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The Geometry of b^k Manifolds

Geoffrey Scott

Let Z be a hypersurface of a manifold M. The b-tangent bundle of (M, Z), whose sections are vector fields tangent to Z, is used to study pseudodifferential operators and stable Poisson structures on M. In this paper we introduce the b^k-tangent bundle, whose sections are vector fields with "order k tangency" to Z. We describe the geometry of this bundle and its dual, generalize the celebrated Mazzeo-Melrose theorem of the de Rham theory of b-manifolds, and apply these tools to classify certain Poisson structures on compact oriented surfaces.

1. Introduction

Melrose developed the b-calculus to study pseudodifferential operators on noncompact manifolds ([Me], [G]). Considering the manifold in question as the interior of a manifold M with boundary, he constructed the *b-tangent bundle* ${}^bTM$ whose sections are vector fields on M tangent to ∂M, and the *b-cotangent bundle* ${}^bT^*M$, whose sections are differential forms with a specific kind of order-one singularity at ∂M. The authors of [GMP2] applied these ideas to study global Poisson geometry; in this context, ${}^bTM$ and ${}^bT^*M$ are defined on a manifold M with a distinguished hypersurface Z rather than on a manifold with boundary[1], and sections of ${}^bTM$ (and ${}^bT^*M$) are vector fields (and differential forms) tangent to Z (or singular at Z). In this paper, we generalize this construction so that vector fields and differential forms with higher order tangency and higher order singularity may also be realized as sections of bundles.

The construction of these bundles in Section 2 is subtle: although we wish to begin by defining a b^k-vector field as a vector field with an "order

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[1]These competing perspectives can be reconciled by viewing a boundary of a manifold M as a hypersurface in the double of M. In this paper, we follow the precedent of [GMP2] and define our bundles over manifolds with distinguished hypersurfaces.

k tangency to Z," there is no straightforward way to rigorously define this notion. To do so, we must include in the definition of a b^k-manifold the data of a $(k-1)$-jet of Z. We then define a b^k-vector field as a vector field v such that $\mathcal{L}_v(f)$ vanishes to order k for functions f that represent the jet data, the b^k-tangent bundle ${}^kTM$ as the vector bundle whose sections are b^k-vector fields, and the b^k-cotangent bundle ${}^kT^*M$ as its dual. When $k=1$, these are the familiar constructions from [Me] and [GMP2].

In Section 3 we define a differential on the complex of *b^k-forms* (sections of $\wedge^*({}^kT^*M)$) and prove a Mazzeo-Melrose type theorem for the cohomology ${}^kH^*(M)$ of this complex.

$$ {}^kH^p(M) \cong H^p(M) \oplus \left(H^{p-1}(Z)\right)^k \tag{1} $$

However, this isomorphism is non-canonical. By defining the *Laurent Series* of a b^k-form, which expresses a b^k-form as a sum of simpler b^ℓ-forms (for $\ell \leq k$), we show that there *is* a way to construct the isomorphism in Equation 1 so that the $\left(H^{p-1}(Z)\right)^k$ summand of a b^k-cohomology class *is* canonically defined.

In Section 4, we study b^k-forms of top degree. We show that the asymptotic behavior near Z of a b^k-form of top degree can be encoded by a polynomial, and we define the *Liouville volume* of the form to be the constant term of this polynomial — this definition agrees with the definition of Liouville volume studied in [R] when $k=1$. We can also take the Liouville volume of a degree p b^k-form along any p dimensional submanifold of M. Citing Poincaré duality, we define the *smooth part* of a b^k cohomology class $[\omega]$ to be the de Rham cohomology class whose integrals along p-cycles equal the Liouville volumes of ω along these cycles. Using these tools, we realize the abstract isomorphism in Equation 1 with an explicit canonical map. The image of a b^k-form under this map is its *Liouville-Laurent decomposition*.

In Section 5, we define a symplectic b^k-form as a closed b^k-form of degree two having full rank (when $k=1$, these are also called *log symplectic* forms), and prove the classic Moser theorems in the b^k category. We also revisit the classification theorems of stable Poisson structures on compact oriented surfaces from [R] and [GMP2]. The author of [R] classifies stable Poisson structures using geometric data, while the authors of [GMP2] use cohomological data; in this paper, we show how the Liouville-Laurent decomposition relates the geometric data to the cohomological data.

This paper ends in Section 6 with an example of how the theory of b^k-manifolds can answer questions from outside b^k-geometry. Let Π be a Poisson structure on a manifold M whose rank differs from $\dim(M)$ precisely on a hypersurface Z. We say that Π is *of b^k-type* if it is dual to a symplectic b^k-form for *some* choice of jet data. We give a condition for two such Poisson structures on a compact surface to be isomorphic in terms of the summands in their respective Liouville-Laurent decompositions.

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2. The b^k tangent bundle

All manifolds, maps, and vector fields in this paper are assumed to be smooth. We recall the basic definitions from the theory of b-manifolds. These ideas were introduced by Melrose (e.g. [Me]); see [GMP2] for a exposition of these concepts which emphasizes the ideas relevant to this paper.

Definition 2.1. A b**-manifold** is a pair (M, Z) of a smooth oriented manifold M and an oriented hypersurface $Z \subseteq M$ such that $Z = \{f = 0\}$ for some global defining function $f : M \to \mathbb{R}$. A b**-map** from (M, Z) to (M', Z') is a map $\varphi : M \to M'$ such that $\varphi^{-1}(Z') = Z$ and φ is transverse to Z'. A b**-vector field** on (M, Z) is a vector field v on M such that $v_p \in T_pZ$ for all $p \in Z$. The b**-tangent bundle** ${}^bTM$ on (M, Z) is the vector bundle whose sections are the b-vector fields on (M, Z). The b**-cotangent bundle** ${}^bT^*M$ is the dual bundle of ${}^bTM$.

Sections of $\wedge^*({}^bT^*M)$ are differential forms on M with a certain kind of order-one singularity at Z. To construct bundles of forms with higher-order singularities, we wish to first define a b^k-vector field as a vector field "tangent to order k on Z." However, the next example shows that the naive definition of being "tangent to order k on Z" (as a vector field v such that for a defining function f of Z, $\mathcal{L}_v(f)$ vanishes to order k along Z) is ill-defined.

Example. On $(M, Z) = (\{(x, y) \in \mathbb{R}^2\}, \{y = 0\})$, two defining functions for Z are given by y and $e^x y$. The vector field $v = \frac{\partial}{\partial x}$ satisfies

$$\mathcal{L}_v(y) = 0 \qquad \text{and} \qquad \mathcal{L}_v(e^x y) = e^x y$$

so the order of vanishing of the Lie derivative of a defining function depends on the choice of defining function.

This prevents us from emulating the [GMP2] paper mutatis mutandis; we must endow our b-manifolds with the data of a $(k-1)$-jet along Z to make possible the definition of a b^k-vector field.

Definition 2.2. Let $i : Z \to M$ be the inclusion of a hypersurface into a manifold, let C^∞ be the sheaf of smooth functions on M, and let $\mathcal{I} \subseteq C^\infty$ be the ideal sheaf of Z. A **germ** at Z is a global section of the sheaf $i^{-1}(C^\infty)$, and a k**-jet** at Z is a global section of the sheaf $i^{-1}(C^\infty/\mathcal{I}^{k+1})$.

We will write J^k to denote the k-jets at Z, and I to denote the global sections of $i^{-1}(\mathcal{I})$. We write $[f]^k$ to denote the k-jet represented by a smooth function f defined in a neighborhood of Z. Also, if j is a k-jet, we write $f \in j$ if f is a smooth function in a neighborhood of Z that represents j, and $f \in I^k$ if f represents an element of I^k (equivalently, if $[f]^{k-1} = 0$).

Definition 2.3. For $k \geq 1$, a b^k**-manifold** is a triple (M, Z, j) where M is an oriented manifold, $Z \subseteq M$ is an embedded oriented hypersurface, and j is an element of J^{k-1} that can be represented by a positively oriented local defining function y for Z (that is, if Ω_Z is a positively oriented volume form of Z, then $dy \wedge \Omega_Z$ is positively oriented for M). A b^k**-map** from (M, Z, j) to (M', Z', j') is a map $\varphi : M \to M'$ such that $\varphi^{-1}(Z') = Z$, φ is transverse to Z', and $\varphi^*(j') = j_Z$.

