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A refinement of sutured Floer homology

Akram S. Alishahi and Eaman Eftekhary

We introduce a refinement of the Ozsváth-Szabó complex associated by Juhász [Ju1] to a balanced sutured manifold (X,τ). An algebra $\mathbb{A}_\tau$ is associated to the boundary of a sutured manifold. For a fixed class $\mathfrak{s}$ of a Spin^c structure over the manifold $\overline{X}$, which is obtained from X by filling out the sutures, the Ozsváth-Szabó chain complex $\mathrm{CF}(X,\tau,\mathfrak{s})$ is then defined as a chain complex with coefficients in $\mathbb{A}_\tau$ and filtered by the relative Spin^c classes in $\mathrm{Spin}^c(X,\tau)$. The filtered chain homotopy type of this chain complex is an invariant of (X,τ) and the Spin^c class $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$. The construction generalizes the construction of Juhász. It plays the role of $\mathrm{CF}^-(X,\mathfrak{s})$ when X is a closed three-manifold, and the role of

$$\mathrm{CFK}^-(Y,K;\mathfrak{s})=\bigoplus_{\underline{\mathfrak{s}}\in\mathfrak{s}}\mathrm{CFK}^-(Y,K,\underline{\mathfrak{s}}),$$

when the sutured manifold is obtained from a knot K inside a three-manifold Y. Our invariants thus generalize both the knot invariants of Ozsváth-Szabó and Rasmussen and the link invariants of Ozsváth and Szabó. We study some of the basic properties of the Ozsváth-Szabó complex corresponding to a balanced sutured manifold, including the behaviour under boundary connected sum, some form of stabilization for the complex, and an exact triangle generalizing the surgery exact triangles for knot Floer complexes.

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1. Introduction

1.1. Introduction and background

The introduction of Heegaard Floer homology by Ozsváth and Szabó ([OS5], [OS6]) for closed three dimensional manifolds around the beginning of the millennium resulted in very powerful tools for the study of various structures in low dimensional topology. In particular, invariants for knots (c.f. [OS1], [Ras1] and [Ef3]), for links [OS9], and for contact structures [OS11] were constructed using the fundamental idea of associating a chain complex to a pointed Heegaard diagram. Moreover, four-manifold invariants were constructed as some TQFT type homomorphisms between the homology groups of the chain complexes associated with the positive and negative boundary components [OS12]. The Ozsváth-Szabó complexes associated with a closed three-manifold come in different flavours. These are typically called *hat, minus, plus* and *infinity* modules. The other versions may be re-constructed from the *minus chain complex* if one also keeps track of the so called U-action.

Attempts towards extending the Ozsváth-Szabó invariants to three-manifolds with boundary, at least when the boundary is equipped with some extra structure, have been made through two different approaches. If a parametrization of the boundary surface is fixed, the three-manifold is called a bordered three manifold. Lipshitz, Ozsváth and Thurston generalize the hat version of the Ozsváth-Szabó complex for bordered three-manifold by first constructing a graded differential algebra corresponding to the parameterized boundary, and then associating the bordered Floer modules of types A and D to the bordered manifold, which are respectively an $\mathcal{A}_\infty$ module and a module over the differential graded algebra (see [LOT1], [LOT2]). Gluing of bordered three-manifolds for constructing closed three-manifolds is translated to an appropriate tensor product construction on the corresponding bordered Floer modules.

