT, RUHR-UNIVERSITÄT BOCHUM  
44780 BOCHUM, GERMANY  
*E-mail address:* `stefan@mi.ras.ru`

RECEIVED AUGUST 23, 2013