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On hofer energy of J-holomorphic curves for asymptotically cylindrical J

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In this paper, we provide a bound for the generalized Hofer energy of punctured J-holomorphic curves in almost complex manifolds with asymptotically cylindrical ends. As an application, we prove a version of Gromov's Monotonicity Theorem with multiplicity. Namely, for a closed symplectic manifold (M, ω')[1]with a compatible almost complex structure J and a ball B in M, there exists a constant $\hbar > 0$, such that any J-holomorphic curve $\tilde{u}$ passing through the center of B for k times (counted with multiplicity) with boundary mapped to ∂B has symplectic area $\int_{\tilde{u}^{-1}(B)} \tilde{u}^*\omega' > k\hbar$, where the constant $\hbar$ depends only on (M, ω', J) and the radius of B. As a consequence, the number of times that any closed J-holomorphic curve in M passes through a point is bounded by a constant depending only on (M, ω', J) and the symplectic area of $\tilde{u}$. Here J is any $\omega'-$compatible smooth almost complex structure on M. In particular, we do not require J to be integrable.

1. Introduction

Hofer energy is introduced in [10] for J-holomorphic curves in symplectization of contact manifolds, and is generalized in [5] for J-holomorphic curves in the "almost complex manifolds with cylindrical ends". Here "cylindrical" means that the almost complex structure J is invariant under translation. Hofer energy plays an essential role in the study of J-holomorphic curves in Symplectic Field Theory mainly because of the following two properties: (A) the asymptotic behavior of a J-holomorphic curve in a noncompact symplectic manifold can be controlled by requiring its Hofer energy to be finite,

[1]Following the notation in [5] we save ω for something else.

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and hence a uniform Hofer energy bound gives a Symplectic Field Theory type of compactification of moduli spaces of J-holomorphic curves; on the other hand, (B) a uniform Hofer energy bound can be obtained by specifying the behavior the J-holomorphic curves at infinity and bounding their symplectic areas (see [5, 10]). In [2] the notion of Hofer energy and Property (A) are further generalized to include J-holomorphic curves in "almost complex manifolds with asymptotically cylindrical ends". Here "asymptotically cylindrical" means that the difference between the almost complex structure J and a translation invariant one is exponentially small. In this paper, we prove Property (B) in this setting. Property (A) and property (B) together imply the expected useful compactness results in Symplectic Field Theory.

One of the main advantages of this generalization is that the asymptotically cylindrical J arises naturally. As an application, we prove a version of Gromov's Monotonicity Theorem with multiplicity[2], namely for a closed symplectic manifold (M, ω') with a compatible almost complex structure J and a ball B in M, there exists a constant $\hbar > 0$, such that any J-holomorphic curve $\tilde{u}$ passing through the center of B k times (counted with multiplicity) with the boundary mapped to ∂B has symplectic area $\int_{\tilde{u}^{-1}(B)} \tilde{u}^*\omega' > k\hbar$, where the constant $\hbar$ depends only on (M, ω', J) and the radius of B.

The inequality $k < \frac{1}{\hbar} \int_{\tilde{u}^{-1}(B)} \tilde{u}^*\omega'$ is closely related to a question asked in [6], where they study J-holomorphic curves with boundaries lying inside two clean intersecting Lagrangian submanifolds, and prove that the number of "boundary switches" at the intersecting loci is uniformly bounded by the Hofer Energy. Their proof in an essential way relies on the additional requirement that the almost complex structure J is integrable near the intersecting loci. They ask to what extent their results are still true without assuming the integrability of J. In this paper, we provide a simple proof for the closed version of their result for arbitrary J. Namely, the J-holomorphic curves we consider in this paper have no boundaries. In this case, "boundary switches" just means that the J-holomorphic curve passes a fixed point in M. Furthermore, the analysis developed in [2] and this paper can be carried out to include Lagrangians without difficulty (see for example section 5 in [2] for the setup).

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[2]This can also be derived from [8]. See Remark 14.

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2. Asymptotically cylindrical almost complex structure

Let V_- be a smooth closed oriented manifold of dimension $2N-1$, and J be a smooth almost complex structure on $W_- = \mathbb{R}^- \times V_-$ such that the orientation of W_- induced from J conincides with the one induced from the standard of orientation of $\mathbb{R}^-$ and the orientation of V_-. Let $\mathbf{R}$ be the smooth vector field on W_- defined by $\mathbf{R} := J\left(\frac{\partial}{\partial r}\right)$, and ξ be the subbundle of the tangent bundle TW_- defined by $\xi_{(r,v)} = (JT_v(\{r\} \times V_-)) \cap (T_v(\{r\} \times V_-))$, for $(r,v) \in W_-$. Then the tangent bundle TW_- splits as $TW_- = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi$. Define the 1-forms λ and σ on W_- respectively by

$$\lambda(\xi) = 0 \qquad \lambda\left(\frac{\partial}{\partial r}\right) = 0 \qquad \lambda(\mathbf{R}) = 1, \tag{1}$$

$$\sigma(\xi) = 0 \qquad \sigma\left(\frac{\partial}{\partial r}\right) = 1 \qquad \sigma(\mathbf{R}) = 0. \tag{2}$$

Let $f_s : W_- \to W_-$ be the translation $f_s(r,v) = (r+s, v)$, for $s \leqq 0$. We call a tensor on W_- translationally invariant if it is invariant under f_s.

Definition 1. Under the above notations, J is called asymptotically cylindrical at negative infinity, if J satisfies (ACC1)–(ACC5):

- (ACC1) There exist a smooth translationally invariant almost complex structure $J_{-\infty}$ on W_- and constants $K_l, \delta_l > 0$, such that restricted to the region $(-\infty, r] \times V_-$

$$\|J - J_{-\infty}\|_l \leqq K_l e^{\delta_l r} \tag{3}$$

  for all $r \leqq 0$ and $l \in \mathbb{Z}_{\geqq 0}$, where $\|\cdot\|_k$ is the C^k-norm defined by $\|\varphi\|_k := \sup\limits_w \sum_{i=0}^k \left|\nabla^i \varphi(w)\right|$ and $|\cdot|$ is computed using a translationally invariant metric g_{W_-} on W_-, for example $g_{W_-} = dr^2 + g_{V_-}$, and ∇ is the corresponding Levi-Civita connection.

- (ACC2) $i(\mathbf{R}_{-\infty})d\lambda_{-\infty} = 0$, where $\mathbf{R}_{-\infty} := \lim\limits_{s\to-\infty} f_s^* \mathbf{R}$, $\lambda_{-\infty} := \lim\limits_{s\to-\infty} f_s^* \lambda$, and both limits exist by (ACC1).

- (ACC3) $\mathbf{R}_{-\infty}(r,v) \in T_v(\{r\} \times V_-)$, i.e. $\mathbf{R}_{-\infty}$ is tangent to the level sets.