If $j \in J^{k-1}$ for $k > 1$, and some $f \in j$ is a positively oriented local defining function for Z, then every $f \in j$ is a positively oriented local defining function for Z. When $k = 1$, the jet data is vacuous, so the definition of a b^1-manifold agrees with that of a b-manifold.

Lemma 2.4. *Given an embedded hypersurface $Z \subseteq M$, a function $f \in C^\infty(M)$, and a vector field v on M satisfying $v_p \in T_pZ$ for all $p \in Z$, the jet $[\mathcal{L}_v(f)]^{k-1}$ depends only on $[f]^{k-1}$.*

Proof. If $[f_2]^{k-1} = [f_1]^{k-1}$, then $f_2 - f_1 = y^k g$ for a local defining function y and some smooth g. For a vector field v satisfying $v_p \in T_pZ$,

$$[\mathcal{L}_v(f_2)]^{k-1} = [\mathcal{L}_v(f_1) + y^k \mathcal{L}_v(g) + kgy^{k-1}\mathcal{L}_v(y)]^{k-1} = [\mathcal{L}_v(f_1)]^{k-1}. \qquad \square$$

Lemma 2.4 shows that the following definition makes sense.

Definition 2.5. A **b^k-vector field** on (M, Z, j_Z) is a vector field v with $v_p \in T_p(Z)$ for $p \in Z$ such that for any $f \in j_Z$, $\mathcal{L}_v(f) \in I^k$.

To check whether a vector field v is a b^k-vector field, it suffices (by Lemma 2.4) to check that $\mathcal{L}_v(f) \in I^k$ for just one local defining function $f \in j_Z$.

Example. On the b^k-manifold $(\mathbb{R}^n, Z = \{x_n = 0\}, [x_n]^{k-1})$, a vector field $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$ is a b^k-vector field iff $\mathcal{L}_v(x_n) \in I^k$, which occurs iff $v_n \in I^k$. That is, the b^k-vector fields are precisely those of the form

$$\phi_n x_n^k \frac{\partial}{\partial x_n} + \sum_{i=1}^{n-1} \phi_i \frac{\partial}{\partial x_i}$$

for smooth functions ϕ_i.

On a b^k-manifold (M, Z, j), each $p \notin Z$ is contained in a coordinate neighborhood $(U, \{x_1, \dots, x_n\})$ on which the vector fields $\{\frac{\partial}{\partial x_i}\}$ generate the space of b^k-vector fields over U as a free $C^\infty(U)$-module. For points $p \in Z$, Example 2 shows that on a coordinate neighborhood $(U, \{x_1, \dots, x_n\})$ of p with $x_n \in j$, the vector fields

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}, x_n^k \frac{\partial}{\partial x_n}\right\}$$

generate the space of b^k-vector fields over U as a $C^\infty(U)$-module. Consequently, b^k-vector fields form a projective C^∞ module over M, as well as a Lie subalgebra of the algebra of vector fields on M, so we can realize b^k-vector fields as the sections of a bundle on M.

We call this bundle ${}^kTM$ the **b^k-tangent bundle**. The dual of this bundle ${}^kT^*M$ is the **b^k-cotangent bundle**. When $k = 1$ we recover the classic definitions of a b-vector field and the b-(co)tangent bundle. We write ${}^k\Omega^p(M)$ for sections of $\wedge^p({}^kT^*M)$. Elements of ${}^k\Omega^p(M)$ are **differential b^k-forms**. Similar to the b-manifold case, there are maps between the (co)tangent bundles of M and the b^k-(co)tangent bundles of M.

$$ {}^kTM \twoheadrightarrow TM \qquad {}^kT^*M \hookleftarrow T^*M \tag{2}$$

The bundle ${}^kTM$ is a Lie algebroid over M with bracket given by the standard Lie bracket of vector fields, and anchor map given by Map 2.

In a coordinate neighborhood $(U, \{x_1, \ldots, x_n\})$ of $p \in Z$ with $x_n \in j$, although $x_n^{-k} dx_n$ is not defined on Z as a section of T^*X, its pairing with any b^k-vector field extends smoothly over Z, so $x_n^{-k} dx_n$ is an everywhere-defined section of ${}^kT^*M$. The b^k-cotangent bundle is trivialized by the sections

$$\left\{ dx_1, \ldots, dx_{n-1} \frac{dx_n}{x_n^k} \right\}.$$

By writing any b^k-form using this trivialization, and collecting all terms containing $x_n^{-k} dx_n$, the following result follows.

Remark 2.6. Let ω be a b^k-form of degree $p > 0$, and let $y \in j$ be any local defining function for Z. There are $\alpha \in \Omega^{p-1}$ and $\beta \in \Omega^p$ such that

$$\omega = \frac{dy}{y^k} \wedge \alpha + \beta \tag{3}$$

The choice of α and β are not unique.

3. De Rham theory and Laurent series of b^k-forms

3.1. The residue map

Although the forms α and β appearing in Equation 3 are not uniquely defined by ω, we will show that $i^*(\alpha)$ is independent of the choice of y, α and β, where $i : Z \to M$ is the inclusion.

Proposition 3.1. *On a b^k-manifold, if $f_1, f_2 \in j$ are local defining functions for Z, then in a neighborhood U of Z*

$$\frac{df_1}{f_1^k} = \frac{df_2}{f_2^k} + \beta$$

for some $\beta \in \Omega^1(U)$.

Proof. The case $k = 1$ was covered in [GMP2], so we assume $k \geq 2$. Because $[f_1]^{k-1} = [f_2]^{k-1}$, we have $f_1 = f_2(1 + g f_2^{k-1})$ for a smooth function g. Note that $(1+gf_2^{k-1})^{-1} = (1+g'f_2^{k-1})$ for $g' = -g(1+gf_2^{k-1})^{-1}$. Then

$$\begin{aligned}\frac{df_1}{f_1^k} &= (1+g'f_2^{k-1})^{k-1}\frac{df_2}{f_2^k} + (k-1)g(1+g'f_2^{k-1})^k\frac{df_2}{f_2} + \beta' \\ &= \frac{df_2}{f_2^k} + (k-1)g'\frac{df_2}{f_2} + (k-1)g\frac{df_2}{f_2} + \beta'' = \frac{df_2}{f_2^k} + \beta\end{aligned}$$

where

$$\begin{aligned}\beta' &= (1+g'f_2^{k-1})^k dg \\ \beta'' &= \beta' + \sum_{i=2}^{k-1}\binom{k-1}{i}(g'f_2^{k-1})^i\frac{df_2}{f_2^k} + (k-1)g\sum_{i=1} k\binom{k}{i}(g'f_2^{k-1})^i\frac{df_2}{f_2} \\ \beta &= \beta'' - (k-1)gg'f_2^{k-1}\frac{df_2}{f_2}\end{aligned}$$

□

Corollary 3.2. *Given a decomposition of $\omega \in {}^k\Omega(M)$ as in Equation 3, $i^*(\alpha)$ is independent of the decomposition.*

Proof. Let α_1 and α_2 be the α terms of two such decompositions. Setting the decompositions equal and applying Proposition 3.1 shows that $y^{-k}dy \wedge (\alpha_2 - \alpha_1)$ is a smooth form for some local defining function $y \in j$, so $i^*(\alpha_2 - \alpha_1) = 0$. □

This proves the well-definedness of the map

$$\begin{aligned}\text{res} : {}^k\Omega^p(M) &\to \Omega^{p-1}(Z) \\ \frac{dy}{y^k}\wedge\alpha + \beta &\mapsto i^*(\alpha)\end{aligned} \tag{4}$$

3.2. Viewing a b^ℓ-form as a b^k-form

For any $0 < \ell \leq k$, the natural map $J^{k-1} \to J^{\ell-1}$ allows us to canonically endow a b^k-manifold (M, Z, j) with a b^ℓ-manifold structure. Defining ${}^0TM := TM$ and ${}^0T^*M := T^*M$ for notational convenience, a b^k-manifold structure on M defines $2k+2$ different bundles ${}^\ell TM$, ${}^\ell T^*M$ over M for $0 \leq \ell \leq k$. A b^k-vector field will also be a b^ℓ-vector field for the induced b^ℓ-manifold structure. This induces maps

$$ {}^kTM \to {}^\ell TM \qquad {}^\ell T^*M \to {}^kT^*M, \tag{5}$$

the latter of which can be described explicitly in terms of the decompositions from Equation 3 as

$$\frac{dy}{y^\ell} \wedge \alpha + \beta \mapsto \frac{dy}{y^k} \wedge (y^{k-\ell}\alpha) + \beta.$$

3.3. The b^k-de Rham complex

We define a differential $d : {}^k\Omega^p(M) \to {}^k\Omega^{p+1}(M)$ by

$$d\left(\frac{dy}{y^k} \wedge \alpha + \beta\right) = \frac{dy}{y^k} \wedge d\alpha + d\beta.$$

This definition does not depend on the decomposition. Indeed, $d(\omega)$ is the unique extension of the image of the classic de Rham differential $d(\omega\big|_{M\backslash Z}) \in \Omega^p(M\backslash Z) \cong {}^k\Omega^p(M\backslash Z)$ over Z. The b^k**-de Rham complex** is $({}^k\Omega^p(M), d)$, with ${}^k\Omega^0(M) := C^\infty(M)$, and the b^k**-cohomology** ${}^kH^*(M)$ is its cohomology.