In a different direction, if the boundary of a three-manifold X is decorated with a set τ of sutures, Juhász associates a complex, the so called *sutured Floer complex*, to the sutured manifold (X,τ) [Ju1], provided that (X,τ) is balanced. The complex generalizes the hat versions of the Ozsváth-Szabó complexes associated with closed three-manifolds and links inside three-manifolds. The theory of sutured manifolds was introduced in [Gab1] and developed in [Gab2] and [Gab3] by D. Gabai in order to study the existence of taut foliations on three-manifolds. Sutured manifolds are oriented three-manifolds with boundary, together with a set of oriented simple closed curves (the sutures) that divide the boundary into positive and negative parts. Gabai defined the so called *sutured manifold decomposition* which consists of cutting the manifold along a properly embedded oriented surface R and adding one side of R to the plus boundary and the other side to the minus boundary. He showed that a sutured manifold carries a taut foliation if and only if there is a sequence of decompositions that result in a product sutured manifold. Honda, Kazez, and Matić generalized the theory of sutured manifold decomposition for the study of tight contact structures on three-manifolds, and developed the convex decomposition theory [HKM1]. In addition to the introduction of the sutured Floer complex, Juhász described how his sutured Floer complex changes through sutured manifold decomposition [Ju2]. As a consequence, he showed that a sutured manifold (X,τ) is taut if and only if the sutured Floer homology group $\mathrm{SFH}(X,\tau)$ is non- trivial and X is irreducible. These results suggested a deep connection between sutured Floer theory of Juhász and the sutured manifold decomposition theory of Gabai, as well as the contact geometry of three-manifolds. Subsequent developments included the study of the sutured Floer polytope by Juhász [Ju3] and introduction of contact invariants for contact three-manifolds with convex boundary by Honda, Kazez and Matić [HKM2]. This last invariant generalizes the contact invariant of Ozsváth and Szabó for a closed contact three-manifold defined in [OS11].

1.2. Main results

In this paper, we extend the construction of Juhász and construct a *minus theory* associated with a sutured manifold, providing an answer to the first two questions in problem 1 from [Ju4]. Moreover, we present a connected sum formula for the sutured Floer complex, which provides partial answer to the third question in the aforementioned problem. More precisely, let (X,τ) be a sutured manifold, and assume that the set of sutures does not contain any toridal components. Let $\tau=\{\tau_1,\dots,\tau_\kappa\}$ be the set of sutures. We will denote $\partial X-\tau$ by $\mathfrak{R}(\tau)=\mathfrak{R}^+(\tau)\cup\mathfrak{R}^-(\tau)$, where $\mathfrak{R}^+(\tau)$ and $\mathfrak{R}^-(\tau)$ are the positive and the negative parts of the boundary, respectively. We will further assume that the Euler characteristics $\chi(\mathfrak{R}^+(\tau))$ and $\chi(\mathfrak{R}^-(\tau))$ agree. Such sutured manifolds will be called *weakly balanced*. Our assumptions for being balanced are thus weaker than those of Juhász [Ju1].

We first associate an algebra $\mathbb{A}=\mathbb{A}_\tau$ to the boundary of X as follows. Let us assume that

$$\mathfrak{R}^-(\tau)=\bigcup_{i=1}^{k}R_i^- \quad\text{and}\quad \mathfrak{R}^+(\tau)=\bigcup_{j=1}^{l}R_j^+,$$

where $\{R_i^-\}_i$ and $\{R_j^+\}_j$ are the connected components of $\mathfrak{R}^-(\tau)$ and $\mathfrak{R}^+(\tau)$, respectively. Let g_i^- denote the genus of R_i^- and g_j^+ denote the genus of R_j^+. Consider the elements

$$\mathfrak{u}_i^- := \prod_{\tau_j\subset\partial R_i^-} \mathfrak{u}_j, \quad i=1,\dots,k \quad \text{and} \quad \mathfrak{u}_i^+ := \prod_{\tau_j\subset\partial R_i^+} \mathfrak{u}_j, \quad i=1,\dots,l,$$

in the free $\mathbb{Z}$-algebra $\mathbb{Z}[\kappa] := \langle \mathfrak{u}_1,\dots,\mathfrak{u}_\kappa\rangle$ generated by $\mathfrak{u}_1,\dots,\mathfrak{u}_\kappa$. Let $\mathbb{A}=\mathbb{A}_\tau$ denote the algebra

$$\mathbb{A} := \frac{\langle \mathfrak{u}_1,\dots,\mathfrak{u}_\kappa\rangle_{\mathbb{Z}}}{\langle \mathfrak{u}^+(\tau)-\mathfrak{u}^-(\tau)\rangle + \langle \mathfrak{u}_i^+ \mid g_i^+>0\rangle + \langle \mathfrak{u}_j^- \mid g_j^->0\rangle} = \frac{\mathbb{Z}[\kappa]}{\mathcal{I}(\tau)},$$
$$\text{where} \quad \mathfrak{u}^-(\tau) = \sum_{i=1}^k \mathfrak{u}_i^- \quad \text{and} \quad \mathfrak{u}^+(\tau) = \sum_{i=1}^l \mathfrak{u}_i^+.$$