There exists a translationally invariant closed 2-form $\omega_{-\infty}$ on W_- such that

- (ACC4) $i\left(\frac{\partial}{\partial r}\right)\omega_{-\infty} = 0 = i(\mathbf{R}_{-\infty})\omega_{-\infty}$.
- (ACC5) $\omega_{-\infty}|_{\xi_{-\infty}}(\cdot, J_{-\infty}\cdot)$ is a metric on $\xi_{-\infty} := \lim\limits_{s\to-\infty} f_s^*\xi$.

When we say J is asymptotically cylindrical, we choose $\omega_{-\infty}$ without mentioning.

Similarly, we could define the notion of J being asymptotically cylindrical at positive infinity for $W_+ = \mathbb{R}^+ \times V_+$.

Notice that this definition is equivalent to the definition given in [2]. In [2] for J being asymptotically cylindrical, besides (ACC1)-(ACC5) we require that there exists a 2-form ω on W_- such that

- (a) $i\left(\frac{\partial}{\partial r}\right)\omega = 0 = i(\mathbf{R})\omega$.
- (b) $\omega|_\xi(\cdot, J\cdot)$ is a metric on ξ.
- (c) There exist constants $K_l, \delta_l \geqq 0$, such that

$$\left\| (\omega - \omega_{-\infty})|_{(-\infty,r]\times V_-} \right\|_l \leqq K_l e^{\delta_l r} \tag{4}$$

  for all $r \leqq 0$ and $l \in \mathbb{Z}_{\geqq 0}$.

Indeed, take

$$\omega(x,y) = \frac{1}{2}\left[\omega_{-\infty}(\pi_\xi x, \pi_\xi y) + \omega_{-\infty}(J\pi_\xi x, J\pi_\xi y)\right]$$

for $x, y \in T_{(r,v)}W^-$. Then (a) is satisfied. From (ACC1) and (ACC4) we can see that (c) is satisfied. Notice

$$\begin{aligned}
\omega(Jx,Jy) &= \frac{1}{2}\left[\omega_{-\infty}(\pi_\xi Jx, \pi_\xi Jy) + \omega_{-\infty}(J\pi_\xi Jx, J\pi_\xi Jy)\right]\\
&= \frac{1}{2}\left[\omega_{-\infty}(J\pi_\xi x, J\pi_\xi y) + \omega_{-\infty}(\pi_\xi x, \pi_\xi y)\right]\\
&= \omega(x,y).
\end{aligned}$$

Hence $\omega|_\xi(\cdot, J\cdot)$ is symmetric. For $x \in \xi_{(r,v)}$, we have

$$\begin{aligned}\omega(x, Jx) &= \frac{1}{2}\left[\omega_{-\infty}(\pi_\xi x, \pi_\xi Jx) + \omega_{-\infty}(J\pi_\xi x, J\pi_\xi Jx)\right] \\ &= \frac{1}{2}\left[\omega_{-\infty}(x, Jx) + \omega_{-\infty}(Jx, -x)\right] \\ &= \omega_{-\infty}(x, Jx).\end{aligned}$$

Because that $\omega_{-\infty}(x, J_{-\infty}x)$ is positive on every nonzero vector $x \in \xi_{-\infty}$, we have $\omega(\cdot, J_{-\infty}\cdot)|_S > \varpi > 0$, for some ϖ, where

$$S := \left\{ (x, y) \in \xi_{-\infty} \times \xi_{-\infty} |\ \|x\|_{g_{W_-}} = 1, y = J_{-\infty}x \right\}.$$

When r is sufficiently negative, by (ACC1), (x, Jx) is uniformly close to S, for all $x \in \xi_{(r,v)}$. Therefore, for $0 \neq x \in \xi_{(r,v)}$, we obtain $\omega(x, Jx) = \omega_{-\infty}(x, Jx) > 0$, and hence (b). Since we restrict ourselves to the behaviors of J-holomorphic curves near infinity, for the purpose of simplifying the notations, we assume ω satisfies (b) for $r \leqq 0$.

Remark 2. (ACC1)-(ACC5) imply that $(V_-, \omega_{-\infty})$ is a stable hamiltonian structure and $(\lambda_{-\infty}, J_{-\infty})$ is a framing of the stable hamiltonian structure (See [7] for the definition of stable hamiltonian structure. In this paper we do not need it).

Definition 3. We say an asymptotically cylindrical almost complex structure J is of contact type if $\omega_{-\infty} = d\lambda_{-\infty}$.

The following definition is the case considered in [4, 5, 10, 11].

Definition 4. We say J is a cylindrical almost complex structure, if J is an asymptotically cylindrical almost complex structure and translationally invariant.

By (ACC2) and (ACC3) we can see that $\mathbf{R}_{-\infty}$ is a translationally invariant vector field on W_- and it is tangent to each level set $\{r\} \times V_-$, so we can view $\mathbf{R}_{-\infty}$ as a vector field on V_-. Let ϕ^t be the flow of $\mathbf{R}_{-\infty}$ on V_-, i.e. $\phi^t : V_- \to V_-$ satisfies $\frac{d}{dt}\phi^t = \mathbf{R}_{-\infty} \circ \phi^t$. Then we have

$$\frac{d}{dt}[(\phi^t)^*\lambda_{-\infty}] = (\phi^t)^*(i_{\mathbf{R}_{-\infty}} d\lambda_{-\infty} + d i_{\mathbf{R}_{-\infty}} \lambda_{-\infty}) = 0.$$

Thus ϕ^t preserves $\lambda_{-\infty}$ and hence $\xi_{-\infty}$. Similarly ϕ^t preserves $\omega_{-\infty}$.

Let's denote by $\mathcal{P}_-$ the set of periodic trajectories, counting their multiples, of the vector field $\mathbf{R}_{-\infty}$ restricting to V_-. Notice that any smooth family of periodic trajectories from $\mathcal{P}_-$ have the same period by Stokes' Theorem and (ACC2).

Definition 5. We say that an asymptotically cylindrical J is Morse-Bott if, for every $T > 0$ the subset $N_T \subseteq V_-$ formed by the closed trajectories from $\mathcal{P}_-$ of period T is a smooth closed submanifold of V_-, such that the rank of $\omega_{-\infty}|_{N_T}$ is locally constant and $T_pN_T = \ker\left(d\phi^T - Id\right)_p$.

In this paper, we assume that J is Morse-Bott. The Morse-Bott condition is the condition assumed in [2] to guarantee Theorem 6, Lemma 7 and Theorem 9. For the application in section 4, it is easy to check that this requirement is satisfied.

