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Deforming symplectomorphism of certain irreducible Hermitian symmetric spaces of compact type by mean curvature flow

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In this paper, we generalize Medos-Wang's arguments and results on the mean curvature flow deformations of symplectomorphisms of $\mathbb{C}P^n$ in [MeWa] to complex Grassmann manifold $G(n, n+m; \mathbb{C})$ and compact totally geodesic Kähler-Einstein submanifolds of it. We also give an abstract result and discuss the case of complex tori.

1. Introduction

A symplectic manifold (M, ω) is said to be Kähler if there exists an integrable almost complex structure J on M such that the bilinear form $g(X, Y) = \langle X, Y\rangle := \omega(X, JY)$ defines a Riemannian metric on M. The triple (ω, J, g) is called a Kähler structure on M, g and ω are called a Kähler metric and a Kähler form, respectively. Such a Kähler manifold is called a *Kähler-Einstein manifold* if the Ricci form $\rho_\omega \equiv \rho_g$ of g satisfies $\rho_\omega = c\omega$ for some constant $c \in \mathbb{R}$. For a Kähler manifold (M, J, g, ω) let $\mathrm{Symp}(M, \omega)$ and $\mathrm{Aut}(M, J)$ denote the group of symplectomorphisms of the symplectic manifold (M, ω) and the group of biholomorphisms of the complex manifold (M, J), respectively. Their intersection is equal to the group of isometries of the Kähler manifold (M, J, g, ω), $\mathrm{I}(M, J, g) := \{\phi \in \mathrm{Aut}(M, J) \,|\, \phi^* g = g\}$.

Assume that M is closed (i.e. compact and without boundary). It is well-known that $\mathrm{Symp}(M, \omega)$ is an infinite dimensional Lie group whose Lie algebra is the space of symplectic vector fields. A lot of symplectic topology information of (M, ω) is contained in $\mathrm{Symp}(M, \omega)$. (See beautiful books [Ban, HoZe, McSa, Po] for detailed study). On the other hand $\mathrm{I}(M, J, g)$ is a finite dimensional Lie subgroup of $\mathrm{Symp}(M, \omega)$. Hence in order to understand topology of $\mathrm{Symp}(M, \omega)$, e.g. its homotopy groups, it is helpful to study the topology properties of the inclusion $\mathrm{I}(M, J, g) \hookrightarrow \mathrm{Symp}(M, \omega)$. Let $g^{(n)}_{\mathrm{FS}}$ and $\omega^{(n)}_{\mathrm{FS}}$ denote, up to multiplying a positive number, the Fubini-Study metric and the associated Kähler form on the complex projective spaces $\mathbb{C}P^n$ respectively, and let i be the standard complex structure on $\mathbb{C}P^n$. In his famous paper [Gr] Gromov invented a powerful pseudo-holomorphic curve theory to study symplectic topology and got:

- For any two area forms ω_1 and ω_2 on $\mathbb{C}P^1$ with $\int_{\mathbb{C}P^1} \omega_1 = \int_{\mathbb{C}P^1} \omega_2$, $\mathrm{Symp}(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_1 \oplus \omega_2)$ contracts onto $\mathrm{I}(\mathbb{C}P^1 \times \mathbb{C}P^1, i \times i, g^{(1)}_{\mathrm{FS}} \oplus g^{(1)}_{\mathrm{FS}}) = \mathbb{Z}/2\mathbb{Z}$ **extension of** $SO(3) \times SO(3)$ ([Gr, §2.4.A_1]), and $\mathrm{Symp}(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_1 \oplus \omega_2)$ cannot contract onto $SO(3) \times SO(3)$ if $\int_{\mathbb{C}P^1} \omega_1 \neq \int_{\mathbb{C}P^1} \omega_2$ ([Gr, §2.4.C_2]). (A simple application of Moser theorem can reduce these to the case $\omega_1 = a\omega^{(1)}_{\mathrm{FS}}$ and $\omega_2 = b\omega^{(1)}_{\mathrm{FS}}$ for nonzero $a, b \in \mathbb{R}$).
- $\mathrm{Symp}(\mathbb{C}P^2, \omega^{(2)}_{\mathrm{FS}})$ contracts onto $\mathbf{I}(\mathbb{C}P^2, i, g^{(2)}_{\mathrm{FS}})$ ([Gr, §2.4.B'_3]).

For $\mathrm{Symp}(S^2 \times S^2, \omega_{\mathrm{FS}}^{(1)} \oplus \lambda\omega_{\mathrm{FS}}^{(1)})$ with $\int_{S^2} \omega_{\mathrm{FS}}^{(1)} = 1$ and $\lambda \neq 1$, so far some deep results were made by Abreu [Ab], Abreu and McDuff [AbMc], Anjos and Granja [AnGr] and others following an approach suggested by Gromov [Gr, §2.4.C_2]. (See McDuff's survey [Mc] for recent developments).

In past ten years a new method (*mean curvature flow* (MCF) method) to the above question was developed by Smoczyk and Mu-Tao Wang [Smo2, SmoWa, Wa1, Wa2, Wa3, Wa4, Wa5, TsWa, MeWa]. For compact Riemann surfaces they obtained the desired results (cf. [Wa4, Wa5, Smo2]). Recently Ivana Medos and Mu-Tao Wang [MeWa] applied the MCF to deform symplectomorphisms of $\mathbb{C}P^n$ for each dimension n, and obtained a constant $\Lambda_0(n) \in (1, +\infty]$ only depending on $n \in \mathbb{N}$, (see (3.7) for its definition), such that any Λ-pinched symplectomorphism of $\mathbb{C}P^n$ with

(1.1)
$$1 \le \Lambda \le \Lambda_1(n) := \left[\frac{1}{2}\left(\Lambda_0(n) + \frac{1}{\Lambda_0(n)}\right)\right]^{\frac{1}{n}} + \sqrt{\left[\frac{1}{2}\left(\Lambda_0(n) + \frac{1}{\Lambda_0(n)}\right)\right]^{\frac{2}{n}} - 1}$$

is symplectically isotopic to a biholomorphic isometry ([MeWa, Cor.5]). Here a symplectomorphism φ of the Kähler manifold (M, ω, J, g) is called Λ-*pinched* if

$$\frac{1}{\Lambda^2} g \le \varphi^* g \le \Lambda^2 g$$

(cf. [MeWa, Def.1]). The constant $\Lambda_0(n)$ was introduced above Remark 2 of [MeWa, p.322], and it was shown that $\Lambda_0(1) = \infty$ there. For $n \in \mathbb{N}$ we define an increasing function $[1, \infty) \ni \Lambda \mapsto \Lambda'_n$ by

$$\Lambda'_n := \left[\frac{1}{2}\left(\Lambda + \frac{1}{\Lambda}\right)\right]^n + \sqrt{\left[\frac{1}{2}\left(\Lambda + \frac{1}{\Lambda}\right)\right]^{2n} - 1}. \tag{1.2}$$

(This is obtained from [MeWa, (3.11)] when Λ_1 in [MeWa, (3.10)] is replaced by Λ.) Then $\Lambda'_n = \Lambda_0(n)$ if $\Lambda = \Lambda_1(n)$ by the proof of [MeWa, Cor.5].

By Cartan's classification, in addition to two exceptional spaces $E_6/(\mathrm{Spin}(10) \times \mathrm{SO}(n+2))$ and $E_7/(E_6 \times \mathrm{SO}(2))$, all irreducible Hermitian symmetric spaces of compact type (IHSSCT) have the following form of four types (in the terminology of [He, p. 518]):

$$U(n+m)/U(n) \times U(m),\ n, m \ge 1, \qquad SO(2n)/U(n),\ n \ge 2,$$
$$Sp(n)/U(n)\ n \ge 2, \qquad SO(n+2)/SO(n) \times SO(2),\ n \ge 3.$$

They are, respectively, holomorphically equivalent to:

$G^{\mathrm{I}}(n,n+m)=G(n,n+m;\mathbb{C})$ the complex Grassmann manifold which may be defined as the quotient $M(n,n+m;\mathbb{C})/\mathrm{GL}(n;\mathbb{C})$, where $\mathrm{GL}(n;\mathbb{C}) = \{Q\in\mathbb{C}^{n\times n}\,|\,\det Q\neq 0\}$ acts on $M(n+m,n;\mathbb{C}) := \{A\in\mathbb{C}^{n\times(n+m)}\mid \mathrm{rank}A = n\}$ freely from the left by matrix multiplication;

$$G^{\mathrm{II}}(n,2n)=\left\{[A]\in G(n,2n;\mathbb{C})\;\middle|\;\exists A\in[A] \text{ s.t } A\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix}A'=0\right\};$$
$$G^{\mathrm{III}}(n,2n)=\left\{[A]\in G(n,2n;\mathbb{C})\;\middle|\;\exists A\in[A] \text{ s.t } A\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}A'=0\right\};$$
$$G^{\mathrm{IV}}(1,n+1)=\left\{[(z_1,\ldots,z_{n+2})]\in\mathbb{C}P^{n+1}\;\middle|\;\sum_{j=1}^{n}z_j^2-z_{n+1}^2-z_{n+2}^2=0\right\}$$

(cf. [CaVe] and [Lu1, Lu2, Lu3]), which are the compact duals (or extended spaces) of the classical domains $\mathrm{D}^{\mathrm{I}}_{n,m}$, $\mathrm{D}^{\mathrm{II}}_{n}$, $\mathrm{D}^{\mathrm{III}}_{n}$ and $\mathrm{D}^{\mathrm{IV}}_{n}$, respectively. Let h and h_{I} be the canonical Kähler metrics on $G(n,n+m;\mathbb{C})$ and $G^{\mathrm{I}}(n,2n)$, respectively. Denote by h_{II} and h_{III} the induced metrics on $G^{\mathrm{II}}(n,2n)$ and $G^{\mathrm{III}}(n,2n)$, respectively. Then both $(G^{\mathrm{II}}(n,2n),h_{\mathrm{II}})$ and $(G^{\mathrm{III}}(n,2n),h_{\mathrm{III}})$ are totally geodesic Kähler-Einstein submanifolds of $(G^{\mathrm{I}}(n,2n),h_{\mathrm{I}})$. (See the claim on the page 136 of [Mok] and the proof of Lemma 1 on the page 85 of [Mok]).

Theorem 1.1. *Let ω be the Kähler form corresponding with the canonical metric h on $G(n,n+m;\mathbb{C})$, $g=\mathrm{Re}(h)$ and J the standard complex structure. Then for every Λ-pinched symplectomorphism $\varphi\in\mathrm{Symp}(G(n,n+m;\mathbb{C}),\omega)$ with $\Lambda\in[1,\Lambda_1(mn)]\setminus\{\infty\}$ the following holds:*

- (i) *The mean curvature flow Σ_t of the graph of φ in $G(n,n+m;\mathbb{C})\times G(n,n+m;\mathbb{C})$ exists for all $t>0$.*
- (ii) *Σ_t is the graph of a symplectomorphism φ_t for each $t>0$, and φ_t is Λ'_{mn}-pinched along the mean curvature flow, where Λ'_{mn} is defined by (1.2).*
- (iii) *φ_t converges smoothly to a biholomorphic isometry of $(G(n,n+m;\mathbb{C}),J,g)$ as $t\to\infty$.*

Consequently, each such Λ-pinched symplectomorphism $\varphi\in\mathrm{Symp}(G(n,n+m;\mathbb{C}),\omega)$ is symplectically isotopic to a biholomorphic isometry of $(G(n,n+m;\mathbb{C}),J,g)$.

Theorem 1.2. *Let (M, ω, J, g) be a compact Kähler-Einstein submanifold of $(G(n, n+m; \mathbb{C}), h)$ which is totally geodesic. Set* $\dim M = 2N$. *Then for every Λ-pinched symplectomorphism $\varphi \in \mathrm{Symp}(M, \omega)$ with $\Lambda \in [1, \Lambda_1(N)] \setminus \{\infty\}$ the following holds:*

(i) *The mean curvature flow Σ_t of the graph of φ in $M \times M$ exists for all $t > 0$.*

(ii) *Σ_t is the graph of a symplectomorphism φ_t for each $t > 0$, and φ_t is Λ'_N-pinched along the mean curvature flow, where Λ'_N is defined by (1.2).*

(iii) *φ_t converges smoothly to a biholomorphic isometry of (M, J, g) as $t \to \infty$.*

Consequently, each such Λ-pinched symplectomorphism $\varphi : (M, \omega) \to (M, \omega)$ is symplectically isotopic to a biholomorphic isometry of (M, J, g).

In particular, this theorem holds for $(G^{\mathrm{II}}(n, 2n), h_{\mathrm{II}})$ and $(G^{\mathrm{III}}(n, 2n), h_{\mathrm{III}})$ (or $SO(2n)/U(n)$ and $Sp(n)/U(n)$ in the terminology of [He, p. 518]).

Recall that a complex torus of complex dimension n is the quotient space $T^n = \mathbb{C}^n/\Gamma$, where Γ is a lattice in $\mathbb{C}^n$ generated by $2n$ vectors $\{u_1, \ldots, u_{2n}\}$ in $\mathbb{C}^n$ which are linearly independent over $\mathbb{R}$. It has a natural flat Kähler metric induced from the flat metric of $\mathbb{C}^n$. By Bieberbach theorem ([Ch, page 65]), any compact flat Kähler manifold is holomorphically covered by a complex torus ([Be, Example 2.60]). From this and Calabi-Yau theorem it follows that any compact Kähler manifold M with the first and the second (real) Chern class vanishing must be (holomorphically) covered by a complex torus ([Be, Cor. 11.27]). Unfortunately, for complex tori we cannot obtain the corresponding result with (iii) of Theorems 1.1 and 1.2 yet though other conclusions are proved under the weaker pinching condition.

Theorem 1.3. *Let (M, ω, J, g) and $(\widetilde{M}, \tilde{\omega}, \tilde{J}, \tilde{g})$ be two real $2n$-dimensional compact Kähler-Einstein manifolds of* constant zero holomorphic sectional curvature. *Then for every Λ-pinched symplectomorphism $\varphi : M \to \widetilde{M}$ with $\Lambda \in (1, \Lambda_0(n))$ there hold:*

(i) *The mean curvature flow Σ_t of the graph of φ in $M \times \widetilde{M}$ exists smoothly for all $t > 0$;*

(ii) *Σ_t is the graph of a symplectomorphism φ_t for each $t > 0$, and φ_t is still $\Lambda_0(n)$-pinched along the mean curvature flow.*

(iii) *If* $\Lambda < \widehat{\Lambda}_1$ *for some* $\Lambda_1 \in (\Lambda, \Lambda_0(n))$, *where* $\widehat{\Lambda}_1 > 1$ *is a constant determined by* Λ_1 *and* n *(see Lemma 4.2), then the flow converges to a totally geodesic submanifold of* $M \times \widetilde{M}$ *as* $t \to \infty$. *(In addition* $\widehat{\Lambda}_1$ *is more than or equal to*

$$\left(2\exp\left(\frac{0.141446\delta_{\Lambda_1}}{5n}\right)+2\exp\left(\frac{0.141446\delta_{\Lambda_1}}{10n}\right)\sqrt{\exp\left(\frac{0.141446\delta_{\Lambda_1}}{5n}\right)-1}-1\right)^{\frac{1}{2}},$$

where δ_{Λ_1} *is defined by (3.6)).*

It is easily seen that the convergence assertion in Theorem 1.3 cannot be derived from [Wa3, Theorem B]. Moreover, it was pointed out in [Wa2, Remark 8.1] that when M is locally a product of two Riemannian surfaces of nonpositive curvature the uniform convergence of the flow can also be proved with the method in [Wa4]. Related to the result K.Smoczyk and M.-T. Wang [SmoWa] treated the Lagrangian mean curvature flow of symplectomorphisms between flat tori in case of a length decreasing (hence pinching) property.

It is possible to generalize the above three theorems to a larger class of manifolds — compact homogeneous Kähler-Einstein manifolds. (See Theorem 5.1). Recall that a Kähler manifold (M, ω, J, g) is called *homogeneous* if $\mathrm{I}(M, J, g)$ acts transitively on M. In particular, a simply-connected compact homogeneous Kähler manifold is called a *Kähler C-space* in [W] (or a *generalized flag manifold*). However, except the manifolds contained in the three theorems above we do not find an example satisfying the conditions of Theorem 5.1.

In this paper we follow [KoNo] to define the curvature tensor R of a Kähler manifold (M, ω, J, g) by

$$R(X, Y, Z, W) = g(R(X, Y)W, Z) = g(R(Z, W)Y, X)$$

for $X, Y, Z, W \in \Gamma(TM)$. Then the *holomorphic sectional curvature* in the direction $X \in TM \setminus \{0\}$ is defined by

$$H(X) = R(X, JX, X, JX)/[g(X, X)]^2.$$

(After extending g and R by $\mathbb{C}$-linearity to $TM \otimes_{\mathbb{R}} \mathbb{C}$, $H(X)$ is equal to $-R(Z, \overline{Z}, Z, \overline{Z})/[g(Z, \overline{Z})]^2$ for $Z = (X - \sqrt{-1}JX)/2 \in T^{(1,0)}M$).

The paper is organized as follows. In Section 2 we review differential geometry of Grassmann manifolds, the key Proposition 2.3 seems to be new.

Section 3 is our technical core, where we study evolution along the mean curvature flow under different pinching conditions for different cases. In Section 4 we prove Theorems 1.1, 1.2 and 1.3. Finally, Section 5 gives a general result under stronger assumptions as a concluding remark.

2. Differential geometry of Grassmann manifolds

2.1. Curvatures

For increasing integers $1 \leq \alpha_1 < \cdots < \alpha_n \leq n+m$ let $\{\alpha_{n+1}, \dots, \alpha_{n+m}\}$ be the complement of $\{\alpha_1, \dots, \alpha_n\}$ in the set $\{1, 2, \dots, n+m\}$. For $[A] \in G(n, n+m; \mathbb{C}) = M(n, n+m; \mathbb{C})/\mathrm{GL}(n; \mathbb{C})$ write A as $(A_1, \dots, A_{n+m})$, where $A_1, \dots, A_{n+m}$ are $n \times 1$ matrices. Set $A_{\alpha_1 \cdots \alpha_n} = (A_{\alpha_1}, \dots, A_{\alpha_n}) \in \mathbb{C}^{n\times n}$, $A_{\alpha_{n+1} \cdots \alpha_{n+m}} = (A_{\alpha_{n+1}}, \dots, A_{\alpha_{n+m}}) \in \mathbb{C}^{n\times m}$. Define $U_{\alpha_1, \dots, \alpha_n} = \{[A] \in G(n, n+m; \mathbb{C}) \,|\, \det A_{\alpha_1 \cdots \alpha_n} \neq 0\,\}$ and $\Theta_{\alpha_1 \cdots \alpha_n} : U_{\alpha_1 \cdots \alpha_n} \to \mathbb{C}^{n\times m} \equiv \mathbb{C}^{nm}$ by

$$[A] \to Z = (A_{\alpha_1 \cdots \alpha_n})^{-1} A_{\alpha_{n+1} \cdots \alpha_{n+m}}.$$

We call Z the *local coordinate* of $[A] \in G(n, n+m; \mathbb{C})$, and

$$\left\{ \left(U_{\alpha_1 \cdots \alpha_n},\, \Theta_{\alpha_1 \cdots \alpha_n} \right) \mid 1 \leq \alpha_1 < \cdots < \alpha_n \leq n \right\}$$

the *canonical atlas* on $G(n, n+m; \mathbb{C})$ ([Le, Lu1, Wo2]). The canonical Kähler-Einstein h on $G(n, n+m; \mathbb{C})$ is given by

$$h = \partial\bar{\partial} \log\det(I + Z\overline{Z}') \tag{2.1}$$

in the local chart $(U_{1\cdots n}, Z = \Theta_{1\cdots n})$ as above, where $\overline{Z}'$ and $\overline{dZ}'$ are the conjugate transposes of Z and dZ respectively, and $\partial = \sum_{i,\alpha} dZ^{i\alpha} \frac{\partial}{\partial Z^{i\alpha}}$ and $\bar{\partial} = \sum_{i,\alpha} \overline{dZ^{i\alpha}} \frac{\partial}{\partial \overline{Z^{i\alpha}}}$. (See [Lu1, Lu2]).