We will denote the set of monomials $\prod_{i=1}^\kappa \mathfrak{u}_i^{a_i}$ by $G(\mathbb{A})$, which forms a set of generators for $\mathbb{A}$ as a module over $\mathbb{Z}$. One may define a natural map from $G(\mathbb{A})$ to the $\mathbb{Z}$-module $\mathbb{H}=\mathbb{H}_\tau := \mathrm{H}^2(X,\partial X,\mathbb{Z})$ by

$$\chi : G(\mathbb{A}) \longrightarrow \mathbb{H} = \mathrm{H}^2(X,\partial X;\mathbb{Z}),$$
$$\chi\left(\prod_{i=1}^\kappa \mathfrak{u}_i^{a_i}\right) := a_1\mathrm{PD}[\tau_1]+\dots+a_\kappa\mathrm{PD}[\tau_\kappa], \quad \forall\, a_1,\dots,a_\kappa\in\mathbb{Z}^{\geq 0}.$$

Note that $\mathbb{H}_\tau$ only depends on X, and τ may thus be dropped from the notation. However, we will sometimes keep it to highlight the connection of the the $\mathbb{Z}$-module $\mathbb{H}$ to X (where the sutures in τ live).

Let $\overline{X}=\overline{X}^\tau$ denote the three-manifold (with boundary) obtained by filling the sutures of (X,τ) by attaching 2-handles to the sutures in τ. Fix a (relative) Spin^c class $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$. Suppose that $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ is a Heegaard diagram for the sutured manifold (X,τ), which is admissible in an appropriate sense (see Section 4 for a precise definition of admissibility). Thus Σ is a closed Riemann surface, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are ℓ-tuples of disjoint simple closed curves, and $\mathbf{z}$ is a set of κ marked points on Σ. If $\Sigma^\circ=\Sigma-\mathrm{nd}(\mathbf{z})$ is the complement of a neighbourhood of $\mathbf{z}$, X is obtained from $\Sigma^\circ\times[-1,1]$ by attaching 2-handles to $\boldsymbol{\alpha}\times\{-1\}$ and $\boldsymbol{\beta}\times\{1\}$, while τ is obtained as $(\partial\Sigma^\circ)\times\{0\}$. The Ozsváth-Szabó chain complex $\mathrm{CF}(X,\tau,\mathfrak{s})$ is then generated, as a free $\mathbb{A}$-module, by those intersection points of the tori $\mathbb{T}_\alpha,\mathbb{T}_\beta\subset\mathrm{Sym}^\ell(\Sigma)$ associated with $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ which correspond to the Spin^c class $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$. The set $\pi_2^+(\mathbf{x},\mathbf{y})$ of positive homotopy classes of Whitney disks connecting the generators $\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ is defined as usual, and we have a map

$$\begin{aligned}\mathfrak{u}_{\mathbf{z}}:&\coprod_{\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\pi_2^+(\mathbf{x},\mathbf{y})\longrightarrow G(\mathbb{A})\\ \mathfrak{u}_{\mathbf{z}}(\phi):=&\prod_{i=1}^{\kappa}\mathfrak{u}_i^{n_{z_i}(\phi)},\quad\forall\,\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta,\ \ \forall\,\phi\in\pi_2^+(\mathbf{x},\mathbf{y}).\end{aligned}$$

Here $n_{z_i}(\phi)$ denotes the coefficient of z_i in the domain $\mathcal{D}(\phi)$ associated with the Whitney disk ϕ. We will abuse the notation and denote the class $[\mathfrak{u}_{\mathbf{z}}(\phi)]\in\mathbb{Z}[\kappa]/\mathcal{I}(\tau)$ by $\mathfrak{u}_{\mathbf{z}}(\phi)$ as well. The differential ∂ of the complex $\mathrm{CF}(X,\tau,\mathfrak{s})$ is defined by counting holomorphic disks ϕ of Maslov index 1 connecting the generators $\mathbf{x}$ and $\mathbf{y}$ of the complex, with an appropriate sign and with the weight $\mathfrak{u}_{\mathbf{z}}(\phi)\in\mathbb{A}$. The assignment of relative Spin^c structures to the intersection points $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ using $\mathbf{z}$ gives $\mathrm{CF}(X,\tau,\mathfrak{s})$ the structure of a filtered $(\mathbb{A},\mathbb{H})$ chain complex (see Section 3 for a precise definition). The following is the main result of this paper.