Proposition 3.3. *The sequence below, with g given by Map* (5), *is exact*

$$0 \to {}^{k-1}\Omega^p(M) \xrightarrow{g} {}^k\Omega^p(M) \xrightarrow{res} \Omega^{p-1}(Z) \to 0. \tag{6}$$

Moreover, for any closed $\alpha \in \Omega^{p-1}(Z)$ and collar neighborhood $(y, \pi) : U \to \mathbb{R} \times Z$ of Z with $y \in j$, there is a closed form $\omega \in res^{-1}(\alpha)$ such that in a neighborhood of Z,

$$\omega = \frac{dy}{y^k} \wedge \pi^*(\alpha).$$

Proof. The only nontrivial part of the exactness claim is that $\ker(\mathrm{res}) \subseteq \mathrm{im}(g)$. The kernel of res consists precisely of those ω that admit some decomposition $\omega = y^{-k}dy \wedge \alpha + \beta$ in a neighborhood of Z for which $i^*(\alpha) = 0$. Locally around Z, T^*M splits as $T^*Z + \langle dy \rangle$, so we may replace α by a form that vanishes on Z without changing ω. Then $y^{-1}\alpha$ is a smooth form, and

$$\frac{dy}{y^{k-1}} \wedge \frac{\alpha}{y} + \beta$$

extends over M to a b^{k-1} form in $g^{-1}(\omega)$. Therefore, Sequence 6 is exact.

Given a closed $\alpha \in \Omega^{p-1}(Z)$ and a collar neighborhood $(y, \pi) : U \to (-R, R) \times Z$ of Z with $y \in j$, let $\widetilde{y} \in C^\infty(M)$ be a function that agrees with y on $(-R/2, R/2) \times Z$ and is locally constant outside U. Then the b^k-form $\omega = \widetilde{y}^{-k} d\widetilde{y} \wedge \pi^*(\alpha)$ extends to a closed b^k-form on M that vanishes outside U and satisfies $\mathrm{res}(\omega) = \alpha$. In $(-R/2, R/2) \times Z$,

$$\omega = \frac{dy}{y^k} \wedge \pi^*(\alpha).$$

□

The short exact sequence from Proposition 3.3 is a chain map of complexes, hence induces a long exact sequence

$$\cdots \to {}^{k-1}H^*(M) \to {}^kH^*(M) \to H^{*-1}(Z) \to {}^{k-1}H^{*+1}(M) \to \cdots$$

By Proposition 3.3, the maps ${}^kH^*(M) \to H^{*-1}(Z)$ are surjective, so the long exact sequence is a collection of short exact sequences

$$0 \to {}^{k-1}H^p(M) \to {}^kH^p(M) \to H^{p-1}(Z) \to 0 \tag{7}$$

Using induction on k, this proves the following proposition.

Proposition 3.4. *The b^k-cohomology of a b^k-manifold (M, Z, j) satisfies*

$${}^kH^p(M) \cong H^p(M) \oplus \left(H^{p-1}(Z)\right)^k.$$

So far, the isomorphism in Proposition 3.4 is non-canonical. In Subsection 3.4 and Section 4, we will find canonical maps from ${}^kH^p(M)$ to $(H^{p-1}(Z))^k$ and $H^p(M)$, respectively, which will make the isomorphism canonical.

3.4. The Laurent series of a closed b^k-Form

Definition 3.5. A **Laurent Series** of a closed b^k-form ω is an expression for ω in a neighborhood of Z of the form

$$\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta$$

where $y \in j$ is a positively oriented local defining function and each α_{-i} is closed.

Remark 3.6. Every closed b^k-form has a Laurent series. In fact, Proposition 3.3 shows that given a collar neighborhood $(y, \pi) : U \to (-R, R) \times Z$ of Z with $y \in j$, every closed b^k-form ω can be written (in a neighborhood of Z) as the sum of a closed b^{k-1} form and $y^{-k} dy \wedge \pi^*(\mathrm{res}(\omega))$. By applying induction on the b^{k-1} form, we arrive at a Laurent series of the form $\omega = \sum_{i=1}^{k} y^{-i} dy \wedge \pi^*(\gamma_{-i}) + \beta$ for closed forms γ_{-i} on Z.

Example. Consider the b^k-manifold $(S^1 \times S^1, Z_1 \cup Z_2, [y]^{k-1})$ pictured in Figure 1.

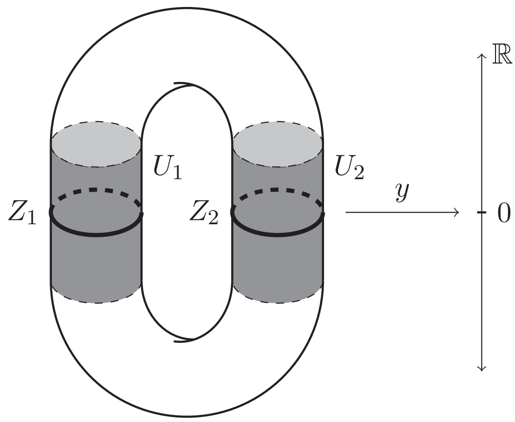

Figure 1: A b^k-manifold with disconnected Z

where a collar neighborhood $U = U_1 \cup U_2$ of Z is shaded. Let $\{(\theta_i, y)\}$ be coordinates on U_i. Then $d\theta_1$ (respectively $d\theta_2$) extends trivially over U_2 (respectively U_1) to a smooth form on all of U. Let ω be a degree two b^k-form on M. On U, it admits a decomposition $\omega = y^{-k} dy \wedge (f d\theta_1 + g d\theta_2) + \beta$ for smooth functions f, g and a smooth form β. Let $\pi : U \to Z$ be the vertical projection, and for $-k \leq i \leq -1$, let

$$f_i := \frac{1}{(k+i)!} \frac{\partial^{k+i} f}{\partial y^{k+i}}\bigg|_Z \qquad g_i := \frac{1}{(k+i)!} \frac{\partial^{k+i} g}{\partial y^{k+i}}\bigg|_Z .$$

Then

$$f = \pi^*(f_{-k}) + \pi^*(f_{-k+1}) y + \cdots + \pi^*(f_{-1}) y^{k-1} + \widetilde{f}$$
$$g = \pi^*(g_{-k}) + \pi^*(g_{-k+1}) y + \cdots + \pi^*(g_{-1}) y^{k-1} + \widetilde{g}$$

for $\widetilde{f}, \widetilde{g} \in I^k$. Then ω has a Laurent series

$$\omega = \sum_{i=1}^{k} y^{-i} dy \wedge (\pi^*(f_i) d\theta_1 + \pi^*(g_i) d\theta_2) + \beta'$$

where β' is smooth form.

Proposition 3.7. *Let $\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta$ be a Laurent series of the closed b^k-form ω. The map*

$$\begin{aligned} {}^k H^p(M) &\to (H^{p-1}(Z))^k \\ [\omega] &\mapsto ([i^*(\alpha_{-1})], [i^*(\alpha_{-2})], \ldots, [i^*(\alpha_{-k})]) \end{aligned} \tag{8}$$

is independent of the choice of Laurent series.