If a (real) tangent vector T at the point $Z \in U_{1\cdots n}$ is represented by their component matrices, i.e., we identify

$$T = \sum_{k,l} \mathrm{Re}(T^{kl}) \frac{\partial}{\partial X^{kl}} + \sum_{k,l} \mathrm{Im}(T^{kl}) \frac{\partial}{\partial Y^{kl}} \tag{2.2}$$

with complex matrices $(T^{kl}) \in \mathbb{C}^{n\times m}$, where $Z^{kl} = X^{kl} + iY^{kl}$, $k = 1, \dots, n$ and $l = 1, \dots, m$, then the Riemannian metric $g := Re(h)$ is given by

$$g_Z(T_1, T_2) = Re\mathrm{Tr}[(I + Z\overline{Z}')^{-1} T_1 (I + \overline{Z}' Z)^{-1} \overline{T_2}'] \tag{2.3}$$

(cf. [Wo2, (2)]). The curvature tensor R_Z of g at Z has the expression

$$\begin{aligned}&R_Z(T_1,T_2)T\\ &= T\big[(I+\overline{Z}'Z)^{-1}\overline{T_2}'(I+Z\overline{Z}')^{-1}T_1-(I+\overline{Z}'Z)^{-1}\overline{T_1}'(I+Z\overline{Z}')^{-1}T_2\big]\\ &\quad+\big[T_1(I+\overline{Z}'Z)^{-1}\overline{T_2}'(I+Z\overline{Z}')^{-1}-T_2(I+\overline{Z}'Z)^{-1}\overline{T_1}'(I+Z\overline{Z}')^{-1}\big]T\end{aligned}$$

(cf. [Wo2, (4)]). Here as above the left is a real tangent vector and the right is the corresponding complex matrix representation of it. Let $p_0\in U_{1\cdots n}$ has coordinate $Z(p_0)=0$. Then

$$\begin{aligned}&R_{p_0}(T_1,T_2,T_3,T_4)\\ &:= g_p(R_p(T_3,T_4)T_2,T_1)\\ &= Re\mathrm{Tr}\big[(T_2\overline{T}'_4T_3\overline{T}_1-T_2\overline{T}'_3T_4\overline{T}'_1+T_3\overline{T}_4T_2\overline{T}_1-T_4\overline{T}_3T_2\overline{T}_1)\big]\end{aligned}\tag{2.4}$$

for any tangent vectors in $T_{p_0}G(n,n+m;\mathbb{C})$ as in (2.2), $T_i,\ i=1,2,3,4$, which are identified with complex matrices $(T_i^{kl})\in\mathbb{C}^{n\times m}$, $i=1,2,3,4$. It follows that the sectional curvature sits between 0 and 4, and that the holomorphic sectional curvature of $G(n,n+m;\mathbb{C})$ at the point $p_0\in U_{1\cdots n}$ in the direction T is given by

$$H(0,T)=\frac{2\mathrm{Tr}(T\overline{T}'T\overline{T}')}{[\mathrm{Tr}(T\overline{T}')]^2}\in[4/\min(n,m),4]\tag{2.5}$$

(cf. [Lu1, (2.11)] and [Wo2, page 77]).

Proposition 2.1. *For the metric h in (2.1) let R be the Riemannian curvature tensor R of the Riemannian metric $g=Re(h)$ (extended to $TG(n,n+m;\mathbb{C})\otimes_{\mathbb{R}}\mathbb{C}$ in a $\mathbb{C}$-linear way). For $1\le i,j,k,h\le n$ and $1\le\alpha,\beta,\gamma,\delta\le m$ let*

$$\begin{aligned}R_{i\alpha,\overline{j\beta},k\gamma,\overline{h\delta}} &= R\left(\frac{\partial}{\partial Z^{i\alpha}}\Big|_0,\frac{\partial}{\partial\overline{Z}^{j\beta}}\Big|_0,\frac{\partial}{\partial Z^{k\gamma}}\Big|_0,\frac{\partial}{\partial\overline{Z}^{h\delta}}\Big|_0\right)\\ &= g\left(R\left(\frac{\partial}{\partial Z^{i\alpha}}\Big|_0,\frac{\partial}{\partial\overline{Z}^{j\beta}}\Big|_0\right)\frac{\partial}{\partial\overline{Z}^{h\delta}}\Big|_0,\frac{\partial}{\partial Z^{k\gamma}}\Big|_0\right)\end{aligned}$$

and others be defined similarly. Then

$$\begin{aligned}R_{i\alpha,\overline{j\beta},k\gamma,\overline{h\delta}} = R_{i\alpha,\overline{h\delta},k\gamma,\overline{j\beta}} &= -R_{i\alpha,\overline{h\delta},\overline{j\beta},k\gamma}\\ &= \frac{1}{2}(-\delta_{ij}\delta_{kh}\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{ih}\delta_{kj}\delta_{\alpha\beta}\delta_{\gamma\delta})\end{aligned}$$

for all $1 \le i, j, k, l \le n$ *and* $1 \le \alpha, \beta, \gamma, \delta \le m$. *These and their complex conjugates are all component types different from zero.*

Proof. By (2.1), for $h = 2\partial\bar{\partial}\Phi(Z)$, where $\Phi(Z) = \frac{1}{2}\ln\det(I + Z\overline{Z}')$, from the well-known formula $\det A = \exp\{\operatorname{Tr}\ln A\}$ we have

$$\begin{aligned}2\Phi(Z) &= \operatorname{Tr}\ln(I + Z\overline{Z}') = \operatorname{Tr}\left(\sum_{q=1}^{\infty}\frac{(-1)^{q+1}}{q}(Z\overline{Z}')^q\right)\\ &= \sum_{i,\alpha}|Z^{i\alpha}|^2 - \frac{1}{2}\sum_{i,j,\alpha,\beta}\overline{Z}^{i\alpha}Z^{i\beta}\overline{Z}^{j\beta}Z^{j\alpha} + (\text{higher order terms})\end{aligned}$$

for $\|Z\overline{Z}'\| < 1$. (See also [CaVe, page 493]). From this and the arguments on the pages 155-159 of [KoNo], it follows that the curvature tensor at $Z = 0$ is given by

$$\begin{aligned}R_{i\alpha,\overline{j\beta},k\gamma,\overline{h\delta}} &= \frac{\partial^4\Phi}{\partial Z^{i\alpha}\partial\overline{Z}^{j\beta}\partial Z^{k\gamma}\partial\overline{Z}^{h\delta}}\Big|_{Z=0}\\ &= \frac{1}{2}(-\delta_{ij}\delta_{kh}\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{ih}\delta_{kj}\delta_{\alpha\beta}\delta_{\gamma\delta})\end{aligned}$$

for all $1 \le i, j, k, l \le n$ and $1 \le \alpha, \beta, \gamma, \delta \le m$. Moreover, from the Bianchi identity and the fact that the curvature tensor R of Kähler manifold is of type $(2, 2)$ it is not hard to derive that

$$R_{i\alpha,\overline{j\beta},k\gamma,\overline{h\delta}} = R_{i\alpha,\overline{h\delta},k\gamma,\overline{j\beta}} = -R_{i\alpha,\overline{h\delta},\overline{j\beta},k\gamma}$$

for all $1 \le i, j, k, l \le n$ and $1 \le \alpha, \beta, \gamma, \delta \le m$. These and their complex conjugates are all component types different from zero. □

Let h_{I} be the canonical Kähler metric on $G^{\mathrm{I}}(n, 2n)$, which in the coordinate chart $U_{\alpha_1\cdots\alpha_n}$ is given by $\partial\bar{\partial}\ln\det(I + Z\overline{Z}')$ as in (2.1). It induces a Kähler metric h_{II} on $G^{\mathrm{II}}(n, 2n)$ which in the induced coordinate system

$$G^{\mathrm{II}}(n, 2n) \cap U_{\alpha_1\cdots\alpha_n} \ni [A] \mapsto \left(Z^{kl}([A])\right)_{k<l} \tag{2.6}$$

is given by

$$h_{\mathrm{II}} = \partial\bar{\partial}\ln\det(I - Z\overline{Z}) \tag{2.7}$$

with $Z \in \mathbb{C}^{n\times n}$ and $Z = -Z'$; moreover h_{I} induces a Kähler metric h_{III} on $G^{\mathrm{III}}(n, 2n)$ which in the induced coordinate system

$$G^{\mathrm{III}}(n, 2n) \cap U_{\alpha_1\cdots\alpha_n} \ni [A] \mapsto \Big(Z^{kl}([A])\Big)_{k\le l} \tag{2.8}$$

is given by

$$h_{\mathrm{III}} = \partial\bar{\partial} \ln\det(I + Z\overline{Z}) \tag{2.9}$$

with $Z \in \mathbb{C}^{n\times n}$ and $Z = Z'$.

Let h_{FS} be the Fubini-Study metric on $\mathbb{C}P^{n+1}$, which is given by

$$h_{\mathrm{FS}} = \partial\bar{\partial} \ln(1 + |\xi_1|^2 + \cdots + |\xi_{n+1}|^2) \tag{2.10}$$

with $\xi_k = \xi_k([z]) = \frac{z_k}{z_{n+2}}$, $k = 1, \ldots, n+1$, $[z] \in U_{n+2} = \{[z_1, \ldots, z_{n+2}] \in \mathbb{C}P^{n+1} \mid z_{n+2} \neq 0\}$. Then $G^{\mathrm{IV}}(1, n+1)$ is a Kähler submanifold of $\mathbb{C}P^{n+1}$ with the induced Kähler metric

$$h_{\mathrm{IV}} = \partial\bar{\partial} \ln(1 + |\xi_1|^2 + \cdots + |\xi_n|^2 + |1 - \xi_1^2 - \cdots - \xi_n^2|) \tag{2.11}$$

on $G^{\mathrm{IV}}(1, n+1) \cap U_{n+2}$ from h_{FS}. If $\mathrm{Im}\xi_{n+1} \neq 0$, in the new coordinate chart on $G^{\mathrm{IV}}(1, n+1)$,

$$(\xi_1, \ldots, \xi_n) \mapsto Z = (Z_1, \ldots, Z_n) = \left(\frac{\xi_1}{\xi_{n+1} + i}, \ldots, \frac{\xi_n}{\xi_{n+1} + i}\right),$$

the metric h_{IV} has the following expression (cf.[Lu1])

$$h_{\mathrm{IV}} = \partial\bar{\partial} \ln(1 + |ZZ'|^2 + 2Z\bar{Z}'). \tag{2.12}$$

All irreducible symmetric spaces of compact type have positive holomorphic sectional curvatures (cf. [Bo, CaVe, Lu1]). As in (2.5) one can give explicit expressions of holomorphic sectional curvatures $H_{\mathrm{II}}(Z, T)$, $H_{\mathrm{III}}(Z, T)$ and $H_{\mathrm{IV}}(0, T)$ under the above coordinate charts too (cf. [Lu1]).

Let R^{I} denote the curvature tensor of the metric $h_{\mathrm{I}} = \partial\bar{\partial} \ln\det(I + Z\overline{Z}')$ on $G^{\mathrm{I}}(n, 2n)$. By Proposition 2.1, at $Z = 0$ we have

$$\begin{aligned} R^{\mathrm{I}}_{i\alpha,\overline{j\beta},k\gamma,\overline{h\delta}} &= R^{\mathrm{I}}_{i\alpha,\overline{h\delta},k\gamma,\overline{j\beta}} = -R^{\mathrm{I}}_{i\alpha,\overline{h\delta},\overline{j\beta},k\gamma} \\ &= \frac{1}{2}(-\delta_{ij}\delta_{kh}\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{ih}\delta_{kj}\delta_{\alpha\beta}\delta_{\gamma\delta}) \end{aligned} \tag{2.13}$$

for all $1 \le i, j, k, l, \alpha, \beta, \gamma, \delta \le n$. These and their complex conjugates are all component types different from zero.

Denote the curvature tensors of $(G^{\mathrm{II}}(n,2n),h_{\mathrm{II}})$ and $(G^{\mathrm{III}}(n,2n),h_{\mathrm{III}})$ by R^{II} and R^{III}, respectively. Note that at $Z=0$ the local coordinate systems $(U_{1\cdots n},Z)$ on $G(n,2n;\mathbb{C})$ and (2.6)–(2.8) are normal coordinates (or complex geodesic coordinates) for the metrics h_{I}, h_{II} and h_{III}. By (2.13) we have

Proposition 2.2. *At $Z=0$ the curvature tensors R^{II} and R^{III} are the restrictions of R^{I}, that is,*

$$\begin{aligned}R^{\mathrm{II}}_{i\alpha,\overline{j\beta},k\gamma,\overline{h\delta}} = R^{\mathrm{II}}_{i\alpha,\overline{h\delta},k\gamma,\overline{j\beta}} &= -R^{\mathrm{II}}_{i\alpha,\overline{h\delta},\overline{j\beta},k\gamma}\\ &= \frac{1}{2}(-\delta_{ij}\delta_{kh}\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{ih}\delta_{kj}\delta_{\alpha\beta}\delta_{\gamma\delta})\end{aligned}$$

for all $1\le i<\alpha\le n, 1\le j<\beta\le n, 1\le k<\gamma\le n, l\le l<\delta\le n$, and

$$\begin{aligned}R^{\mathrm{III}}_{i\alpha,\overline{j\beta},k\gamma,\overline{h\delta}} = R^{\mathrm{III}}_{i\alpha,\overline{h\delta},k\gamma,\overline{j\beta}} &= -R^{\mathrm{III}}_{i\alpha,\overline{h\delta},\overline{j\beta},k\gamma}\\ &= \frac{1}{2}(-\delta_{ij}\delta_{kh}\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{ih}\delta_{kj}\delta_{\alpha\beta}\delta_{\gamma\delta})\end{aligned}$$

for all $1\le i\le\alpha\le n, 1\le j\le\beta\le n, 1\le k\le\gamma\le n, 1\le l\le\delta\le n$.

Now we consider $(G^{\mathrm{IV}}(1,n+1),h_{\mathrm{IV}})$. By (2.12) the Kähler potential function $\Phi(Z)=\frac{1}{2}\ln(1+|ZZ'|^2+2Z\bar{Z}')$ has the following power series expansion

$$\begin{aligned}\frac{1}{2}\ln(1+|ZZ'|^2+2Z\bar{Z}') &= \frac{1}{2}\ln\left(1+2\sum_k |z_k|^2 + |\sum_{k=1}^n z_k^2|^2\right)\\ =\sum_{k=1}^n |Z_k|^2 + \frac{1}{2}\left|\sum_{k=1}^n Z_k^2\right|^2 &- \left(\sum_{k=1}^n |Z_k|^2\right)^2 + \text{higher order terms}\end{aligned}$$

near $Z=0$. Since the coordinates Z_k $(1\le k\le n)$ are normal coordinates, the curvature tensor at $Z=0$ is given by

$$R^{\mathrm{IV}}_{i\bar{j}k\bar{l}} = \left.\frac{\partial^4\Phi}{\partial Z_i\partial\bar{Z}_j\partial Z_k\partial\bar{Z}_l}\right|_{Z=0} = 2(\delta_{ik}\delta_{jl}-\delta_{ij}\delta_{kl}-\delta_{il}\delta_{jk})$$

for all $1\le i,j,k,l\le n$. In particular we get

$$R^{\mathrm{IV}}_{i\bar{i}i\bar{i}} = -2 \quad \forall i \qquad \text{and} \qquad R^{\mathrm{IV}}_{i\bar{j}i\bar{j}} = 2 \quad \forall i\ne j. \tag{2.14}$$

2.2. An expected local coordinate chart

Let J be the standard complex structure on $G(n, n+m; \mathbb{C})$. For $p \in G(n, n+m; \mathbb{C})$, recall that by $\{a_{ij}, b_{ij},\ i=1,\dots,n,\ j=1,\dots,m\}$ being a *unitary base* of $(T_pG(n, n+m), J_p, g_p)$ we mean

$$a_{ij}, b_{ij} = J_p a_{ij} \in T_pG(n, n+m; \mathbb{C}), \quad i=1,\dots,n,\ j=1,\dots,m,$$

is a unit orthogonal base of $(T_pG(n, n+m; \mathbb{C}), g_p)$. To our knowledge the following result seems to be new. It is key for us completing the proofs of Theorems 1.1, 1.2.

Proposition 2.3. *For any $p \in G(n, n+m; \mathbb{C})$ and a unitary base of $(T_pG(n, n+m; \mathbb{C}), J_p, g_p)$,*

$$a_{ij}, b_{ij} := J_p a_{ij} \in T_pG(n, n+m; \mathbb{C}), \quad i=1,\dots,n,\ j=1,\dots,m,$$

there exists a local chart around p on $G(n, n+m; \mathbb{C})$,

$$\mathcal{U} \ni q \to Z(q) = X(q) + iY(q) \in \mathbb{C}^{n\times m} \tag{2.15}$$

satisfying $Z(p) = 0$, such that

(i) *In this chart the metric h and $g = Re(h)$ are given by (2.1) and (2.3), respectively;*

(ii) $a_{ij} = \frac{\partial}{\partial X^{ij}}\Big|_p,\ b_{ij} = \frac{\partial}{\partial Y^{ij}}\Big|_p,\ i=1,\dots,n,\ j=1,\dots,m.$

Proof. Since the isometry group of the Kähler manifold $(G(n, n+m; \mathbb{C}), h)$, $\mathrm{I}(G(n, n+m; \mathbb{C}), h) = SU(n+m)$, acts transitively on $(G(n, n+m; \mathbb{C}), h)$, for any $p \in G(n, n+m; \mathbb{C})$ there exists a $\tau \in \mathrm{I}(G(n, n+m; \mathbb{C}), h)$ such that $\tau(p_0) = p$. Clearly, we get a coordinate chart around p on $G(n, n+m; \mathbb{C})$,

$$W = U + iV : \tau(U_{1\cdots n}) \to \mathbb{C}^{n\times m},\ q \mapsto Z(\tau^{-1}(q)). \tag{2.16}$$

Since τ is a Kähler isometry, using (2.1) one easily shows that the metric h in this chart is given by

$$h = \mathrm{Tr}[(I + W\overline{W}')^{-1} dW (I + \overline{W}'W)^{-1} \overline{dW}'].$$

It follows that the Riemannian metric $g = Re(h)$ is given by

$$g_W(T_1, T_2) = Re\mathrm{Tr}[(I + W\overline{W}')^{-1} T_1 (I + \overline{W}'W)^{-1} \overline{T_2}']$$

for real tangent vectors T_1, T_2 at $W \in \tau(U_{1\cdots n})$,

$$T_1 = \sum_{k,l} \mathrm{Re}(T_1^{kl}) \frac{\partial}{\partial U^{kl}} + \sum_{k,l} \mathrm{Im}(T_1^{kl}) \frac{\partial}{\partial V^{kl}},$$
$$T_2 = \sum_{k,l} \mathrm{Re}(T_2^{kl}) \frac{\partial}{\partial U^{kl}} + \sum_{k,l} \mathrm{Im}(T_2^{kl}) \frac{\partial}{\partial V^{kl}},$$

which are identified with complex matrices $(T_1^{kl}), (T_2^{kl}) \in \mathbb{C}^{n\times m}$, respectively.

Define vectors

$$\begin{aligned}
\overrightarrow{a} &= (a_{11}, a_{12}, \dots, a_{1m}, a_{21}, \dots, a_{2m}, \dots, a_{n1}, \dots, a_{nm}),\\
\overrightarrow{b} &= (b_{11}, b_{12}, \dots, b_{1m}, b_{21}, \dots, b_{2m}, \dots, b_{n1}, \dots, b_{nm}),\\
\overrightarrow{\frac{\partial}{\partial U}\Big|_p} &= \left(\frac{\partial}{\partial U^{11}}\Big|_p, \dots, \frac{\partial}{\partial U^{1m}}\Big|_p, \frac{\partial}{\partial U^{21}}\Big|_p, \dots, \frac{\partial}{\partial U^{2m}}\Big|_p, \dots, \right.\\
&\qquad \left. \frac{\partial}{\partial U^{n1}}\Big|_p, \dots, \frac{\partial}{\partial U^{nm}}\Big|_p \right),\\
\overrightarrow{\frac{\partial}{\partial V}\Big|_p} &= \left(\frac{\partial}{\partial V^{11}}\Big|_p, \dots, \frac{\partial}{\partial V^{1m}}\Big|_p, \frac{\partial}{\partial V^{21}}\Big|_p, \dots, \frac{\partial}{\partial V^{2m}}\Big|_p, \dots, \right.\\
&\qquad \left. \frac{\partial}{\partial V^{n1}}\Big|_p, \dots, \frac{\partial}{\partial V^{nm}}\Big|_p \right).
\end{aligned}$$

Since

$$\left\{ \frac{\partial}{\partial U^{ij}}\Big|_p, \frac{\partial}{\partial V^{ij}}\Big|_p,\ i = 1, \dots, n,\ j = 1, \dots, m \right\}$$

is a unitary base of $(T_pG(n, n+m; \mathbb{C}), J_p, g_p)$, there exists a unique real matrix Θ such that

$$(\overrightarrow{a}, \overrightarrow{b}) = \left(\overrightarrow{\frac{\partial}{\partial U}\Big|_p}, \overrightarrow{\frac{\partial}{\partial V}\Big|_p} \right) \Theta. \tag{2.17}$$

The matrix Θ must have form $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & \mathcal{A} \end{pmatrix}$, where $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{nm\times nm}$ is such that $\mathcal{A} + i\mathcal{B}$ is a unitary matrix (which is equivalent to

$$\mathcal{B}'\mathcal{A} = (\mathcal{A}'\mathcal{B})' = \mathcal{A}'\mathcal{B} \qquad \text{and} \qquad \mathcal{A}'\mathcal{A} + \mathcal{B}'\mathcal{B} = I_{nm\times nm}.)$$

Note that (2.17) is equivalent to

$$\overrightarrow{a} + i\overrightarrow{b} = \left(\overrightarrow{\frac{\partial}{\partial U}\Big|_p} + i\overrightarrow{\frac{\partial}{\partial V}\Big|_p} \right)(\mathcal{A} + i\mathcal{B}). \tag{2.18}$$

Recall that the tensor product or Kronecker product of matrices $A = (a_{ij}) \in \mathbb{C}^{n\times m}$ and $B = (b_{ij}) \in \mathbb{C}^{p\times q}$ is a $(np \times mq)$-matrix given by

$$A \otimes B = [a_{ij}B]_{i,j=1}^{n,m} = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \cdots & \cdots & \cdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix}.$$

Define matrices $\mathbf{a} = (a_{ij})$, $\mathbf{b} = (b_{ij})$ and $\frac{\partial}{\partial U}|_p = (\frac{\partial}{\partial U^{ij}}|_p)$, $\frac{\partial}{\partial V}|_p = (\frac{\partial}{\partial V^{ij}}|_p)$. It follows from (2.18) that there exist unitary matrices $\mathcal{R} \in \mathbb{C}^{n\times n}$ and $\mathcal{S} \in \mathbb{C}^{m\times m}$ such that

$$\mathcal{A} + i\mathcal{B} = \mathcal{R}' \otimes \mathcal{S} \quad \text{and} \quad \mathbf{a} + i\mathbf{b} = \mathcal{R}\left(\frac{\partial}{\partial U}\Big|_p + i\frac{\partial}{\partial V}\Big|_p \right)\mathcal{S}. \tag{2.19}$$