Let $\Sigma := \mathbb{R}^- \times S^1$ be the half cylinder with standard almost complex structure j, and $\tilde{u} = (a, u) : (\Sigma, j) \to (W_-, J)$ be a J-holomorphic curve, i.e. $T\tilde{u} \circ j = J(\tilde{u}) \circ T\tilde{u}$. The ω-energy and λ-energy of $\tilde{u}$ are defined as follows respectively

$$E_\omega(\tilde{u}) = \int_\Sigma \tilde{u}^*\omega,$$
$$E_\lambda(\tilde{u}) = \sup_{\phi\in\mathcal{C}} \int_\Sigma \tilde{u}^*(\phi(r)\sigma \wedge \lambda),$$

where $\mathcal{C} = \{\phi \in C^\infty(\mathbb{R}^-, [0, 1]) | \int_{-\infty}^0 \phi(x)dx = 1\}$, and λ and σ are defined as in (1) and (2). The Hofer energy of $\tilde{u}$ is defined by

$$E(\tilde{u}) = E_\omega(\tilde{u}) + E_\lambda(\tilde{u}).$$

Let's equip $\mathbb{R}^- \times S^1$ with coordinate (s, t). Here we view S^1 as $\mathbb{R}/\mathbb{Z}$. It is easy to check that $\tilde{u}^*\omega$ and $\tilde{u}^*(\phi(r)\sigma \wedge \lambda)$ are non-negative multiples of the volume form $ds \wedge dt$ on $\mathbb{R}^- \times S^1$. Actually,

$$\tilde{u}^*\omega = \omega(\pi_\xi\tilde{u}_s, J(\tilde{u})\pi_\xi\tilde{u}_s)ds \wedge dt, \tag{5}$$

where π_ξ is the projection from $TW_- = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi$ to ξ, and

$$\tilde{u}^*(\phi(r)\sigma \wedge \lambda) = \phi(a)\left[\sigma(\tilde{u}_s)^2 + \lambda(\tilde{u}_s)^2\right] ds \wedge dt. \tag{6}$$

The non-negativity is the main reason that we choose the Hofer energy in this form.

The following theorem is one of the most important theorems in [4, 5, 10, 11] for the case when J is cylindrical, and it is proved in the asymptotically cylindrical setting in [2].

Theorem 6. *Suppose that J is an asymptotically cylindrical almost complex structure on $W_- = \mathbb{R}^- \times V_-$. Let $\tilde{u} = (a, u) : \mathbb{R}^- \times \mathbb{R}/\mathbb{Z} \to W_-$ be a J-holomorphic curve with finite Hofer energy. Suppose that the image of $\tilde{u}$ is unbounded in W_-. Then there exists a periodic orbit γ of $\mathbf{R}_{-\infty}$ of period $|T|$ with $T \neq 0$, such that*

$$\lim_{s\to-\infty} u(s,t) = \gamma(Tt) \tag{7}$$

$$\lim_{s\to-\infty} \frac{a(s,t)}{s} = T \tag{8}$$

in $C^\infty(S^1)$.

On the other hand, we have

Lemma 7. *Suppose that J is an asymptotically cylindrical almost complex structure on $W_- = \mathbb{R}^- \times V_-$, and $\tilde{u} = (a, u) : \mathbb{R}^- \times \mathbb{R}/\mathbb{Z} \to W_-$ is a J-holomorphic curve. Suppose that there exits a periodic orbit γ of $\mathbf{R}_{-\infty}$ of period $|T|$ such that*

$$\lim_{s\to-\infty} a(s,t) = -\infty,$$
$$\lim_{s\to-\infty} u(s,t) = \gamma(Tt).$$

Then

$$\lim_{s\to-\infty} \frac{a(s,t)}{s} = T,$$

and Hofer energy $E(\tilde{u}) < \infty$.

Proof. This follows immediately from the proof of Theorem 2 in [2]. Namely, from the assumption, we could derive that the convergence in (7) and (8) is exponentially fast. Then it follows by definition and direct calculation that $E(\tilde{u}) < \infty$. □

Remark 8. Theorem 6 and Lemma 7 also hold for W_+.

3. Almost complex manifolds with asymptotically cylindrical ends

Now we introduce the notion of almost complex manifolds with asymptotically cylindrical ends.

Let (E, J) be a $2N$ dimensional noncompact almost complex manifold, and $W_\pm$ be an open subset containing the positive (negative) end of E. Assume that $W_\pm$ is diffeomorphic to $\mathbb{R}^\pm \times V_\pm$, where $V_\pm$ is a $2N-1$ dimensional closed manifold. Assume that there exists a J-compatible symplectic form ω' on E, and that $J|_{W_\pm}$ is an asymptotically cylindrical almost complex structure at positive (negative) infinity, then we say (E, J) is an almost complex manifold with asymptotically cylindrical positive (negative) ends.

Let $\tilde{u}$ be a J-holomorphic map from a possibly punctured Riemann surface (Σ, j) to (E, J), and then we define for $a \geqq 0$,

$$E_{symp,a}(\tilde{u}) = \int_{\tilde{u}^{-1}\left(E \setminus W_+^a \bigcup W_-^a\right)} \tilde{u}^*\omega',$$

where $W_+^a := (a, +\infty) \times V_+ \subset W_+$, and $W_-^a := (-\infty, -a) \times V_- \subset W_-$.

$$E_\omega(\tilde{u}) = \int_{\tilde{u}^{-1}(W_+)} \tilde{u}^*\omega + \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega,$$

$$E_\lambda(\tilde{u}) = \sup_{\phi \in \mathcal{C}_+} \int_{\tilde{u}^{-1}(W_+)} \tilde{u}^*(\phi(r)\sigma \wedge \lambda) + \sup_{\phi \in \mathcal{C}_-} \int_{w^{-1}(W_-)} \tilde{u}^*(\phi(r)\sigma \wedge \lambda),$$

where

$$\mathcal{C}_+ = \left\{\phi \in C^\infty(\mathbb{R}^+, [0,1])| \int \phi = 1\right\},$$

$$\mathcal{C}_- = \left\{\phi \in C^\infty(\mathbb{R}^-, [0,1])| \int \phi = 1\right\},$$

and

$$E_a(\tilde{u}) = E_{symp,a}(\tilde{u}) + E_\omega(\tilde{u}) + E_\lambda(\tilde{u}).$$

If $\lim_{a \to +\infty} E_{symp,a}(\tilde{u})$ is finite, we define

$$E_{symp}(\tilde{u}) = \lim_{a \to +\infty} E_{symp,a}(\tilde{u})$$

and

$$E(\tilde{u}) = E_{symp}(\tilde{u}) + E_\omega(\tilde{u}) + E_\lambda(\tilde{u}).$$

To compactify the moduli space of J-holomorphic curves, we need to include holomorphic buildings (see [5]). There is no difference between almost complex manifolds with cylindrical ends and almost complex manifolds with asymptotically cylindrical ends when it comes to the definition of holomorphic buildings and the topology of the moduli space of holomorphic buildings. We also have the expected compactness theorem for the latter case.

