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Symplectic capacities of Hermitian symmetric spaces of compact and noncompact type

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Inspired by the work of G. Lu [35] on pseudo symplectic capacities we obtain several results on the Gromov width and the Hofer–Zehnder capacity of Hermitian symmetric spaces of compact type. Our results and proofs extend those obtained by Lu for complex Grassmannians to Hermitian symmetric spaces of compact type. We also compute the Gromov width and the Hofer–Zehnder capacity for Cartan domains and their products.

1. Introduction

Consider the open ball of radius r,

$$B^{2n}(r) = \left\{ (x, y) \in \mathbb{R}^{2n} \;\middle|\; \sum_{j=1}^{n} x_j^2 + y_j^2 < r^2 \right\} \tag{1}$$

in the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. The Gromov width of a $2n$-dimensional symplectic manifold (M, ω), introduced in [17], is defined as

$$c_G(M, \omega) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M, \omega)\}. \tag{2}$$

By Darboux's theorem $c_G(M, \omega)$ is a positive number or ∞. Computations and estimates of the Gromov width for various examples can be found in [3], [4], [5], [7], [17], [22], [23], [28], [35], [36], [37], [38], [46], [51].

Gromov's width is an example of *symplectic capacity* introduced in [20] (see also [21]). A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds

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of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* if it satisfies the following conditions:

(**monotonicity**) if there exists a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$;

(**conformality**) $c(M, \lambda\omega) = |\lambda| c(M, \omega)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$;

(**nontriviality**) $c(B^{2n}(1), \omega_0) = \pi = c(Z^{2n}(1), \omega_0)$.

Here $B^{2n}(1)$ and $Z^{2n}(1)$ are the open unit ball and the open cylinder in the standard $(\mathbb{R}^{2n}, \omega_0)$, i.e.

$$Z^{2n}(r) = \left\{(x, y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2\right\}. \tag{3}$$

Note that the monotonicity property implies that c is a symplectic invariant. The existence of a capacity is not a trivial matter. It is easily seen that the Gromov width is the smallest symplectic capacity, i.e. $c_G(M, \omega) \leq c(M, \omega)$ for any capacity c. Note that the nontriviality property for c_G comes from the celebrated *Gromov's nonsqueezing theorem* according to which the existence of a symplectic embedding of $B^{2n}(r)$ into $Z^{2n}(R)$ implies $r \leq R$. Actually it is easily seen that the existence of any capacity implies Gromov's nonsqueezing theorem. H. Hofer and E. Zehnder [20] prove the existence of a capacity, denoted by c_{HZ} which is important in many respects, for example it plays an important role in the study of Hofer geometry on the group of symplectomorphisms of a symplectic manifold and in establishing the existence of closed characteristics on or near an energy surface. To compute or estimate c_{HZ} is rather difficult even for closed symplectic manifolds. So far the only known examples are closed surfaces where c_{HZ} is just the area [48], and complex projective spaces and their products. H. Hofer and C. Viterbo [19] proved that $c_{HZ}(\mathbb{C}\mathrm{P}^n, \omega_{FS}) = \pi$ which has been extended by G. Lu to the product of projective spaces (see Theorem 1.21 in [35] or (10) below). Lu's ingenious idea was that of defining and introducing the concept of pseudo symplectic capacity, more flexible than that of symplectic capacity, and studying its link with Gromov-Witten invariants (see Section 3 below). This allows him to obtain several valuable results, e.g. the Gromov width of Grassmannians and their products and a lower bound for the Hofer–Zehnder capacity for the product of any closed symplectic manifold with a Grassmannian. One of the aims of the present paper is to extend Lu's results to the case of Hermitian symmetric spaces of compact type.

Notation: *From now on we shall use the shortening HSSCT to denote a Hermitian symmetric space of compact type. Further, throughout the paper* *we shall denote by ω_{FS} the canonical symplectic (Kähler) form on an irreducible HSSCT normalized so that $\omega_{FS}(B) \in \{-\pi, \pi\}$ when B is a generator of $H_2(M, \mathbb{Z})$, and by A the generator for which $\omega_{FS}(A) = \pi$.*

Moreover, we compute the Gromov width and Hofer–Zehnder capacity of Cartan's domains and their products. In the next section we give a description of our results and the ideas of their proofs.

2. Statements of the main results

The following three theorems state our results about the Gromov width and the Hofer-Zehnder capacity of HSSCT.

Theorem 1. *Let (M, ω_{FS}) be an irreducible HSSCT. Then*

$$c_G(M, \omega_{FS}) = \pi. \tag{4}$$

Theorem 2. *Let (M_i, ω^i_{FS}), $i = 1, \dots, r$, be irreducible HSSCT of complex dimension n_i endowed with the canonical symplectic (Kähler) forms ω^i_{FS} normalized as above. Then*

$$c_G\left(M_1 \times \cdots \times M_r, \omega^1_{FS} \oplus \cdots \oplus \omega^r_{FS}\right) = \pi. \tag{5}$$

Moreover, if $a_1, \dots, a_r$ are nonzero constants, then

$$c_G\left(M_1 \times \cdots \times M_r, a_1\omega^1_{FS} \oplus \cdots \oplus a_r\omega^r_{FS}\right) \leq \min\{|a_1|, \dots, |a_r|\}\pi \tag{6}$$

and

$$c_{HZ}\left(M_1 \times \cdots \times M_r, a_1\omega^1_{FS} \oplus \cdots \oplus a_r\omega^r_{FS}\right) \geq (|a_1| + \cdots + |a_r|)\pi. \tag{7}$$

Theorem 3. *Let (M, ω_{FS}) be an irreducible HSSCT and (N, ω) be any closed symplectic manifold. Then, for any nonzero real number a,*

$$c_G(N \times M, \omega \oplus a\omega_{FS}) \leq |a|\pi. \tag{8}$$

Formulas (4) and (5) extend Theorem 1.15 and formula (22) in [35] respectively (valid for the Grassmannians) to the case of HSSCT. The lower bounds $c_G(M, \omega_{FS}) \geq \pi$ in Theorem 1 and

$$c_G\left(M_1 \times \cdots \times M_r, \omega^1_{FS} \oplus \cdots \oplus \omega^r_{FS}\right) \geq \pi$$

in Theorem 2 are obtained by using the results in [11] which imply the existence of a symplectic embedding of the noncompact dual (Ω, ω_0) of (M, ω_{FS})

into (M, ω_{FS}) (where ω_0 is the standard symplectic form of $\Omega \subset \mathbb{C}^n$, being n the complex dimension of M) and by the existence of a symplectic embedding of $B^{2n}(1)$ into (Ω, ω_0) (see Sections 4 and 5 below for details). The upper bounds $c_G(M, \omega_{FS}) \leq \pi$ and

$$c_G\left(M_1 \times \cdots \times M_r, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r\right) \leq \pi$$

are obtained by the use of Lu's pseudo symplectic capacities and their estimation in terms of Gromov-Witten invariants. The key ingredient to obtain these upper bounds is the non vanishing of some genus-zero three-points Gromov-Witten invariants (cfr. Lemma 16 in Section 6 below). Inequality (6), which extends (21) in [35] to HSSCT, is a consequence of (8) in Theorem 3, which in turn extends [35, Corollary 1.31].