Proof. By Proposition 3.1, we may assume that all our Laurent series are written with respect to the same local defining function $y \in j$. When $k = 1$, then for $\omega \in {}^1\Omega^p(M)$, the class $[i^*(\alpha_{-1})]$ is the image of $[\omega]$ in the map appearing in Equation 7, and therefore depends only on $[\omega]$.

For $k > 1$, assume the proposition is true for $k - 1$, and let $\omega \in {}^k\Omega^p(M)$. Consider Laurent series of two representatives of $[\omega]$,

$$\omega_0 = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta \quad \text{and} \quad \omega_1 = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha'_{-i} + \beta'$$

Both $[i^*(\alpha_{-k})]$ and $[i^*(\alpha'_{-k})]$ are the image of $[\omega]$ in Equation 7, so are equal. If we can show that

$$\sum_{i=1}^{k-1} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta \quad \text{and} \quad \sum_{i=1}^{k-1} \frac{dy}{y^i} \wedge \alpha'_{-i} + \beta'$$

are cohomologous b^{k-1}-forms, then we will be done by induction. That is, we must show that

$$\omega_1 - \frac{dy}{y^k} \wedge \alpha'_{-k} - \left(\omega_0 - \frac{dy}{y^k} \wedge \alpha_{-k}\right) \tag{9}$$

is an exact b^{k-1}-form. Because $[\omega_0] = [\omega_1]$, there is a b^k-form η with $d\eta = \omega_1 - \omega_0$. Moreover, because $\alpha_{-k} - \alpha'_{-k}$ is a closed form with $i^*(\alpha_{-k} - \alpha'_{-k})$ exact, the relative Poincaré lemma implies that it has a primitive μ. Then

$\eta + \frac{dy}{y^k} \wedge \mu$ is a primitive for the form (9). However, this primitive is a b^k-form; to prove that (9) is exact as a b^{k-1}-form (and in doing so complete the induction), observe that the map ${}^{k-1}H^p(M) \to {}^kH^p(M)$ from Sequence 7 is injective, so any b^{k-1}-form exact as a b^k-form is also exact as a b^{k-1}-form. □

Definition 3.8. Given a b^k-form ω, the image of $[\omega]$ under Map 8 is the **Laurent Decomposition** of $[\omega]$.

The result below strengthens Theorem 3.4.

Theorem 3.9. *The sequence below, with g, f described by Maps 5 and 8 respectively, is exact.*

$$0 \to H^p(M) \xrightarrow{g} {}^kH^p(M) \xrightarrow{f} (H^{p-1}(Z))^k \to 0. \tag{10}$$

Proof. The map g is a composition of the inclusions ${}^{\ell-1}H^n(M) \to {}^{\ell}H^n(M)$ appearing in the short exact sequence (7) for $\ell \leq k$. Therefore, it itself is an inclusion. The proof that f is surjective follows from the same trick used to create a preimage of a closed $\alpha \in \Omega^{p-1}(Z)$ in the proof of Proposition 3.3. Exactness at the middle is straightforward. □

4. Volume forms on a b^k-manifold

Let (M, Z, j) be a compact b^k-manifold, and let $\omega \in {}^k\Omega^{\dim(M)}(M)$. Because ω "blows up" along Z, we cannot expect its integral to be finite. If we remove from M a neighborhood of Z, then the integral of ω over the remainder is finite, but obviously depends on the choice of neighborhood. In this section, we extract a useful invariant of ω by studying the behavior of this integral as the size of the removed neighborhood shrinks. The results from this section apply even to non-compact manifolds; so that we may state these results in full generality, we begin by introducing notation for compactly supported de Rham theory.

Definition 4.1. The subset ${}^k\Omega^p_c(M) \subseteq {}^k\Omega^p(M)$ consists of b^k-forms with compact support. They form a subcomplex of the b^k-de Rham complex, the cohomology of which is denoted ${}^kH^*_c(M)$

4.1. Liouville volume of a b^k-form

Definition 4.2. Let (M, Z, j) be an n-dimensional b^k-manifold. Given $\omega \in {}^k\Omega^n_c(M)$, $\epsilon > 0$ small, and a local defining function $y \in j$, define $U_{y,\epsilon} = y^{-1}([-\epsilon, \epsilon])$ and $\mathrm{vol}_{y,\epsilon}(\omega) = \int_{M \setminus U_{y,\epsilon}} \omega$

In [R], the author proves that for $k = 1$, $\lim_{\epsilon\to 0}\mathrm{vol}_{y,\epsilon}(\omega)$ converges and is independent of y. When $k > 1$, this limit will not necessarily converge to a number, but rather to a polynomial in ϵ^{-1}.

Theorem 4.3. *For a fixed $[\omega] \in {}^kH^n_c(M)$ on a b^k-manifold (M, Z, j) with Z compact, there is a polynomial $P_{[\omega]}$ for which*

$$\lim_{\epsilon\to 0} \left(P_{[\omega]}\left(\epsilon^{-1}\right) - \mathrm{vol}_{y,\epsilon}(\omega)\right) = 0 \tag{11}$$

for any $y \in j$ and any ω representing $[\omega]$.

The proof of Theorem 4.3 will use the following lemma.

Lemma 4.4. *If $f(x) : \mathbb{R} \to \mathbb{R}$ satisfies $[f]^{k-1} = [x]^{k-1}$ (where the bracket denotes the jet with respect to the hypersurface $\{0\}$ of $\mathbb{R}$) and has inverse $h : (-\epsilon, \epsilon) \to \mathbb{R}$, then for all $i < k$*

$$\frac{1}{x^i} - \frac{1}{(-x)^i} + \frac{1}{h(-x)^i} - \frac{1}{h(x)^i} \tag{12}$$

is a smooth function that vanishes at 0.

Proof. Because $[f]^{k-1} = [x]^{k-1}$, $f = (x + g(x)x^k)$ for some smooth g. Then because $h(x)$ vanishes at 0 and $x = f(h(x)) = h(x) + g(h(x))h(x)^k$, it follows that $[h]^{k-1} = [x]^{k-1}$, so $h(x) = x + \tilde{g}x^k$ for some smooth $\tilde{g}$. Then

$$s(x) := \frac{1}{x^i} - \frac{1}{h(x)^i} = \frac{(1 + \tilde{g}x^{k-1})^i - 1}{x^i(1 + \tilde{g}x^{k-1})^i} = \frac{\left(\sum_{j=1}^{i} \binom{i}{j}\tilde{g}^j x^{j(k-1)-i}\right)}{(1 + \tilde{g}x^{k-1})^i}$$

is a smooth function. Because Equation 12 equals $s(x) - s(-x)$, it is a smooth odd function, hence vanishes at zero. □

Proof. (of Theorem 4.3) We first prove that there is a polynomial $P_{[\omega]}$ that satisfies Equation 11 for a specific y and ω, then we prove that the polynomial is independent of y, then that the polynomial vanishes for exact ω (so depends only on the b^k-cohomology class of ω).

Fix a local defining function $y \in j$ and a closed collar neighborhood $(y, \pi) : U \to [-R, R] \times Z$ of Z. Since ω is compactly supported, $\int_{M \setminus U} \omega < \infty$, so to prove the existence of $P_{[\omega]}$ it suffices to construct a polynomial for the case $M = U$. By Remark 3.6, there exists a Laurent series of ω of the form $\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \pi^*(\alpha_{-i}) + \beta$ where $\alpha_{-i} \in \Omega^{n-1}(Z)$. Then

$$\mathrm{vol}_{y,\epsilon}(\omega) = \int_{U \setminus U_{y,\epsilon}} \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \pi^*(\alpha_{-i}) + \int_{U \setminus U_{y,\epsilon}} \beta.$$

Applying Fubini's theorem (and cancelling log terms), we find that the polynomial

$$P(t) = \left(\int_U \beta + \sum_{\substack{i=2 \\ i \text{ even}}}^{k} \left(\frac{-2R^{1-i}}{i-1} \right) \int_Z \alpha_{-i} \right) + \sum_{\substack{i=2 \\ i \text{ even}}}^{k} \left(\frac{2}{i-1} \int_Z \alpha_{-i} \right) t^{i-1}$$

satisfies Equation 11 for this specific choice of y and ω.