Let $\mathcal{R} = R_1 + iR_2$ with $R_1, R_2 \in \mathbb{R}^{n\times n}$, and $\mathcal{S} = S_1 + iS_2$ with $S_1, S_2 \in \mathbb{R}^{m\times m}$. Then

$$\left.\begin{array}{l} (R_1'R_2)' = R_1'R_2 \quad \text{and} \quad R_1'R_1 + R_2'R_2 = I_{n\times n}, \\ (S_1'S_2)' = S_1'S_2 \quad \text{and} \quad S_1'S_1 + S_2'S_2 = I_{m\times m}. \end{array}\right\}$$

Moreover, the first equality in (2.19) implies

$$\mathcal{A} = R_1' \otimes S_1 - R_2' \otimes S_2 \quad \text{and} \quad \mathcal{B} = R_2' \otimes S_1 + R_1' \otimes S_2.$$

From the local chart $(\tau(U_{1\cdots n}), W)$ in (2.16), we define a new chart

$$\mathcal{U} \to \mathbb{C}^{n\times m},\ q \mapsto G(q) = E(q) + iF(q) := \mathcal{R}^{-1}W(q)\mathcal{S}^{-1}. \tag{2.20}$$

Then $G(p) = W(p) = 0$. Define vectors

$$\begin{aligned} \overrightarrow{W} &= (W^{11}, W^{12}, \ldots, W^{1m}, Z^{21}, \ldots, W^{2m}, \ldots, W^{n1}, \ldots, W^{nm}), \\ \overrightarrow{G} &= (G^{11}, G^{12}, \ldots, G^{1m}, G^{21}, \ldots, G^{2m}, \ldots, G^{n1}, \ldots, G^{nm}). \end{aligned}$$

By [Lu2, page 364, (6)] we get

$$\frac{\partial G}{\partial W} = \frac{\partial \overrightarrow{G}}{\partial \overrightarrow{W}} = (\mathcal{R}^{-1})' \otimes \mathcal{S}^{-1} = (\mathcal{R}' \otimes \mathcal{S})^{-1} = (\mathcal{A} + i\mathcal{B})^{-1}. \tag{2.21}$$

Writing $G = \Phi(W)$ and

$$\begin{aligned}\overrightarrow{\frac{\partial}{\partial W}\Big|_p} = \Bigg(&\frac{\partial}{\partial W^{11}}\Big|_p, \ldots, \frac{\partial}{\partial W^{1m}}\Big|_p, \frac{\partial}{\partial W^{21}}\Big|_p, \ldots, \frac{\partial}{\partial W^{2m}}\Big|_p, \ldots, \\ &\frac{\partial}{\partial W^{n1}}\Big|_p, \ldots, \frac{\partial}{\partial W^{nm}}\Big|_p\Bigg),\end{aligned}$$

$$\begin{aligned}\Phi_*\left(\overrightarrow{\frac{\partial}{\partial W}\Big|_p}\right) = \Bigg(&\Phi_*\left(\frac{\partial}{\partial W^{11}}\Big|_p\right), \ldots, \Phi_*\left(\frac{\partial}{\partial W^{1m}}\Big|_p\right), \Phi_*\left(\frac{\partial}{\partial W^{21}}\Big|_p\right), \ldots, \\ &\Phi_*\left(\frac{\partial}{\partial W^{2m}}\Big|_p\right), \ldots, \Phi_*\left(\frac{\partial}{\partial W^{n1}}\Big|_p\right), \ldots, \\ &\Phi_*\left(\frac{\partial}{\partial W^{nm}}\Big|_p\right)\Bigg),\end{aligned}$$

since $\overrightarrow{\frac{\partial}{\partial U}|_p} + i\overrightarrow{\frac{\partial}{\partial V}|_p} = \overrightarrow{\frac{\partial}{\partial W}|_p}$, by (2.18) and (2.21) we get

$$\begin{aligned}\overrightarrow{\frac{\partial}{\partial G}\Big|_p} &= \Phi_*\left(\overrightarrow{\frac{\partial}{\partial W}\Big|_p}\right) = \overrightarrow{\frac{\partial}{\partial W}\Big|_p}\frac{\partial\overrightarrow{W}}{\partial\overrightarrow{G}} \\ &= \overrightarrow{\frac{\partial}{\partial W}\Big|_p}\left(\frac{\partial\overrightarrow{G}}{\partial\overrightarrow{W}}\right)^{-1} = \overrightarrow{\frac{\partial}{\partial W}\Big|_p}(\mathcal{A} + i\mathcal{B}) = \overrightarrow{a} + i\,\overrightarrow{b}.\end{aligned}$$

That is, the coordinate chart in (2.20), $\mathcal{U} \to \mathbb{C}^{n\times m},\ q \mapsto G(q)$, satisfies

$$a_{ij} = \frac{\partial}{\partial E^{ij}}\Big|_p, \quad b_{ij} = \frac{\partial}{\partial F^{ij}}\Big|_p, \quad i = 1, \ldots, n,\ j = 1, \ldots, m.$$

It remains to prove that the transformation

$$\mathbb{C}^{n\times m} \to \mathbb{C}^{n\times m},\ W \mapsto G = \Phi(W)$$

preserves the Kähler metric

$$ds^2 = \mathrm{Tr}[(I + W\overline{W}')^{-1}dW(I + \overline{W}'W)^{-1}\overline{dW}']$$

on $\mathbb{C}^{n\times m}$. In fact, since

$$\begin{aligned}(I+G\overline{G}')^{-1}dG &= (I+\mathcal{R}^{-1}W\mathcal{S}^{-1}\overline{\mathcal{R}^{-1}W\mathcal{S}^{-1}}')\mathcal{R}^{-1}dW\mathcal{S}^{-1}\\ &= (I+\mathcal{R}^{-1}W\overline{W}'\overline{\mathcal{R}^{-1}}')\mathcal{R}^{-1}dW\mathcal{S}^{-1}\\ &= (\mathcal{R}^{-1}\overline{\mathcal{R}^{-1}}'+\mathcal{R}^{-1}W\overline{W}'\overline{\mathcal{R}^{-1}}')\mathcal{R}^{-1}dW\mathcal{S}^{-1}\\ &= \mathcal{R}^{-1}(I+W\overline{W}')dW\mathcal{S}^{-1},\\ (I+\overline{G}'G)^{-1}d\overline{G}' &= (I+\overline{\mathcal{R}^{-1}W\mathcal{S}^{-1}}'\mathcal{R}^{-1}W\mathcal{S}^{-1})\overline{\mathcal{R}^{-1}dW\mathcal{S}^{-1}}'\\ &= (I+\overline{\mathcal{S}^{-1}}'\overline{W}'\overline{\mathcal{R}^{-1}}'\mathcal{R}^{-1}W\mathcal{S}^{-1})\overline{\mathcal{S}^{-1}}'\overline{dW}'\overline{\mathcal{R}^{-1}}'\\ &= \overline{\mathcal{S}^{-1}}'(I+\overline{W}'W)\overline{dW}'\overline{\mathcal{R}^{-1}}'\end{aligned}$$

we get

$$\begin{aligned}&\mathrm{Tr}[(I+G\overline{G}')^{-1}dG(I+\overline{G}'G)^{-1}\overline{dG}']\\ &= \mathrm{Tr}\big[(I+\Phi(W)\overline{\Phi(W)}')^{-1}d\Phi(W)(I+\Phi(W)\overline{\Phi(W)}'\Phi(W))^{-1}\overline{d\Phi(W)}'\big]\\ &= \mathrm{Tr}[(I+W\overline{W}')^{-1}dW(I+\overline{W}'W)^{-1}\overline{dW}'].\end{aligned}$$

Hence the coordinate chart in (2.20) satisfies the desired requirements. □

Corollary 2.4. *For any $p,q\in G(n,n+m;\mathbb{C})$, let*

$$\begin{aligned}&\{a_{ij}, b_{ij} := J_p a_{ij},\ i=1,\dots,n,\ j=1,\dots,m\} \quad \textit{and}\\ &\{a'_{ij}, b'_{ij} := J_q a'_{ij},\ i=1,\dots,n,\ j=1,\dots,m\}\end{aligned}$$

be unitary bases of $(T_pG(n,n+m;\mathbb{C}),J_p,g_p)$ and $(T_qG(n,n+m;\mathbb{C}),J_q,g_q)$, respectively. Consider the sequence $u_1,\dots,u_{2nm}$ whose all odd (resp. even) terms are given by

$$\begin{aligned}&a_{11},a_{12},\dots,a_{1m},a_{21},\dots,a_{2m},\dots,a_{n1},\dots,a_{nm},\\ \textit{(resp.}\quad &b_{11},b_{12},\dots,b_{1m},b_{21},\dots,b_{2m},\dots,b_{n1},\dots,b_{nm}.\textit{)}\end{aligned}$$

Similarly let the sequence $u'_1,\dots,u'_{2nm}$ be given by $\{a'_{ij}, b'_{ij} := J_q a'_{ij},\ i=1,\dots,n,\ j=1,\dots,m\}$. Then the curvature tensor R of $(G(n,n+m;\mathbb{C}),g)$ satisfies

$$R_p(u_\alpha,u_\beta,u_\gamma,u_\delta) = R_q(u'_\alpha,u'_\beta,u'_\gamma,u'_\delta) \tag{2.22}$$

for any $\alpha,\beta,\gamma,\delta\in\{1,\dots,2nm\}$.

Proof. This can be directly derived from Propositions 2.1, 2.3. We here give another proof of it with (2.4). Let $(\mathcal{U}, Z)$ be a local chart around p as in (2.15). Then $a_{ij} = \frac{\partial}{\partial X^{ij}}|_p$, $b_{ij} = \frac{\partial}{\partial Y^{ij}}|_p$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Let $(\mathcal{V}, W = U + \sqrt{-1}V)$ be a local chart around q as in (2.15). Then $a'_{ij} = \frac{\partial}{\partial U^{ij}}|_q$, $b'_{ij} = \frac{\partial}{\partial V^{ij}}|_q$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Note that according to the above correspondence the tangent vectors $\frac{\partial}{\partial X^{kl}}|_p$ and $\frac{\partial}{\partial Y^{st}}|_p$ have matrices representations

$$S_{(k,l)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{(k,l)} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{n\times m} \text{ and } T_{(s,t)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i_{(s,t)} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{n\times m} \tag{2.23}$$

respectively, where the first index (k, l) means that 1 is in the k-th row and l-th array of the matrix and similarly for other indexes in the sequel. Clearly, the tangent vectors $\frac{\partial}{\partial U^{kl}}|_p$ and $\frac{\partial}{\partial V^{st}}|_p$ are also represented by these two matrices. So for any $\alpha \in \{1, \ldots, 2nm\}$ both u_α and u'_α have the same matrix representations. The desired conclusions follow from (2.4) immediately. □

This corollary and Proposition 2.1 immediately lead to

Corollary 2.5. *Let (M, ω^M, J^M, g^M) be a compact Kähler-Einstein submanifold of $(G(n, n+m; \mathbb{C}), h)$ which is totally geodesic (e.g. $(G^{\mathrm{II}}(n, 2n), h_{\mathrm{II}})$ and $(G^{\mathrm{III}}(n, 2n), h_{\mathrm{III}})$ are such submanifolds of $(G(n, 2n; \mathbb{C}), h_{\mathrm{I}})$). Set $\dim M = 2N$. For any $p, q \in M$, let*

$$\{a_{2i-1}, a_{2i} := J^M_p a_{2i-1},\ i = 1, \ldots, N\} \quad \textit{and}$$
$$\{a'_{2i-1}, a'_{2i} := J^M_q a'_{2i-1},\ i = 1, \ldots, N\}$$

be unitary bases of (T_pM, g^M_p, J^M_p) and (T_qM, g^M_q, J^M_q), respectively. Then the curvature tensor R^M of (M, g) satisfies

$$R^M_p(a_\alpha, a_\beta, a_\gamma, a_\delta) = R^M_q(a'_\alpha, a'_\beta, a'_\gamma, a'_\delta)$$

for any $\alpha, \beta, \gamma, \delta \in \{1, \ldots, \dim M\}$.

Proof. Since (T_pM, g^M_p, J^M_p) and (T_qM, g^M_q, J^M_q) are Hermitian subspaces of $(T_pG(n, n+m; \mathbb{C}), h_p)$ and $(T_qG(n, n+m; \mathbb{C}), h_q)$, respectively, we may extend $\{a_1, \ldots, a_{2N}\}$ and $\{a'_1, \ldots, a'_{2N}\}$ into unitary bases

$$\{a_1, \ldots, a_{2nm}\} \quad \text{and} \quad \{a'_1, \ldots, a'_{2nm}\}$$

of $(T_pG(n, n+m; \mathbb{C}), h_p)$ and $(T_qG(n, n+m; \mathbb{C}), h_q)$, respectively. By the assumptions (M, ω^M, J^M, g^M) is a totally geodesic submanifold of $(G(n, n+$

$m;\mathbb{C}),h)$. R^M is equal to the restriction of R to M. Hence the desired conclusion follows from (2.22). (Of course it may also be obtained from Proposition 2.2 for $(G^{\mathrm{II}}(n,2n),h_{\mathrm{II}})$ and $(G^{\mathrm{III}}(n,2n),h_{\mathrm{III}})$). $\square$

Let $(\mathcal{U},Z)$ be the local chart around p on $G(n,n+m;\mathbb{C})$ as in Proposition 2.3.

Proposition 2.6. *For any $1\le k,s,\mu\le n, 1\le l,t,\nu\le m$ we have*

$$R\left(\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial X^{st}}\Big|_p,\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial Y^{\mu\nu}}\Big|_p\right)=0,$$

$$R\left(\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial X^{st}}\Big|_p,\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial X^{\mu\nu}}\Big|_p\right)=\begin{cases}1 & \text{if } \mu=s\neq k, l=t=\nu,\\ 1 & \text{if } \mu=s=k, l\neq t=\nu,\\ 0 & \text{otherwise,}\end{cases}$$

$$R\left(\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial Y^{st}}\Big|_p,\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial Y^{\mu\nu}}\Big|_p\right)=\begin{cases}1 & \text{if } \mu=s\neq k, l=t=\nu,\\ 1 & \text{if } \mu=s=k, l\neq t=\nu,\\ 4 & \text{if } \mu=s=k, l=t=\nu,\\ 0 & \text{otherwise.}\end{cases}$$

Consequently, for $S_{(k,l)}$ and $T_{(s,t)}$ in (2.23) we get the sectional curvatures

$$\begin{aligned}K_p(S_{(k,l)},T_{(s,t)}) &:= R\left(\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial Y^{st}}\Big|_p,\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial Y^{st}}\Big|_p\right)\\ &=\begin{cases}1 & \text{if } k=s, l\neq t,\\ 1 & \text{if } k\neq s, l=t,\\ 4 & \text{if } k=s, l=t,\\ 0 & \text{if } k\neq s, l\neq t,\end{cases}\end{aligned}$$

$$\begin{aligned}K_p(S_{(k,l)},S_{(s,t)}) &:= R\left(\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial X^{st}}\Big|_p,\frac{\partial}{\partial X^{kl}}\Big|_p,\frac{\partial}{\partial X^{st}}\Big|_p\right)\\ &=\begin{cases}1 & \text{if } k=s, l\neq t,\\ 1 & \text{if } k\neq s, l=t,\\ 0 & \text{if } k=s, l=t,\\ 0 & \text{if } k\neq s, l\neq t.\end{cases}\end{aligned}$$

Proof. Since the only possible non-vanishing terms of the curvature components are of the form $R_{i\alpha,\overline{j\beta},k\gamma,\overline{h\delta}}$ and those obtained from the universal symmetries of the curvature tensor, a direct computation leads to

$$
\begin{aligned}
&R\left(\frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial X^{st}}\Big|_p, \frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial X^{\mu\nu}}\Big|_p\right)\\
&= R\left(\frac{\partial}{\partial Z^{kl}}\Big|_p + \frac{\partial}{\partial \overline{Z}^{kl}}\Big|_p, \frac{\partial}{\partial Z^{st}}\Big|_p + \frac{\partial}{\partial \overline{Z}^{st}}\Big|_p,\right.\\
&\qquad \left.\frac{\partial}{\partial Z^{kl}}\Big|_p + \frac{\partial}{\partial \overline{Z}^{kl}}\Big|_p, \frac{\partial}{\partial Z^{\mu\nu}}\Big|_p + \frac{\partial}{\partial \overline{Z}^{\mu\nu}}\Big|_p\right)\\
&= R\left(\frac{\partial}{\partial Z^{kl}}\Big|_p, \frac{\partial}{\partial \overline{Z}^{st}}\Big|_p, \frac{\partial}{\partial Z^{kl}}\Big|_p, \frac{\partial}{\partial \overline{Z}^{\mu\nu}}\Big|_p\right)\\
&\quad + R\left(\frac{\partial}{\partial Z^{kl}}\Big|_p, \frac{\partial}{\partial \overline{Z}^{st}}\Big|_p, \frac{\partial}{\partial \overline{Z}^{kl}}\Big|_p, \frac{\partial}{\partial Z^{\mu\nu}}\Big|_p\right)\\
&\quad + R\left(\frac{\partial}{\partial \overline{Z}^{kl}}\Big|_p, \frac{\partial}{\partial Z^{st}}\Big|_p, \frac{\partial}{\partial Z^{kl}}\Big|_p, \frac{\partial}{\partial \overline{Z}^{\mu\nu}}\Big|_p\right)\\
&\quad + R\left(\frac{\partial}{\partial \overline{Z}^{kl}}\Big|_p, \frac{\partial}{\partial Z^{st}}\Big|_p, \frac{\partial}{\partial \overline{Z}^{kl}}\Big|_p, \frac{\partial}{\partial Z^{\mu\nu}}\Big|_p\right)\\
&= R_{kl,\overline{st},kl,\overline{\mu\nu}} - R_{kl,\overline{st},\mu\nu,\overline{kl}} - R_{st,\overline{kl},kl,\overline{\mu\nu}} + R_{st,\overline{kl},\mu\nu,\overline{kl}}\\
&= \frac{1}{2}[-\delta_{ks}\delta_{k\mu}\delta_{l\nu}\delta_{tl} - \delta_{k\mu}\delta_{sk}\delta_{tl}\delta_{l\nu}] + \frac{1}{2}[\delta_{ks}\delta_{k\mu}\delta_{ll}\delta_{t\nu} + \delta_{kk}\delta_{s\mu}\delta_{tl}\delta_{l\nu}]\\
&\quad + \frac{1}{2}[\delta_{ks}\delta_{k\mu}\delta_{t\nu}\delta_{ll} + \delta_{s\mu}\delta_{kk}\delta_{tl}\delta_{l\nu}] + \frac{1}{2}[-\delta_{ks}\delta_{k\mu}\delta_{tl}\delta_{l\nu} - \delta_{sk}\delta_{k\mu}\delta_{tl}\delta_{l\nu}]\\
&= -2\delta_{sk}\delta_{k\mu}\delta_{tl}\delta_{l\nu} + \delta_{ks}\delta_{k\mu}\delta_{t\nu} + \delta_{s\mu}\delta_{tl}\delta_{l\nu},
\end{aligned}
$$

where the final equality comes from Proposition 2.1. So we get

$$
R\left(\frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial X^{st}}\Big|_p, \frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial X^{\mu\nu}}\Big|_p\right) = \begin{cases} 1 & \text{if } \mu = s \neq k, l = t = \nu, \\ 1 & \text{if } \mu = s = k, l \neq t = \nu, \\ 0 & \text{otherwise.} \end{cases}
$$

Similarly we may obtain

$$
\begin{aligned}
&R\left(\frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial X^{st}}\Big|_p, \frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial Y^{\mu\nu}}\Big|_p\right) = 0,\\
&R\left(\frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial Y^{st}}\Big|_p, \frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial Y^{\mu\nu}}\Big|_p\right)\\
&= R\left(\frac{\partial}{\partial Z^{kl}}\Big|_p + \frac{\partial}{\partial \overline{Z}^{kl}}\Big|_p, i\frac{\partial}{\partial Z^{st}}\Big|_p - i\frac{\partial}{\partial \overline{Z}^{st}}\Big|_p,\right.\\
&\qquad \left.\frac{\partial}{\partial Z^{kl}}\Big|_p + \frac{\partial}{\partial \overline{Z}^{kl}}\Big|_p, i\frac{\partial}{\partial Z^{\mu\nu}}\Big|_p - i\frac{\partial}{\partial \overline{Z}^{\mu\nu}}\Big|_p\right)\\
&= 2\delta_{sk}\delta_{k\mu}\delta_{tl}\delta_{l\nu} + \delta_{ks}\delta_{k\mu}\delta_{t\nu} + \delta_{s\mu}\delta_{tl}\delta_{l\nu}
\end{aligned}
$$

and therefore

$$R\left(\frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial Y^{st}}\Big|_p, \frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial Y^{\mu\nu}}\Big|_p\right) = \begin{cases} 1 & \text{if } \mu = s \neq k, l = t = \nu, \\ 1 & \text{if } \mu = s = k, l \neq t = \nu, \\ 4 & \text{if } \mu = s = k, l = t = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

□

3. Evolution along the mean curvature flow

3.1. Preliminaries

For convenience we review results in [MeWa, §2]. A real $2N$-dimensional Hermitian vector space is a real $2N$-dimensional vector space V equipped with a Hermitian structure, i.e. a triple (ω, J, g) consisting of a symplectic bilinear form $\omega : V \times V \to \mathbb{R}$, an inner product g and an complex structure J on V satisfying $g = \omega \circ (\mathrm{Id} \times J)$. A *Hermitian isomorphism* from (V, ω, J, g) to another Hermitian vector space $(\widetilde{V}, \tilde{\omega}, \tilde{J}, \tilde{g})$ of real $2n$ dimension is a linear isomorphism $L : V \to \widetilde{V}$ satisfying: $LJ = \tilde{J}L$, $L^*\tilde{\omega} = \omega$ and $L^*\tilde{g} = g$. Proposition 1 and Corollary 2 in Section 2.1 of [MeWa] can be summarized as follows.