Theorem 9. *([5] for cylindrical case; [2]) For any $a \geqq 0$, the moduli space of stable holomorphic buildings with uniformly bounded Hofer energy E_a, whose domains have a fixed number of arithmetic genus and a fixed number of marked points, is compact.*

The following theorem shows that in the contact case Hofer energy $E_a(\tilde{u})$ can be uniformly bounded by the Symplectic area $E_{symp,a}(\tilde{u})$ and the periods of the periodic orbits of $\mathbf{R}_{\pm\infty}$ that $\tilde{u}$ converges to at infinity (compare to 9.2 in [5]).

Theorem 10. *Suppose (E, J) is an almost complex manifold with asymptotically cylindrical ends of contact type. There exist positive constants C, C', and a such that for any finitely punctured Riemann surface (Σ, j) and any non-constant J-holomorphic curve $\tilde{u} : \Sigma \to E$ which converges to periodic orbits $\gamma_\pm$'s of $\mathbf{R}_{\pm\infty}$ around the punctures of Σ, we have*

$$E_a(\tilde{u}) \leqq C\left(2\sum\int \gamma_+^*\lambda_{+\infty} - \sum\int \gamma_-^*\lambda_{-\infty}\right) + C'E_{symp,a}(\tilde{u}),$$

where the summations are taken over all the periodic orbits $\gamma_\pm$'s of $\mathbf{R}_{\pm\infty}$ to which $\tilde{u}$ converges respectively.

The proof of this theorem is given in the appendix. Roughly speaking, it follows from Stokes' theorem.

4. An application to closed symplectic manifolds with a compatible J

Now we would like to apply the previous results to study the moduli space of J-holomorphic curves passing through a fixed point in a closed symplectic manifold. This generalizes some results in [3].

Let M be a closed smooth symplectic manifold of dimension $2N$ with symplectic form ω', and J be a compatible almost complex structure. For a sufficiently small neighborhood U of $p \in M$, there exists a Darboux coordinate chart $\varphi : U \to B(O,\epsilon) \subseteq \mathbb{C}^N$ such that $\varphi(p) = O$, $\varphi_* J|_O = i|_O$ and $\varphi^*\omega_{st} = \omega'$, where O is the origin, $B(O,\epsilon) := \left\{ z \in \mathbb{C}^N \middle| \, |z| < \epsilon \right\}$ and i is the standard complex structure on $\mathbb{C}^N$, and $\omega_{st} := \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k = \sum_{k=1}^n dx_k \wedge dy_k$ is the standard symplectic structure on $\mathbb{C}^N$. We identify $B(O,\epsilon)\backslash O$ with $W_- := \mathbb{R}^- \times S^{2N-1}$ via the map $\psi(z) = (\log|z| - \log\epsilon, \frac{z}{|z|})$. Let us simplify the notation $(\psi \circ \varphi)_* J$ by J when there is no confusion. This gives $(M\backslash p, J)$ the structure of an almost complex manifold with one asymptotically cylindrical negative end.

Indeed, we define ξ, $\mathbf{R}$, and λ as before. Then $\lambda_{-\infty} := \lim_{s\to-\infty} f_s^*\lambda = \Pi^*\lambda_{st}$, where

$$\lambda_{st} = \frac{1}{2}\sum_{k=1}^{N}(x_k dy_k - y_k dx_k)\Bigg|_{S^{2N-1}}$$

is the standard contact 1-form on the unit sphere $S^{2N-1} \subseteq \mathbb{C}^N$, and $\Pi : \mathbb{R}^- \times S^{2N-1} \to S^{2N-1}$ is the projection. We choose $\omega_{-\infty} = d\lambda_{-\infty}$.

Notice that $\mathbf{R}_{-\infty} := \lim_{s\to-\infty} f_s^*\mathbf{R}$ restricted to S^{2N-1} is exactly the standard Reeb vector field on S^{2N-1}, so we can see that J is Morse-Bott.

Let (Σ, j) be a Riemann surface with finitely many punctures and $\tilde{u} : \Sigma \to M\backslash p$ be a J-holomorphic curve, i.e. $J(\tilde{u}) \circ T\tilde{u} = T\tilde{u} \circ j$.

We say a puncture q of Σ is removable if around q, $\tilde{u}$ converges to a point in $M\backslash p$. Otherwise, we say q is non-removable. To clarify the relations between different concepts we state the following lemma.

Lemma 11. *Suppose that all the punctures of Σ are non-removable. Then the following statements are equivalent.*

1) $\tilde{u}$ converges to some Reeb orbits of $\mathbf{R}_{-\infty}$ at negative infinity around the punctures of Σ.
2) $E_a(\tilde{u})$ is finite for all $a \geqq 0$.
3) $E_a(\tilde{u})$ is finite for some $a \geqq 0$.
4) $\lim_{a\to+\infty} E_{symp,a}(\tilde{u})$ is finite.
5) If we view $\tilde{u}$ as a map from Σ to M, then $\tilde{u}$ extends smoothly over S, where S is the smooth Riemann surface associated to Σ.

Proof. It is obvious that (2) $\Longleftrightarrow$ (3). Lemma 7 says (1) $\Longrightarrow$ (3). From Theorem 6 and Removable Singularity Theorem, we get (3) $\Longrightarrow$ (1). (1) $\Longrightarrow$ (4)

follows from direct calculation. (4) $\Longrightarrow$ (5) is true by the Removable Singularity Theorem. Finally, (5) $\Longrightarrow$ (1) is guaranteed by Theorem B[3] in [14]. $\square$

Assuming any of the (1)-(5) is true, then by (4) and (5) we have

$$E_{symp}(\tilde{u}) = \lim_{a\to+\infty} E_{symp,a}(\tilde{u}) = \lim_{a\to+\infty} \int_{\tilde{u}^{-1}\left(E\backslash W_{-}^{a}\right)} \tilde{u}^*\omega' = \int_S \tilde{u}^*\omega' < +\infty.$$

Thus, $E(\tilde{u}) = E_{symp}(\tilde{u}) + E_\omega(\tilde{u}) + E_\lambda(\tilde{u})$ is well defined.

The multiplicity of a Reeb orbit γ is the degree of γ as a cover of a simple Reeb orbit. For each non-removable punctures q of Σ, we can associate a positive integer which is the multiplicity of the corresponding Reeb orbit that $\tilde{u}$ converges to around q.

Let $\tilde{u}$ be a non-constant J-holomorphic curve from a smooth Riemann surface (S, j) to M. By the Carleman Similarity principle, we know $\tilde{u}^{-1}(p)$ is discrete, and hence finite. Let (Σ, j) be the punctured Riemann surface $(S\backslash\tilde{u}^{-1}(p), j)$. Now $\tilde{u}$ can be viewed as a J-holomorphic curve from Σ to $M\backslash p$. This means that the condition (4) in Lemma 11 is satisfied, so we have (1)-(5). An easy modification of the proof of Theorem 10 leads to the next theorem.