When $M_j = \mathbb{C}\mathrm{P}^1$ for all $j = 1, \ldots, r$, inequality (6) is indeed an equality, i.e.

$$c_G(\mathbb{C}\mathrm{P}^1 \times \cdots \times \mathbb{C}\mathrm{P}^1, a_1\omega_{FS} \oplus \cdots \oplus a_r\omega_{FS}) = \min\{|a_1|, \ldots, |a_r|\}\pi. \tag{9}$$

(see [40, Example 12.5] for a proof). We do not know the exact value of

$$c_G(\mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_r}, a_1\omega_{FS}^1 \oplus \cdots \oplus a_r\omega_{FS}^r)$$

if $n_i > 1$ or $a_j \neq 1$ for some $i = 1, \ldots, r$ or $j = 1, \ldots, r$.

When the M_j's are projective spaces it was proved in Theorem 1.21 of [35] that the inequality (7) is an equality, namely

$$c_{HZ}(\mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_r}, a_1\omega_{FS}^1 \oplus \cdots \oplus a_r\omega_{FS}^r) = (|a_1| + \cdots + |a_r|)\pi. \tag{10}$$

In fact, Lu [35] was able to prove that

$$c_{HZ}(\mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_r}, a_1\omega_{FS}^1 \oplus \cdots \oplus a_r\omega_{FS}^r) \leq (|a_1| + \cdots + |a_r|)\pi \tag{11}$$

which, combined with (7), yields (10). To the authors' best knowledge no upper bound of $c_{HZ}(M, \omega_{FS})$ is known for HSSCT (M, ω_{FS}), even for the case of the complex Grassmannians (different from the projective space). The idea's of Lu's proof of the upper bound (11) is sketched in Remark 23, where we also explain why his argument cannot be used to achieve a similar upper bound for HSSCT.

We summarize our knowledge on Gromov width and Hofer–Zehnder capacity of Cartan domains in the following two theorems.

Theorem 4. *Let* (Ω, ω_0) *be a Cartan domain. Then*

$$c_G(\Omega, \omega_0) = \pi \tag{12}$$

and

$$c_{HZ}(\Omega, \omega_0) = \pi. \tag{13}$$

Moreover, if $\Omega_i \subset \mathbb{C}^{n_i}$, $i = 1, \ldots, r$ *are Cartan domains of complex dimension* n_i *equipped with the standard symplectic form* ω_0^i *of* $\mathbb{R}^{2n_i} = \mathbb{C}^{n_i}$, *then*

$$c_G\left(\Omega_1 \times \cdots \times \Omega_r, \omega_0^1 \oplus \cdots \oplus \omega_0^r\right) = \pi. \tag{14}$$

If $a_1, \ldots, a_r$ *are nonzero constants, then*

$$c_G\left(\Omega_1 \times \cdots \times \Omega_r, a_1\omega_0^1 \oplus \cdots \oplus a_r\omega_0^r\right) \leq \min\{|a_1|, \ldots, |a_r|\}\pi \tag{15}$$

Theorem 5. *Let* (Ω, ω_0) *be a Cartan domain and let* (N, ω) *be any closed symplectic manifold. Then*

$$c_{HZ}(N \times \Omega, \omega \oplus \omega_0) = \pi. \tag{16}$$

The proof of Theorem 4 is based (together with the inclusion $B^{2n}(1) \subset (\Omega, \omega_0)$) on the fact that any n-dimensional Cartan domain (Ω, ω_0) symplectically embeds into the cylinder $(Z^{2n}(1), \omega_0)$ (see Sections 4 and 5 for details). Our result extends to all Cartan domains, including the exceptional ones, the results in [36].

Remark 6. Notice that the *Cartan domains* in this paper are linearly equivalent to the Cartan domains in the classical terminology (see Section 5 below). Thus, if one compares our results with those in [36] one has to pay attention to the multiplicative constants involved. For example, the Gromov width of $(R_{IV}(4), \omega_0)$ (the fourth Cartan domain in the classical terminology) as computed in [36] turns out to be equal to $\frac{\pi}{2}$ and the corresponding Cartan domain Ω (in our terminology) is given by $\Omega = \sqrt{2}R_{IV}(4)$, in accordance with our Theorem 4.

The organization of the paper is as follows. In Section 3 we summarize the above mentioned Lu's work and some of his results needed in this paper. In Section 4 we briefly recall some tools on Hermitian positive Jordan triple systems which will be used in Section 5 to construct the above mentioned embeddings a) of a Cartan domain into its compact dual, b) of the unit ball into a Cartan domain and c) of a Cartan domain into the unitary cylinder. Moreover, in Subsection 5.1 we show how these symplectic embeddings could be used to estimate the minimal number of Darboux charts needed to cover a HSSCT. Finally, Section 6 is dedicated to the (conclusion of the) proofs of our theorems.

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3. Lu's pseudo symplectic capacities and Gromov–Witten invariants

G. Lu [35] defines the concept of *pseudo symplectic capacity* by weakening the requirements for a symplectic capacity (see the Introduction) in such a way that this new concept depends on the homology classes of the symplectic manifold in question (for more details the reader is referred to [35]). More precisely, if one denotes by $\mathcal{C}(2n,k)$ the set of all tuples $(M,\omega;\alpha_1,\dots,\alpha_k)$ consisting of a $2n$-dimensional connected symplectic manifold (M,ω) and k nonzero homology classes $\alpha_i\in H_*(M;\mathbb{Q})$, $i=1,\dots,k$, a map $c^{(k)}$ from $\mathcal{C}(2n,k)$ to $[0,+\infty]$ is called a *k-pseudo symplectic capacity* if it satisfies the following properties:

(**pseudo monotonicity**) if there exists a symplectic embedding $\varphi:(M_1,\omega_1)\to(M_2,\omega_2)$ then, for any $\alpha_i\in H_*(M_1;\mathbb{Q})$, $i=1,\dots,k$,

$$c^{(k)}(M_1,\omega_1;\alpha_1,\dots,\alpha_k)\le c^{(k)}(M_2,\omega_2;\varphi_*(\alpha_1),\dots,\varphi_*(\alpha_k));$$

(**conformality**) $c^{(k)}(M,\lambda\omega;\alpha_1,\dots,\alpha_k)=|\lambda|c^{(k)}(M,\omega;\alpha_1,\dots,\alpha_k)$, for every $\lambda\in\mathbb{R}\setminus\{0\}$ and all homology classes $\alpha_i\in H_*(M;\mathbb{Q})\setminus\{0\}$, $i=1,\dots,k$;

(**nontriviality**) $c(B^{2n}(1),\omega_0;pt,\dots,pt)=\pi=c(Z^{2n}(1),\omega_0;pt,\dots,pt)$, where pt denotes the homology class of a point.

Note that if $k>1$ a $(k-1)$-pseudo symplectic capacity is defined by

$$c^{(k-1)}(M,\omega;\alpha_1,\dots,\alpha_{k-1}):=c^{(k)}(M,\omega;pt,\alpha_1,\dots,\alpha_{k-1})$$

and any $c^{(k)}$ induces a true symplectic capacity

$$c^{(0)}(M,\omega) := c^{(k)}(M,\omega;pt,\dots,pt).$$

Observe also that (unlike symplectic capacities) pseudo symplectic capacities do not define symplectic invariants.