Proposition 3.1. *For any linear symplectic isomorphism L from the real $2N$-dimensional Hermitian (V, ω, J, g) to $(\widetilde{V}, \tilde{\omega}, \tilde{J}, \tilde{g})$, let $L^\star : \widetilde{V} \to V$ be the adjoint of L determined by $g(L^\star\tilde{u}, v) = \tilde{g}(\tilde{u}, Lv)$. Then $L^\star L : V \to V$ is positive definite, and $E := L(L^\star L)^{-1/2}$ gives rise to a Hermitian isomorphism from (V, ω, J, g) to $(\widetilde{V}, \tilde{\omega}, \tilde{J}, \tilde{g})$. Moreover, there exists an* unitary basis $\{v_1, \ldots, v_{2N}\}$ *of (V, ω, J, g), i.e.,*

$$g(v_i, v_j) = \delta_{ij} \quad \textit{and} \quad Jv_{2k-1} = v_{2k},\ k = 1, \ldots, N,$$

(and hence an unitary basis of $(\widetilde{V}, \tilde{\omega}, \tilde{J}, \tilde{g})$, $\{\tilde{v}_1, \ldots, \tilde{v}_{2N}\}$, where $\tilde{v}_k = E(v_k)$, $k = 1, \ldots, 2N$), such that

(i) *The matrix representations of J and $\tilde{J}$ under them are all J_0 given by*

$$J_0(x_1, y_1, \ldots, x_N, y_N)^t = (y_1, -x_1, \ldots, y_N, -x_N)^t. \tag{3.1}$$

(ii) *The map* $(L^\star L)^{1/2}$ *has the matrix representation under the basis* $\{v_1,\dots,v_{2N}\}$,

$$(L^\star L)^{1/2}=\mathrm{Diag}(\lambda_1,\lambda_2,\dots,\lambda_{2N-1},\lambda_{2N}),$$

where $\lambda_{2i-1}\lambda_{2i}=1$ *and* $\lambda_{2i-1}\le 1\le\lambda_{2i}$, $i=1,\dots,N$.

(iii) *Under the bases* $\{v_1,\dots,v_{2N}\}$ *and* $\{\tilde v_1,\dots,\tilde v_{2N}\}$ *the map* L *has the matrix representation* $L=\mathrm{Diag}(\lambda_1,\lambda_2,\dots,\lambda_{2N-1},\lambda_{2N})$.

Remark 3.2. From the arguments in [MeWa] one can also choose the $\{v_1,\dots,v_{2N}\}$ such that λ_k, $k=1,\dots,2N$ in Proposition 3.1(ii) satisfy: $\lambda_{2i}\le 1\le\lambda_{2i-1}$, $i=1,\dots,N$.

Let (M,ω,J,g) and $(\widetilde M,\tilde\omega,\tilde J,\tilde g)$ be two real $2N$-dimensional Kähler-Einstein manifolds, and let $\pi_1: M\times\widetilde M\to M$ and $\pi_2: M\times\widetilde M\to\widetilde M$ be two natural projections. We have a product Kähler manifold $(M\times\widetilde M,\pi_1^*\omega-\pi_2^*\tilde\omega,\mathcal J,G)$, where $G=\pi_1^*g+\pi_2^*\tilde g$ and $\mathcal J(u,v)=(Ju,-\tilde Jv)$ for $(u,v)\in T(M\times\widetilde M)$.

For a symplectomorphism $\varphi:(M,\omega)\to(\widetilde M,\tilde\omega)$ let

$$\Sigma=\mathrm{Graph}(\varphi)=\{(p,\varphi(p))\mid p\in M\},$$

and let Σ_t be the mean curvature flow of Σ in $M\times\widetilde M$.

Denote by $\Omega:=\pi_1^*\omega^N$, and by $*\Omega$ the Hodge star of $\Omega|_{\Sigma_t}$ with respect to the induced metric on Σ_t by G. Then $*\Omega$ is the Jacobian of the projection from Σ_t onto M, and $*\Omega(q)=\Omega(e^1,\dots,e^{2N})$ for $q\in\Sigma_t$ and any oriented orthogonal basis $\{e^1,\dots,e^{2N}\}$ of $T_q\Sigma_t$. The implicit function theorem implies that $*\Omega(q)>0$ if and only if Σ_t is locally a graph over M at q.

Let $q=(p,\varphi_t(p))\in\Sigma_t\subset M\times\widetilde M$. Set $L:=D_p\varphi_t: T_pM\to T_{\varphi_t(p)}\widetilde M$ and $E:=D_p\varphi_t[(D_p\varphi_t)^\star D_p\varphi_t]^{-\frac12}: T_pM\to T_{\varphi_t(p)}\widetilde M$. Since L^*L is a positive definite matrix, by the above arguments one can choose a holomorphic local coordinate system $\{z^1,\dots,z^N\}$ around p, $z^j=x^j+iy^j$, $j=1,\dots,N$, such that

(i) $\{\frac{\partial}{\partial x^1}|_p,\dots,\frac{\partial}{\partial x^N}|_p,\frac{\partial}{\partial y^1}|_p,\dots,\frac{\partial}{\partial y^N}|_p\}$ is an orthogonal basis of the real $2N$-dimensional vector space T_pM,

(ii) The complex structure J_p is given by the matrix J_0 in (3.1) with respect to the base $\frac{\partial}{\partial x^1},\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial x^N},\frac{\partial}{\partial y^N}$.

(iii) $L^*L = \mathrm{Diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_{2N-1}^2, \lambda_{2N}^2)$ with respect to these basis, where $\lambda_{2i-1}\lambda_{2i} = 1$, $\lambda_{2i-1} \leq 1 \leq \lambda_{2i}$ for $i = 1, \ldots, N$. Obviously $\frac{\partial}{\partial x^j} = \frac{\partial}{\partial z^j} + \frac{\partial}{\partial \bar{z}^j}$, $\frac{\partial}{\partial y^j} = \frac{1}{\sqrt{-1}}(\frac{\partial}{\partial \bar{z}^j} - \frac{\partial}{\partial z^j})$.

(iv) There exists a Hermitian vector space isomorphism

$$E : (T_pM, \omega_p, J_p, g_p) \to (T_{\varphi_t(p)}\widetilde{M}, \tilde{\omega}_{\varphi_t(p)}, \tilde{J}_{\varphi_t(p)}, \tilde{g}_{\varphi_t(p)})$$

such that under the orthogonal basis of $(T_{\varphi_t(p)}\widetilde{M}, \tilde{g}_{\varphi_t(p)})$,

$$\left\{E\left(\frac{\partial}{\partial x_1}\Big|_p\right), E\left(\frac{\partial}{\partial y_1}\Big|_p\right), \ldots, E\left(\frac{\partial}{\partial x_N}\Big|_p\right), \ldots, E\left(\frac{\partial}{\partial y_N}\Big|_p\right)\right\},$$

$\tilde{J}_{\varphi_t(p)}$ is also given by the matrix J_0 in (3.1).

By the choose of basis, we have

$$\begin{aligned}
&g\left(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p\right) = g\left(\frac{\partial}{\partial y^i}\Big|_p, \frac{\partial}{\partial y^j}|_p\right) = \delta_{ij},\\
&g\left(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial y^j}\Big|_p\right) = g\left(\frac{\partial}{\partial y^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p\right) = 0,\\
&g_{l\bar{d}} = g\left(\frac{\partial}{\partial z^l}\Big|_p, \frac{\partial}{\partial \bar{z}^d}\Big|_p\right) = g_{\bar{d}l} = g_{d\bar{l}} = g_{\bar{l}d} = \frac{\delta_{ld}}{2},\\
&g_{\overline{ld}} = g_{ld} = 0.
\end{aligned}$$

For $j = 1, \ldots, N$, set

$$a^{2j-1} = \frac{\partial}{\partial x_j}\Big|_p \quad \text{and} \quad a^{2j} = \frac{\partial}{\partial y_j}\Big|_p. \tag{3.2}$$

Then by (**ii**) above it holds that

$$J_p(a^{2j-1}) = a^{2j} \quad \text{and} \quad J_p(a^{2j}) = -a^{2j-1}, \quad j = 1, \ldots, N.$$

Let $s' = s + (-1)^{s+1}$, $s = 1, \ldots, 2N$, and let $J_{rs} := g(Ja_s, a_r)$. It follows that

$$J_{s's} = -J_{ss'} \quad \text{and} \quad J_{rs} = \begin{cases} 0 & \text{if } r \neq s', \\ (-1)^{s+1} & \text{if } r = s'. \end{cases}$$

For $i = 1, \dots, 2N$, let

$$\begin{aligned} e^i &= \frac{1}{\sqrt{1+\lambda_i^2}}(a^i, \lambda_i E(a^i)) \quad \text{and} \\ e^{2N+i} &= \frac{1}{\sqrt{1+\lambda_i^2}}(J_p a^i, -\lambda_i E(J_p a^i)). \end{aligned} \tag{3.3}$$

They form an orthogonal basis of $T_q(M \times \widetilde{M})$, and

$$T_q\Sigma_t = \operatorname{span}(\{e^1, \dots, e^{2N})\} \quad \text{and} \quad N_q\Sigma_t = \operatorname{span}(\{e^{2N+1}, \dots, e^{4N}\})$$

and $*\Omega = \Omega(e^1, \dots, e^{2N}) = 1/\sqrt{\prod_{j=1}^{2N}(1+\lambda_j^2)}$.

Proposition 3.3 ([MeWa, Prop. 2]). *Let (M, g, J, ω) and $(\widetilde{M}, \tilde{g}, \tilde{J}, \widetilde{\omega})$ be two compact Kähler-Einstein manifolds of real dimension $2N$, and let Σ_t be the mean curvature flow of the graph Σ of a symplectomorphism $\varphi : (M, \omega) \to (\widetilde{M}, \widetilde{\omega})$. Then $*\Omega$ at each point $q \in \Sigma_t$ satisfies the following equation:*

$$\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega \left\{ Q(\lambda_i, h_{jkl}) + \sum_k \sum_{i \neq k} \frac{\lambda_i (R_{ikik} - \lambda_k^2 \widetilde{R}_{ikik})}{(1+\lambda_k^2)(\lambda_i + \lambda_{i'})} \right\}. \tag{3.4}$$

where

$$Q(\lambda_i, h_{jkl}) = \sum_{i,j,k} h_{ijk}^2 - 2 \sum_k \sum_{i<j} (-1)^{i+j} \lambda_i \lambda_j (h_{i'ik} h_{j'jk} - h_{i'jk} h_{j'ik})$$

with $i' = i + (-1)^{i+1}$, and $R_{ijkl} = R(a^i, a^j, a^k, a^l)$ and $\widetilde{R}_{ijkl} = \widetilde{R}(E(a^i), E(a^j), E(a^k), E(a^l))$ are, respectively, the coefficients of the curvature tensors R and $\widetilde{R}$ with respect to the chosen bases of T_pM and $T_{f(p)}\widetilde{M}$ as in Proposition 3.1.

For $\vec{\lambda} = (\lambda_1, \dots, \lambda_{2N}) \in \mathbb{R}^{2N}$, according to [MeWa, p.322] let

$$\delta_{\vec{\lambda}} := \inf\Big\{ Q(\lambda_i, h_{jkl}) \,\Big|\, h_{ijk} \in \mathbb{R},\, 1 \le i,j,k \le 2N,\, \sum_{i,j,k} h_{ijk}^2 = 1 \Big\}, \tag{3.5}$$

that is, the smallest eigenvalue of Q at $\vec{\lambda}$, and for $\Lambda \in [1, \infty)$ let

$$\delta_\Lambda := \inf \left\{ \delta_{\vec{\lambda}} \,\Big|\, \frac{1}{\Lambda} \le \lambda_i \le \Lambda \text{ for } i = 1, \dots, 2N \right\}, \tag{3.6}$$

$$\Lambda_0(N) := \sup\{\Lambda \mid \Lambda \ge 1 \text{ and } \delta_\Lambda > 0\}. \tag{3.7}$$

By Remark 2 and Lemma 4 in [MeWa] (or the proof of [MeWa, Prop. 3]), $\Lambda_0(1) = \infty$, and

$$Q((1,\ldots,1), h_{ijk}) \geq \frac{3-\sqrt{5}}{6}|II|^2 = \frac{3-\sqrt{5}}{6}\sum_{i,j,k} h_{ijk}^2.$$

Clearly, $\delta_{\vec{\lambda}}$ is continuous in $\vec{\lambda}$, and $[1,\infty) \ni \Lambda \to \delta_\Lambda$ is nonincreasing. They imply $\Lambda_0(N) > 1$. Note that $\delta_{\Lambda'} > 0$ for every $\Lambda' \in [1, \Lambda_0(N))$. Indeed, by the definition of supremum we have a $\Lambda \in (\Lambda', \Lambda_0(N))$ with $\delta_\Lambda > 0$. So $\delta_{\Lambda'} \geq \delta_\Lambda > 0$. In addition, (3.5) and (3.6) imply

$$\begin{aligned} &\inf\Big\{Q(\lambda_i, h_{jkl}) \;\Big|\; h_{ijk} \in \mathbb{R},\ \sum_{i,j,k} h_{ijk}^2 = 1,\ \frac{1}{\Lambda'} \leq \lambda_i \leq \Lambda'\Big\} \\ &= \inf\Big\{\delta_{\vec{\lambda}} \;\Big|\; \frac{1}{\Lambda'} \leq \lambda_i \leq \Lambda'\Big\} = \delta_{\Lambda'} \end{aligned}$$

for every $\Lambda' \in [1, \Lambda_0(N))$. Hence we get:

Proposition 3.4. ([MeWa, Prop. 3]) *Let $Q(\lambda_i, h_{jkl})$ be the the quadratic form defined in Proposition 3.3. Then for the constant $\Lambda_0(N) \in (1, +\infty]$ in (3.7), which only depends on $2N = \dim M$, $Q(\lambda_i, h_{jkl})$ is nonnegative whenever $\frac{1}{\Lambda_0(N)} \leq \lambda_i \leq \Lambda_0(N)$ for $i = 1, \ldots, 2N$. Moreover, for any $\Lambda' \in [1, \Lambda_0(N))$ it holds that*

$$Q(\lambda_i, h_{jkl}) \geq \delta_{\Lambda'} \sum_{ijk} h_{jkl}^2$$

whenever $\frac{1}{\Lambda'} \leq \lambda_i \leq \Lambda'$ for $i = 1, \ldots, 2N$.

3.2. The case of Grassmann manifolds

Let $\varphi : M = G(n, n+m; \mathbb{C}) \to \widetilde{M} = G(n, n+m; \mathbb{C})$ be a Λ-pinched symplectomorphism and $\Sigma = \mathrm{Graph}(\varphi)$. For $(p, \varphi_t(p)) \in \Sigma_t$, let a^j, $j = 1, \ldots, nm$, be the chosen unitary base of $(T_pG(n, n+m; \mathbb{C}), J_p, g_p)$ as in Proposition 3.1. Then

$$R_{ijkl} = R(a^i, a^j, a^k, a^l), \tag{3.8}$$
$$\widetilde{R}_{ijkl} = \widetilde{R}(E(a^i), E(a^j), E(a^k), E(a^l)) \tag{3.9}$$

are, respectively, the coefficients of the curvature tensors R and $\widetilde{R}$ with respect to the chosen unitary bases of $T_pG(n, n+m; \mathbb{C})$ and $T_{\varphi_t(p)}G(n, n+m; \mathbb{C})$.

From Corollary 2.4 it follows that

$$R_{ijkl} = \widetilde{R}_{ijkl} \quad \forall 1 \le i, j, k, l \le 2nm. \tag{3.10}$$

Let $(\mathcal{U}, Z)$ be the local chart around p on $G(n, n+m; \mathbb{C})$ as in Proposition 2.3. The final two equalities in Proposition 2.6 show

$$\begin{aligned} &R\Big(\frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial Y^{st}}\Big|_p, \frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial Y^{st}}\Big|_p\Big) \\ &- R\Big(\frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial X^{st}}\Big|_p, \frac{\partial}{\partial X^{kl}}\Big|_p, \frac{\partial}{\partial X^{st}}\Big|_p\Big) = 4\delta_{ks}\delta_{lt}. \end{aligned} \tag{3.11}$$

Writing $Z^{11}, Z^{12}, \dots, Z^{1m}, Z^{21}, \dots, Z^{2m}, \dots, Z^{n1}, \dots, Z^{nm}$ into $z^1, z^2, \dots, z^{nm}$ we have

$$e_k := a^{2k-1} = \frac{\partial}{\partial x^k}\Big|_p \quad \text{and} \quad f_k := a^{2k} = \frac{\partial}{\partial y^k}\Big|_p \tag{3.12}$$

for $k = 1, \dots, nm$. Then (3.11) can be written as

$$\begin{aligned} &R\big(e_{(k-1)m+l}, f_{(s-1)m+t}, e_{(k-1)m+l}, f_{(s-1)m+t}\big) \\ &- R\big(e_{(k-1)m+l}, e_{(s-1)m+t}, e_{(k-1)m+l}, e_{(s-1)m+t}\big) = 4\delta_{ks}\delta_{lt} \end{aligned} \tag{3.13}$$

for any $1 \le k, s \le n$ and $1 \le l, t \le m$. Clearly, this is equivalent to

$$R(e_i, f_j, e_i, f_j) - R(e_i, e_j, e_i, e_j) = 4\delta_{ij} \quad \forall 1 \le i, j \le nm. \tag{3.14}$$

Now for $M = \widetilde{M} = G(n, n+m; \mathbb{C})$, by (3.10) we may rewrite the second term in the big bracket of (3.4) as follows:

$$
\begin{aligned}
(3.15)\qquad & \sum_k \sum_{i\neq k} \frac{\lambda_i(R_{ikik}-\lambda_k^2 \widetilde{R}_{ikik})}{(1+\lambda_k^2)(\lambda_i+\lambda_{i'})} = \sum_k \sum_{i\neq k} \frac{\lambda_i(1-\lambda_k^2)R_{ikik}}{(1+\lambda_k^2)(\lambda_i+\lambda_{i'})} \\
&= \sum_{k=2r-1,i=2s-1,r\neq s} \frac{\lambda_{2s-1}(1-\lambda_{2r-1}^2)R(e_s,e_r,e_s,e_r)}{(1+\lambda_{2r-1}^2)(\lambda_{2s-1}+\lambda_{2s})} \\
&\quad + \sum_{k=2r-1,i=2s} \frac{\lambda_{2s}(1-\lambda_{2r-1}^2)R(f_s,e_r,f_s,e_r)}{(1+\lambda_{2r-1}^2)(\lambda_{2s-1}+\lambda_{2s})} \\
&\quad + \sum_{k=2r,i=2s-1} \frac{\lambda_{2s-1}(1-\lambda_{2r}^2)R(e_s,f_r,e_s,f_r)}{(1+\lambda_{2r}^2)(\lambda_{2s-1}+\lambda_{2s})} \\
&\quad + \sum_{k=2r,i=2s,r\neq s} \frac{\lambda_{2s}(1-\lambda_{2r}^2)R(f_s,f_r,f_s,f_r)}{(1+\lambda_{2r}^2)(\lambda_{2s-1}+\lambda_{2s})} \\
&= \sum_{r\neq s} \frac{R(e_s,e_r,e_s,e_r)}{(\lambda_{2s-1}+\lambda_{2s})}\left[\frac{\lambda_{2s-1}(1-\lambda_{2r-1}^2)}{(1+\lambda_{2r-1}^2)} + \frac{\lambda_{2s}(1-\lambda_{2r}^2)}{(1+\lambda_{2r}^2)}\right] \\
&\quad + \sum_{r,s} \frac{R(e_s,f_r,e_s,f_r)(\lambda_{2r}^2-1)(\lambda_{2s}-\lambda_{2s-1})}{(\lambda_{2s-1}+\lambda_{2s})(1+\lambda_{2r}^2)} \\
&= \sum_{r,s} \frac{(\lambda_{2r}^2-1)(\lambda_{2s}-\lambda_{2s-1})}{(\lambda_{2s-1}+\lambda_{2s})(1+\lambda_{2r}^2)}\left[R(e_s,f_r,e_s,f_r)-R(e_s,e_r,e_s,e_r)\right] \\
&= \sum_{r,s} \frac{(\lambda_{2r}^2-1)(\lambda_{2s}^2-1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)}\left[R(e_s,f_r,e_s,f_r)-R(e_s,e_r,e_s,e_r)\right] \\
&\overset{(3.14)}{=} \sum_{r=s} 4\frac{(\lambda_{2r}^2-1)(\lambda_{2s}^2-1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)} \\
&= 4\sum_s \frac{(\lambda_{2s}^2-1)^2}{(1+\lambda_{2s}^2)^2}.
\end{aligned}
$$

Hence in the present case (3.4) becomes

$$
\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega \left\{ Q(\lambda_i, h_{jkl}) + 4\sum_{s=1}^{nm} \frac{(\lambda_{2s}^2-1)^2}{(1+\lambda_{2s}^2)^2} \right\}. \tag{3.16}
$$

This and Proposition 3.4 immediately lead to the following generalization of [MeWa, §3,Cor.4].