Theorem 1.1. *Suppose that (X,τ) is a weakly balanced sutured manifold, $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$ is a Spin^c structure on $\overline{X}=\overline{X}^\tau$, and that $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ is an $\mathfrak{s}$-admissible Heegaard diagram for (X,τ). Then $\mathrm{CF}(X,\tau)$, as defined above, is a filtered $(\mathbb{A}_\tau,\mathbb{H}_\tau)$ chain complex. The filtered $(\mathbb{A}_\tau,\mathbb{H}_\tau)$ chain homotopy type of the filtered $(\mathbb{A}_\tau,\mathbb{H}_\tau)$ chain complex $\mathrm{CF}(X,\tau,\mathfrak{s})$ is an invariant of the weakly balanced sutured manifold (X,τ) and the Spin^c class $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$. In particular, for any $\underline{\mathfrak{s}}\in\mathfrak{s}\subset\mathrm{Spin}^c(X,\tau)$ the chain homotopy type of the summand*

$$\mathrm{CF}(X,\tau,\underline{\mathfrak{s}})\subset\mathrm{CF}(X,\tau,\mathfrak{s})=\bigoplus_{\underline{\mathfrak{s}}\in\mathfrak{s}}\mathrm{CF}(X,\tau,\underline{\mathfrak{s}})$$

is also an invariant of $(X,\tau,\underline{\mathfrak{s}})$.

For a *test ring* $\mathbb{B}$, i.e. a coefficient ring which has the structure of a module over $\mathbb{A}$, the chain homotopy type of the complex

$$\mathrm{CF}(X,\tau,\mathfrak{s};\mathbb{B})=\mathrm{CF}(X,\tau,\mathfrak{s})\otimes_{\mathbb{A}}\mathbb{B}$$

is thus an invariant of the sutured manifold (X,τ) as well. If $\mathbb{B}$ admits a filtration by $\mathbb{H}$ and the action of $\mathbb{A}$ on the $\mathbb{A}$-module $\mathbb{B}$ respects the filtration of the monomials of $\mathbb{A}$ by the elements of $\mathbb{H}$, the above complex is equipped with a filtration by $\mathrm{Spin}^c(X,\tau)$. In this case, it makes sense to talk about the following decomposition of $\mathrm{CF}(X,\tau,\mathfrak{s};\mathbb{B})$:

$$\mathrm{CF}(X,\tau,\mathfrak{s};\mathbb{B})=\bigoplus_{\underline{\mathfrak{s}}\in\mathfrak{s}\subset\mathrm{Spin}^c(X,\tau)}\mathrm{CF}(X,\tau,\underline{\mathfrak{s}};\mathbb{B}).$$

In particular, the homology groups

$$\mathrm{HF}(X,\tau,\underline{\mathfrak{s}};\mathbb{B})=H_*\left(\mathrm{CF}(X,\tau,\underline{\mathfrak{s}};\mathbb{B}),\partial\right),\quad \forall\ \underline{\mathfrak{s}}\in\mathrm{Spin}^c(X,\tau)$$

may be defined, and are invariants of the sutured manifold and the relative Spin^c class $\underline{\mathfrak{s}}\in\mathrm{Spin}^c(X,\tau)$. As a special case, if the sutured manifold is balanced in the sense of [Ju1], we may take $\mathbb{B}=\mathbb{Z}$ and let the non-trivial monomials of $\mathbb{A}$ act trivially on $\mathbb{Z}$. We thus obtain a natural $\mathbb{A}$-module structure on $\mathbb{Z}$. The sutured Floer homology of Juhász is then recovered as:

$$\mathrm{SFH}(X,\tau,\underline{\mathfrak{s}})=\mathrm{HF}(X,\tau,\underline{\mathfrak{s}};\mathbb{Z}),\quad \forall\ \underline{\mathfrak{s}}\in\mathrm{Spin}^c(X,\tau).$$

Consider the quotient $\mathbb{B}_\tau$ of $\mathbb{A}_\tau$ defined by setting

$$\mathbb{B}_\tau=\frac{\left\langle \mathfrak{u}_1,\dots,\mathfrak{u}_\kappa\right\rangle_{\mathbb{Z}}}{\left\langle \mathfrak{u}\in G(\mathbb{A})\setminus\{1\}\ \middle|\ \chi(\mathfrak{u})\text{ is torsion}\right\rangle_{\mathbb{Z}}}.$$

We call the monomials in the denominator of the above quotient the *homologically trivial monomials*, when they are considered as elements in $\mathbb{A}_\tau$. Multiplication by homologically trivial monomials respects the filtration by relative Spin^c structures. Clearly, there is a quotient map $\rho_\tau:\mathbb{A}_\tau\to\mathbb{B}_\tau$ giving $\mathbb{B}_\tau$ the structure of an $\mathbb{A}_\tau$ module.