In [35] G. Lu was able to construct two 2-pseudo symplectic capacities denoted by $C^{(2)}_{HZ}(M,\omega;\alpha_1,\alpha_2)$ and $C^{(2o)}_{HZ}(M,\omega;\alpha_1,\alpha_2)$ respectively (see Definition 1.3 and Theorem 1.5 in [35]), where α_1 and α_2 are homology classes[1] in $H_*(M;\mathbb{Q})$. The $C^{(2)}_{HZ}$ and $C^{(2o)}_{HZ}$ are called by Lu *pseudo symplectic capacities of Hofer–Zehnder type.*

Denote by

$$C_{HZ}(M,\omega) := C^{(2)}_{HZ}(M,\omega;pt,pt)$$

(resp. $C^{o}_{HZ}(M,\omega) := C^{(2o)}_{HZ}(M,\omega;pt,pt)$) the corresponding true symplectic capacities associated to Lu's pseudo symplectic capacities. The next lemma summarizes some properties of the concepts involved so far.

Lemma 7. *Let (M,ω) be any symplectic manifold. Then, for arbitrary homology classes $\alpha_1,\alpha_2 \in H_*(M;\mathbb{Q})$ and for a nonzero homology class α, with $\dim\alpha \leq \dim M - 1$, the following inequalities hold true:*

$$C^{(2)}_{HZ}(M,\omega;\alpha_1,\alpha_2) \leq C^{(2o)}_{HZ}(M,\omega;\alpha_1,\alpha_2) \tag{17}$$

$$C^{(2)}_{HZ}(M,\omega;\alpha_1,\alpha_2) \leq C_{HZ}(M,\omega) \leq c_{HZ}(M,\omega) \tag{18}$$

$$C^{(2o)}_{HZ}(M,\omega;\alpha_1,\alpha_2) \leq C^{o}_{HZ}(M,\omega) \leq c^{o}_{HZ}(M,\omega) \tag{19}$$

$$c_G(M,\omega) \leq C^{(2)}_{HZ}(M,\omega;pt,\alpha), \tag{20}$$

where $c^{o}_{HZ}(M,\omega)$ is the π_1-sensitive Hofer–Zehnder capacity introduced in [47] (and independently in [34]), $c_{HZ}(M,\omega)$ is the Hofer-Zehnder capacity and $c_G(M,\omega)$ is the Gromov width of (M,ω). Furthermore, if M is closed then

$$C_{HZ}(M,\omega) = c_{HZ}(M,\omega)$$

and

$$C^{o}_{HZ}(M,\omega) = c^{o}_{HZ}(M,\omega).$$

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[1] In the notations of [35] the generic classes α_1 (resp. α_2) are called α_0 (resp. α_∞). The reason for this notation comes from the concept of hypersurface $S \subset M$ separating the homology classes α_0 and α_∞ (see Definition 1.3 and the (α_0,α_∞)-Weinstein conjecture at p.6 of [35]).

Proof. See Lemma 1.4 and (12) in [35]. □

Remark 8. The last two equalities together with (17) imply that for a closed symplectic manifold (M, ω)

$$c_{HZ}(M, \omega) \leq c^o_{HZ}(M, \omega).$$

It follows that inequality (7) in Theorem 2 holds true also when we replace c_{HZ} with c^o_{HZ}.

When the symplectic manifold M is closed the pseudo symplectic capacities $C^{(2)}_{HZ}(M, \omega; \alpha_1, \alpha_2)$ and $C^{(2o)}_{HZ}(M, \omega; \alpha_1, \alpha_2)$ can be estimated by other two pseudo symplectic capacities $GW(M, \omega; \alpha_1, \alpha_2)$ and $GW_0(M, \omega; \alpha_1, \alpha_2)$. These GW and GW_0 are defined in terms of *Liu–Tian type Gromov-Witten invariants* as follows. Let $B \in H_2(M, \mathbb{Z})$: the Liu–Tian type Gromov–Witten invariant of genus g and with k marked points is a homomorphism

$$\Psi^M_{B,g,k} : H_*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q}) \times H_*(M; \mathbb{Q})^k \to \mathbb{Q},\ 2g + k \geq 3$$

where $\overline{\mathcal{M}}_{g,k}$ is the space of isomorphism classes of genus g stable curves with k marked points. When there is no risk of confusion, we will omit the superscript M in $\Psi^M_{B,g,k}$. Roughly speaking, one can think of $\Psi^M_{B,g,k}(\mathcal{C}; \alpha_1, \dots, \alpha_k)$ as counting, for suitable generic ω-tame almost complex structure J on M, the number of J-holomorphic curves of genus g representing B, with k marked points p_i which pass through cycles X_i representing α_i, and such that the image of the curve belongs to a cycle representing $\mathcal{C}$ (for details the reader is referred to the Appendix in [35] and references therein).
In fact, several different constructions of Gromov-Witten invariants appear in the literature and the question whether they agree is not trivial (see [35] and also Chapter 7 in [39]). The Gromov–Witten invariants described in the book of D. McDuff and D. Salamon [39] are the most commonly used: these are homomorphisms

$$\Psi_{B,g,m+2} : H_*(M; \mathbb{Q})^{m+2} \to \mathbb{Q},\ m \geq 1$$

which play an important role in the proofs of this paper. The conditions under which these invariants agree with the ones considered by Lu are given in Lemma 10 below.

Let $\alpha_1, \alpha_2 \in H_*(M, \mathbb{Q})$. Following [35], one defines $GW_g(M, \omega; \alpha_1, \alpha_2) \in (0, +\infty]$ as the infimum of the ω-areas $\omega(B)$ of the homology classes $B \in$

$H_2(M, \mathbb{Z})$ for which the Liu–Tian Gromov–Witten invariant $\Psi_{B,g,m+2}(C; \alpha_1, \alpha_2, \beta_1, \ldots, \beta_m) \neq 0$ for some homology classes $\beta_1, \ldots, \beta_m \in H_*(M, \mathbb{Q})$ and $C \in H_*(\overline{\mathcal{M}}_{g,m+2}; \mathbb{Q})$ and integer $m \geq 1$ (we use the convention $\inf \emptyset = +\infty$). The positivity of GW_g reflects the fact that $\Psi_{B,g,m+2} = 0$ if $\omega(B) < 0$ (see, for example, Section 7.5 in [39]). Set

$$GW(M, \omega; \alpha_1, \alpha_2) := \inf\{GW_g(M, \omega; \alpha_1, \alpha_2) \mid g \geq 0\} \in [0, +\infty]. \tag{21}$$

Lemma 9. *Let (M, ω) be a closed symplectic manifold. Then*

$$0 \leq GW(M, \omega; \alpha_1, \alpha_2) \leq GW_0(M, \omega; \alpha_1, \alpha_2).$$

Moreover $GW(M, \omega; \alpha_1, \alpha_2)$ and $GW_0(M, \omega; \alpha_1, \alpha_2)$ are pseudo symplectic capacities and, if $\dim M \geq 4$ then, for nonzero homology classes α_1, α_2, we have

$$C_{HZ}^{(2)}(M, \omega; \alpha_1, \alpha_2) \leq GW(M, \omega; \alpha_1, \alpha_2)$$
$$C_{HZ}^{(2o)}(M, \omega; \alpha_1, \alpha_2) \leq GW_0(M, \omega; \alpha_1, \alpha_2).$$

In particular, for every nonzero homology class $\alpha \in H_*(M, \mathbb{Q})$,

$$C_{HZ}^{(2)}(M, \omega; pt, \alpha) \leq GW(M, \omega; pt, \alpha) \tag{22}$$

$$C_{HZ}^{(2o)}(M, \omega; pt, \alpha) \leq GW_0(M, \omega; pt, \alpha). \tag{23}$$

Proof. See Theorems 1.10 and 1.13 in [35]. □

We end this section with the following lemmata fundamental for the proof of our results. Recall that a closed symplectic manifold is *monotone* if there exists a number $\lambda > 0$ such that $\omega(B) = \lambda c_1(B)$ for B spherical (a homology class is called spherical if it is in the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M, \mathbb{Z})$). Further, a homology class $B \in H_2(M, \mathbb{Z})$ is *indecomposable* if it cannot be decomposed as a sum $B = B_1 + \cdots + B_k$, $k \geq 2$, of classes which are spherical and satisfy $\omega(B_i) > 0$ for $i = 1, \ldots, k$.