Theorem 12. *(Gromov's Monotonicity Theorem with multiplicity) For a closed symplectic manifold (M, ω') with a compatible almost complex structure J, there exists a constant $r_0 > 0$ and a function $\hbar(r) > 0$ such that for any point $p \in M$, and any J-holomorphic curve $\tilde{u}$ from a Riemann surface (with boundary) $\mathcal{S}$ mapped to M that passes through the point p for k times (counted with multiplicity), and satisfies $\tilde{u}(\partial\mathcal{S}) \cap B_r(p) = \emptyset$, for $0 < r < r_0$, the following is true.*

$$\int_{\tilde{u}^{-1}(B_r(p))} \tilde{u}^*\omega' > k\hbar(r),$$

where $B_r(p)$ is a ball of radius r centered at p inside M.

The proof of Theorem 12 is given in the appendix. Now it follows immediately that

Corollary 13. *There exists a constant $C > 0$ depending only on (M, ω, J) such that for any Riemann surface (S, j) and any non-constant J-holomorphic curve $\tilde{u} : S \to M$ passing through a point p for k times, we have $k \leqq C E_{symp}(\tilde{u})$.*

[3]Theorem B is stated for the case of a J-holomorphic strip with Lagrangian boundary condition, but it is easy to see that it is also true in this closed case.

Remark 14. After the submission of the arXiv version 1 of this paper, we were informed that Corollary 13 could also be derived from Corollary 3.6 and the remarks below Corollary 3.6 in [8]. It is very interesting to see that the methods used in [8] and this paper are quite different. In [8] the technics from minimal surfaces is used, and a stronger result than Theorem 12 is achieved. In particular, [8] implies that $\hbar(r)$ is proportional to r^2. While, in this paper, we view $M\backslash p$ as a manifold with asymptotically end, and Theorem 12 follows roughly from Stokes' Theorem immediately. (Also see [3] for a slightly different proof.) However, using this method it is not clear why $\hbar(r)$ is proportional to r^2.

Let $\mathcal{M}_g(M, J, Q)$ be the moduli space of stable J-holomorphic curves $\tilde{u}$ in M with genus g and $E_{symp}(\tilde{u}) \leqq Q$. From Corollary 13 and Theorem 9, we can compactify $\mathcal{M}_g(M, J, Q)$ by including holomorphic buildings (See [3] for more discussions).

It will be very interesting and useful to generalize the results in this paper by replacing the fixed point p with an almost complex submanifold.

5. Appendix: Proof of Theorem 10 and Theorem 12

For convenience let us introduce the following terminology.

Definition 15. We say that a $2-$form Δ defined on $(-\infty, -R] \times V_-$ is J-positive (or non-negative), if for a sufficiently large R, Δ is positive (or non-negative) on any J-complex planes of $TW_{-R} := T\left((-\infty, -R] \times V_-\right)$. In other words, $\Delta(x, Jx) > 0$ (or $\geqq 0$), for all $x \in TW_{-R}$.

Definition 16. We say that a $2-$form Δ defined on $(-\infty, -R] \times V_-$ is J-positively bounded away from 0, if $\inf \Delta(x, Jx) > 0$, where the infimum is taken over all the $x \in TW_{-R}$. with norm $\|x\|_{g_{W_-}} = 1$ (Recall that g_{W_-} is a translational invariant metric).

Proof. (Theorem 10) Let us deal with the negative end W_- first.

For any $R > 0$, we pick $-\mathfrak{r} \in [-2R, -R]$ such that $-\mathfrak{r}$ is a regular value of $r \circ \tilde{u}$, where $r : W_- \to (-\infty, 0)$ is the projection map. Denote $A := \tilde{u}^{-1}((-\infty, -\mathfrak{r}] \times V_-) \subseteq \Sigma$ and $B_1 := \tilde{u}^{-1}(\{-\mathfrak{r}\} \times V_-)$. Let $\hat{A}$ be the oriented blow up of A around all the punctures of A, i.e. $\hat{A} = A \sqcup B_2$ with $B_2 := \sqcup S^1$ being the disjoint union of circles introduced by the oriented blow up. Hence we have $\partial \hat{A} = B_1 \sqcup B_2$. We choose the orientation of B_1 to be the boundary orientation from $\hat{A}$, while we choose the orientation of B_2 to be the reverse orientation of the boundary orientation from $\hat{A}$.

For $x \in TW_- = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}_{-\infty}) \oplus \xi_{-\infty}$, we can write x as $x = dr(x)\frac{\partial}{\partial r} + \lambda(x)\mathbf{R}_{-\infty} + \pi_{\xi_{-\infty}}x$. Then for any constants $P, Q > 0$, we have

$$\begin{aligned}&[Pd\lambda_{-\infty} + Qdr \wedge \lambda_{-\infty}]\,(x, J_{-\infty}x)\\ &= Pd\lambda_{-\infty}(\pi_{\xi_{-\infty}}x, J_{-\infty}\pi_{\xi_{-\infty}}x) + Q\,[dr(x)]^2 + Q\,[\lambda(x)]^2\,.\end{aligned}$$

Because $d\lambda_{-\infty}(\cdot, J_{-\infty}\cdot)$ defines a metric on $\xi_{-\infty}$, we get $Pd\lambda_{-\infty} + Qdr \wedge \lambda_{-\infty}$ is $J_{-\infty}-$positively bounded away from 0. Denote

$$\mathcal{S} := \Big\{(x, y) \in TW_- \times TW_- |\, \|x\|_{g_{W_-}} = 1, y = J_{-\infty}x\Big\}$$

and

$$\mathcal{T}_{-R} := \Big\{(x, y) \in TW_{-R} \times TW_{-R}|\, \|x\|_{g_{W_-}} = 1, y = Jx\Big\}\,.$$

Let Δ be the smooth map $TW_- \times TW_- \to \mathbb{R}$ defined by applying $Pd\lambda_{-\infty} + Qdr \wedge \lambda_{-\infty}$. The fact that $Pd\lambda_{-\infty} + Qdr \wedge \lambda_{-\infty}$ is $J_{-\infty}-$positively bounded away from 0 means that $\Delta|_{\mathcal{S}} > \varpi > 0$ for some enough small ϖ. By (ACC1) there exists R large enough, such that $\Delta|_{\mathcal{T}_{-R}} > \frac{1}{2}\varpi > 0$. Therefore, we get that $Pd\lambda_{-\infty} + Qdr \wedge \lambda_{-\infty}$ is J-positively bounded away from 0.