Proposition 1.2. *An irreducible balanced sutured manifold (X,τ) is taut if and only if the filtered $(\mathbb{B}_\tau,\mathbb{H}_\tau)$ chain homotopy type of the complex*

$$\mathrm{CF}(X,\tau;\mathbb{B}_\tau)=\bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})}\mathrm{CF}(X,\tau;\mathfrak{s};\mathbb{B}_\tau)$$

is non-trivial.

The above proposition is a refinement of Juhász's Theorem 1.4 from [Ju2] in the following sense. If the irreducible balanced sutured manifold (X,τ) is not taut, not only $\mathrm{SFH}(X,\tau)=0$ by Theorem 1.4 from [Ju2], but also the chain homotopy type of the complex $\mathrm{CF}(X,\tau;\mathbb{B}_\tau)$ is trivial.

We find it useful to illustrate some of the above constructions in a few cases, before moving to the statement of some of the properties of the sutured Floer complex constructed in this paper.

Example 1. Suppose that Y is a closed three-manifold and that the sutured manifold $(X, \tau = \{\tau_1, \dots, \tau_n\}) = Y(n)$ is obtained by removing n disjoint balls from Y and placing a suture on each one of the resulting sphere boundary components. In this case we have

$$k = l = n, \quad g_i^+ = g_i^- = 0 \quad \text{and} \quad \mathfrak{u}_i^+ = \mathfrak{u}_i^- = \mathfrak{u}_i, \quad i = 1, \dots, n.$$

Thus the algebra $\mathbb{A}$ is equal to $\mathbb{Z}[\mathfrak{u}_1, \dots, \mathfrak{u}_n]$ in this case, and we recover the multi-pointed Ozsváth-Szabó complex associated with the closed three-manifold Y. Each $\mathfrak{u}_i$ is a homologically trivial monomial, and the action of $\{\mathfrak{u}_i\}_i$ on the complex corresponds to the so-called U-action. Note that the above construction refines the construction of Ozsváth and Szabó in [OS9] by giving a complex with coefficients in $\mathbb{Z}$ rather than $\mathbb{Z}/2\mathbb{Z}$.

Example 2. For a link L inside a closed three-manifold Y which has n connected components, the boundary of the corresponding sutured manifold $(X = Y - \text{nd}(L), \tau = \{\tau_1, \dots, \tau_{2n}\})$ consists of n tori and τ consists of a pair of parallel sutures $\{\tau_{2i-1}, \tau_{2i}\}$ on the i-th torus. With the above notation,

$$k = l = n, \quad g_i^+ = g_i^- = 0 \quad \text{and} \quad \mathfrak{u}_i^+ = \mathfrak{u}_i^- = \mathfrak{u}_{2i-1}\mathfrak{u}_{2i}, \quad \forall\, i = 1, \dots, n.$$

Thus, the algebra $\mathbb{A}$ is equal to $\mathbb{Z}[\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_{2n}]$. In particular, when L is a knot (i.e. the number of components is just 1) this gives the $\mathbb{Z} \oplus \mathbb{Z}$ filtration associated with the knot L inside the three-manifold Y. Note that $\mathfrak{u}_{2i-1}\mathfrak{u}_{2i}$ is a homologically trivial monomial for $i = 1, \dots, n$. Multiplication by these monomials gives the U-action on the knot Floer complex. Once again, this generalizes the construction of [OS9] in several ways (including the fact that the coefficient ring is improved to $\mathbb{Z}$).