Lemma 10. *Let (M, ω) be a closed monotone symplectic manifold. Let $B \in H_2(M, \mathbb{Z})$ be an indecomposable spherical class, let pt denote the class of a point in $H_*(\overline{\mathcal{M}}_{g,m+2}; \mathbb{Q})$ and let $\alpha_i \in H_*(M, \mathbb{Z})$, $i = 1, 2, 3$. Then the Liu–Tian Gromov–Witten invariant $\Psi_{B,0,3}(pt; \alpha_1, \alpha_2, \alpha_3)$ agrees with the Gromov–Witten invariant $\Psi_{B,0,3}(\alpha_1, \alpha_2, \alpha_3)$.*

Proof. See [35, Proposition 7.6]. □

Lemma 11. *Let (N_1, ω_1) and (N_2, ω_2) be two closed symplectic manifolds. Then for every integer $k \geq 3$ and homology classes $A_2 \in H_2(N_2; \mathbb{Z})$ and $\beta_i \in H_*(N_2; \mathbb{Z})$, $i = 1, \ldots, k$,*

$$\Psi^{N_1 \times N_2}_{0 \oplus A_2, 0, k}(pt; [N_1] \otimes \beta_1, \ldots, [N_1] \otimes \beta_{k-1}, pt \otimes \beta_k) = \Psi^{N_2}_{A_2, 0, k}(pt; \beta_1, \ldots, \beta_k).$$

Proof. See [35, Proposition 7.4]. □

4. Hermitian positive Jordan triple system

We refer the reader to [43] (see also [33]) for more details on Hermitian symmetric spaces of noncompact type (HSSNT) and Hermitian positive Jordan triple systems (HPJTS).

Definitions and notations. A Hermitian Jordan triple system is a pair $(\mathcal{M}, \{,,\})$, where $\mathcal{M}$ is a complex vector space and $\{,,\}$ is a map

$$\begin{aligned} \{,,\} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} &\to \mathcal{M} \\ (u, v, w) &\mapsto \{u, v, w\} \end{aligned}$$

which is $\mathbb{C}$-bilinear and symmetric in u and w, $\mathbb{C}$-antilinear in v and such that the following *Jordan identity* holds:

$$\begin{aligned} &\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} \\ &= \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}. \end{aligned}$$

For $x, y, z \in \mathcal{M}$ consider the operators

$$\begin{aligned} T(x, y) z &= \{x, y, z\} \\ Q(x, z) y &= \{x, y, z\} \\ Q(x, x) &= 2 Q(x) \\ B(x, y) &= \mathrm{id}_{\mathcal{M}} - T(x, y) + Q(x) Q(y). \end{aligned}$$

The operator $B(x, y)$ is called the *Bergman operator*. A Hermitian Jordan triple system is called *positive* if the sesquilinear form

$$(u \mid v) = \frac{1}{\gamma} \operatorname{tr} T(u, v) \tag{24}$$

is a Hermitian product, where γ is a positive constant called the *genus* of $(\mathcal{M}, \{,,\})$.

HSSNT associated to HPJTS. M. Koecher ([25], [26]) discovered that to every HPJTS $(\mathcal{M}, \{,,\})$ one can associate an Hermitian symmetric space of noncompact type, in its realization as circled[2] bounded symmetric domain $\Omega_{\mathcal{M}}$ centered at the origin $0 \in \mathcal{M}$. More precisely, $\Omega_{\mathcal{M}}$ is defined as the connected component containing the origin of the set of all $u \in \mathcal{M}$ such that $B(u, u)$ is positive definite with respect to the Hermitian product (24).

HPJTS associated to HSSNT. The HPJTS $(\mathcal{M}, \{,,\})$ can be recovered from its associated HSSNT $\Omega_{\mathcal{M}}$ by defining $\mathcal{M} = T_0\Omega_{\mathcal{M}}$ (the tangent space to the origin of $\Omega_{\mathcal{M}}$) and

$$\{u,\, v,\, w\} = -\frac{1}{2}\left(R_0(u,\, v)\; w + J_0\, R_0(u,\; J_0\, v)\, w\right),$$

where R_0 (resp. J_0) is the curvature tensor of the Bergman metric (resp. the complex structure) of $\Omega_{\mathcal{M}}$ evaluated at the origin. The reader is referred to Proposition III.2.7 in [2] for details. To learn more on the correspondence between HPJTS and HSSNT we refer to p. 85 of Satake's book [45].

0.3cm **Spectral decomposition.** Let $(\mathcal{M}, \{,,\})$ be a HPJTS. An element $c \in \mathcal{M}$ is called *tripotent* if $\{c,\, c,\, c\} = 2\, c$. Two tripotents c_1 and c_2 are called *(strongly) orthogonal* if $T(c_1,\, c_2) = 0$. Each element $v \in \mathcal{M}$ has a unique *spectral decomposition*

$$v = \lambda_1\, c_1 + \cdots + \lambda_s\, c_s \qquad (\lambda_1 > \cdots > \lambda_s > 0)\,,$$

where $(c_1, \ldots,\, c_s)$ is a sequence of pairwise orthogonal (with respect to (24)) tripotents and the λ_j's are real numbers called eigenvalues of v. The integer s is called the *rank* of v and is denoted by $\mathrm{rk}(v)$. The *rank* of $\mathcal{M}$ is the positive integer r defined as $r = \max\{\mathrm{rk}(z) \mid z \in \mathcal{M}\}$. The elements $z \in \mathcal{M}$ such that $\mathrm{rk}(z) = r$ are called *regular*.

Let us denote by $\|v\|_{\max}$ the largest eigenvalue of v. Due to the convexity of $\Omega_{\mathcal{M}}$, $\|v\|_{\max}$ is a norm on $\mathcal{M}$, called the *spectral norm*. The following proposition provides a description of the domain $\Omega_{\mathcal{M}}$ in terms of its spectral norm.