Since $J - J_{-\infty}$ is exponentially small by (ACC1), there exist constants $C_1, \kappa_1 > 0$ such that

$$|d\lambda_{-\infty}(x, (J - J_{-\infty})x)| \leqq \frac{1}{2}C_1 e^{\kappa_1 r}, \tag{9}$$

for all $x \in TW_{-R}$ with $||x||_{g_{W_-}} = 1$.

From now on let us pick g_{W_-} to be

$$\langle x, y\rangle_{g_{W_-}} = (dr \wedge \lambda_{-\infty} + d\lambda_{-\infty})\,(x, J_{-\infty}y)$$

for convenience. Notice that by (ACC1) again, for all $x \in TW_{-R}$ with $||x||_{g_{W_-}} = 1$, we get

$$\begin{aligned}&(dr \wedge \lambda_{-\infty} + d\lambda_{-\infty})\,(x, Jx)\\ &= (dr \wedge \lambda_{-\infty} + d\lambda_{-\infty})\,(x, J_{-\infty}x)\\ &\quad + (dr \wedge \lambda_{-\infty} + d\lambda_{-\infty})\,(x, (J - J_{-\infty})x)\\ &\geqq \|x\|_{g_{W_-}} - K_0 e^{\delta_0 r}\\ &> \frac{1}{2}.\end{aligned} \tag{10}$$

Hence (9) and (10) imply

$$\begin{aligned}
&\left[d\lambda_{-\infty} + C_1 e^{\kappa_1 r}\left(dr \wedge \lambda_{-\infty} + d\lambda_{-\infty}\right)\right](x, Jx) \\
&> d\lambda_{-\infty}(x, J_{-\infty}x) + d\lambda_{-\infty}(x, (J - J_{-\infty})x) + \frac{1}{2}C_1 e^{\kappa_1 r} \\
&\geqq d\lambda_{-\infty}(x, J_{-\infty}x) \\
&\geqq 0,
\end{aligned}$$

where the last inequality comes from the fact that $d\lambda_{-\infty}$ is $J_{-\infty}-$non-negative. Hence

$$d\lambda_{-\infty} + C_1 e^{\kappa_1 r}\left(dr \wedge \lambda_{-\infty} + d\lambda_{-\infty}\right)$$

is J-positive, so is

$$d\lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty}.$$

Similarly, by varying C_1 and κ_1 if necessary, we can get that

$$|dr \wedge \lambda_{-\infty}(x, (J - J_{-\infty})x)| \leqq \frac{1}{2}C_1 e^{\kappa_1 r},$$

for all $x \in TW_{-R}$ with $||x||_{g_{W_-}} = 1$. As before, we have

$$\begin{aligned}
&\left[dr \wedge \lambda_{-\infty} + C_1 e^{\kappa_1 r}\left(dr \wedge \lambda_{-\infty} + d\lambda_{-\infty}\right)\right](x, Jx) \\
&> dr \wedge \lambda_{-\infty}(x, J_{-\infty}x) + dr \wedge \lambda_{-\infty}(x, (J - J_{-\infty})x) + \frac{1}{2}C_1 e^{\kappa_1 r} \\
&\geqq dr \wedge \lambda_{-\infty}(x, J_{-\infty}x) \\
&\geqq 0.
\end{aligned}$$

This implies

$$dr \wedge \lambda_{-\infty} + C_1 e^{\kappa_1 r}\left(dr \wedge \lambda_{-\infty} + d\lambda_{-\infty}\right)$$

is J-positive, so is

$$dr \wedge \lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} d\lambda_{-\infty}.$$

From Equation (4) and the fact $\omega_{-\infty} = d\lambda_{-\infty}$, we get $\omega - d\lambda_{-\infty}$ is exponentially small. Because $J - J_{-\infty}$ is also exponentially small, by varying C_1 and κ_1 if necessary, we can get that

$$|\left(\omega - d\lambda_{-\infty}\right)(x, Jx)| \leqq \frac{1}{2}C_1 e^{\kappa_1 r}, \tag{11}$$

for all $x \in TW_{-R}$ with $||x||_{g_{W_-}} = 1$.

Therefore, by (4) and (10) we have

$$\omega(x, Jx) \leqq d\lambda_{-\infty}(x, Jx) + C_1 e^{\kappa_1 r}\left(dr \wedge \lambda_{-\infty} + d\lambda_{-\infty}\right)(x, Jx).$$

When restricted to J-complex planes in TW_{-R} for large R, we get

$$\begin{aligned}\omega &\leqq d\lambda_{-\infty} + C_1 e^{\kappa_1 r}\left(dr \wedge \lambda_{-\infty} + d\lambda_{-\infty}\right) \\ &\leqq \left(1 + C_1 e^{\kappa_1 r}\right)\left(d\lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty}\right) \\ &\leqq C_2\left(d\lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty}\right),\end{aligned} \tag{12}$$

where $C_2 = 1 + C_1 e^{-\kappa_1 R} < 2$.

Similarly, for all $x \in TW_{-R}$ with $||x||_{g_{W_-}} = 1$, we have

$$\left|\left(\sigma \wedge \lambda - dr \wedge \lambda_{-\infty}\right)(x, Jx)\right| \leqq \frac{1}{2} C_1 e^{\kappa_1 r}. \tag{13}$$

Hence when restricted to J-complex planes in TW_{-R} for large R, by (10) and (13) we have

$$\begin{aligned}\sigma \wedge \lambda &\leqq dr \wedge \lambda_{-\infty} + C_1 e^{\kappa_1 r}\left(dr \wedge \lambda_{-\infty} + d\lambda_{-\infty}\right) \\ &\leqq C_2\left(\frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} d\lambda_{-\infty} + dr \wedge \lambda_{-\infty}\right).\end{aligned} \tag{14}$$

On the other hand, since $\omega + \sigma \wedge \lambda$ is J - positively bounded away from 0, when restricted on J-complex planes in TW_{-R} for large R, we get

$$\begin{aligned}\left|dr \wedge \lambda_{-\infty}\right| &\leqq \left|dr \wedge \lambda_{-\infty} - \sigma \wedge \lambda\right| + \sigma \wedge \lambda \\ &\leqq C_1 e^{\kappa_1 r}\left(\omega + \sigma \wedge \lambda\right) + \sigma \wedge \lambda \\ &\leqq C_1 e^{\kappa_1 r} \omega + C_2 \sigma \wedge \lambda\end{aligned} \tag{15}$$

and

$$\begin{aligned}\left|d\lambda_{-\infty}\right| &\leqq \left|d\lambda_{-\infty} - \omega\right| + \omega \\ &\leqq C_1 e^{\kappa_1 r}\left(\omega + \sigma \wedge \lambda\right) + \omega \\ &\leqq C_2 \omega + C_1 e^{\kappa_1 r} \sigma \wedge \lambda,\end{aligned} \tag{16}$$

by modifying C_1 and κ_1.