To show that this polynomial does not depend on our choice of y, let U be a tubular neighborhood $(y, \pi) : U \to [-R, R] \times Z$, with $y \in j$, and let $h \in j$ be another local defining function. It suffices to show that

$$\lim_{\epsilon \to 0} \left(\mathrm{vol}_{h,\epsilon}(\omega) - \mathrm{vol}_{y,\epsilon}(\omega) \right) = 0$$

for the case $M = U$. To do so, let $y_{h,z} : \mathbb{R} \to \mathbb{R}$ be the function, defined near zero, inverse to $h\big|_{[-R,R] \times \{z\}}$. That is, for sufficiently small ϵ, $h(y_{h,z}(\epsilon), z) = \epsilon$ and $U_{h,\epsilon} = \{(y, z) \in [-R, R] \times Z \mid y_{h,z}(-\epsilon) \le y \le y_{h,z}(\epsilon)\}$. Then

$$\mathrm{vol}_{h,\epsilon}(\omega) - \mathrm{vol}_{y,\epsilon}(\omega) = \left(\int_{U \setminus U_{h,\epsilon}} - \int_{U \setminus U_{y,\epsilon}} \right) \omega$$

$$= \left(\int_{U \setminus U_{h,\epsilon}} - \int_{U \setminus U_{y,\epsilon}} \right) \beta + \int_Z \left(\log \left| \frac{y_{h,z}(-\epsilon)}{y_{h,z}(\epsilon)} \right| + \log \left| \frac{\epsilon}{-\epsilon} \right| \right) \pi^*(\alpha_{-1}) +$$

$$+ \sum_{i=2}^{k} \frac{1}{1-i} \int_Z \left(y_{h,z}(-\epsilon)^{1-i} - (y_{h,z}(\epsilon))^{1-i} + \epsilon^{1-i} - (-\epsilon)^{1-i} \right) \pi^*(\alpha_{-i}).$$

Applying Lemma 4.4 to the functions y and h shows that the limit as $\epsilon \to 0$ of the above expression is zero, which proves that the volume polynomials for y and h are equal.

To show that the polynomial associated to any exact form is trivial, suppose ω is exact and let $\eta = \sum_{i=1}^{k} y^{-i} dy \wedge \pi^* \eta_{-i} + \beta_\eta$ be a Laurent series of a primitive of ω. Then

$$\int_{M\setminus U_{y,\epsilon}} \omega = \int_{\partial(M\setminus U_{y,\epsilon})} \eta = \int_{\partial(M\setminus U_{y,\epsilon})} \beta_\eta$$

which approaches 0 as $\epsilon \to 0$. $\square$

Definition 4.5. The polynomial $P_{[\omega]}$ described in Theorem 4.3 is the **volume polynomial** of $[\omega]$. Its constant term $P_{[\omega]}(0)$ is the **Liouville volume** of $[\omega]$.

Given a b^k-form ω of degree $p < \dim(M)$ and a compact p-dimensional submanifold $Y \subseteq M$ transverse to Z, the pullback of ω will be a b^k-form of top degree for the induced b^k-structure on Y, and therefore has a Louville volume. By Poincaré duality, this remark inspires the definition of the *smooth part* of a b^k-form.

Definition 4.6. Let $[\omega] \in {}^kH^p(M)$. The image of $[\omega]$ under the map

$$\begin{aligned} {}^kH^p(M) &\to (H^{n-p}_c(M))^* \cong H^p(M) \\ [\omega] &\mapsto \big([\eta] \mapsto P_{[\omega\wedge\eta]}(0)\big) \end{aligned} \tag{13}$$

is its **smooth part** $[\omega_{\text{sm}}] \in H^p(M)$.

If $[\omega]$ is smooth (that is, $[\omega] \in H^n(M) \subseteq {}^kH^n(M)$), then $P_{[\omega\wedge\eta]}(0)$ equals $\int_M \omega\wedge\eta$, so $[\omega] = [\omega_{\text{sm}}]$. This shows that Equation 13 splits the short exact sequence from Equation 10, which yields a canonical isomorphism that realizes the (abstract) isomorphism from Proposition 3.4.

$$\begin{aligned} \varphi : {}^kH^n(M) &\cong H^n(M) \oplus \big(H^{n-1}(Z)\big)^k \\ [\omega] &\mapsto ([\omega_{\text{sm}}], [\alpha_{-1}], \dots, [\alpha_{-k}]) \end{aligned} \tag{14}$$

Definition 4.7. Let ω be a b^k-form of top degree. The **Liouville-Laurent** decomposition of $[\omega]$ is its image under Equation 14, $([\omega_{\text{sm}}], [\alpha_{-1}], \dots, [\alpha_{-k}])$.

The next proposition shows that taking the Liouville-Laurent decomposition of a b^k-form commutes with taking its pullback under a b^k-map.

Proposition 4.8. *Let $\varphi : (M, Z, j) \to (M', Z', j')$ be a b^k-map, and let $[\omega'] \in {}^kH^p(M')$ have Liouville-Laurent decomposition $([\omega'_{sm}], [\alpha'_{-1}], \dots, [\alpha'_{-k}])$. Then $[\varphi^*(\omega')]$ has Liouville-Laurent decomposition*

$$([\varphi^*(\omega'_{sm})], [\varphi\big|_Z{}^*(\alpha'_{-1})], \dots, [\varphi\big|_Z{}^*(\alpha'_{-k})]).$$

Proof. Let $y' \in j'$, and $i_Z : Z \to M$, $i_{Z'} : Z' \to M'$ be the inclusions. By the definition of a b^k-map, $y := \varphi^*(y')$ represents j. Near Z', ω' can be written in the form $\omega' = \sum_{i=1}^k y'^{-i} dy' \wedge \pi^*\alpha'_{-i} + \beta'$, and $\varphi^*(\omega)$ in the form $\varphi^*(\omega') = \sum_{i=1}^k y^{-i} dy \wedge \varphi^*(\pi^*\alpha'_{-i}) + \varphi^*(\beta')$, and we see that $[\varphi^*(\omega')]$ has Laurent decomposition

$$([i_Z^*(\varphi^*(\pi^*\alpha'_{-1}))], \ldots, [i_Z^*(\varphi^*(\pi^*\alpha'_{-k}))]) = ([\varphi\big|_Z{}^*(\alpha'_{-1})], \ldots, [\varphi\big|_Z{}^*(\alpha'_{-k})]).$$

To prove $[\varphi^*(\omega')_{\mathrm{sm}}] = [\varphi^*(\omega'_{\mathrm{sm}})]$, we show that for all $[\eta] \in H_c^{n-p}(M)$,

$$P_{[\varphi^*(\omega')\wedge\eta]}(0) = \int_M \varphi^*(\omega'_{\mathrm{sm}}) \wedge \eta. \tag{15}$$

First, we introduce an auxiliary family of smooth closed differential forms $\omega'_\epsilon \in \Omega^p(M')$ with the property that the Liouville volume of $\varphi^*(\omega') \wedge \eta$ can be calculated in terms of the asymptotic behavior of $\int_M \varphi^*(\omega'_\epsilon) \wedge \eta$ instead of $\int_{M\setminus U_{y,\epsilon}} \varphi^*(\omega') \wedge \eta$. For $\epsilon > 0$ small, let $f_\epsilon : \mathbb{R} \to [0,1]$ be a smooth function such that

$$f_\epsilon\big|_{\mathbb{R}\setminus(-\epsilon,\epsilon)} = 1 \quad \text{and} \quad f_\epsilon\big|_{(-\epsilon+\exp(-\epsilon^{-1}),\epsilon-\exp(-\epsilon^{-1}))} = 0$$

and assume that f_ϵ varies smoothly with ϵ. Define

$$\omega'_\epsilon = \sum_{i=1}^k f_\epsilon(y') \frac{dy'}{y'} \wedge \pi^*\alpha'_{-i} + \beta'$$

and observe that ω'_ϵ is closed and that $\int_M \varphi^*(\omega'_\epsilon) \wedge \eta$ approaches $\mathrm{vol}_{y,\epsilon}(\varphi^*(\omega') \wedge \eta)$ as $\epsilon \to 0$.