Proposition 12. *Let $\Omega_{\mathcal{M}} \subset \mathcal{M}$ be the HSSNT associated to $(\mathcal{M}, \{,,\})$. Then*

$$\Omega_{\mathcal{M}} = \{v \mid \|v\|_{\max} < 1\}. \tag{25}$$

Proof. See [33, Corollary 3.15]. □

[2]The domain $\Omega \subset \mathcal{M}$ is circled if $e^{i\theta} \cdot \Omega = \Omega$

5. Cartan domains, their compact duals and some symplectic embeddings

Let $(\mathcal{M}, \{,,\})$ be a HPJTS and $\Omega_{\mathcal{M}}$ be its associated HSSNT. Let n be the complex dimension of $\mathcal{M}$. By fixing an orthonormal basis $\underline{e} = \{e_1, \ldots, e_n\}$ of $(\mathcal{M}, (\cdot \mid \cdot))$ we get the identification

$$\mathcal{M} \to \mathbb{C}^n, \quad v \overset{e}{\mapsto} z = (z_1, \ldots, z_n), \quad v = z_1 e_1 + \cdots + z_n e_n, \tag{26}$$

which induces an isometry between $(\mathcal{M}, (\cdot \mid \cdot))$ and $(\mathbb{C}^n, h_0)$, where h_0 is the canonical Hermitian product on $\mathbb{C}^n$. Under the identification

$$(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$$

between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ we have $h_0 = g_0 + i\omega_0$, where $g_0 = \sum_{j=1}^n dx_j^2 + dy_j^2$ is the standard scalar product on $\mathbb{R}^{2n}$ and ω_0 is the canonical symplectic form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$ on $\mathbb{C}^n = \mathbb{R}^{2n}$. From now on we assume $\mathcal{M}$ to be *simple*, which is equivalent to the irreducibility of $\Omega_{\mathcal{M}}$. Then, under the previous identification, the HSSNT $\Omega_{\mathcal{M}}$ corresponds to a bounded symmetric domain $\Omega = \overline{e}(\Omega_{\mathcal{M}}) \subset \mathbb{C}^n$. The complex and Riemannian geometry of these domains is well-known (see, e.g. [24]). Indeed, each of these domains is linearly equivalent to a Cartan domain (see, e.g. [26, Chapter V] for a proof).

Terminology: *In the present paper, with a slight abuse of terminology, the domain $\Omega = \overline{e}(\Omega_{\mathcal{M}})$ has been called a Cartan domain.*

We describe below some symplectic-geometric properties of Cartan domains and their compact duals which are needed in this paper (for the concept of compact dual see [18] or [11] and references therein).

Let $\Omega \subset \mathbb{C}^n$ be a Cartan domain and let M be its compact dual. Then M is an n-dimensional HSSCT. Denote by

$$BW : M \to \mathbb{C}P^N \tag{27}$$

the *Borel–Weil* (holomorphic) embedding. It is well-known (see e.g. [50]) that the pull-back $BW^*\omega_{\mathrm{FS}}$ of the Fubini–Study form ω_{FS} of $\mathbb{C}P^N$ is a homogeneous Kähler-Einstein form on M (ω_{FS} is the Kähler form which, in the homogeneous coordinates $[z_0, \ldots, z_N]$ on $\mathbb{C}P^N$, is given by $\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar\partial \log(|z_0|^2 + \cdots + |z_N|^2)$). Here we denote (with a slight abuse of notation and terminology) by ω_{FS} the form $BW^*\omega_{\mathrm{FS}}$ and call it the *Fubini–Study form* on M. The symplectic form ω_{FS} can be equivalently described as the symmetric or canonical form on M normalized so that $\omega_{FS}(B) = \pm\pi$ when B is a generator of $H_2(M, \mathbb{Z})$.

The domain (Ω, ω_0) **can be embedded into** $(M, \omega_{\mathrm{FS}})$**.**
Let (Ω, ω_0), $\Omega \subset \mathbb{C}^n$, be a Cartan domain equipped with the canonical symplectic form ω_0 of $\mathbb{R}^{2n}$ and let (M, ω_{FS}) be its compact dual. In [11] the first author in collaboration with A. J. Di Scala, by an use of HPJTS, construct an embedding

$$\Phi_\Omega : \Omega \to M \tag{28}$$

such that $\Phi_\Omega^* \omega_{FS} = \omega_0$.

Actually in [11] much more is proved. In particular it is shown that the embedding Φ_Ω induces a global symplectomorphism

$$\Phi_\Omega : (\Omega, \omega_0) \to (M \setminus \mathrm{Cut}_0(M), \omega_{FS})$$

where $\mathrm{Cut}_0(M)$ is the cut locus of (M, ω_{FS}) with respect to a fixed point $0 \in M$ (see [11, Theorem 1.1]). This diffeomorphism has been christened in [11] as a *symplectic duality* due to the fact that, amongst other properties, it also satisfies $\Phi_\Omega^* \omega_0 = \omega_{hyp}$, where ω_0 denotes the standard form on $\mathbb{C}^n \cong M \setminus \mathrm{Cut}_0(M)$ and ω_{hyp} is the hyperbolic metric on Ω (see either [11] or [12] for details and also [9], [13], [30], [31], [29], [32] and [41] for the construction of explicit symplectic coordinates.

Remark 13. In [35, Lemma 4.1 in Section 4] it is shown the existence of a symplectic embedding

$$\Phi_{\Omega_I[k,n]} : \Omega_I[k, n] \to G(k, n) \tag{29}$$

from the first Cartan domain $\Omega_I[k, n] \subset \mathbb{C}^{k(n-k)}$ into its compact dual $G(k, n)$ (where $G(k, n)$ denotes the complex Grassmannian of k dimensional subspaces of $\mathbb{C}^n$). Our result (28) extends Lu's results to all HSSCT.

The unitary ball $(B^{2n}(1), \omega_0)$ **can be embedded into** (Ω, ω_0).
Let $v = \lambda_1 c_1 + \cdots + \lambda_r c_r$ be the spectral decomposition of a regular point $v \in \Omega_{\mathcal{M}} \subset \mathcal{M}$, then the distance $\mathrm{d}_0(0, v)$ from the origin $0 \in \mathcal{M}$ to v is given by

$$\mathrm{d}_0(0, v) = (v \mid v)^{\frac{1}{2}} = \sqrt{\sum_{j=1}^{r} \lambda_j^2}, \tag{30}$$

(see [43, Proposition VI.3.6] for a proof). Since the set of regular points of $\mathcal{M}$ is dense ([43, Proposition IV.3.1]) we conclude, by (25) and by the identification $\Omega_{\mathcal{M}} \cong \Omega$ (induced by $(\mathcal{M}, (\cdot \mid \cdot)) \cong (\mathbb{C}^n, h_0)$) that

$$(B^{2n}(1), \omega_0) \subset (\Omega, \omega_0). \tag{31}$$

□

Remark 14. The inclusion (31) has been obtained in [35, Lemma 4.2, Section 4] for the case of the first Cartan domain, namely $B^{2k(n-k)}(1) \subset \Omega_I[k,n]$ (see also [36] for the case of classical Cartan domains). Combining this with the symplectic embedding (29) Lu was able (see [35, Theorem 1.35]) to obtain the upper bound

$$F(G(k,n), \omega_{FS}) \leq [n/k],$$

where $F(N, \omega)$ denotes the *Fefferman invariant* of a closed symplectic manifold (N, ω), namely the largest integer p for which there exists a symplectic packing by p open unit balls, and $[n/k]$ is the largest integer less than or equal to n/k. The authors believe it is an intriguing problem to give a similar upper bound for all HSSCT by using the techniques of this paper.