Therefore, we have

$$\begin{aligned}(17)\qquad &\int_{\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega \\ &\leqq \int_{\hat{A}} \tilde{u}^*\omega + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega \\ &\leqq C_2 \int_{\hat{A}} \tilde{u}^* \left(d\lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1+C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega \\ &= C_2 \int_{B_1} \tilde{u}^*\lambda_{-\infty} - C_2 \int_{B_2} \tilde{u}^*\lambda_{-\infty} \\ &\quad + C_2 \int_{\hat{A}} \tilde{u}^* \left(\frac{C_1 e^{\kappa_1 r}}{1+C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega.\end{aligned}$$

While,

$$\begin{aligned}(18)\qquad &\left| \int_{\hat{A}} \tilde{u}^* \left(\frac{C_1 e^{\kappa_1 r}}{1+C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \right| \\ &\leqq \int_{\hat{A}} \left| \tilde{u}^* \left(\frac{C_1 e^{\kappa_1 r}}{1+C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \right| \\ &\leqq C_1 \int_{\hat{A}} \left| \tilde{u}^* e^{\kappa_1 r} \left(C_1 e^{\kappa_1 r}\omega + C_2 \sigma \wedge \lambda \right) \right| \\ &\leqq \frac{1}{4} E_\omega(\tilde{u}|_{W_-}) + C_1 C_2 \kappa_1^{-1} e^{-\kappa_1 \mathfrak{r}} \int_{\hat{A}} \tilde{u}^* \left(\kappa_1 e^{\kappa_1(\mathfrak{r}+r)} \sigma \wedge \lambda \right).\end{aligned}$$

Since $\int_{-\infty}^{-\mathfrak{r}} \kappa_1 e^{\kappa_1(\mathfrak{r}+r)} dr = 1$, we have

$$\int_{\hat{A}} \tilde{u}^* \left(\kappa_1 e^{\kappa_1(\mathfrak{r}+r)} \sigma \wedge \lambda \right) \leqq E_\lambda(\tilde{u}).$$

Therefore, by picking R sufficiently large, we can make $\mathfrak{r}$ sufficiently large, and then (18) implies

$$(19)\qquad \left| \int_{\hat{A}} \tilde{u}^* \left(\frac{C_1 e^{\delta_1 r}}{1+C_1 e^{\delta_1 r}} dr \wedge \lambda_{-\infty} \right) \right| \leqq \frac{1}{4} E_\omega(\tilde{u}|_{W_-}) + \frac{1}{4} E_\lambda(\tilde{u}|_{W_-}).$$

Let $\Phi(r) = \int_{-\infty}^{r} \phi(t)dt$, for $\phi \in \mathcal{C}$, and then we get

$$
\begin{aligned}
(20)\qquad & \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^* \left(\phi(r)\sigma \wedge \lambda\right) \\
\leqq & \int_{\hat{A}} \tilde{u}^* \left(\phi(r)\sigma \wedge \lambda\right) + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^* \left(\phi(r)\sigma \wedge \lambda\right) \\
\leqq & C_2 \int_{\hat{A}} \tilde{u}^* \left(\phi(r)dr \wedge \lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r) d\lambda_{-\infty}\right) \\
& + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^* \left(\phi(r)\sigma \wedge \lambda\right) \\
\leqq & C_2 \int_{\hat{A}} \tilde{u}^* d(\Phi(r)\lambda_{-\infty}) - C_2 \int_{\hat{A}} \tilde{u}^* (\Phi(r) d\lambda_{-\infty}) \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r) d\lambda_{-\infty}\right) \\
& + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^* \left(\phi(r)\sigma \wedge \lambda\right) \\
\leqq & C_2 \int_{\hat{A}} \tilde{u}^* d(\Phi(r)\lambda_{-\infty}) \\
& - C_2 \left\{ \int_{\hat{A}} \tilde{u}^* \left[\Phi(r) d\lambda_{-\infty} + \Phi(r) \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty}\right]\right\} \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\Phi(r) \frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty}\right) \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r) d\lambda_{-\infty}\right) \\
& + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^* \left(\phi(r)\sigma \wedge \lambda\right) \\
\leqq & C_2 \int_{\hat{A}} \tilde{u}^* d(\Phi(r)\lambda_{-\infty}) + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\Phi(r) \frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty}\right) \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r) d\lambda_{-\infty}\right) \\
& + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W)} \tilde{u}^* \left(\phi(r)\sigma \wedge \lambda\right) \\
= & C_2 \int_{B_1} \tilde{u}^* (\Phi(-\mathfrak{r})\lambda_{-\infty}) + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\Phi(r) \frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty}\right) \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r) d\lambda_{-\infty}\right) \\
& + \int_{\{\Sigma\backslash A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^* \left(\phi(r)\sigma \wedge \lambda\right),
\end{aligned}
$$

where the last inequality follows from the fact that $\Phi(r)d\lambda_{-\infty} + \Phi(r)\frac{C_1 e^{\kappa_1 r}}{1+C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty}$ is J- positive.

While we have

$$\begin{aligned}
&\left| C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\Phi(r) \frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \right| \\
&\leqq C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left| e^{\kappa_1 r} dr \wedge \lambda_{-\infty} \right| \\
&\leqq \frac{1}{4} E_\omega(\tilde{u}|_{W_-}) + \frac{1}{4} E_\lambda(\tilde{u}|_{W_-}),
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\left| C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r) d\lambda_{-\infty} \right) \right| \\
&\leqq C_2 C_1 \int_{\hat{A}} \tilde{u}^* e^{\kappa_1 r} \left(C_2 \omega + C_1 e^{\kappa_1 r} \sigma \wedge \lambda \right) \\
&\leqq \frac{1}{4} E_\omega(\tilde{u}|_{W_-}) + \frac{1}{4} E_\lambda(\tilde{u}|_{W_-}).
\end{aligned} \tag{22}$$

Therefore, from (17), (19), (20), (21), and (22), we get

$$\begin{aligned}
E(\tilde{u}|_{W_-}) &:= E_\omega(\tilde{u}|_{W_-}) + E_\lambda(\tilde{u}|_{W_-}) \\
&\leqq 2C_2 \int_{B_1} \tilde{u}^* \lambda_{-\infty} - C_2 \int_{B_2} \tilde{u}^* \lambda_{-\infty} + \frac{3}{4} E_\omega(\tilde{u}|_{W_-}) + \frac{3}{4} E_\lambda(\tilde{u}|_{W_-}) \\
&\quad + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \left(\phi(r) \sigma \wedge \lambda \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
E(\tilde{u}|_{W_-}) &\leqq 4C_2 \left(2 \int_{B_1} \tilde{u}^* \lambda_{-\infty} - \int_{B_2} \tilde{u}^* \lambda_{-\infty} \right) + 4 \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega \\
&\quad + 4 \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \left(\phi(r) \sigma \wedge \lambda \right).
\end{aligned} \tag{23}$$