Proposition 3.5. *Let $\Lambda_0 = \Lambda_0(nm) > 1$ be the constant defined by (3.7). For any $\Lambda \in [1, \Lambda_0)$ it holds that*

$$\left(\frac{d}{dt} - \Delta\right) * \Omega \geq \delta_\Lambda * \Omega |II|^2 + 4 * \Omega \sum_{s=1}^{nm} \frac{(1-\lambda_{2s}^2)^2}{(1+\lambda_{2s}^2)^2}$$

whenever $\frac{1}{\Lambda} \leq \lambda_i \leq \Lambda$ for $i = 1, \dots, 2nm$. Here $|II|$ is the norm of the second fundamental form of Σ_t.

Recall that $*\Omega = 1/\sqrt{\prod_{j=1}^{2mn}(1+\lambda_j^2)} = 1/\prod_{i\,\mathrm{odd}} \frac{1}{\lambda_i + \lambda_{i'}}$ on Σ_t, where $i' = i + (-1)^{i+1}$ for $i = 1, \dots, 2nm$. For $\Lambda > 1$ and $0 < \epsilon < 1/2^{nm}$ set

$$\begin{aligned} \epsilon(mn, \Lambda) &= \frac{1}{2^{mn}} - \frac{1}{(\Lambda + \frac{1}{\Lambda})^{mn}}, \\ \Lambda(mn, \epsilon) &= \frac{2^{-mn}}{2^{-mn} - \epsilon} + \sqrt{\left(\frac{2^{-mn}}{2^{-mn} - \epsilon}\right)^2 - 1}. \end{aligned}$$

Then $\epsilon(mn, \Lambda) > 0$ and $\Lambda(mn, \epsilon) > 1$. Lemmas 5 and 6 in [MeWa] showed

$$\begin{aligned} \frac{1}{\Lambda} \leq \lambda_i \leq \Lambda \quad \forall i \quad &\Longrightarrow \quad \frac{1}{2^{mn}} - \epsilon(mn, \Lambda) \leq *\Omega, \\ \frac{1}{2^{mn}} - \epsilon \leq *\Omega \quad &\Longrightarrow \quad \frac{1}{\Lambda(mn, \epsilon)} \leq \lambda_i \leq \Lambda(mn, \epsilon) \quad \forall i. \end{aligned}$$

From these and Proposition 3.5 we may repeat the proofs of Proposition 4 and Corollary 5 in [MeWa] to obtain the following generalization of them.

Proposition 3.6. *For some $T > 0$ let $[0, T) \ni t \to \Sigma_t$ be the mean curvature flow of the graph Σ of a symplectomorphism $\varphi : G(n, n+m; \mathbb{C}) \to G(n, n+m; \mathbb{C})$, where $G(n, n+m; \mathbb{C})$ is equipped with the unique (up to $\times$ nonzero factor) invariant Kähler-Einstein metric. Let $*\Omega(t)$ be the Jacobian of the projection $\pi_1 : \Sigma_t \to G(n, n+m; \mathbb{C})$. Suppose for some $\Lambda \in (1, \Lambda_0(nm))$ that*

$$\frac{1}{2^{mn}} - \epsilon = \frac{1}{2^{mn}} - \frac{1}{2^{mn}}\left(1 - \frac{2\Lambda}{\Lambda^2+1}\right) = \frac{1}{2^{mn-1}} \frac{\Lambda}{\Lambda^2+1} \leq *\Omega(0).$$

Then along the mean curvature flow $*\Omega$ satisfies

$$\left(\frac{d}{dt} - \Delta\right) * \Omega \geq \delta_\Lambda * \Omega |II|^2 + 4 * \Omega \sum_{s=1}^{nm} \frac{(1-\lambda_{2s}^2)^2}{(1+\lambda_{2s}^2)^2},$$

where δ_Λ is given in (3.6), and so $\min_{\Sigma_t} *\Omega$ *is nondecreasing as a function in t and Σ_t is the graph of a symplectomorphism $\varphi_t : G(n, n+m;\mathbb{C}) \to G(n, n+m;\mathbb{C})$. In particular, if φ is Λ-pinched for some $\Lambda \in (1, \Lambda_1(mn)] \setminus \{\infty\}$, then each φ_t is Λ'_{mn}-pinched along the mean curvature flow, where Λ'_{mn} is defined by (1.2).* (Note: $\Lambda'_{mn} = \Lambda_0(mn)$ *if* $\Lambda = \Lambda_1(mn) < \infty$.)

Remark 3.7. Let (M, ω, J, g) be a compact totally geodesic Kähler-Einstein submanifold of $(G(n, n+m;\mathbb{C}), h)$ (e.g. $(G^{\mathrm{II}}(n, 2n), h_{\mathrm{II}})$ and $(G^{\mathrm{III}}(n, 2n), h_{\mathrm{III}})$ are such submanifolds of $(G(n, 2n;\mathbb{C}), h_{\mathrm{I}})$), $\dim M = 2N$. By Corollary 2.5 we immediately obtain corresponding results with Propositions 3.5 and 3.6.

3.3. The case of flat complex tori

The following proposition is actually contained in the proof of Corollary 3 of [MeWa, p.320]. We still give its proof.

Proposition 3.8. *If M and $\widetilde{M}$ are real $2n$-dimensional Kähler manifolds with constant holomorphic sectional curvature* $c \geq 0$ *(hence are Einstein and have the same scalar curvature), then*

$$\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega \left\{ Q(\lambda_i, h_{jkl}) + c \sum_{k \text{ odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2} \right\}.$$

Proof. With the choice of bases of T_pM and $T_{f(p)}\widetilde{M}$, (we shall suppress $|_p$ in $\frac{\partial}{\partial x^r}|_p$ and $\frac{\partial}{\partial y^r}|_p$, $r = 1, \ldots, n$ for simplicity), it is easily computed that

$$\begin{aligned} R_{ikik} &= R(a^i, a^j, a^k, a^l) \\ &= \begin{cases} R(\frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}) & \text{if } i = 2r-1, k = 2s-1, \\ R(\frac{\partial}{\partial x^r}, \frac{\partial}{\partial y^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial y^s}) & \text{if } i = 2r-1, k = 2s, \\ R(\frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}) & \text{if } i = 2r, k = 2s-1, \\ R(\frac{\partial}{\partial y^r}, \frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial y^s}) & \text{if } i = 2r, k = 2s. \end{cases} \end{aligned}$$

Plugging $\frac{\partial}{\partial x^j} = \frac{\partial}{\partial z^j} + \frac{\partial}{\partial \bar{z}^j}$, $\frac{\partial}{\partial y^j} = \frac{1}{i}(\frac{\partial}{\partial \bar{z}^j} - \frac{\partial}{\partial z^j})$ into the above equalities we get

$$R_{ikik} = R_{r\bar{s}r\bar{s}} + R_{s\bar{r}s\bar{r}} - R_{r\bar{s}s\bar{r}} - R_{s\bar{r}r\bar{s}} \tag{3.17}$$

if $(i, k) = (2r-1, 2s-1)$ or $(i, k) = (2r, 2s)$, and

$$R_{ikik} = -(R_{r\bar{s}r\bar{s}} + R_{s\bar{r}s\bar{r}} + R_{r\bar{s}s\bar{r}} + R_{s\bar{r}r\bar{s}})$$

if $(i,k)=(2r-1,2s)$ or $(i,k)=(2r,2s-1)$. Note that

$$g_{l\bar{d}}=g\left(\frac{\partial}{\partial z^l},\frac{\partial}{\partial \bar{z}^d}\right)=g_{\bar{d}l}=g_{d\bar{l}}=g_{\bar{l}d}=\frac{\delta_{ld}}{2},\quad g_{\bar{l}\bar{d}}=g_{ld}=0$$

and that the nonzero components of the Riemannian curvature in the complex local system $z^1,\dots,z^n$ are exactly $R_{i\bar{j}k\bar{l}}$ and $R_{\bar{i}j\bar{k}l}$. Moreover,

$$R_{i\bar{j}k\bar{l}}=-\frac{c}{2}(g_{i\bar{j}}g_{k\bar{l}}+g_{i\bar{l}}g_{\bar{j}k})$$

on the Kähler manifolds of constant holomorphic sectional curvature c (by Proposition 7.6 of [KoNo, p. 169]). From (3.17) we derive

$$R_{ikik}=\begin{cases}-\frac{c}{4}(\delta_{rs}-1) & \text{if } (i,k)=(2r-1,2s-1) \text{ or } (i,k)=(2r,2s),\\ \frac{c}{4}(3\delta_{rs}+1) & \text{if } (i,k)=(2r-1,2s) \text{ or } (i,k)=(2r,2s-1).\end{cases}$$

This shows that

$$R_{ikik}=\frac{c}{4}(3\delta_{ik'}+1)\quad \forall i\neq k.$$

Plugging into (3.4) yields

$$\begin{aligned}\frac{d}{dt}*\Omega&=\Delta*\Omega+*\Omega\left\{Q(\lambda_i,h_{jkl})+\frac{c}{4}\sum_k\sum_{i\neq k}\frac{\lambda_i(1-\lambda_k^2)(1+3\delta_{ik'})}{(1+\lambda_k^2)(\lambda_i+\lambda_{i'})}\right\}\\&=\Delta*\Omega+*\Omega\left\{Q(\lambda_i,h_{jkl})+c\sum_{k\text{ odd}}\frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2}\right\}.\end{aligned}$$

□

As in the proof of [MeWa, §3,Cor.4], from this and Proposition 3.4 we immediately get the following result.

Proposition 3.9. *Under the assumptions of Proposition 3.8, for any* $\Lambda\in[1,\Lambda_0(n))$ *it holds that*

$$\left(\frac{d}{dt}-\Delta\right)*\Omega\geq\delta_\Lambda*\Omega|II|^2+c*\Omega\sum_{k\text{ odd}}\frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2}\tag{3.18}$$

whenever $\frac{1}{\Lambda}\leq\lambda_i\leq\Lambda$ *for* $i=1,\dots,2n$. *Here* $|II|$ *is the norm of the second fundamental form of* Σ_t.

From now on we shall assume $c=0$. In this case we can improve the pinching condition.

Proposition 3.10. *Under the assumptions of Proposition 3.8, if $c = 0$ and φ is Λ-pinched with $\Lambda \in [1,\infty)$ then φ_t is still Λ-pinched on $[0,T)$, i.e.*

$$\left.\begin{array}{r}\frac{1}{\Lambda} \le \lambda_i(0) \le \Lambda \\ \forall i = 1,\dots,2n\end{array}\right\} \implies \left\{\begin{array}{l}\frac{1}{\Lambda} \le \lambda_i(t) \le \Lambda \\ \forall i = 1,\dots,2n \quad \textit{and} \quad \forall t \in [0,T).\end{array}\right.$$

Here $[0,T)$ is the maximal existence interval of the mean curvature flow, and $T > 0$ or $T = \infty$.

Proof. Since λ_i, $i = 1, \dots$, are singular values of a linear symplectic map, we have $\frac{1}{\lambda_i} \in \{\lambda_1,\dots,\lambda_{2n}\}$ for $i = 1,\dots,2n$. (See Lemma 3 of [MeWa]). So the question is reduced to prove

$$\left.\begin{array}{r}\lambda_i(0) \le \Lambda \\ \forall i = 1,\dots,2n\end{array}\right\} \implies \left\{\begin{array}{l}\lambda_i(t) \le \Lambda \\ \forall i = 1,\dots,2n \quad \text{and} \quad \forall t \in [0,T).\end{array}\right.$$

We shall use the method in [TsWa, Section 4] and [Smo3] to prove this.

Let a^j, $j = 1,\dots,n$ be as in Proposition 3.3 with $N = n$. Set

$$e^i = \frac{1}{\sqrt{1+\lambda_i^2}}(a^i, \lambda_i E(a^i)) \quad \text{and} \quad e^{2n+i} = \frac{1}{\sqrt{1+\lambda_i^2}}(J_p a^i, -\lambda_i E(J_p a^i))$$

for $i = 1,\dots,2n$. Identifying the tangent space of $M \times \widetilde{M}$ with $TM \oplus T\widetilde{M}$, let π_1 and π_2 denote the projection onto the first and second factors in the splitting. Then

$$\pi_1(e^i) = \frac{a^i}{\sqrt{1+\lambda_i^2}}, \quad \pi_2(e^i) = \frac{\lambda_i E(a^i)}{\sqrt{1+\lambda_i^2}};$$
$$\pi_1(e^{2n+i}) = \frac{Ja^i}{\sqrt{1+\lambda_i^2}}, \quad \pi_2(e^{2n+i}) = \frac{-\lambda_i E(Ja^i)}{\sqrt{1+\lambda_i^2}}$$

for $i = 1,\dots,2n$. Let us define the following parallel symmetric two-tensor S by

$$S(X,Y) = \frac{\Lambda^2\langle \pi_1(X), \pi_1(Y)\rangle - \langle \pi_2(X), \pi_2(Y)\rangle}{\Lambda^{2+\Xi}}$$

for any $X, Y \in T(M \times \widetilde{M})$, where $\Xi > 0$ is a parameter determined later. Then

$$
\begin{aligned}
S_{ij} &:= S(e^i, e^j) = \frac{(\Lambda^2 - \lambda_i\lambda_j)\delta_{ij}}{\Lambda^{2+\Xi}\cdot\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}},\\
S_{r(2n+j)} &:= S(e^r, e^{2n+j}) = \frac{(\Lambda^2 + \lambda_r\lambda_j)\delta_{rj'}(-1)^{j+1}}{\Lambda^{2+\Xi}\cdot\sqrt{(1+\lambda_r^2)(1+\lambda_j^2)}},\\
S_{(2n+i)(2n+j)} &:= S(e^{2n+i}, e^{2n+j}) = \frac{(\Lambda^2 - \lambda_i\lambda_j)\delta_{ij}}{\Lambda^{2+\Xi}\cdot\sqrt{(1+\lambda_i^2)(1+\lambda_j^2)}}
\end{aligned}
$$

for $i, j, r = 1, \dots, 2n$, and the matrix $S = (S_{kl})_{1\le k,l\le 4n}$ can be written in the block form

$$
\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}
$$

where $A = D = \mathrm{Diag}\Big(\frac{(\Lambda^2-\lambda_1^2)}{\Lambda^{2+\Xi}\cdot(1+\lambda_1^2)}, \dots, \frac{(\Lambda^2-\lambda_{2n}^2)}{\Lambda^{2+\Xi}\cdot(1+\lambda_{2n}^2)}\Big)$. So

$$
\begin{aligned}
&A \text{ is positive definite on } \Sigma_t \text{ if and only if}\\
&\Lambda^2 - \lambda_i^2 > 0, \quad i = 1, \dots, 2n.
\end{aligned}
\tag{3.19}
$$

Obverse that $e^1, \dots, e^{2n}$ forms an orthogonal basis for the tangent space of Σ_t. As in [TsWa, Prop. 3.2], the pullback of S to Σ_t satisfies the equation

$$
\begin{aligned}
(\frac{d}{dt} - \Delta)S_{ij} = &-h_{\alpha li}H_\alpha S_{lj} - h_{\alpha jl}H_\alpha S_{li} + \mathcal{R}_{kik\alpha}S_{\alpha j} + \mathcal{R}_{kjk\alpha}S_{\alpha i}\\
&+ h_{\alpha kl}h_{\alpha ki}S_{lj} + h_{\alpha kl}h_{\alpha kj}S_{li} - 2h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}
\end{aligned}
\tag{3.20}
$$

for $i, j = 1, \dots, 2n$, where Δ is the rough Laplacian on 2-tensors over Σ_t, $h_{ijk} = G(\nabla^{M\times\widetilde{M}}_{e^i} e^j, \mathcal{J}e^k)$, and $\mathcal{R}_{kik\alpha} = \mathcal{R}(e^k, e^i, e^k, e^\alpha)$ is the component of the curvature tensor $\mathcal{R}$ of $(M\times\widetilde{M}, G)$ with $\mathcal{J}$ and $G = \pi_1^*g + \pi_2^*\tilde{g}$ as in Section 3.1.

Consider the $2n\times 2n$ matrix $(S_{ij}) := (S(e^i, e^j)_{1\le i,j\le 2n}$. By (3.19) we only need to prove

$$
(S_{ij}) > 0 \text{ at } t = 0 \Longrightarrow (S_{ij}) > 0 \text{ in } [0, T).
$$

This can be directly derived from the following analogue of [TsWa, Lemma 4.1]. $\square$

Proposition 3.11. *Let* $x^{n+i} = y^i$*,* $i = 1, \ldots, n$*, and* $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$*,* $i, j = 1, \ldots, 2n$*. For any given* $\epsilon > 0$*, there exists a parameter* $\Xi > 0$ *such that the condition* $(T_{ij}) := (S_{ij}) - \epsilon(g_{ij}) > 0$ *is preserved along the mean curvature flow.*

Proof. Let $\alpha = 2n + \mu$ and $\beta = 2n + \nu$, $\mu, \nu = 1, \ldots, 2n$. As in [TsWa], (3.20) yields

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right) T_{ij} = & -h_{\alpha li} H_\alpha T_{lj} - h_{\alpha jl} H_\alpha T_{li} \\ & + \mathcal{R}_{kik\alpha} S_{\alpha j} + \mathcal{R}_{kjk\alpha} S_{\alpha i} \\ & + h_{\alpha kl} h_{\alpha ki} T_{lj} + h_{\alpha kl} h_{\alpha kj} T_{li} \\ & + 2\epsilon h_{\alpha ki} h_{\alpha kj} - 2h_{\alpha ki} h_{\beta kj} S_{\alpha\beta}. \end{aligned} \tag{3.21}$$

Let N_{ij} denote the right hand side of (3.21). A vector $V = (V^1, \ldots, V^{2n})$ is called a null eigenvector V of the matrix (T_{ij}) if $\sum_j T_{ij} V^j = 0 \; \forall i$. By the Hamilton's maximum principle [Ha, Theorem 9.1], if we may prove

$$\sum_{ij} N_{ij} V^i V^j \geq 0$$

for any null eigenvector V of the matrix (T_{ij}), then the fact that $(T_{ij}) \geq 0$ at $t = 0$ implies that $(T_{ij}) \geq 0$ on $[0, T)$, i.e. Proposition 3.11 holds.

By a direct computation we only need to prove that at $t = 0$

$$\begin{aligned} \sum_{ij} N_{ij} V^i V^j = & \sum_{i,j,k,\alpha} \left[2\epsilon h_{\alpha ki} h_{\alpha kj} V^i V^j - 2 \sum_\beta h_{\alpha ki} h_{\beta kj} S_{\alpha\beta} V^i V^j \right] \\ & + 2 \sum_{i,j,k,\alpha} \mathcal{R}_{kik\alpha} S_{\alpha j} V^i V^j \\ \geq & \; 0 \end{aligned} \tag{3.22}$$

for any null eigenvector $V = (V^1, \ldots, V^{2n})$ of the matrix (T_{ij}). It is easily estimated that

$$\begin{aligned}
&2\sum_{i,j,k,\alpha,\beta} h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}V^iV^j\\
&= 2\sum_{i,j,k,\mu,\nu} h_{2n+\mu,ki}h_{2n+\nu,kj}S_{2n+\mu,2n+\nu}V^iV^j\\
&= 2\sum_{i,j,k,\mu,\nu} \frac{h_{2n+\mu,ki}h_{2n+\nu,kj}(\Lambda^2-\lambda_\mu\lambda_\nu)\delta_{\mu\nu}V^iV^j}{\Lambda^{2+\Xi}\cdot\sqrt{(1+\lambda_\mu^2)(1+\lambda_\nu^2)}}\\
&= 2\sum_k\sum_\mu\left(\sum_{i,j} h_{2n+\mu,ki}h_{2n+\mu,kj}V^iV^j\right)\frac{\Lambda^2-\lambda_\mu^2}{\Lambda^{2+\Xi}\cdot(1+\lambda_\mu^2)}\\
&\le 2\sum_\mu\sum_k\left(\sum_{i,j} h_{2n+\mu,ki}h_{2n+\mu,kj}V^iV^j\right)\sum_\nu\frac{\Lambda^2-\lambda_\nu^2}{\Lambda^{2+\Xi}\cdot(1+\lambda_\nu^2)}\\
&\le \frac{4n}{\Lambda^\Xi}\sum_{i,j,k,\mu} h_{2n+\mu,ki}h_{2n+\mu,kj}V^iV^j.
\end{aligned}$$

Here in the first inequality we used the facts

- $\sum_i(a_ib_i)\le(\sum_i a_i)(\sum_i b_i)$ for $a_i\ge 0, b_i\ge 0$, and
- $\sum_{i,j} h_{2n+\mu,ki}h_{2n+\mu,kj}V^iV^j=(\sum_i h_{2n+\mu,ki}V^i)^2\ge 0$,

and the second one comes from the inequality

$$\sum_\nu\frac{\Lambda^2-\lambda_\nu^2}{\Lambda^{2+\Xi}\cdot(1+\lambda_\nu^2)}\le\sum_\nu\frac{\Lambda^2}{\Lambda^{2+\Xi}}\le\frac{2n}{\Lambda^\Xi}.$$

So the first sum in the right side of (3.22) becomes

$$\begin{aligned}
&\sum_{i,j,k,\alpha}\left[2\epsilon h_{\alpha ki}h_{\alpha kj}V^iV^j-2\sum_\beta h_{\alpha ki}h_{\beta kj}S_{\alpha\beta}V^iV^j\right]\\
&\ge\sum_{i,j,k,\mu} h_{2n+\mu,ki}h_{2n+\mu,kj}V^iV^j\Big(2\epsilon-\frac{4n}{\Lambda^\Xi}\Big)
\end{aligned}$$

because $\alpha=2n+\mu$ and $\beta=2n+\nu$, $\mu,\nu=1,\dots,2n$.