The domain (Ω, ω_0) can be embedded into $(Z^{2n}(1), \omega_0)$.
Let $Z^{2n}(1) = \{(x, y) \mid x_1^2 + y_1^2 < 1\}$ be the unitary cylinder in $\mathbb{R}^{2n}$. Let $v = \lambda_1 c_1 + \cdots + \lambda_r c_r$ be the spectral decomposition of a regular point $v \in \Omega_{\mathcal{M}} \subset \mathcal{M}$. By (30) and by the continuity of d_0 (the distance function from the origin $0 \in \mathcal{M}$) we see that $\mathrm{d}_0(0, c_1) = 1$. Set $c := c_1$, by [33, Corollary 4.8] $c \in \partial\Omega_{\mathcal{M}}$. Since $\Omega_{\mathcal{M}}$ is convex ([33, Corollary 4.7]), by the supporting hyperplane property there exists a real hyperplane π of $\mathcal{M}$ through c not intersecting $\Omega_{\mathcal{M}}$. Denote by $p = \overline{e}(c) \in \partial\Omega$ the image of the tripotent c by the isometry (26). Hence $p \in S^{2n-1}$, where $S^{2n-1} = \partial B^{2n}(1)$ is the $(2n-1)$-dimensional unit sphere centered at the origin of $\mathbb{R}^{2n}$. By (31), $B^{2n}(1) \subset \Omega = \overline{e}(\Omega_{\mathcal{M}})$ and hence $\overline{e}(\pi) = T_p S^{2n-1}$. By applying the same argument to any tripotent $c_\theta := e^{i\theta} \cdot c$, we see that Ω is contained in the cylinder $\tilde{Z}$ bounded by the envelope of the family of real hyperplanes $\left\{T_{p_\theta} S^{2n-1}, p_\theta = \overline{e}(c_\theta)\right\}_{\theta \in \mathbb{R}}$. Let $W \in U(n)$ such that

$$W \cdot p = (z_1, 0, \ldots, 0)$$

for some $z_1 \in \mathbb{C}$, $\|z_1\| = 1$. It follows that $W \cdot \tilde{Z} = Z^{2n}(1)$ and the desired symplectic embedding of (Ω, ω_0) into $(Z^{2n}(1), \omega_0)$ is given by

$$\Omega \subset \tilde{Z} \stackrel{W}{\to} Z^{2n}(1). \tag{32}$$

Remark 15. In [36] a similar (symplectic) embedding $(\Omega, \omega_0) \hookrightarrow (Z^{2n}(1), \omega_0)$ has been considered for the classical Cartan domains.

5.1. Minimal symplectic atlases of HSSCT

Consider a closed symplectic manifold (M, ω). In [44] Yu. B. Rudyak and F. Schlenk have introduced the *symplectic Lustermik-Schnirelmann category* $S(M, \omega)$, defined as

$$S(M, \omega) = \min\{k \mid M = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k\}$$

where each $\mathcal{U}_i$ is the image $\Phi_i(U_i)$ of a symplectic embedding $\Phi_i : U_i \to \mathcal{U}_i \subset M$ of a bounded subset U_i of $(\mathbb{R}^{2n}, \omega_0)$ diffeomorphic to an open ball in $\mathbb{R}^{2n}$. From our results one obtains the upper bound

$$S(M, \omega_{FS}) \leq N + 1 \tag{33}$$

for Lustermik-Schnirelmann category of a Hermitian symmetric space of compact type (M, ω_{FS}), where N is the dimension of the complex projective space $\mathbb{C}P^N$ where the manifold can be Kähler embedded via the Borel–Weil embedding $BW : M \to \mathbb{C}P^N$ (see (27)). Indeed, as in the case of the complex Grassmannian $G(k, n)$ (where the Borel–Weil embedding is given by the Plücker embedding $P : G(k, n) \to \mathbb{C}\mathrm{P}^{\binom{n}{k}-1}$), one can define a *canonical atlas* on (M, ω_{FS}) using the $N+1$ holomorphic charts $\Omega_0, \ldots, \Omega_N$ defined as $\Omega_j = M \setminus \{BW^{-1}(Z_j = 0)\}$, and $Z_j = 0$, $j = 0, \ldots, N$, is the standard hyperplane of $\mathbb{C}\mathrm{P}^N$. Each $\Omega_j \subset \mathbb{C}^n$, $j = 0, \ldots, N$, is biholomorphic to the noncompact dual Ω of M. It follows by (28) that (Ω_j, ω_0) can be symplectically embedded into (M, ω_{FS}) for $j = 1, \ldots, N$. On the other hand, each Ω is a bounded domain diffeomorphic to the ball in $\mathbb{R}^{2n}$ and so (33) follows. Our knowledge of the Gromov width of any HSSCT (M, ω_{FS}) can be used to estimate and compute the minimal numbers of Darboux charts needed to cover M. This number, introduced in [44] and denoted there by $S_B(M, \omega)$, has been computed and estimated for various symplectic manifolds including the complex Grassmannian (see [44, Corollary 5.10]). Using the results of this section, similar computations and related problems (which will appear in a forthcoming paper) can be done for all HSSCT.

6. The proofs of Theorems 1, 2, 3, 4 and 5

The following lemma is the key ingredient to achieve the upper bound of Gromov width in Theorems 1, 2 and 3.

Lemma 16. *Let (M, ω_{FS}) be an irreducible HSSCT of complex dimension n. Then there exist $\alpha(M, \omega_{FS})$ and $\beta(M, \omega_{FS})$ in $H_*(M, \mathbb{Z})$ such that*

$$\dim \alpha(M, \omega_{FS}) + \dim \beta(M, \omega_{FS}) = 4n - 2c_1(A)$$

and

$$\Psi_{A,0,3}(pt; \alpha(M, \omega_{FS}), \beta(M, \omega_{FS}), pt) \neq 0. \tag{34}$$

Proof. Since the canonical symplectic form ω_{FS} is Kähler-Einstein, it follows that (M, ω_{FS}) is monotone, so that Lemma 10 applies under our assumptions. We need then to show the existence, for every irreducible HSSCT, of a non-vanishing Gromov-Witten invariant $\Psi_{A,0,3}(\alpha(M, \omega_{FS}), \beta(M, \omega_{FS}), pt)$. This follows from the results about the *quantum cohomology* of these spaces proved in [1], [8], [27], [42], [49]. Let us recall that the quantum cohomology ring of M is the product $H_*(M) \otimes \mathbb{Z}[q]$ endowed with the *quantum cup product*, defined for any two homology classes $\alpha, \beta \in H_*(M)$ as

$$\alpha * \beta = \sum_{\gamma,d} \Psi_{dA,0,3}(\alpha, \beta, \gamma)\gamma^* q^d, \tag{35}$$

the sum running over $d \in \mathbb{Z}$ and γ such that $\dim(\alpha) + \dim(\beta) + \dim(\gamma) = 4n - 2dc_1(A)$, where γ^* denotes the dual class of γ.