Now we define a function τ by $\tau(r) = \frac{R+r}{R-\mathfrak{r}}$ for $-\mathfrak{r} \leqq r \leqq -R$. Since $\tau(-\mathfrak{r}) = 1$ and $\tau(-R) = 0$, by Stokes' Theorem we get

$$\left|\int_{B_1} \tilde{u}^*\lambda_{-\infty}\right| = \left|\int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}([-\mathfrak{r},-R]\times V_-)} \tilde{u}^* d\left(\tau(r)\lambda_{-\infty}\right)\right|$$

$$\leqq \int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}([-\mathfrak{r},-R]\times V_-)} \left|\tilde{u}^* d\left(\tau(r)\lambda_{-\infty}\right)\right|$$

$$\leqq C_3 \int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}([-\mathfrak{r},-R]\times V_-)} \tilde{u}^*\omega' \tag{24}$$

$$\leqq C_3 \int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega' \tag{25}$$

where C_3 is a constant depending on R, and the second inequality follows from the fact that on any J-complex planes the symplectic form ω' is positive. For the same reason, by modifying C_3 if necessary, we also have

$$\int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega \leqq C_3 \int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega' \tag{26}$$

and

$$\int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\left(\sigma\wedge\lambda\right) \leqq C_3 \int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega'. \tag{27}$$

Then (23), (25), (26), (27), and $\int_{\{\Sigma\setminus A\}\cap\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega' \leqq E_{symp,2R}(\tilde{u})$ together imply

$$E(\tilde{u}|_{W_-}) \leqq C_4 E_{symp,2R}(\tilde{u}) - 4C_2 \sum \int \gamma_-^*\lambda_{-\infty}, \tag{28}$$

where $C_4 = 8(C_2+1)C_3$ is a constant independent of $\tilde{u}$, and the summation is taken over all the periodic orbits γ_-'s of $\mathbf{R}_{-\infty}$ to which $\tilde{u}$ converges at negative infinity.

For positive end W_+, the estimates are very similar. The only difference comes from the fact that the orientation of V_+ agrees with the boundary orientation of $\{+\infty\}\times V_+$, and the orientation of V_- disagrees with the boundary orientation of $\{-\infty\}\times V_-$. One can easily adjust the above estimates to W_+ case. For example, in (17) the main part is $\int_{\hat{A}} \tilde{u}^* d\lambda_{-\infty} = \int_{B_1} \tilde{u}^*\lambda_{-\infty} - \int_{B_2} \tilde{u}^*\lambda_{-\infty}$, and in W_+-version we replace it by

$$\int_{\tilde{u}^{-1}(\hat{A}_+)} \tilde{u}^* d\lambda_{+\infty} = \int_{B_{2+}} \tilde{u}^*\lambda_{+\infty} - \int_{B_{1+}} \tilde{u}^*\lambda_{+\infty},$$

where $B_{1+} := \tilde{u}^{-1}(\{\mathfrak{r}_+\} \times V_+)$ and $B_{2+} := \tilde{u}^{-1}(\{+\infty\} \times V_+)$; in (20) the main part is $\int_{\hat{A}} \tilde{u}^* d(\Phi(r)\lambda_{-\infty}) = \int_{B_1} \tilde{u}^*(\Phi(-\mathfrak{r})\lambda_{-\infty})$, and in W_+-version we replace it by

$$\int_{\tilde{u}^{-1}(\hat{A}_+)} \tilde{u}^* d(\Phi_+(r)\lambda_{+\infty}) \leqq \int_{B_{2+}} \tilde{u}^*\lambda_{+\infty} - \int_{B_{1+}} \tilde{u}^*(\Phi_+(\mathfrak{r}_+)\lambda_{+\infty}).$$

Then a similar estimate as in (25) shows that all the error terms including $-\int_{B_{1+}} \tilde{u}^*\lambda_{+\infty}$ and $-\int_{B_{1+}} \tilde{u}^*(\Phi_+(\mathfrak{r}_+)\lambda_{+\infty})$ can be bounded by a multiple of $E_{symp,2R}(\tilde{u})$. Indeed, one can show that

$$\begin{aligned} E(\tilde{u}|_{W_+}) &= E_\omega(\tilde{u}|_{W_+}) + E_\lambda(\tilde{u}|_{W_+}) \\ &\leqq 8C_2 \sum \int \gamma_+^*\lambda_{+\infty} + C_4 E_{symp,2R}(\tilde{u}), \end{aligned} \tag{29}$$

where the summations are taken over all the periodic orbits γ_+'s of $\mathbf{R}_{+\infty}$ to which $\tilde{u}$ converges at positive infinity.

By (29) and (28), we have $E_a(\tilde{u}) \leqq C\left(2\sum\int \gamma_+^*\lambda_{+\infty} - \sum\int \gamma_-^*\lambda_{-\infty}\right) + C' E_{symp,a}(\tilde{u})$, where $C = 4C_2$ and $C' = 2C_4$. □

Proof. (Theorem 12) We view $(M\backslash p, J)$ as an almost complex manifold with asymptotically cylindrical negative end W_- as described in the beginning of Section 3, with W_- biholomorphic to $B_r(p)\backslash\{p\}$. Notice that all the estimates before formula (24) in the proof of Theorem 10 are local, i.e. inside W_-. Thus, we get

$$\begin{aligned} E(\tilde{u}|_{W_-}) &\leqq C_4 \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega' - 4C_2 \sum \int \gamma_-^*\lambda_{-\infty} \\ &= C_4 \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega' - 4C_2(2k\pi). \end{aligned}$$

From the fact that $E(\tilde{u}|_{W_-}) > 0$, we get

$$\int_{\tilde{u}^{-1}(B_r(p))} \tilde{u}^*\omega' = \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^*\omega' > \frac{4C_2(2k\pi)}{C_4}.$$

Now we show that the constant $\frac{4C_2(2\pi)}{C_4}$ can be chosen to be independent of p. For each point $p \in M$, we can choose a Darboux chart whose size is uniformly bounded away from 0 and the almost complex structure J at p coincides with the standard one defined by $\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial y} \mapsto -\frac{\partial}{\partial x}$. Identifying this neighborhood minus p with the half infinite cylinder as described in the beginning of Section 4, we get (ACC1)-(ACC5) are satisfied with constants

K_l bounded by the C^l-norm of J and the norm of the Nijenhuis tensor of J (We only need K_0 in this theorem). Since we assume that M is compact, and ω' and J are smooth, we can make K_l independent of p. Following the proofs of Theorem 10 and Theorem 12 carefully, we can see that the constant C_2 can be chosen to be close to 1 and C_4 can be bounded using K_0. Therefore, we can make $\frac{4C_2(2\pi)}{C_4}$ independent of p. □

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