For a given $\epsilon>0$ we can choose $\Xi>0$ so large that $\epsilon-\frac{2n}{\Lambda^\Xi}>0$. Then (3.22) is proved if we show

$$\sum_{i,j,k,\alpha} \mathcal{R}_{kik\alpha} S_{\alpha j} V^i V^j \geq 0$$

for any null eigenvector V of the matrix (T_{ij}). But this is obvious because $(M \times \widetilde{M}, G)$ is flat and hence $\mathcal{R} = 0$. $\square$

From Propositions 3.9 and 3.10 we immediately obtain the following strengthen analogue of Proposition 3.9.

Proposition 3.12. *For some $T > 0$ let $[0, T) \ni t \to \Sigma_t$ be the mean curvature flow of the graph Σ of a symplectomorphism $\varphi : M \to \widetilde{M}$, where M and $\widetilde{M}$ are Kähler-Einstein manifolds of constant holomorphic sectional curvature 0. Let $*\Omega(t)$ be the Jacobian of the projection $\pi_1 : \Sigma_t \to M$. For the constant $\Lambda_0(n)$ in (3.7) and any $\Lambda \in [1, \Lambda_0(n))$, if φ is Λ-pinched initially, then $*\Omega$ satisfies*

$$\left(\frac{d}{dt} - \Delta\right) * \Omega \geq \delta_\Lambda * \Omega |II|^2$$

along the mean curvature flow, where δ_Λ is given in (3.6). In particular, $\min_{\Sigma_t} *\Omega$ is nondecreasing as a function in t.

4. Proofs of Theorems 1.1, 1.2 and 1.3

4.1. Proofs of Theorems 1.1, 1.2

Using Propositions 3.5 and 3.6 (resp. Remark 3.7) and almost repeating the arguments in §3.3, §3.4 of [MeWa] we can complete the proof of Theorem 1.1 (resp. Theorem 1.2).

4.2. Proof of Theorem 1.3

4.2.1. The long-time existence. Embedding $M \times \widetilde{M}$ into some $\mathbb{R}^N$ isometrically, as in [MeWa] the mean curvature flow equation can be written as $\frac{d}{dt}F(x,t) = H = \overline{H} + V$ in terms of the coordinate function $F(x,t) \in \mathbb{R}^N$, where $H \in T_{\Sigma_t}(M \times \widetilde{M})/T\Sigma_t$ and $\overline{H} \in T_{\Sigma_t}\mathbb{R}^N/T\Sigma_t$ are the mean curvature vectors of Σ_t in $M \times \widetilde{M}$ and $\mathbb{R}^N$, respectively, and $V = -II_M(e_a, e_a)$. Suppose by a contradiction that there is a singularity at space time point

$(y_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$. Let $d\mu_t$ denote the volume form of Σ_t, and let

$$\rho_{(y_0,t_0)}(y,t) = \frac{1}{(4\pi(t_0-t))^n} \exp\left(\frac{-|y-y_0|^2}{4(t_0-t)}\right)$$

be the backward heat kernel of $\rho_{(y_0,t_0)}$ at (y_0, t_0). Under our present assumptions, as in [MeWa, page 328] we can still use Proposition 3.12 to derive the corresponding inequality of [MeWa, page 328], that is,

$$\begin{aligned}
&\frac{d}{dt}\int (1-*\Omega)\rho_{(y_0,t_0)} d\mu_t \\
&\leq -\delta_\Lambda \int *\Omega \|II\|^2 \rho_{(y_0,t_0)} d\mu_t + \int (1-*\Omega)\rho_{(y_0,t_0)} \frac{\|V\|^2}{4} d\mu_t \\
&\quad - \int (1-*\Omega)\rho_{(y_0,t_0)} \left\| \frac{F^\perp}{2(t_0-t)} + \overline{H} + \frac{V}{2} \right\|^2 d\mu_t.
\end{aligned}$$

Then the expected long-time existence can be obtained by repeating the remain arguments on the pages 328–330 of [MeWa].

4.2.2. The convergence. Let $\varphi : M \to \widetilde{M}$ be a Λ-pinched symplectomorphism with $\Lambda \in (1, \Lambda_0(n))$. Take an arbitrary $\Lambda_1 \in (\Lambda, \Lambda_0(n))$.

Lemma 4.1. (**Djokovic inequality**)*:*

$$\tan x \begin{cases} > x + \frac{1}{3}x^3, & \text{if } 0 < x < \frac{\pi}{2}, \\ < x + f(\alpha)x^3, & \text{if } 0 < x < \alpha < \frac{\pi}{2}, \end{cases}$$

where $f(\alpha) = \frac{\tan\alpha - \alpha}{\alpha^3}$, *in particular* $f(\frac{\pi}{6}) < \frac{4}{9}$.

The following lemma is key for us.

Lemma 4.2. *For every* $\Lambda_1 \in [1, \Lambda_0(n))$ *there exists a* $\widehat{\Lambda}_1 > 1$ *such that for every* $\Lambda \in (1, \widehat{\Lambda}_1)$ *we have* $k, l > 0$ *to satisfy*

$$\frac{\pi}{2} \cdot 2^{nl} > \sqrt{(\sqrt{21}-3)/2} \cdot \left(\Lambda + \frac{1}{\Lambda}\right)^{nl}, \tag{4.1}$$

$$\frac{l\delta_{\Lambda_1}}{10} \geq \frac{\tan\left(k(\frac{1}{2^n})^l\right)}{k(\frac{1}{2^n})^l}, \tag{4.2}$$

$$\frac{\pi}{2} > k \cdot \left(\frac{1}{2^n}\right)^l > k \cdot \left(\frac{1}{(\Lambda + \frac{1}{\Lambda})^n}\right)^l \geq \sqrt{(\sqrt{21}-3)/2}. \tag{4.3}$$

Moreover $\widehat{\Lambda}_1$ is more than or equal to

$$\left(2\exp\left(\frac{0.141446\delta_{\Lambda_1}}{5n}\right)+2\exp\left(\frac{0.141446\delta_{\Lambda_1}}{10n}\right)\sqrt{\exp\left(\frac{0.141446\delta_{\Lambda_1}}{5n}\right)-1}-1\right)^{\frac{1}{2}}.$$

Its proof will be given at the end of this section.

By the assumption of Theorem 1.3 we have $\Lambda_1 \in (\Lambda, \Lambda_0)$ such that $\Lambda < \widehat{\Lambda}_1$. Fix this Λ_1 below. By Proposition 3.12 we have

$$\frac{d}{dt} * \Omega \geq \Delta * \Omega + \delta_{\Lambda_1} \cdot *\Omega \cdot |II|^2. \tag{4.4}$$

From [Wa2, Section 7] we also know that

$$\begin{aligned}\frac{d}{dt}|II|^2 = \Delta|II|^2 &- 2|\nabla II|^2 + 2\left[(\overline{\nabla}_{\partial_k}\overline{R})_{\underline{s}ijk} + (\overline{\nabla}_{\partial_j}\overline{R})_{\underline{s}kik}\right] h_{sij} \\ &- 4\overline{R}_{lijk}h_{slk}h_{sij} + 8\overline{R}_{\underline{s}\ \underline{t}jk}h_{tik}h_{sij} \\ &- 4\overline{R}_{lkik}h_{slj}h_{sij} + 2\overline{R}_{\underline{s}k\underline{t}k}h_{tij}h_{sij} \\ &+ 2\sum_{s,t,i.m}\left(\sum_k (h_{sik}h_{tmk} - h_{smk}h_{tik})\right)^2 \\ &+ 2\sum_{i,j,m,k}\left(\sum_s h_{sij}h_{smk}\right)^2,\end{aligned} \tag{4.5}$$

where $\overline{R}$ is the curvature tensor and $\overline{\nabla}$ is the covariant derivative of the ambient space, $\underline{s} = 2n + s$. Now on one hand

$$\begin{aligned}&2\sum_{s,t,i,m}\left(\sum_k (h_{sik}h_{tmk} - h_{smk}h_{tik})\right)^2 + 2\sum_{i,j,m,k}\left(\sum_s h_{sij}h_{smk}\right)^2 \\ &\leq 4\sum_{s,t,i,m}\left[\left(\sum_k |h_{sik}|^2\right)\left(\sum_k |h_{tmk}|^2\right) + \left(\sum_k |h_{smk}|^2\right)\left(\sum_k |h_{tik}|^2\right)\right] \\ &\quad + 2\sum_{i,j,m,k}\left(\sum_s h_{sij}^2\right)\left(\sum_s h_{smk}^2\right) \\ &= 8\sum_{s,i,k} h_{sik}^2 \sum_{t,m,k} h_{tmk}^2 + 2\left(\sum_{s,i,j} h_{sij}^2\right)\left(\sum_{s,m,k} h_{smk}^2\right) \\ &= 8|II|^4 + 2|II|^4 = 10|II|^4,\end{aligned} \tag{4.6}$$

where the first inequality comes from

$$\left(\sum_k (h_{sik}h_{tmk} - h_{smk}h_{tik})\right)^2 \le \left(\sum_k (|h_{sik}h_{tmk}| + |h_{smk}h_{tik}|)\right)^2$$
$$\le \left(\left(\sum_k |h_{sik}|^2\right)^{\frac{1}{2}} \left(\sum_k |h_{tmk}|^2\right)^{\frac{1}{2}} + \left(\sum_k |h_{smk}|^2\right)^{\frac{1}{2}} \left(\sum_k |h_{tik}|^2\right)^{\frac{1}{2}}\right)^2$$
$$\le 2\left[\left(\sum_k |h_{sik}|^2\right)\left(\sum_k |h_{tmk}|^2\right) + \left(\sum_k |h_{smk}|^2\right)\left(\sum_k |h_{tik}|^2\right)\right].$$

This and (4.5)–(4.6) lead to

$$\frac{d}{dt}|II|^2 \le \Delta|II|^2 - 2|\nabla II|^2 + 10|II|^4. \tag{4.7}$$

We hope to prove that $\max_{\Sigma_t} |II|^2 \to 0$ as $t \to \infty$. To this goal, for positive numbers k, l, s determined later let us compute the evolution equation of $\frac{|II|^2}{[\sin(k(*\Omega)^l)]^s}$ as follows:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^s}\right)\\
&= \frac{1}{[\sin(k(*\Omega)^l)]^s}\frac{d|II|^2}{dt} - \frac{s\cdot k\cdot l(*\Omega)^{l-1}|II|^2\cos(k(*\Omega)^l)}{[\sin(k(*\Omega)^l)]^{s+1}}\frac{d*\Omega}{dt},
\end{aligned}$$

$$\begin{aligned}
&\Delta\left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^s}\right)\\
&= \frac{\Delta|II|^2}{[\sin(k(*\Omega)^l)]^s} - \frac{s\cdot k\cdot l\cdot |II|^2\cdot (*\Omega)^{l-1}\cdot \cos(k(*\Omega)^l)\cdot \Delta*\Omega}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad - \frac{2s\cdot k\cdot l\cdot \nabla|II|^2\cdot (*\Omega)^{l-1}\cdot \cos(k(*\Omega)^l)\cdot \nabla*\Omega}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad + \frac{s\cdot k^2\cdot l^2\cdot |II|^2\cdot (*\Omega)^{2l-2}\cdot \sin(k(*\Omega)^l)\cdot |\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad - \frac{s\cdot k\cdot l\cdot (l-1)\cdot |II|^2\cdot (*\Omega)^{l-2}\cdot \cos(k(*\Omega)^l)\cdot |\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad + \frac{s\cdot (s+1)\cdot k^2\cdot l^2\cdot |II|^2\cdot (*\Omega)^{2l-2}\cdot (\cos(k(*\Omega)^l))^2\cdot |\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+2}}
\end{aligned}$$

and hence

$$
\begin{aligned}
(4.8)\qquad & \left(\frac{d}{dt}-\Delta\right)\left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^s}\right)\\
&= \frac{1}{[\sin(k(*\Omega)^l)]^s}\left(\frac{d}{dt}-\Delta\right)|II|^2\\
&\quad - \frac{s\cdot k\cdot l\cdot |II|^2\cdot(*\Omega)^{l-1}\cdot\cos(k(*\Omega)^l)}{[\sin(k(*\Omega)^l)]^{s+1}}\left(\frac{d}{dt}-\Delta\right)*\Omega\\
&\quad + \frac{4\cdot s\cdot k\cdot l\cdot |II|\cdot\nabla|II|\cdot(*\Omega)^{l-1}\cdot\cos(k(*\Omega)^l)\cdot\nabla*\Omega}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad - \frac{s\cdot k^2\cdot l^2\cdot|II|^2\cdot(*\Omega)^{2l-2}\cdot\sin(k(*\Omega)^l)\cdot|\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad + \frac{s\cdot k\cdot l\cdot(l-1)\cdot|II|^2\cdot(*\Omega)^{l-2}\cdot\cos(k(*\Omega)^l)\cdot|\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad - \frac{s\cdot(s+1)\cdot k^2\cdot l^2\cdot|II|^2\cdot(*\Omega)^{2l-2}\cdot(\cos(k(*\Omega)^l))^2\cdot|\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+2}}\\
&\overset{(4.4)}{\leq} \frac{-2|\nabla II|^2+10|II|^4}{[\sin(k(*\Omega)^l)]^s} - \frac{s\cdot k\cdot l\cdot\delta_{\Lambda_1}\cdot|II|^4\cdot(*\Omega)^l\cdot\cos(k(*\Omega)^l)}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad + \frac{4\cdot s\cdot k\cdot l\cdot |II|\cdot\nabla|II|\cdot(*\Omega)^{l-1}\cdot\cos(k(*\Omega)^l)\cdot\nabla*\Omega}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad + \text{(the last three terms)}\\
&= \frac{-2|\nabla II|^2}{[\sin(k(*\Omega)^l)]^s} + 10[\sin(k(*\Omega)^l)]^s\left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^s}\right)^2\\
&\quad - s\cdot k\cdot l\cdot\delta_{\Lambda_1}\cdot(*\Omega)^l\cdot\cos(k(*\Omega)^l)\cdot[\sin(k(*\Omega)^l)]^{s-1}\left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^s}\right)^2\\
&\quad + \frac{4\cdot s\cdot k\cdot l\cdot |II|\cdot\nabla|II|\cdot(*\Omega)^{l-1}\cdot\cos(k(*\Omega)^l)\cdot\nabla*\Omega}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad + \text{(the last three terms).}
\end{aligned}
$$

Note that the Cauchy-Schwarz inequality implies

$$
\begin{aligned}
|\nabla|II||^2 &= \sum_{i=1}^{2n}\left(\nabla_i\sqrt{\sum_{j,k,l=1}^{2n}h_{jkl}^2}\right)^2 = \sum_{i=1}^{2n}\left(\frac{2\sum_{j,k,l}h_{jkl}\partial_i h_{jkl}}{2|II|}\right)^2\\
&\leq \sum_{i=1}^{2n}\left(\sum_{j,k,l}\frac{h_{jkl}^2}{|II|^2}\sum_{j,k,l}(\partial_i h_{jkl})^2\right) \leq \sum_{i,j,k,l}(\partial_i h_{jkl})^2 = |\nabla II|^2.
\end{aligned}
$$

The term in (4.8) becomes

$$
\begin{aligned}
&\frac{4\cdot s\cdot k\cdot l\cdot |II|\cdot \nabla|II|\cdot (*\Omega)^{l-1}\cdot \cos(k(*\Omega)^l)\cdot \nabla*\Omega}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\leq \frac{4\cdot s\cdot k\cdot l\cdot |II|\cdot |\nabla|II||\cdot (*\Omega)^{l-1}\cdot \cos(k(*\Omega)^l)\cdot |\nabla*\Omega|}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\leq \frac{4\cdot s\cdot k\cdot l\cdot |II|\cdot |\nabla II|\cdot (*\Omega)^{l-1}\cdot \cos(k(*\Omega)^l)\cdot |\nabla*\Omega|}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&= \frac{4}{[\sin(k(*\Omega)^l)]^{s}}\left[\frac{s\cdot k\cdot l\cdot |II|\cdot (*\Omega)^{l-1}\cdot \cos(k(*\Omega)^l)\cdot |\nabla*\Omega|}{\sin(k(*\Omega)^l)}\right]\cdot|\nabla II|\\
&\leq \frac{2}{[\sin(k(*\Omega)^l)]^{s}}\Bigg[\frac{s^2\cdot k^2\cdot l^2\cdot |II|^2\cdot (*\Omega)^{2l-2}\cdot (\cos(k(*\omega)^l))^2\cdot |\nabla*\Omega|^2}{(\sin(k(*\omega)^l))^2}\\
&\qquad\qquad + |\nabla II|^2\Bigg]\\
&= \frac{2\cdot s^2\cdot k^2\cdot l^2\cdot |II|^2\cdot (*\Omega)^{2l-2}\cdot (\cos(k(*\Omega)^l))^2\cdot |\nabla*\Omega|^2}{(\sin(k(*\Omega)^l))^{s+2}}\\
&\quad + \frac{2|\nabla II|^2}{[\sin(k(*\Omega)^l)]^{s}}.
\end{aligned}
$$

Hence we arrive at

$$
\begin{aligned}
&\left(\frac{d}{dt}-\Delta\right)\left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^{s}}\right)\\
&\leq 10[\sin(k(*\Omega)^l)]^{s}\left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^{s}}\right)^2\\
&\quad - s\cdot k\cdot l\cdot \delta_{\Lambda_1}\cdot (*\Omega)^l\cdot \cos(k(*\Omega)^l)\cdot [\sin(k(*\Omega)^l)]^{s-1}\left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^{s}}\right)^2\\
&\quad + \frac{2\cdot s^2\cdot k^2\cdot l^2\cdot |II|^2\cdot (*\Omega)^{2l-2}\cdot (\cos(k(*\Omega)^l))^2\cdot |\nabla*\Omega|^2}{(\sin(k(*\Omega)^l))^{s+2}}\\
&\quad - \frac{s\cdot k^2\cdot l^2\cdot |II|^2\cdot (*\Omega)^{2l-2}\cdot \sin(k(*\Omega)^l)\cdot |\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad + \frac{s\cdot k\cdot l\cdot (l-1)\cdot |II|^2\cdot (*\Omega)^{l-2}\cdot \cos(k(*\Omega)^l)\cdot |\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+1}}\\
&\quad - \frac{s\cdot (s+1)\cdot k^2\cdot l^2\cdot |II|^2\cdot (*\Omega)^{2l-2}\cdot (\cos(k(*\Omega)^l))^2\cdot |\nabla*\Omega|^2}{[\sin(k(*\Omega)^l)]^{s+2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{|II|^2}{[\sin(k(*\Omega)^l)]^s}\right)^2 \cdot [\sin(k(*\Omega)^l)]^{s-1} \\
&\quad \cdot \big[10 \cdot \sin(k(*\Omega)^l) - s \cdot k \cdot l \cdot \delta_{\Lambda_1} \cdot (*\Omega)^l \cdot \cos(k(*\Omega)^l)\big] \\
&\quad + \frac{s \cdot k \cdot l \cdot (*\Omega)^{l-2}|II|^2|\nabla * \Omega|^2}{[\sin(k(*\Omega)^l)]^{s+2}}[(s-1) \cdot k \cdot l \cdot (*\Omega)^l \cdot (\cos(k(*\Omega)^l))^2 \\
&\quad - k \cdot l \cdot (*\Omega)^l \cdot (\sin(k(*\Omega)^l))^2 + (l-1)\cos(k(*\Omega)^l)\sin(k(*\Omega)^l)].
\end{aligned}
$$

Take $s = 1$ we obtain

$$
\begin{aligned}
&\left(\frac{d}{dt} - \Delta\right)\left(\frac{|II|^2}{\sin(k(*\Omega)^l)}\right) \\
&\quad \le \left(\frac{|II|^2}{\sin(k(*\Omega)^l)}\right)^2 \cdot \big[10 \cdot \sin(k(*\Omega)^l) - k \cdot l \cdot \delta_{\Lambda_1} \cdot (*\Omega)^l \cdot \cos(k(*\Omega)^l)\big] \\
&\qquad + \frac{k \cdot l \cdot (*\Omega)^{l-2}|II|^2|\nabla * \Omega|^2}{[\sin(k(*\Omega)^l)]^3}\big[-k \cdot l \cdot (*\Omega)^l \cdot (\sin(k(*\Omega)^l))^2 \\
&\qquad\qquad + (l-1)\cos(k(*\Omega)^l)\sin(k(*\Omega)^l)\big].
\end{aligned} \tag{4.9}
$$

Claim 4.3. *If the positive numbers k, l satisfy (4.1)–(4.3) in Lemma 4.2, then*

$$10\sin(k(*\Omega)^l) - k \cdot l \cdot \delta_{\Lambda_1} \cdot (*\Omega)^l \cdot \cos(k(*\Omega)^l) < 0$$

and

$$(l-1) \cdot \cos(k(*\Omega)^l) - k \cdot l \cdot (*\Omega)^l \sin(k(*\Omega)^l) < 0,$$

that is

$$\frac{l-1}{l \cdot k(*\Omega)^l} < \tan(k(*\Omega)^l) < \frac{l \cdot \delta_{\Lambda_1} \cdot k \cdot (*\Omega)^l}{10} \tag{4.10}$$

for any $*\Omega \in [\frac{1}{2^n}, \frac{1}{\Lambda + \frac{1}{\Lambda}}]$ *with* $1 < \Lambda < \widehat{\Lambda}_1$.