Looking at the formulas for the quantum product proved in the above-mentioned references, it is not hard to find a Gromov-Witten invariant $\Psi_{A,0,3}(\alpha, \beta, pt)$ which does not vanish for some classes α, β. In more details, when M is the Grassmannian $G(k, n)$, by [49] there exist $\alpha \in H_{2k(n-1)}(M)$ and $\beta \in H_{2n(k-1)}(M)$ such that this holds; by [42] the same is true for suitable $\alpha = \beta \in H_{(n-1)(n-2)}(SO(2n)/U(n))$; by Corollary 8 in [27] α and β can be taken of codimension n and 1 when M is the Lagrangian Grassmannian $LG(n, 2n)$; in [8] (see the formulas in Sections 5.1 and 5.2) it is shown that for the Cayley plane (resp. for the Freudenthal variety) one can take for example α and β of codimensions 8 and 4, (resp. of codimensions 13 and 5). Finally, in [1] is studied the quantum cohomology of complete intersections, which in particular gives a non-vanishing Gromov-Witten invariant for the complex quadric. □

Remark 17. Formulas for quantum products in the homogeneous spaces, expressed in terms of the combinatorial invariants of the Lie algebra of the symmetry group of the space (Dynkin diagram and Weyl group), can be found in [16] (see also [15], [14]) and could be also used to prove the above Lemma.

We are now in the position to prove Theorem 1.

Proof of Theorem 1. In order to use Lemma 9 we can assume, without loss of generality, that $\dim M \geq 4$. Indeed the only irreducible HSSCT of dimension < 4 is $(\mathbb{C}\mathrm{P}^1, \omega_{FS})$ whose Gromov width is well-known to be equal to π. Let $A = [\mathbb{C}P^1]$ be the generator of $H_2(M, \mathbb{Z})$ as in the Notation at page 1050. Then the value $\omega_{FS}(A) = \pi$ is clearly the infimum of the ω_{FS}-areas $\omega_{FS}(B)$ of the homology classes $B \in H_2(M, \mathbb{Z})$ for which $\omega_{FS}(B) > 0$.
By Lemma 16 we have $\Psi_{A,0,3}(pt; pt, \alpha, \beta) \neq 0$, with $\alpha = \alpha(M, \omega_{FS})$ and $\beta = \beta(M, \omega_{FS})$, and hence, by definition of GW_g,

$$GW(M, \omega_{FS}; pt, \gamma) = GW_0(M, \omega_{FS}; pt, \gamma) = \pi \tag{36}$$

with $\gamma = \alpha(M, \omega_{FS})$ or $\gamma = \beta(M, \omega_{FS})$. It follows by the inequalities (17), (20), (22) and (23) that

$$c_G(M, \omega_{FS}) \leq C^{(2)}_{HZ}(M, \omega_{FS}; pt, \gamma) \leq C^{(2o)}_{HZ}(M, \omega_{FS}; pt, \gamma) \leq \pi \tag{37}$$

with $\gamma = \alpha(M, \omega_{FS})$ or $\gamma = \beta(M, \omega_{FS})$. Combining this with the lower bound $c_G(M, \omega_{FS}) \geq \pi$ coming from the inclusion $B^{2n}(1) \subset (\Omega, \omega_0)$ (cfr. (31)), the symplectic embedding $\Phi_\Omega : (\Omega, \omega_0) \to (M, \omega_{FS})$ (cfr. (28)) and the monotonicity and nontriviality of c_G, one gets:

$$c_G(M, \omega_{FS}) = C^{(2)}_{HZ}(M, \omega_{FS}; pt, \gamma) = C^{(2o)}_{HZ}(M, \omega_{FS}; pt, \gamma) = \pi \tag{38}$$

with $\gamma = \alpha(M, \omega_{FS})$ or $\gamma = \beta(M, \omega_{FS})$. This concludes the proof of Theorem 1. □

Remark 18. Observe that we have proven more than stated in Theorem 1. Indeed, we have computed the value of Lu's pseudo symplectic capacities evaluated at the homology class of a point and at $\alpha(M, \omega_{FS})$ (or $\beta(M, \omega_{FS})$), namely

$$\begin{aligned} c_G(M, \omega) &= C^{(2)}_{HZ}(M, \omega; pt, \alpha(M, \omega_{FS})) = C^{(2o)}_{HZ}(M, \omega; pt, \alpha(M, \omega_{FS})) \\ &= C^{(2)}_{HZ}(M, \omega; pt, \beta(M, \omega_{FS})) = C^{(2o)}_{HZ}(M, \omega; pt, \beta(M, \omega_{FS})) = \pi. \end{aligned}$$

This extends the result obtained by G. Lu for the complex Grassmannian (cfr. [35, Theorem 1.15] for details) to HSSCT.

Remark 19. An alternative proof of the upper bound $c_G(M, \omega_{FS}) \leq \pi$ in Theorem 1 can be achieved by combining Lemma 16 with [23, Proposition 4.1] which asserts that if (M, ω) is a symplectic manifold of (real) dimension $2n$, $B \in H_2(M, \mathbb{Z})$ is an indecomposable spherical class and $\Phi_{B,0,3}(pt, \alpha_0, \beta_0) \neq 0$, for suitable α_0 and β_0 in $H_*(M, \mathbb{Z})$ (which necessarily satisfy $\dim \alpha_0 + \dim \beta_0 = 4n - 2c_1(B)$) then $c_G(M, \omega) \leq \omega(B)$.
Actually, the GW invariant $\Phi_{B,0,3}(pt, \alpha_0, \beta_0) \neq 0$ for some $B \in H_2(M, \mathbb{Z})$ implies that there exists a rational curve of class B through a generic point in M and hence the inequality $c_G(M, \omega) \leq \omega(B)$ by the Gromov's arguments in [17]; see [23], [7] for details.

In order to prove Theorem 2 we need the following lemma, interesting on its own sake, which extends Lu's formula (20) in [35, Theorem 1.16] (for the Grassmannian) to the case of HSSCT.

Lemma 20. *Let (M, ω_{FS}) be a HSSCT and let (N, ω) be any closed symplectic manifold. Then*

$$C_{HZ}^{(2o)}(N \times M, \omega \oplus a\omega_{FS}; pt, [N] \times \gamma) \leq |a|\pi \tag{39}$$

for any $a \in \mathbb{R} \setminus \{0\}$ and $\gamma = \alpha(M, \omega_{FS})$ or $\gamma = \beta(M, \omega_{FS})$, with $\alpha(M, \omega_{FS})$ and $\beta(M, \omega_{FS})$ given by Lemma 16.

Proof. Since by (34) we have $\Psi^M_{A,0,3}(pt; \alpha, \beta, pt) \neq 0$, with $\alpha = \alpha(M, \omega_{FS})$ and $\beta = \beta(M, \omega_{FS})$, it follows by Lemma 11 that

$$\Psi^{N \times M}_{B,0,3}(pt; [N] \times \alpha(M, \omega_{FS}), [N] \times \beta(M, \omega_{FS}), pt) \neq 0$$

for $B = 0 \times A$, where 0 denotes the zero class in $H_2(N, \mathbb{Z})$. Hence (39) easily follows from (23) in Lemma 9. □

Proof of Theorem 2. To see (5) we assume $r > 1$ because of the result in Theorem 1. It immediately follows from (17) and (20) in Lemma 7 and by (39) that

$$c_G\left(M_1 \times \cdots \times M_r, \omega^1_{FS} \oplus \cdots \oplus \omega^r_{FS}\right) \leq \pi.$$

On the other hand, we have the symplectic embeddings

$$\times_{j=1}^r B^{2n_j}(1) \subset \times_{j=1}^r \Omega_j \overset{\Phi_{\Omega_1} \times \cdots \times \Phi_{\Omega_r}}{\longrightarrow} \times_{j=1}^r M_j$$

(induced by (31) and (28) respectively) and the natural inclusion

$$B^{2n_1+\cdots+2n_r}(1) \subset \times_{j=1}^{r} B^{2n_j}(1). \tag{40}$$

Thus, it follows by the monotonicity and nontriviality of c_G that

$$c_G\left(M_1 \times \cdots \times M_r, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r\right) \geq \pi.$$

Hence (5) follows. As we have already pointed out in the Introduction, inequality (6) is a straightforward consequence of (8) in Theorem 3.