Next, recall that the pullback map in de Rham cohomology induces (by Poincaré duality) a pushforward map in compactly supported cohomology; we will use the notation $\varphi_*\eta$ for a representative of the pushforward of $[\eta] \in H_c^{n-p}(M)$. Using this notation,

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0} \left(P_{[\varphi^*(\omega')\wedge\eta]}(\epsilon^{-1}) - \mathrm{vol}_{y,\epsilon}(\varphi^*(\omega') \wedge \eta)\right) \\
&= \lim_{\epsilon\to 0} \left(P_{[\varphi^*(\omega')\wedge\eta]}(\epsilon^{-1}) - \int_{M'} \omega'_\epsilon \wedge \varphi_*\eta\right) \\
&= \lim_{\epsilon\to 0} \left(P_{[\varphi^*(\omega')\wedge\eta]}(\epsilon^{-1}) - P_{[\omega'\wedge\varphi_*\eta]}(\epsilon^{-1})\right)
\end{aligned}$$

so

$$P_{[\varphi^*(\omega')\wedge\eta]}(0) = \int_{M'} \omega'_{\mathrm{sm}} \wedge \varphi_*\eta = \int_M \varphi^*(\omega'_{\mathrm{sm}}) \wedge \eta$$

which proves Equation 15. □

Although $P_{[\omega]}$ depends only on $[\omega]$ and not on ω itself, it *does* depend on j. That is, if ω is a volume form on $M\backslash Z$ and j, j' are two jets for which ω extends to a global b^k-form, the terms appearing in $P_{[\omega]}$ will depend on whether j or j' is used to define the b^k-structure. However, the residue term of $P_{[\omega]}$ is unchanged — we study this in Section 6.

4.2. b^k orientation

The notion of orientability of a smooth manifold generalizes in a natural way to the b^k-world.

Definition 4.9. A **volume b^k-form** on a b^k-manifold is a nowhere vanishing b^k-form of top degree. A connected b^k-manifold is **b^k-orientable** if it admits a volume b^k-form, and a **b^k-orientation** is a choice of one of the two connected components of the space of volume b^k-forms.

For example, if $Z \subseteq M$ is a meridian of the torus $S^1 \times S^1$, the corresponding b^1-manifold is not b^1-orientable even though M is orientable. Given a volume b^k-form ω, $\mathrm{res}(\omega)$ is a smooth volume form on Z. In this way, a b^k-orientation on (M, Z, j) induces an orientation on Z which may or may not agree with the orientation of Z given in the data of a b^k-manifold.

Definition 4.10. Let ω be a volume b^k-form on (M, Z, j). If the smooth form $\mathrm{res}(\omega)$ is positively oriented, we say that ω is a **positively oriented** volume b^k-form.

If two volume b^k-forms ω_0, ω_1 are cohomologous, then $[\mathrm{res}(\omega_0)]$ and $[\mathrm{res}(\omega_1)]$ are cohomologous volume forms on Z, hence induce the same orientation on Z. This implies that ω_0 and ω_1 are in the same connected component of $\wedge^n({}^kT_p^*M)\backslash\{0\}$ for all $p \in Z$ (and therefore, for all $p \in M$). Therefore, cohomologous volume b^k-forms induce the same b^k-orientation on M.

5. Symplectic and Poisson geometry of b^k-Forms

In this section we introduce the notion of a symplectic b^k-form, prove Moser's theorems in the b^k-category, classify symplectic b^k-surfaces, and show how the Liouville-Laurent decomposition of a b-symplectic form on a surface reconciles a classification theorem from [GMP2] with one from [R].

Definition 5.1. A **symplectic b^k-form** on a b^k-manifold is a closed degree two b^k-form having maximal rank at every $p \in M$. A **symplectic b^k-manifold** (M, Z, j, ω) is a b^k-manifold (M, Z, j) with a symplectic b^k-form ω. A **b^k-symplectomorphism** $\varphi : (M, Z, j, \omega) \to (M', Z', j_{Z'}, \omega')$ is a b^k-map satisfying $\varphi^*(\omega') = \omega$.

Theorem 5.2. *(relative Moser's theorem) If ω_0, ω_1 are symplectic b^k-forms on (M, Z, j) with Z compact, $\omega_0\big|_Z = \omega_1\big|_Z$, and $[\omega_0] = [\omega_1]$, then there are neighborhoods U_0, U_1 of Z and a b^k-symplectomorphism $\varphi : (U_0, Z, j, \omega_0) \to (U_1, Z, j, \omega_1)$ such that $\varphi\big|_Z = \mathrm{id}$.*

Proof. Pick a local defining function $y \in j$ and Laurent series of ω_0, ω_1

$$\omega_0 = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta, \qquad \omega_1 = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha'_{-i} + \beta'.$$

Then each $i^*(\alpha'_{-i} - \alpha_{-i}) \in \Omega^1(Z)$ is exact, and $i^*(\alpha'_{-k} - \alpha_{-k}) = i^*(\beta' - \beta) = 0$. By the relative Poincaré lemma there are primitives μ_i of $(\alpha'_{-i} - \alpha_{-i})$ and μ_β of $(\beta' - \beta)$ with $\mu_{-k}\big|_Z = \mu_\beta\big|_Z = 0$. Then $\omega_1 - \omega_0 = d\mu$, where

$$\mu = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \mu_{-i} + \mu_\beta.$$

Let $\omega_t = t\omega_1 + (1-t)\omega_0$, and observe that $d\omega_t/dt = d\mu$. By shrinking our neighborhood around Z, we can assume that ω_t has full rank for all t, giving a pairing between b^k-vector fields and degree one b^k-forms. Because μ vanishes on Z (since $\mu_{-k}\big|_Z = 0$ and $\mu_\beta\big|_Z = 0$), the vector field v_t defined by Moser's equation $\iota_{v_t}\omega_t = -\mu$ is a b^k-vector field that vanishes on Z, the time-one flow of which is the desired b^k-symplectomorphism. $\square$

Theorem 5.3. *(global Moser's theorem) Let (M, Z, j) be a compact b^k-manifold, and $\omega_t := t\omega_1 + (1-t)\omega_0$ a symplectic b^k-form for $t \in [0,1]$, with $[\omega_0] = [\omega_1]$. Then there is an isotopy ρ_t of b^k-maps with $\rho_t^*(\omega_t) = \omega_0$ for $t \in [0,1]$.*

Proof. Because $d\omega_t/dt = \omega_1 - \omega_0$ is exact, there is a smooth b^k-form μ such that $d\mu = \omega_1 - \omega_0$. Because ω_t is a b^k-form, it defines an pairing between b^k-vector fields and degree one b^k-forms. Therefore, the vector field v_t defined by Moser's equation

$$\iota_{v_t}\omega_t = -\mu$$

is a b^k-vector field, so its flow defines an isotopy ρ_t of b^k-maps with $\rho_t^*(\omega_t) = \omega_0$. $\square$

In [R], the author classifies the space of stable Poisson structures on a connected, compact surface in terms of geometric data. In [GMP2], the authors demonstrate a correspondence between stable Poisson structures and b-symplectic forms on a manifold, and classify b-symplectic forms on a connected, compact surface in terms of their b-cohomology class. Proposition 5.4 exhibits a direct connection between the cohomological classification data in [GMP2] and the geometric classification data in [R].

Proposition 5.4. *Let $[\omega] = ([\omega_{sm}], [\alpha_{-1}])$ be the Liouville-Laurent decomposition of a positively oriented b-symplectic form on a connected compact surface. Let $\{\gamma_r\}$ be the oriented circles that constitute Z. Then the Liouville volume of ω is $\int_M \omega_{sm}$, and the period of the modular vector field on γ_r is $\int_{\gamma_r} \alpha_{-1}$*

Proof. The fact that the Liouville volume of ω equals $\int_M \omega_{\mathrm{sm}}$ follows from the definition of the smooth part of a b^k-form. Let γ_i be a connected component of Z. We can find a collar neighborhood

$$U = \{(y,\theta), |y| < R, \theta \in [0,1]/\sim\} \qquad R > 0$$

such that on $U, \omega = c\frac{dy}{y} \wedge d\theta$ for some $c > 0$, where $d\theta$ is a positively-oriented volume form on Z. From [R], we know that the period of the modular vector field is c, and we calculate that $\int_{\gamma_i} \alpha_{-1} = \int_{\gamma_i} c d\theta = c$. $\square$

Theorem 5.5. *Let ω_0, ω_1 be symplectic b^k-forms on a compact connected b^k-surface (M, Z, j). The following are equivalent*

1) *There is a b^k-symplectomorphism $\varphi : (M, Z, j, \omega_0) \to (M, Z, j, \omega_1)$.*
2) $[\omega_0] = [\omega_1]$
3) *The Liouville volumes of ω_0 and ω_1 agree, as do the numbers $\int_{\gamma_r} \alpha_{-i}$ for all connected components $\gamma_r \subseteq Z$ and all $1 \le i \le k$, where α_{-i} are the terms appearing in the Laurent decomposition of the two forms.*

Proof.