We put off its proof. Then (4.9) becomes

$$
\begin{aligned}
&\left(\frac{d}{dt} - \Delta\right)\left(\frac{|II|^2}{\sin(k(*\Omega)^l)}\right) \\
&\le \left(\frac{|II|^2}{\sin(k(*\Omega)^l)}\right)^2 \big[10 \cdot \sin(k(*\Omega)^l) - k \cdot l \cdot \delta_{\Lambda_1} \cdot (*\Omega)^l \cdot \cos(k(*\Omega)^l)\big].
\end{aligned}
$$

Let $g = \frac{|II|^2}{\sin(k(*\Omega)^l)}$ and

$$K_1 := \max_{*\Omega \in [\frac{1}{(\Lambda+\frac{1}{\Lambda})^n}, \frac{1}{2^n}]} \left[10 \cdot \sin(k(*\Omega)^l) - k \cdot l \cdot \delta_{\Lambda_1} \cdot (*\Omega)^l \cdot \cos(k(*\Omega)^l)\right].$$

By Claim 4.3, $K_1 < 0$ and

$$\left(\frac{d}{dt} - \Delta\right) g \leq K_1 \cdot g^2. \tag{4.11}$$

Consider the initial value problem

$$\frac{d}{dt} y = K_1 \cdot y^2 \quad \text{and} \quad y(0) = \max_{\Sigma_0} \mathrm{g}. \tag{4.12}$$

The unique solution of it is given by $y(t) = \frac{y(0)}{1-y(0)K_1 t}$. By (4.11)–(4.12) the comparison principle for parabolic equations yields

$$g = \frac{|II|^2}{\sin(k(*\Omega)^l)} \leq y(t) \quad \forall t > 0.$$

Since (4.3) implies that the function

$$\left[\frac{1}{(\Lambda + \frac{1}{\Lambda})^n}, \frac{1}{2^n}\right] \ni *\Omega \to \sin(k(*\Omega)^l)$$

is bounded away from zero, we derive

$$\max_{\Sigma_t} |II|^2 \leq \sin\left(k(\frac{1}{2^n})^l\right) \cdot \frac{y(0)}{1 - y(0)K_1 t} \to 0, \quad t \to \infty.$$

The desired claim is proved. So up to proofs of Lemma 4.2 and Claim 4.3, we have proved that the flow converges to a totally geodesic Lagrangian submanifold at infinity.

Proof of Claim 4.3. Fix the positive numbers k, l satisfying (4.1)–(4.3) in Lemma 4.2. By (4.3) we have

$$\frac{\pi}{2} > k \cdot \left(\frac{1}{2^n}\right)^l \geq k \cdot (*\Omega)^l \geq k \cdot \left(\frac{1}{(\Lambda + \frac{1}{\Lambda})^n}\right)^l \geq \sqrt{(\sqrt{21} - 3)/2}$$

because $*\Omega \in [\frac{1}{(\Lambda+\frac{1}{\Lambda})^n}, \frac{1}{2^n}]$. Note that

$$\sqrt{(\sqrt{21} - 3)/2} = \inf\left\{x(x + \frac{1}{3}x^3) \geq 1 \;\middle|\; 0 < x < \pi/2\right\} \approx 0.8895436175241$$

sits in $[\frac{\pi}{3.5317}, \frac{\pi}{3.5316}]$. By Lemma 4.1 (the Djokovic inequality) we get

$$k\cdot(*\Omega)^l(\tan(k(*\Omega)^l)) > k\cdot(*\Omega)^l(k\cdot(*\Omega)^l+\frac{1}{3}(k\cdot(*\Omega)^l)^3)\geq 1>\frac{l-1}{l},$$

that is, the first inequality in (4.10). Similarly, the second inequality in (4.10) follows from (4.2). Claim 4.3 is proved. □

Proof of Lemma 4.2. For conveniences we set $\tau := \tau(\Lambda) = \Lambda + \frac{1}{\Lambda}$, which is larger than 2 because $\Lambda > 1$. Since $\frac{\pi}{2} > \sqrt{(\sqrt{21}-3)/2}$ we may fix a small $\epsilon > 0$ such that

$$\frac{\pi}{2} > \frac{\pi}{2} - \epsilon > \sqrt{(\sqrt{21}-3)/2}.$$

Set $\alpha = \frac{\pi}{2} - \epsilon$. Then (4.1) holds for any

$$l \leq \frac{\ln\left(\alpha/\sqrt{(\sqrt{21}-3)/2}\right)}{n\ln\frac{\tau}{2}}. \tag{4.13}$$

More precisely, such a l satisfies

$$\alpha\cdot 2^{nl} \geq \sqrt{(\sqrt{21}-3)/2}\cdot\tau^{nl}.$$

Hence we can always take $k = k_l > 0$ such that

$$\sqrt{(\sqrt{21}-3)/2}\cdot\tau^{nl} \leq k \leq \alpha\cdot 2^{nl}$$

or equivalently

$$\frac{\pi}{2} > \alpha \geq k\cdot\left(\frac{1}{2^n}\right)^l > k\cdot\left(\frac{1}{(\Lambda+\frac{1}{\Lambda})^n}\right)^l \geq \sqrt{(\sqrt{21}-3)/2}.$$

By the Djokovic inequality

$$\frac{\tan\left(k(\frac{1}{2^n})^l\right)}{k\left(\frac{1}{2^n}\right)^l} \leq 1 + f(\alpha)\left(k\left(\frac{1}{2^n}\right)^l\right)^2$$

if $k\cdot(\frac{1}{2^n})^l \leq \alpha$. So (4.2) holds if $k > 0$ and $l > 0$ are chosen to satisfy

$$\frac{l\delta_{\Lambda_1}}{10} \geq 1 + f(\alpha)\alpha^2 \geq 1 + f(\alpha)\left(k\left(\frac{1}{2^n}\right)^l\right)^2$$

or equivalently

$$l \geq \frac{10}{\delta_{\Lambda_1}} \cdot \left(1 + f(\alpha)\alpha^2\right). \tag{4.14}$$

Hence we can take $l > 0$ to satisfy (4.13) and (4.14) if

$$\frac{\ln\left(\alpha/\sqrt{(\sqrt{21}-3)/2}\right)}{n \ln \frac{\tau}{2}} \geq \frac{10}{\delta_{\Lambda_1}} \cdot \left(1 + f(\alpha)\alpha^2\right). \tag{4.15}$$

Since the function

$$(1, \infty) \to \mathbb{R}, \ \Lambda \mapsto \Lambda + \frac{1}{\Lambda}$$

is strictly increasing, $\log \frac{\tau}{2} \to 0^+$ as $\Lambda \to 1^+$. Hence for a given

$$\frac{\pi}{2} > \alpha > \sqrt{(\sqrt{21}-3)/2},$$

there exists the largest $\Lambda_1^{(\alpha)} > 1$ such that (4.15) holds for $\tau = \tau_\alpha = \Lambda_1^{(\alpha)} + 1/\Lambda_1^{(\alpha)}$, i.e.

$$g(\alpha) := \frac{\alpha \ln\left(\alpha/\sqrt{(\sqrt{21}-3)/2}\right)}{\tan \alpha} \geq \frac{10n}{\delta_{\Lambda_1}} \cdot \ln \frac{\tau_\alpha}{2}. \tag{4.16}$$

Of course, (4.16) also holds for for every $\tau = \Lambda + \frac{1}{\Lambda}$ with $\Lambda \in (1, \Lambda_1^{(\alpha)})$. Then

$$\widehat{\Lambda}_1 = \sup\left\{\Lambda_1^{(\alpha)} \;\middle|\; \sqrt{(\sqrt{21}-3)/2} < \alpha < \frac{\pi}{2} \text{ and (4.15) holds for } \tau = \tau_\alpha\right\}$$

satisfies the desired condition. In Appendix A we shall prove

Claim 4.4. *There exists a unique* $\alpha_0 \in (\sqrt{(\sqrt{21}-3)/2}, \frac{\pi}{2})$ *such that*

$$g(\alpha_0) = \sup\left\{g(\alpha) \;\middle|\; \sqrt{(\sqrt{21}-3)/2} < \alpha < \frac{\pi}{2}\right\}.$$

Moreover $\alpha_0 \approx 1.238756$ *and* $g(\alpha_0) \approx 0.141446$.

Hence $\widehat{\Lambda}_1 \geq \Lambda_1^{(\alpha_0)}$, where $\Lambda_1^{(\alpha_0)}$ is determined by

$$g(\alpha_0) = \frac{10n}{\delta_{\Lambda_1}} \cdot \left[\ln \left(\Lambda_1^{(\alpha_0)} + \frac{1}{\Lambda_1^{(\alpha_0)}} \right) - \ln 2 \right],$$

or more precisely

$$\Lambda_1^{(\alpha_0)} = \left(2\exp\left(\tfrac{g(\alpha_0)\delta_{\Lambda_1}}{5n}\right) + 2\exp\left(\tfrac{g(\alpha_0)\delta_{\Lambda_1}}{10n}\right)\sqrt{\exp\left(\tfrac{g(\alpha_0)\delta_{\Lambda_1}}{5n}\right) - 1} - 1 \right)^{\frac{1}{2}}$$
$$\approx \left(2\exp\left(\tfrac{0.141446\delta_{\Lambda_1}}{5n}\right) + 2\exp\left(\tfrac{0.141446\delta_{\Lambda_1}}{10n}\right)\sqrt{\exp\left(\tfrac{0.141446\delta_{\Lambda_1}}{5n}\right) - 1} - 1 \right)^{\frac{1}{2}}.$$

This completes the proof of Lemma 4.2. □

In summary the proof of Theorem 1.3 is complete.

5. A concluding remark

Carefully checking the proofs of Theorems 1.1, 1.2 we find that our real $2n$-dimensional compact Kähler-Einstein manifolds (M, ω, J, g) all satisfy the following three conditions (**A**), (**B**) and (**C**):

(A) The curvature tensor R is constant on subbundle

$$\{(X, JX, Y, JY) \,|\, g(X, Y) = 0,\ g(X, JY) = 0,\ g(X, X) = 1 = g(Y, Y)\}.$$

In other words, for any $p, q \in M$ and any unit orthogonal bases of (T_pM, J_p, g_p) and (T_qM, J_q, g_q), $\{a_1, \ldots, a_{2n}\}$ and $\{a'_1, \ldots, a'_{2n}\}$ with $a_{2k} = J_p a_{2k-1}$ and $a'_{2k} = J_q a'_{2k-1}$, $k = 1, \ldots, n$, it holds that

$$R(a_i, a_k, a_i, a_k) = R(a'_i, a'_k, a'_i, a'_k) \quad \forall 1 \leq i, k \leq 2n.$$

If (M, ω, J, g) is also homogeneous, this is equivalent to the following weaker

(A') For any $p \in M$ and any unit orthogonal bases of (T_pM, J_p, g_p), $\{a_1, \ldots, a_{2n}\}$ and $\{a'_1, \ldots, a'_{2n}\}$ with $a_{2k} = J_p a_{2k-1}$ and $a'_{2k} = J_q a'_{2k-1}$, $k = 1, \ldots, n$, it holds that $R(a_i, a_k, a_i, a_k) = R(a'_i, a'_k, a'_i, a'_k)$ for all $1 \leq i, k \leq 2n$.

(B) $\mathrm{Re}(R(X, \overline{Y}, X, \overline{Y})) \leq 0$ for any $X, Y \in T^{(1,0)}M$.

(C) The holomorphic sectional curvature is positive, i.e. $\exists\, c_0 > 0$ such that

$$\begin{aligned} &R(u, Ju, u, Ju)\\ &= -4R\Big(\frac{u-\sqrt{-1}Ju}{2}, \frac{u+\sqrt{-1}Ju}{2}, \frac{u-\sqrt{-1}Ju}{2}, \frac{u+\sqrt{-1}Ju}{2}\Big) \geq c_0 \end{aligned}$$

for any unit vector $u \in TM$.

By Propositions 2.1, 2.2 and Corollaries 2.4, 2.5, the manifolds $(G(n, n + m; \mathbb{C}), h)$, and $(G^{\mathrm{II}}(n, 2n), h_{\mathrm{II}})$ and $(G^{\mathrm{III}}(n, 2n), h_{\mathrm{III}})$ satisfy these conditions. On the other hand, from (2.14) we see that $(G^{\mathrm{IV}}(1, n + 1), h_{\mathrm{IV}})$ does not satisfy the condition (**B**) though the condition (**C**) holds for it. Actually, in addition to irreducible Hermitian symmetric spaces of compact type, there also exist countably Kähler C-spaces associated with a complex simple Lie algebra of classical type that have positive holomorphic sectional curvature.

We may obtain the following theorem, which generalizes Theorems 1.1 and 1.2, but partially contains 1.3.

Theorem 5.1. *Let (M, ω, J, g) and $(\widetilde{M}, \tilde{\omega}, \tilde{J}, \tilde{g})$ be two real $2n$-dimensional compact Kähler-Einstein manifolds satisfying the above conditions (**A**) and (**B**). Then for any Λ-pinched symplectomorphism $\varphi : (M, \omega) \to (\widetilde{M}, \widetilde{\omega})$ with $\Lambda \in [1, \Lambda_1(n)] \setminus \{\infty\}$, where $\Lambda_1(n)$ is given by (1.1), the following conclusions hold:*

(i) *The mean curvature flow Σ_t of the graph of φ in $M \times \widetilde{M}$ exists smoothly for all $t > 0$.*

(ii) *Σ_t is the graph of a symplectomorphism φ_t for each $t > 0$, and φ_t is Λ'_n-pinched along the mean curvature flow, where Λ'_n is defined by (1.2).*

(iii) *If $\Lambda < \widehat{\Lambda}_1$ for some $\Lambda_1 \in (\Lambda, \Lambda_1(n)] \setminus \{\infty\}$, where $\widehat{\Lambda}_1 > 1$ is a constant determined by Λ_1 and n (see Lemma 4.2), then the flow converges to a Lagrangian submanifold of $M \times \widetilde{M}$ as $t \to \infty$.*

(iv) *The flow converges to a totally geodesic Lagrangian submanifold of $M \times \widetilde{M}$ and φ_t converges smoothly to a biholomorphic isometry from M to $\widetilde{M}$ as $t \to \infty$ provided additionally that (M, ω, J, g) and $(\widetilde{M}, \tilde{\omega}, \tilde{J}, \tilde{g})$ satisfy the condition (**C**). Consequently, the symplectomorphism $\varphi : M \to \widetilde{M}$ is symplectically isotopic to a biholomorphic isometry.*

In order to prove it we start with two simple lemmas.

Lemma 5.2. *Let R be the curvature tensor of a Kähler manifold (M,g,J,ω) of real dimension $2N$. For any local holomorphic coordinate system $(z^1,\ldots,z^n)$*

on it, let $R_{r\bar{s}r\bar{s}} = R(\frac{\partial}{\partial z^r}, \frac{\partial}{\partial \bar{z}^s}, \frac{\partial}{\partial z^r}, \frac{\partial}{\partial \bar{z}^s})$ *and* $z^s = x^s + \sqrt{-1}y^s$, $s = 1, \dots, n$. *Then*

$$R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}\right) - R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}\right) = -4\mathrm{Re}(R_{r\bar{s}r\bar{s}})$$

for all r, s. *In particular, we have*

$$R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^s}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^s}\right) = -4R_{s\bar{s}s\bar{s}} \quad \forall s,$$

i.e., the holomorphic sectional curvature in the direction $\frac{\partial}{\partial x^s}$ *is given by*

$$H\left(\frac{\partial}{\partial x^s}\right) = -\frac{4R_{s\bar{s}s\bar{s}}}{[g(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^s})]^2}.$$

Proof. Since the only possible non-vanishing terms of the curvature components are of the form $R_{i\bar{j}k\bar{l}}$ and those obtained from the universal symmetries of the curvature tensor, it is not hard to prove that

$$\begin{aligned} &R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}\right) \\ &= R\left(\frac{\partial}{\partial z^s}+\frac{\partial}{\partial \bar{z}^s}, \sqrt{-1}\left(\frac{\partial}{\partial z^r}-\frac{\partial}{\partial \bar{z}^r}\right), \frac{\partial}{\partial z^s}+\frac{\partial}{\partial \bar{z}^s}, \sqrt{-1}\left(\frac{\partial}{\partial z^r}-\frac{\partial}{\partial \bar{z}^r}\right)\right) \\ &= -(R_{r\bar{s}r\bar{s}} + R_{s\bar{r}s\bar{r}} + R_{r\bar{s}s\bar{r}} + R_{s\bar{r}r\bar{s}}) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}\right) \\ &= R\left(\frac{\partial}{\partial z^s} + \frac{\partial}{\partial \bar{z}^s}, \frac{\partial}{\partial z^r} + \frac{\partial}{\partial \bar{z}^r}, \frac{\partial}{\partial z^s} + \frac{\partial}{\partial \bar{z}^s}, \frac{\partial}{\partial z^r} + \frac{\partial}{\partial \bar{z}^r}\right) \\ &= R_{r\bar{s}r\bar{s}} + R_{s\bar{r}s\bar{r}} - R_{r\bar{s}s\bar{r}} - R_{s\bar{r}r\bar{s}}. \end{aligned} \tag{5.2}$$

Note that $R_{r\bar{s}r\bar{s}} = R_{\bar{s}r\bar{s}r} = \overline{R_{s\bar{r}s\bar{r}}}$. It follows from this and (5.1)–(5.2) that

$$\begin{aligned} &R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}\right) - R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}\right) \\ &= -2R_{r\bar{s}r\bar{s}} - 2R_{s\bar{r}s\bar{r}} = -4\mathrm{Re}(R_{r\bar{s}r\bar{s}}). \end{aligned}$$

The second equality may be derived from (5.1) directly. Lemma 5.2 is proved. $\square$

Lemma 5.3. *Under the assumptions of Lemma 5.2, if (M, g, J, ω) also satisfies the condition* (**B**)*, then* $\mathrm{Re}(R_{r\bar{s}r\bar{s}}) \leq 0$ *for all* $1 \leq r, s \leq n$.

Proof. Set $X = \sum_{i=1}^n u_i \frac{\partial}{\partial z^i}$ and $Y = \sum_{j=1}^n v_j \frac{\partial}{\partial z^j}$ with $u_i, v_j \in \mathbb{C}$. Then

$$\begin{aligned} R(X, \overline{Y}, X, \overline{Y}) &= \sum_{i,j,k,l=1}^n R\left(u_i \frac{\partial}{\partial z^i}, \overline{v}_j \frac{\partial}{\partial \overline{z}^j}, u_k \frac{\partial}{\partial z^k}, \overline{v}_l \frac{\partial}{\partial \overline{z}^l}\right) \\ &= \sum_{i,j,k,l=1}^n u_i u_k \overline{v}_j \overline{v}_l R_{i\bar{j}k\bar{l}} \end{aligned}$$

and

$$\begin{aligned} R(Y, \overline{X}, Y, \overline{X}) &= \sum_{i,j,k,l=1}^n R\left(v_j \frac{\partial}{\partial z^j}, \overline{u}_i \frac{\partial}{\partial \overline{z}^i}, v_l \frac{\partial}{\partial z^l}, \overline{u}_k \frac{\partial}{\partial \overline{z}^k}\right) \\ &= \sum_{i,j,k,l=1}^n v_j v_l \overline{u}_i \overline{u}_k R_{j\bar{i}l\bar{k}}. \end{aligned}$$

Since $R_{j\bar{i}l\bar{k}} = \overline{R_{i\bar{j}k\bar{l}}}$ we get

$$\begin{aligned} & R(X, \overline{Y}, X, \overline{Y}) + R(Y, \overline{X}, Y, \overline{X}) \\ = & \sum_{i,j,k,l=1}^n \left(u_i u_k \overline{v}_j \overline{v}_l R_{i\bar{j}k\bar{l}} + \overline{u_i u_k \overline{v}_j \overline{v}_l R_{i\bar{j}k\bar{l}}}\right) \\ = & R(X, \overline{Y}, X, \overline{Y}) + \overline{R(X, \overline{Y}, X, \overline{Y})} \\ = & 2\mathrm{Re}(R(X, \overline{Y}, X, \overline{Y})). \end{aligned}$$

Taking $X = \frac{\partial}{\partial z^r}$, $Y = \frac{\partial}{\partial z^s}$, the desired results are obtained. □

The following proposition implies Theorem 5.1(i) and (ii).