Inequality (7) follows by (4), by the monotonicity of c_{HZ} and from the fact that for two compact symplectic manifolds (N_1, ω_1) and (N_2, ω_2)

$$c_{HZ}(N_1 \times N_2, \omega_1 \oplus \omega_2) \geq c_{HZ}(N_1, \omega_1) + c_{HZ}(N_2, \omega_2) \tag{41}$$

(see [35, Lemma 4.3, p. 43] for a proof). This concludes the proof of Theorem 2. □

Remark 21. The upper bound

$$c_G\left(M_1 \times \cdots \times M_r, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r\right) \leq \pi$$

obtained in the proof of Theorem 2 can also be achieved by using the fact that HSSCT and their products are uniruled manifolds (see Definition 1.14, Theorem 1.27 in [35] and the remark following this theorem).

Remark 22. Note that another interesting result shown in [35, Theorem 1.16] is formula (21) in [35]. One can prove the analogous of this formula using the techniques developed so far. That is

$$C_{HZ}^{(2o)}(\times_{j=1}^{r} M_j, \oplus_{j=1}^{r} a_j \omega_{FS}^j; pt, \times_{j=1}^{r} \alpha_j) \leq (|a_1| + \cdots + |a_r|)\pi,$$

for all $a_j \in \mathbb{R} \setminus \{0\}$ and $\alpha_j = \alpha_j(M_j, \omega_{FS}^j)$ or $\beta_j = \beta_j(M_j, \omega_{FS}^j)$.

Remark 23. We do not know if the inequality

$$c_{HZ}\left(M_1 \times \cdots \times M_r, a_1\omega_{FS}^1 \oplus \cdots \oplus a_r\omega_{FS}^r\right) \leq (|a_1| + \cdots + |a_r|)\pi.$$

holds true. Unfortunately, the proof given by Lu in the case of product of projective spaces [35, Theorem 1.21] does not extend to the general case of HSSCT. Indeed the Gromov–Witten invariant $\Psi_{B,0,m+2}(pt, pt, \beta_1, \ldots, \beta_m)$ of $M = M_1 \times \cdots \times M_r$ does not vanish (for some homology classes $\beta_1, \ldots, \beta_m$)

if and only if all the M_j's are projective spaces, since it is easily checked that the dimension condition $\sum_{j=1}^{m}\deg(\beta_j)=2(c_1(B)-\dim(M)-1+m)$, necessary for the Gromov-Witten invariant to be nonzero ([39], p. 11), is satisfied only in this case. For comments and conjectures related to this problem the reader is referred to [35, Corollary 1.19 and Example 1.20]).

Proof of Theorem 3. From (17) and (20) in Lemma 7 and by (39) it follows that

$$c_G(N\times M,\omega\oplus a\omega_{FS})\le C_{HZ}^{(2o)}(N\times M,\omega\oplus a\omega_{FS};pt,[N]\times\gamma)\le |a|\pi,$$

where $\gamma=\alpha(M,\omega_{FS})$ (or $\gamma=\beta(M,\omega_{FS})$), which yields the desired inequality (8). □

Proof of Theorem 4. By

$$(B^{2n}(1),\omega_0)\subset(\Omega,\omega_0)\stackrel{W}{\to}(Z^{2n}(1),\omega_0),$$

(given by (31) and (32) respectively) and the monotonicity and nontriviality of c_G and c_{HZ} we get $c_G(\Omega,\omega_0)=c_{HZ}(\Omega,\omega_0)=\pi$, namely (12) and (13). Analogously, let us denote M_j the compact dual of Ω_j: by (6) and by the symplectic embedding

$$(\times_{j=1}^{r}\Omega_j,\oplus_{j=1}^{r}a_j\omega_0^j)\stackrel{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}}{\longrightarrow}(\times_{j=1}^{r}M_j,\oplus_{j=1}^{r}a_j\omega_{FS}^j)$$

induced by (28) one obtains (15) which, together with the symplectic embedding $\times_{j=1}^{r}B^{2n_j}(1)\subset\times_{j=1}^{r}\Omega_j$ (induced by (31)) and (40) yields (14). □

In order to prove Theorem 5 we need the following interesting result of Lu.

Lemma 24. *Let (N,ω) be any closed symplectic manifold. Then, for any $r>0$ one has*

$$c_{HZ}(N\times B^{2n}(r),\omega\oplus\omega_0)=c_{HZ}(N\times Z^{2n}(r),\omega\oplus\omega_0)=\pi r^2.$$

where $Z^{2n}(r)$ is given by (3).

Proof. See [35, Theorem 1.17, p.14]. □

Proof of Theorem 5. By $(B^{2n}(1),\omega_0)\subset(\Omega,\omega_0)\stackrel{W}{\to}(Z^{2n}(1),\omega_0)$ one has the embeddings

$$(N\times B^{2n}(1),\omega\oplus\omega_0)\subset(N\times\Omega,\omega\oplus\omega_0)\stackrel{id_N\times W}{\to}(N\times Z^{2n}(1),\omega\oplus\omega_0)$$

and so the desired (16), i.e. $c_{HZ}(N\times\Omega,\omega\oplus\omega_0)=\pi$, follows by Lemma 24 and the monotonicity of c_{HZ}. □

Final remarks on Seshadri constants

Our knowledge of the Gromov width of a HSSCT allows us to obtain an upper bound of the Seshadri constant of an ample line bundle over a HSSCT (M,ω_{FS}). Recall that given a compact complex manifold (N,J) and a holomorphic line bundle $L\to N$ the *Seshadri constant* of L at a point $x\in N$ is defined as the nonnegative real number

$$\epsilon(L,x)=\inf_{C\ni x}\frac{\int_C c_1(L)}{\mathrm{mult}_x C},$$

where the infimum is taken over all irreducible holomorphic curves C passing through the point x and $\mathrm{mult}_x C$ is the multiplicity of C at x (see [10] for details). The (global) Seshadri constant is defined by

$$\epsilon(L)=\inf_{x\in N}\epsilon(L,x).$$

Note that Seshadri's criterion for ampleness says that L is ample if and only if $\epsilon(L)>0$. P. Biran and K. Cieliebak [6, Prop. 6.2.1] have shown that

$$\epsilon(L)\le c_G(N,\omega_L),$$

where ω_L is any Kähler form which represents the first Chern class of L, i.e. $c_1(L)=[\omega_L]$. Consider now an irreducible HSSCT (M,ω_{FS}) and the line bundle $L\to M$ such that $c_1(L)=[\frac{\omega_{FS}}{\pi}]$ (L can be taken as the pull-back via the Borel–Weil embedding (27) of the universal bundle of $\mathbb{C}\mathrm{P}^N$). Therefore, by using the upper bound $c_G(M,\omega_{FS})\le\pi$ and the conformality of c_G we get:

Corollary 25. *Let (M,ω_{FS}) be an irreducible HSSCT and let $L\to M$ as above. Then $\epsilon(L)\le 1$.*

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