(1) $\iff$ (2)**:** This follows from Theorem 5.3 in dimension 2, using the fact that two cohomologous volume b^k-forms induce the same b^k-orientation to show that the ω_t appearing in Theorem 5.3 are non-degenerate.

(2) $\iff$ (3)**:** The cohomology class of a volume b^k form is determined by its Liouville-Laurent decomposition, which in turn is determined by its Liouville volume and the integrals $\int_{\gamma_r} \alpha_{-i}$.

□

6. Symplectic and Poisson structures of b^k type

When the authors of [GMP2] studied the Poisson structures dual to symplectic b-forms, they found that b-symplectomorphisms are precisely Poisson isomorphisms of the dual Poisson manifolds. This observation does not generalize to the b^k case: although every symplectic b^k-form is dual to a Poisson bivector, not every Poisson isomorphism (with respect to this bivector) is realized by a b^k-map. Similarly, if (M, Z, j, ω) and (M, Z, j', ω') are two symplectic b^k-manifolds, there may be a diffeomorphism of (M, Z) that restricts to a symplectomorphism $(M\backslash Z, \omega) \to (M\backslash Z, \omega')$ even if there is no b^k-symplectomorphism $(M, Z, j, \omega) \to (M, Z, j', \omega')$. In this section, we show how to use b^k-manifolds to prove statements about objects outside of the b^k-category. We begin by defining the notion of a Poisson (and symplectic) structure of b^k type — these are the Poisson (and symplectic) structures that are dual to (or equal to) a symplectic b^k-form for *some* choice of jet data. Then we apply the theory of symplectic b^k-forms to classify these structures on compact connected surfaces.

Definition 6.1. Let Z be an oriented hypersurface of an oriented manifold M. Let Π be a Poisson structure on M having full rank on $M\backslash Z$, and let $\omega \in \Omega^2(M\backslash Z)$ be the symplectic form dual to $\Pi\big|_{M\backslash Z}$. We say that Π and ω are **of b^k type** if there is some $j \in J^{k-1}$ for which (M, Z, j) is a b^k-manifold on which ω extends to a symplectic b^k-form.

Remark 6.2. If Π is a Poisson structure of b^k type on (M^{2n}, Z) with dual form ω, then there will be several distinct jets with respect to which ω is a symplectic b^k-form. For example, if $j = [y]$ is one such jet and $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$ and $f'(0) > 0$, then the jet $j' := [f \circ y]$ defines exactly the same b^k-(co)tangent bundles as j. As such, ω is a symplectic b^k-form with respect to both j' and j. However, the condition of ω^n being positively oriented (as a volume b^k-form in the sense of Definition 4.10) does *not* depend upon the chosen jet. We say that Π (or ω) is a **positively oriented** Poisson structure (or symplectic form) of b^k type if ω extends to a positively oriented volume b^k-form for *any* choice of jet j for which ω extends to a b^k form.

To study Poisson and symplectic structures of b^k type, we must understand how a b^k-form behaves under diffeomorphisms of (M, Z) that are not necessarily b^k-maps. Of particular interest are diffeomorphisms of M that restrict to $(z, y) \mapsto (z, P(y))$ in a collar neighborhood $Z \times \mathbb{R}$ of Z, where P is a polynomial.

Proposition 6.3. *Let P be a polynomial with $P(0) = 0$ and $P'(0) > 0$. Let (M, Z, j) be a b^k-manifold with positively oriented local defining function $y \in j$, and let $\varphi : M \to M$ be a diffeomorphism given by* $\mathrm{id} \times P(y)$ *in a collar neighborhood $(\pi, y) : U \to Z \times \mathbb{R}$ of Z. Then*

- *If ω is a b^k-form, then $\varphi^*(\omega)$ is also a b^k-form on (M, Z, j).*
- *If $[\omega]$ has Liouville-Laurent decomposition $([\omega_{sm}], [\alpha_{-1}], \ldots, [\alpha_{-k}])$ and $[\varphi^*(\omega)]$ has Liouville-Laurent decomposition $([\omega'_{sm}], [\alpha'_{-1}], \ldots, [\alpha'_{-k}])$, then $[\varphi^*(\omega_{sm})] = [\omega'_{sm}]$ and $[\alpha_{-1}] = [\alpha'_{-1}]$.*

Proof. In a collar neighborhood, let $\omega = \sum_{i=1}^{k} y^{-i} dy \wedge \pi^*(\alpha_{-i}) + \beta$ be a Laurent decomposition of ω. Then

$$\varphi^*(\omega) = \sum_{i=1}^{k} \frac{P'(y)dy}{P(y)^i} \wedge \pi^*(\alpha_{-i}) + \varphi^*\beta. \tag{16}$$

Notice that each term $\frac{P'(y)}{P(y)^i}$ must have a Laurent series with no exponents less than $-i$, so the first claim is proved by replacing each $\frac{P'(y)}{P(y)^i}$ in Equation 16 with its Laurent series. To prove the second claim, first observe that for $i \neq 1$,

$$\frac{P'(y)dy}{P(y)^i} = d\left(\frac{1}{-i+1} P(y)^{-i+1}\right)$$

so the meromorphic function $P'(y)P(y)^{-i}$ has no residue. For $i = 1$ the function $P'(y)P(y)^{-1}$ has a Laurent series with principal part $1/y$. Therefore, by replacing the $P'(y)P(y)^{-i}$ terms in Equation 16 with their Laurent series in the variable y, we arrive at a Laurent series of $\varphi^*(\omega)$ that has

$y^{-1}dy \wedge \pi^*(\alpha_{-1})$ as its residue term, proving that $[\alpha_{-1}] = [\alpha'_{-1}]$. To prove that $[\varphi^*(\omega_{\mathrm{sm}})] = [\omega'_{\mathrm{sm}}]$, let $[\eta] \in {}^k H_c^{n-p}(M)$, where p is the degree of ω and $n = \dim(M)$. It suffices to show that

$$P_{[\varphi^*(\omega)\wedge\eta]}(0) = \int_M \varphi^*(\omega_{\mathrm{sm}}) \wedge \eta. \tag{17}$$

Observe that for $\epsilon > 0$ small, $\varphi(U_{y,\epsilon}) = U_{y,P(\epsilon)}$, so $\mathrm{vol}_{y,\epsilon}(\varphi^*(\omega \wedge (\varphi^{-1})^*\eta)) = \mathrm{vol}_{y,P(\epsilon)}(\omega \wedge (\varphi^{-1})^*\eta))$. Then letting

$$\omega \wedge (\varphi^{-1})^*\eta = \sum_{i=1}^{k} \frac{dy}{y} \wedge \pi^*(\widetilde{\alpha}_{-i}) + \widetilde{\beta}$$

be a Laurent series of $\omega \wedge (\varphi^{-1})^*\eta$,

$$\begin{aligned}
&\mathrm{vol}_{y,\epsilon}(\varphi^*(\omega) \wedge \eta) - \mathrm{vol}_{y,\epsilon}(\omega \wedge (\varphi^{-1})^*\eta) \\
&= \left(\int_{M\setminus U_{y,P(\epsilon)}} - \int_{M\setminus U_{y,\epsilon}} \right) \omega \wedge (\varphi^{-1})^*\eta \\
&= \int_Z \left(\int_{P(\epsilon)}^{\epsilon} - \int_{P(-\epsilon)}^{-\epsilon} \right) \sum_{i=1}^{k} \frac{dy}{y^i} \pi^*(\widetilde{\alpha}_{-i}) + \left(\int_{M\setminus U_{y,P(\epsilon)}} - \int_{M\setminus U_{y,\epsilon}} \right) \widetilde{\beta}.
\end{aligned}$$

As $\epsilon \to 0$, this limit approaches an odd function of ϵ. This proves that $P_{[\varphi^*(\omega)\wedge\eta]}(0) = P_{[\omega\wedge(\varphi^{-1})^*\eta]}(0)$, from which Equation 17 follows. □

Lemma 6.4. *Let $(a_{-1}, \dots, a_{-k}) \in \mathbb{R}^k$ with $a_{-k} > 0$. There is a polynomial $P(y) = \sum p_i y^i$ with $p_0 = 0$ and $p_1 > 0$ satisfying*

$$\sum_{i=1}^{k} a_{-i} \frac{P'(y)}{P(y)^i} = \frac{1}{y^k} + \frac{a_{-1}}{y} + Q(y)$$

where $Q(y)$ is a polynomial.