Proposition 5.4. *Let (M, ω, J, g) be a real $2n$-dimensional compact Kähler-Einstein manifold satisfying the conditions* (**A**) *and* (**B**)*. Then for any symplectomorphism $\varphi : M \to \widetilde{M}$ it holds that*

$$\frac{d}{dt} * \Omega \geq \Delta * \Omega + *\Omega \cdot Q(\lambda_i, h_{jkl}), \tag{5.3}$$

along the mean curvature flow Σ_t of the graph Σ of φ. Furthermore, if φ is Λ-pinched for some $\Lambda \in (1, \Lambda_1(n))$, then the symplectomorphism $\varphi_t : M \to \widetilde{M}$, *whose graph is* Σ_t*, is* Λ'_n*-pinched and*

$$\frac{d}{dt} * \Omega \geq \Delta * \Omega + \delta_\Lambda \cdot *\Omega |II|^2 \tag{5.4}$$

along the mean curvature flow. In particular, $\min_{\Sigma_t} *\Omega$ *is nondecreasing as a function in* t*.*

Proof. By $(\mathbf{A})$, $R_{ikik} = \widetilde{R}_{ikik}$ $\forall i, k$. Hence the second term in the big bracket of (3.4) can be written as follows (omitting $|_p$ in $\frac{\partial}{\partial x^t}|_p$ and $\frac{\partial}{\partial y^t}|_p$),

$$\begin{aligned}
&\sum_k \sum_{i\neq k} \frac{\lambda_i (R_{ikik} - \lambda_k^2 \widetilde{R}_{ikik})}{(1+\lambda_k^2)(\lambda_i + \lambda_{i'})} = \sum_k \sum_{i \neq k} \frac{\lambda_i (1-\lambda_k^2) R_{ikik}}{(1+\lambda_k^2)(\lambda_i + \lambda_{i'})} \\
&= \sum_{k=2r-1, i=2s-1, r\neq s} \frac{\lambda_{2s-1}(1-\lambda_{2r-1}^2) R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}\right)}{(1+\lambda_{2r-1}^2)(\lambda_{2s-1}+\lambda_{2s})} \\
&\quad + \sum_{k=2r-1, i=2s} \frac{\lambda_{2s}(1-\lambda_{2r-1}^2) R\left(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial y^s}, \frac{\partial}{\partial x^r}\right)}{(1+\lambda_{2r-1}^2)(\lambda_{2s-1}+\lambda_{2s})} \\
&\quad + \sum_{k=2r, i=2s-1} \frac{\lambda_{2s-1}(1-\lambda_{2r}^2) R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}\right)}{(1+\lambda_{2r}^2)(\lambda_{2s-1}+\lambda_{2s})} \\
&\quad + \sum_{k=2r, i=2s, r\neq s} \frac{\lambda_{2s}(1-\lambda_{2r}^2) R\left(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^r}\right)}{(1+\lambda_{2r}^2)(\lambda_{2s-1}+\lambda_{2s})} \\
&= \sum_{r\neq s} \frac{R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}\right)}{(\lambda_{2s-1}+\lambda_{2s})} \left[\frac{\lambda_{2s-1}(1-\lambda_{2r-1}^2)}{(1+\lambda_{2r-1}^2)} + \frac{\lambda_{2s}(1-\lambda_{2r}^2)}{(1+\lambda_{2r}^2)}\right] \\
&\quad + \sum_{r,s} \frac{R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}\right)(\lambda_{2r}^2 - 1)(\lambda_{2s} - \lambda_{2s-1})}{(\lambda_{2s-1}+\lambda_{2s})(1+\lambda_{2r}^2)} \\
&= \sum_{r,s} \frac{(\lambda_{2r}^2-1)(\lambda_{2s}-\lambda_{2s-1})}{(\lambda_{2s-1}+\lambda_{2s})(1+\lambda_{2r}^2)} \left[R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}\right) \right. \\
&\qquad\qquad \left. - R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}\right)\right] \\
&= \sum_{r,s} \frac{(\lambda_{2r}^2-1)(\lambda_{2s}^2-1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)} \left[R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^r}\right) \right. \\
&\qquad\qquad \left. - R\left(\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r}\right)\right]
\end{aligned} \tag{5.5}$$

$$
\begin{aligned}
&= \sum_{r\neq s}\frac{(\lambda_{2r}^2-1)(\lambda_{2s}^2-1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)}\left[R\left(\frac{\partial}{\partial x^s},\frac{\partial}{\partial y^r},\frac{\partial}{\partial x^s},\frac{\partial}{\partial y^r}\right)\right.\\
&\qquad\qquad\left.-R(\frac{\partial}{\partial x^s},\frac{\partial}{\partial x^r},\frac{\partial}{\partial x^s},\frac{\partial}{\partial x^r})\right]\\
&\quad+\sum_{r=s}\frac{(\lambda_{2r}^2-1)(\lambda_{2s}^2-1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)}\left[R\left(\frac{\partial}{\partial x^s},\frac{\partial}{\partial y^r},\frac{\partial}{\partial x^s},\frac{\partial}{\partial y^r}\right)\right.\\
&\qquad\qquad\left.-R\left(\frac{\partial}{\partial x^s},\frac{\partial}{\partial x^r},\frac{\partial}{\partial x^s},\frac{\partial}{\partial x^r}\right)\right]\\
&= \sum_{r\neq s}\frac{(\lambda_{2r}^2-1)(\lambda_{2s}^2-1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)}[-4\mathrm{Re}(R_{r\bar{s}r\bar{s}})]\\
&\quad+\sum_{r=s}\frac{(\lambda_{2r}^2-1)(\lambda_{2s}^2-1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)}[-4\mathrm{Re}(R_{s\bar{s}s\bar{s}})]\geq 0
\end{aligned}
$$

because of Lemmas 5.2, 5.3 and our choice that $\lambda_{2i-1}\leq 1\leq\lambda_{2i}, i=1,\dots,n$. This leads to (5.3).

Now if φ is Λ-pinched, then $\frac{1}{\Lambda}\leq\lambda_i(0)\leq\Lambda$ for $i=1,\dots,2n$. Since $\Lambda_1(n)<\Lambda_0(n)$ in the case $\Lambda_0(n)<\infty$, by Proposition 3.4 we get

$$
Q(\lambda_i(0),h_{jkl})\geq\delta_\Lambda\sum_{ijk}h_{jkl}^2
$$

and hence $\left(\frac{d}{dt}-\triangle\right)*\Omega\geq 0$ at $t=0$. Note that Lemma 5 of [MeWa] implies that $\frac{1}{2^n}-\epsilon(n,\Lambda)\leq *\Omega$ at $t=0$, where $\epsilon(n,\Lambda)=\frac{1}{2^n}-\frac{1}{(\Lambda+\frac{1}{\Lambda})^n}$. Then repeating the proof of Proposition 4 and Corollary 5 in [MeWa] we may get (5.4). $\square$

Using this proposition we may prove the long-time existence in Theorem 5.1 (i) as in [MeWa, §3.3] (or that of Theorem 1.1).

The proof of Theorem 5.1(iii). The idea is similar to that of Theorem 1.3. All arguments from the beginning of Section 4.2.2 to (4.6) in the proof of convergence in Theorem 1.3 are still valid. Then there exists a positive number K_2 depending on the manifolds M and $\widetilde{M}$ such that

$$
\sum_{s,i,j}\left(\sum_k[(\overline{\nabla}_{\partial_k}\overline{R})_{\underline{s}ijk}+(\overline{\nabla}_{\partial_j}\overline{R})_{\underline{s}kik})]\right)^2\leq K_2
$$

and hence

$$\sum_{s,i,j,k} 2\left[(\overline{\nabla}_{\partial_k}\overline{R})_{\underline{s}ijk} + (\overline{\nabla}_{\partial_j}\overline{R})_{\underline{s}kik})\right] h_{sij} \le K_2 + |II|^2.$$

As there it follows from the boundedness of the curvature that

$$\frac{d}{dt}|II|^2 \le \Delta|II|^2 - 2|\nabla II|^2 + 10|II|^4 + K_1|II|^2 + K_2, \tag{5.6}$$

where K_1 is a nonnegative constant that depends on the dimensions of M and $\widetilde{M}$. With the same proof we may get the corresponding result of (4.9), i.e.

$$\begin{aligned}
&\left(\frac{d}{dt} - \Delta\right)\left(\frac{|II|^2}{\sin(k(*\Omega)^l)}\right) \\
&\le \left(\frac{|II|^2}{\sin(k(*\Omega)^l)}\right)^2 \cdot \left[10 \cdot \sin(k(*\Omega)^l) - k \cdot l \cdot \delta_{\Lambda_1} \cdot (*\Omega)^l \cdot \cos(k(*\Omega)^l)\right] \\
&\quad + \frac{k \cdot l \cdot (*\Omega)^{l-2}|II|^2|\nabla * \Omega|^2}{[\sin(k(*\Omega)^l)]^3}\left[-k \cdot l \cdot (*\Omega)^l \cdot (\sin(k(*\Omega)^l))^2\right. \\
&\qquad\qquad \left. + (l-1)\cos(k(*\Omega)^l)\sin(k(*\Omega)^l)\right] \\
&\quad + K_1\frac{|II|^2}{\sin(k(*\Omega)^l)} + \frac{K_2}{\sin(k(*\Omega)^l)}.
\end{aligned} \tag{5.7}$$

By Claim 4.3, it follows from (5.7) that

$$\begin{aligned}
&\left(\frac{d}{dt} - \Delta\right)\left(\frac{|II|^2}{\sin(k(*\Omega)^l)}\right) \\
&\le \left(\frac{|II|^2}{\sin(k(*\Omega)^l)}\right)^2 \cdot \left[10 \cdot \sin(k(*\Omega)^l) - k \cdot l \cdot \delta_{\Lambda_1} \cdot (*\Omega)^l \cdot \cos(k(*\Omega)^l)\right] \\
&\quad + K_1\frac{|II|^2}{\sin(k(*\Omega)^l)} + \frac{K_2}{\sin(k(*\Omega)^l)}.
\end{aligned}$$

Let $g = \frac{|II|^2}{\sin(k(*\Omega)^l)}$, $K_4 := \max \frac{K_2}{\sin(k(*\Omega)^l)} = \frac{K_2}{\sin\left(k(\frac{1}{(\Lambda+\frac{1}{\Lambda})^n})^l\right)}$ and

$$K_3 := \max_{*\Omega \in \left[\frac{1}{(\Lambda+\frac{1}{\Lambda})^n}, \frac{1}{2^n}\right]} \left[10 \cdot \sin(k(*\Omega)^l) - k \cdot l \cdot \delta_{\Lambda_1} \cdot (*\Omega)^l \cdot \cos(k(*\Omega)^l)\right].$$

By Claim 4.3, $K_3 < 0$ and

$$\left(\frac{d}{dt} - \Delta\right) g \leq K_3 \cdot g^2 + K_1 \cdot g + K_4. \tag{5.8}$$

Consider the initial value problem

$$\frac{d}{dt} y = K_3 \cdot y^2 + K_1 \cdot y + K_4 \quad \text{and} \quad y(0) = \max_{\Sigma_0} \mathrm{g}. \tag{5.9}$$

If $y(0) > \frac{-K_1 - \sqrt{K_1^2 - 4K_3K_4}}{2K_3}$, the unique solution of (5.9) is given by

$$y(t) = \frac{(K_1 + \sqrt{K_1^2 - 4K_3K_4}) \cdot \exp(\sqrt{K_1^2 - 4K_3K_4}t + K_5) - K_1 + \sqrt{K_1^2 - 4K_3K_4}}{-2K_3 \cdot [\exp(\sqrt{K_1^2 - 4K_3K_4}t + K_5) - 1]},$$

where $K_5 = \ln \frac{2K_3y(0) + K_1 - \sqrt{K_1^2 - 4K_3K_4}}{2K_3y(0) + K_1 + \sqrt{K_1^2 - 4K_3K_4}}$. Clearly, $y(t) \to \frac{K_1 + \sqrt{K_1^2 - 4K_3K_4}}{-2K_3}$ as $t \to \infty$.

If $y(0) = \frac{-K_1 - \sqrt{K_1^2 - 4K_3K_4}}{2K_3}$, then $y(t) \equiv \frac{-K_1 - \sqrt{K_1^2 - 4K_3K_4}}{2K_3}$.

If $y(0) < \frac{-K_1 - \sqrt{K_1^2 - 4K_3K_4}}{2K_3}$, then there exists a $T > 0$ such that on $[0, T]$ we have $y(t) - \frac{-K_1 - \sqrt{K_1^2 - 4K_3K_4}}{2K_3} \leq 0$, and therefore

$$\left(y(t) + \frac{K_1}{2K_3}\right)^2 = -\exp(\sqrt{K_1^2 - 4K_3K_4}t + K_5) + \frac{K_1^2 - 4K_3K_4}{4K_3^2} \geq 0$$

where $K_5 = \ln\left(-y(0)^2 - \frac{K_1 y(0)}{K_3} - \frac{K_4}{K_3}\right)$. It follows that

$$T = \frac{\ln(\frac{K_1^2 - 4K_3K_4}{4K_3^2}) - K_5}{\sqrt{K_1^2 - 4K_3K_4}} \geq 0, \quad y(T) = -\frac{K_1}{2K_3} < \frac{-K_1 - \sqrt{K_1^2 - 4K_3K_4}}{2K_3}.$$

Hence we can continue this procedure and get

$$\left(y(t) + \frac{K_1}{2K_3}\right)^2 = -\exp(\sqrt{K_1^2 - 4K_3K_4}t + K_5) + \frac{K_1^2 - 4K_3K_4}{4K_3^2} \geq 0$$

for all time $t \geq 0$. From this we derive

$$\begin{aligned} y(t) &= -\frac{K_1}{2K_3} + \sqrt{-\exp(\sqrt{K_1^2 - 4K_3K_4}t + K_5) + \frac{K_1^2 - 4K_3K_4}{4K_3^2}} \\ &\leq -\frac{K_1}{2K_3} + \sqrt{\frac{K_1^2 - 4K_3K_4}{4K_3^2}} = \frac{-K_1 - \sqrt{K_1^2 - 4K_3K_4}}{2K_3} \end{aligned}$$

if $-\frac{K_1}{2K_3} \le y(0) < \frac{-K_1-\sqrt{K_1^2-4K_3K_4}}{2K_3}$, and

$$\begin{aligned} y(t) &= -\frac{K_1}{2K_3} - \sqrt{-\exp\left(\sqrt{K_1^2-4K_3K_4}t+K_5\right) + \frac{K_1^2-4K_3K_4}{4K_3^2}} \\ &\le -\frac{K_1}{2K_3} \end{aligned}$$

if $0 \le y(0) < -\frac{K_1}{2K_3}$.

By (5.8)–(5.9) the comparison principle for parabolic equations yields

$$g = \frac{|II|^2}{\sin(k(*\Omega)^l)} \le y(t) \quad \forall t > 0.$$

Since (4.3) implies that the function

$$\left[\frac{1}{(\Lambda+\frac{1}{\Lambda})^n}, \frac{1}{2^n}\right] \ni *\Omega \to \sin(k(*\Omega)^l)$$

is bounded away from zero, we derive

$$\max_{\Sigma_t} |II|^2 \le \sin\left(k\left(\frac{1}{2^n}\right)^l\right) \cdot y(t) \le \sin\left(k(\frac{1}{2^n})^l\right) \cdot L,$$

where $L = \frac{-K_1-\sqrt{K_1^2-4K_3K_4}}{2K_3}$ if $y(0) \ge -\frac{K_1}{2K_3}$, and $L = -\frac{K_1}{2K_3}$ if $0 \le y(0) < -\frac{K_1}{2K_3}$. Hence $|II|^2$ is uniformly bounded. Namely, we have proved that the flow converges to a Lagrangian submanifold at infinity provided that the flow exists for all the time. (*Note*: Different from the case of tori we cannot prove $\max_{\Sigma_t} |II|^2 \to 0$ as $t \to \infty$, and hence cannot assert that the limit submanifold is totally geodesic.)

The proof of Theorem 5.1(iv). The idea is similar to that of Theorem 1.1. In the present case we have the following

Proposition 5.5. *Under the assumptions of Proposition 5.4, suppose further that (M, ω, J, g) also satisfies the condition* (**C**)*. Then along the mean curvature flow a similar inequality to that of Proposition 3.5 holds, i.e.*

$$\frac{d}{dt} * \Omega \ge \Delta * \Omega + \delta_\Lambda \cdot *\Omega |II|^2 + c_0 \cdot *\Omega \sum_{k \text{ odd}} \frac{(1-\lambda_k^2)^2}{(1+\lambda_k^2)^2}.$$

Proof. Under the further assumption, by (5.5) we have

$$\begin{aligned}
&\sum_k \sum_{i\neq k} \frac{\lambda_i(R_{ikik} - \lambda_k^2 \widetilde{R}_{ikik})}{(1+\lambda_k^2)(\lambda_i + \lambda_{i'})} \\
&= \sum_{r\neq s} \frac{(\lambda_{2r}^2 - 1)(\lambda_{2s}^2 - 1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)} [-4\mathrm{Re}(R_{r\bar{s}r\bar{s}})] \\
&\quad + \sum_{r=s} \frac{(\lambda_{2r}^2 - 1)(\lambda_{2s}^2 - 1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)} [-4\mathrm{Re}(R_{s\bar{s}s\bar{s}})] \geq c_0 \sum_{r=s} \frac{(\lambda_{2r}^2 - 1)(\lambda_{2s}^2 - 1)}{(1+\lambda_{2s}^2)(1+\lambda_{2r}^2)}.
\end{aligned}$$

This and Propositions 3.3, 3.4 give the desired inequality. □

As in [MeWa], using this we may prove that $\lambda_i \to 1$ and $\max_{\Sigma_t} |II|^2 \to 0$ as $t \to \infty$, and hence that the flow converges to a totally geodesic Lagrangian submanifold of $M \times \widetilde{M}$ as $t \to \infty$ and that φ_t converges smoothly to a biholomorphic isometry $\varphi_\infty : M \to \widetilde{M}$. Theorem 5.1 is proved. □

A theorem by Matsushima and Borel-Remmert claimed that every compact homogeneous Kähler manifold is the Kähler product of a flat complex torus (known as the *Albanese* torus of (M, J)) and a Kähler C-space (cf. [Be, Theorem 8.97]). As a consequence, a compact homogeneous Kähler manifold admits a Kähler-Einstein structure if and only if it is a complex torus or is simply-connected. If we restrict the manifolds in Theorem 5.1 to homogeneous Kähler-Einstein manifolds, then Theorem 5.1 has sense only for simply-connected case (because the better result has been obtained for complex tori).

Appendix A. Proof of Claim 4.4

For simplicity write $L := \sqrt{(\sqrt{21} - 3)/2}$. Then the function $g(\alpha)$ in (4.16) is equal to $\alpha \ln(\alpha/L)/\tan\alpha$. A direct computation yields

$$g'(\alpha) = \frac{1}{(\sin\alpha)^2}\big[\sin\alpha \cdot \cos\alpha \cdot \ln(\alpha/L) + \sin\alpha \cdot \cos\alpha - \alpha \ln(\alpha/L)\big],$$

$$g''(\alpha) = \frac{1}{(\sin\alpha)^3}\left[\frac{(\sin\alpha)^2 \cdot \cos\alpha}{\alpha} + 2\alpha\cos\alpha \cdot \ln(\alpha/L) - 2\sin\alpha - 2\sin\alpha \cdot \ln(\alpha/L)\right]. \tag{A.1}$$

Clearly, $\lim_{\alpha\to\frac{\pi}{2}} g(\alpha) = 0 = g(L)$, and $g(\alpha) > 0$ on $(L, \frac{\pi}{2})$. Moreover,

$$g'(\frac{\pi}{2}) = -\frac{\pi}{2}\ln\big(\frac{\pi}{2L}\big) < 0 \quad \text{and} \quad g'(L) = \frac{1}{\tan L} > 0.$$

(Note that $L \approx 0.8895436175241$ sits between $\frac{\pi}{3.5317}$ and $\frac{\pi}{3.5316}$). Hence $g(\alpha)$ attains its maximum at some point $\alpha_0 \in (L, \pi/2)$ with $g'(\alpha_0) = 0$. Since any zero α of g' in $(L, \pi/2)$ satisfies the following equation

$$\sin\alpha \cdot \cos\alpha \cdot \ln(\alpha/L) + \sin\alpha \cdot \cos\alpha - \alpha\ln(\alpha/L) = 0,$$

plugging this into (A.1) we get

$$\begin{aligned}
g''(\alpha) &= \frac{1}{(\sin\alpha)^3}\Bigg[\frac{(\sin\alpha)^2\cdot\cos\alpha}{\alpha} + 2\alpha\cos\alpha\cdot\ln(\alpha/L) \\
&\qquad - 2\sin\alpha - 2\sin\alpha\cdot\ln(\alpha/L)\Bigg] \\
&= \frac{1}{(\sin\alpha)^3}\Bigg[\frac{(\sin\alpha)^2\cdot\cos\alpha}{\alpha} - 2\sin\alpha - 2\sin\alpha\cdot\ln(\alpha/L) \\
&\qquad + 2\cos\alpha\cdot(\sin\alpha\cdot\cos\alpha)(1+\ln(\alpha/L))\Bigg] \\
&= \frac{1}{(\sin\alpha)^3}\Bigg[\frac{(\sin\alpha)^2\cdot\cos\alpha}{\alpha} - 2(\sin\alpha)^3 - 2(\sin\alpha)^3\ln(\alpha/L)\Bigg] \\
&= \frac{1}{\sin\alpha}\Bigg[\frac{\cos\alpha}{\alpha} - 2\sin\alpha - 2\sin\alpha\cdot\ln(\alpha/L)\Bigg].
\end{aligned}$$

Observe that the function $u(\alpha) = \frac{\cos\alpha}{\alpha}$ is decreasing on $(L, \frac{\pi}{2})$ because of $u'(\alpha) = -\frac{\sin\alpha}{\alpha} - \frac{\cos\alpha}{\alpha^2} < 0$. From $L \approx 0.8895436175241$ we derive

$$\frac{\cos\alpha}{\alpha} - 2\sin\alpha - 2\sin\alpha\cdot\ln(\alpha/L) < \frac{\cos\alpha}{\alpha} - 2\sin\alpha \leq \frac{\cos L}{L} - 2\sin L < 0.$$

That is, $g''(\alpha) < 0$ for any zero α of g' in $(L, \frac{\pi}{2})$. It follows that each zero α of of g' in $(L, \frac{\pi}{2})$ is a local maximum point of g. This implies that g' has a unique zero α_0 in $(L, \frac{\pi}{2})$ and that

$$g(\alpha_0) = (\cos\alpha_0)^2\,(1 + \ln(\alpha_0/L)) = \frac{\alpha_0(\cos\alpha_0)^2}{\alpha_0 - \sin\alpha_0\cdot\cos\alpha_0}$$

is the maximum of g in $(L, \frac{\pi}{2})$. We can compute $\alpha_0 \approx 1.238756$ and $g(\alpha_0) \approx 0.141446$.

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