Proof. Recall from the proof of Proposition 6.3 that for any polynomial P and $i \neq 1$, the expression $P^{-i}P'$ has a Laurent series in y with trivial residue term and no exponents less than $-i$. When $i = 1$, $P^{-i}P'$ has principal part

y^{-1}. Therefore, for any polynomial P,

$$\sum_{i=1}^{k} a_{-i}\frac{P'}{P^i} = \sum_{i=2}^{k} \frac{b_{-i}}{y^i} + \frac{a_{-1}}{y} + Q(y) \tag{18}$$

for some $b_{-i} \in \mathbb{R}$ and some polynomial $Q(y)$. If $P(y) = (a_{-k})^{1/(1-k)}y$, then a straightforward calculation shows that $b_{-k} = 1$ in the expression above. However, we wish to find a polynomial P such that not only does $b_{-k} = 1$, but $(b_{-k}, b_{-k+1}, \ldots, b_{-2}) = (1, 0, \ldots, 0)$. The proof will be inductive: assume that we can pick $P = \sum p_i y^i$ so that $P(0) = 0$, $P'(0) > 0$, and $(b_{-k}, b_{-k+1}, \ldots, b_{-k+j-1}) = (1, 0, \ldots, 0)$ in Equation 18 — we aim to find a new P so that $P(0) = 0$, $P'(0) > 0$, $(b_{-k}, b_{-k+1}, \ldots, b_{-k+j}) = (1, 0, \ldots, 0)$. For $t \in \mathbb{R}$ let $\widetilde{P} = P + tP^{j+1}$. Then for a smooth function g,

$$\begin{aligned}\sum_{i=1}^{k} a_{-i}\frac{\widetilde{P}'}{\widetilde{P}^i} &= \sum_{i=1}^{k} a_{-i}\frac{P'}{P^i}\left(1 + (j+1-i)tp_1^j y^j + gy^{j+1}\right)\\ &= \frac{1}{y^k} + \sum_{i=2}^{k-j}\frac{b_{-i}}{y^i} + \frac{a_{-1}}{y} + Q(y)\\ &\quad + \sum_{i=1}^{k} a_{-i}\frac{P'}{P^i}\left((j+1-i)tp_1^j y^j + gy^{j+1}\right).\end{aligned}$$

If we set $t = -b_{-k+j}p_1^{k-j-1}/(a_{-k}(j+1-k))$, the y^{-k+j} term of the above expression vanishes, completing the induction. □

The two results above are the ingredients we need to prove the main theorem of this section.

Theorem 6.5. *Let Z be an oriented hypersurface of a compact connected oriented surface M. Let Π, Π' be two positively oriented Poisson structures of b^k type on (M, Z), and ω, ω' be the dual b^k-symplectic forms (with respect to possibly different b^k-structures) with Liouville-Laurent decompositions*

$$[\omega] = ([\omega_{sm}], [\alpha_{-1}], \ldots, [\alpha_{-k}]) \qquad [\omega'] = ([\omega'_{sm}], [\alpha'_{-1}], \ldots, [\alpha'_{-k}]).$$

If $[\omega'_{sm}] = [\omega_{sm}] \in H^2(M)$ and $[\alpha'_{-1}] = [\alpha_{-1}] \in H^1(Z)$, then there is a Poisson isomorphism $\varphi : (M, \Pi) \to (M, \Pi')$.

Proof. Let j and j' be the jets of Z with respect to which ω and ω' respectively are b^k-forms with the described Liouville-Laurent decompositions, and let $y \in j, y' \in j'$ be positively oriented local defining functions for Z. Let $\{\gamma_\ell\}$ be the oriented circles that constitute the connected components of Z. If

$$\varphi : U_\ell \to \mathbb{R} \times \mathbb{S}^1 = \{(y, \theta)\} \qquad \varphi' : U_\ell \to \mathbb{R} \times \mathbb{S}^1 = \{(y', \theta)\}$$

are local coordinate charts for a collar neighborhood U_ℓ of γ_ℓ, then the map $(\varphi')^{-1} \circ \varphi$ is an orientation-preserving map in a neighborhood of γ_i, restricts to the identity on γ_i, and pulls j' back to j. As such, the collection of these maps (one for each $\gamma_\ell \subseteq Z$) defines a smooth map in a neighborhood of Z that extends to a b^k-diffeomorphism $(M, Z, j) \to (M, Z, j')$. By replacing ω' with its pullback under this b^k-diffeomorphism and citing Proposition 4.8, we may assume that ω, ω' are b^k-symplectic forms on the *same* b^k-manifold (M, Z, j), and that the Liouville-Laurent decompositions of ω, ω' with respect to this b^k structure are as described by Proposition 4.8.

Let $\pi : U_\ell = \{(y, \theta_\ell)\} \to S^1$ be projection onto the second coordinate. By Theorem 5.3, we may assume that

$$\omega\big|_{U_\ell} = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge a_i \pi^*(d\theta_\ell) + \beta_0$$

where $a_i \in \mathbb{R}$ and $a_{-k} > 0$ (because Π, Π' are positively oriented). We apply Lemma 6.4 to find a polynomial $P_\ell = \sum p_i y^i$ with $p_0 = 0, p_1 > 0$ satisfying

$$\sum_{i=1}^{k} a_{-i} \frac{P'}{P^i} = \frac{1}{y^k} + \frac{a_{-1}}{y} + Q_\ell(y)$$

for some polynomial $Q_\ell(y)$. By replacing ω with its pullback under a diffeomorphism of (M, Z) that is of the form $(y, \theta_\ell) \mapsto (P_\ell(y), \theta_\ell)$ in each U_ℓ, we may assume $[\omega]$ has Liouville-Laurent decomposition

$$([\omega_{\mathrm{sm}}], [\alpha_{-1}], 0, \ldots, 0, [d\theta])$$

where $d\theta$ is the form on Z that restricts to $d\theta_i$ on each γ_i. Similarly, we may replace ω' with a form also having this Liouville-Laurent decomposition. Applying Theorem 5.5 completes the proof. $\square$

References

[Ca] A. Cannas da Silva, *Lectures on Symplectic Geometry* (book), Lecture Notes in Mathematics 1764, Corrected 2nd Printing, Springer, 2008.

[Cl] E. Clader, *Inverse limits of finite topological spaces*, Homology, Homotopy and Applications, **11**, no. 2, 2009, pp. 223–227.

[GMP] V. Guillemin, E. Miranda, and A. R. Pires, *Codimension one symplectic foliations and regular Poisson structures*, Bulletin of the Brazilian Mathematical Society, New Series, **42** (4), 2011, pp. 1–17.

[GMP2] V. Guillemin, E. Miranda, and A. R. Pires, *Symplectic and Poisson Geometry on b-Manifolds*, arXiv, 2012.

[GMPS] V. Guillemin, E. Miranda, A. R. Pires, and G. Scott, *Toric Actions on b-symplectic Manifolds*, preprint, 2013.

[G] D. Grieser, *Basics of the b-Calculus*, arXiv, 2012.

[Me] R. Melrose, *Atiyah-Patodi-Singer Index Theorem* (book), Research Notices in Mathematics, A. K. Peters, Wellesley, 1993.

[R] O. Radko, *A classification of topologically stable Poisson structures on a compact oriented surface*, Journal of Symplectic Geometry, **1**, no. 3, 2002, pp. 523–542.

Department of Mathematics, University of Toronto
Toronto, Ontario, Canada M5S 2E4
E-mail address: `gscott@math.utoronto.edu`

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