Example 3. Let K be a homologically trivial knot inside a closed three-manifold Y, and let S be a connected Seifert surface for K of genus g. The sutured manifold $(X, \tau = \{\tau_1\}) = Y(S)$ is then obtained by removing a product neighbourhood of S from Y and adding a copy of K as the single suture on the boundary of $Y - \text{nd}(S)$. In this case we have

$$k = l = 1, \quad g_1^+ = g_1^- = g \quad \text{and} \quad \mathfrak{u}_1^+ = \mathfrak{u}_1^- = \mathfrak{u}_1.$$

If $g = 0$ and K is thus the unknot, we will have $\mathbb{A} = \mathbb{Z}[\mathfrak{u}_1]$, while for $g > 0$, the corresponding algebra would be $\mathbb{Z}$. In this latter case we recover the complex of Juhász. However, a slightly stronger version of the construction allows us to use the coefficient ring $\mathbb{Z}[\mathfrak{u}_1]/\langle \mathfrak{u}_1^2 \rangle$ when the genus g is bigger than 1 (see Remark 5.9). It is interesting to investigate if the improved invariant would help us with distinguishing different Seifert surfaces of the same knot.

Example 4. Consider three points $0, 1$ and ∞ on the standard sphere $S^2 = \mathbb{CP}^1$, and denote the boundary of small disjoint disks D_0, D_1, D_∞ around these points by C_0, C_1 and C_∞, respectively. Let $\Sigma = \mathbb{CP}^1 \setminus (D_0 \cup D_1)$, and suppose that X denotes the three-manifold with boundary which is obtained from $\Sigma \times [-1,1]$ by attaching a 2-handle to $C_0 \times \{-1\}$ and a 2-handle to $C_\infty \times \{1\}$. The curves $\tau_\bullet = C_\bullet \times \{0\}$ for $\bullet \in \{0,1\}$ define a sutured manifold $(X, \tau = \{\tau_0, \tau_1\})$. It is easy to see that $X = [0,1] \times S^2$, and that the boundary of X consists of a sphere with two parallel sutures on it, which decompose it into two disks and a cylinders, and a sphere without any sutures on it. The algebra associated with this sutured manifold is thus equal to

$$\mathbb{A} = \mathbb{A}_\tau = \frac{\mathbb{Z}[\mathfrak{u}_0, \mathfrak{u}_1]}{\left\langle (\mathfrak{u}_0 - 1)(\mathfrak{u}_1 - 1) \right\rangle}.$$

Note that $(\Sigma = \mathbb{CP}^1, \boldsymbol{\alpha} = \{C_0\}, \boldsymbol{\beta} = \{C_\infty\}, \mathbf{z} = \{0,1\})$ is a Heegaard diagram for this sutured manifold. In order to make this Heegaard diagram admissible, a pair of cancelling intersection points should be created between C_0 and C_∞. The chain complex associated with this Heegaard diagram is thus generated by a pair of generators x and y, corresponding to the copies $\mathbb{A}_x$ and $\mathbb{A}_y$ of $\mathbb{A}$. The differential is defined by

$$\partial(x) = (\mathfrak{u}_1 - 1)y \quad \text{and} \quad \partial(y) = (\mathfrak{u}_0 - 1)x.$$

Thus the chain complex has non-trivial chain homotopy type. This illustrates one of the simplest examples of a situation beyond the framework of [Ju1] where the current construction may be applied and the outcome is non-trivial. Note that in this case (where the boundary has components without any sutures) the ring $\mathbb{Z}$ does not admit the structure of a natural $\mathbb{A}$-module.

Suppose that $(X, \tau = \{\tau_1, \ldots, \tau_\kappa\})$ is a weakly balanced sutured manifold as above, and assume that the boundary $S = \partial X$ is connected. We may let the mapping class group $\mathrm{MCG}(S)$ of S act on the sutures; for any $\phi \in \mathrm{MCG}(S)$, let $\tau_\phi = \{\phi(\tau_1), \ldots, \phi(\tau_\kappa)\}$. Clearly, (X, τ_ϕ) is a new sutured manifold, and $\mathbb{A}_{\tau_\phi} = \mathbb{A}_\tau$. Usually it sounds impossible, however, to relate the complex $\mathrm{CF}(X, \tau_\phi)$ to $\mathrm{CF}(X, \tau)$. Nevertheless, the surgery exact triangle for the Ozsváth-Szabó complexes associated with closed three-manifolds may be generalized to give a partial answer in certain situations.

Let (X, τ) be a sutured manifold and $\tau_1, \tau_2 \in \tau$ be two sutures which belong to the common boundary of genus zero connected components $R_1^+ \subset$

$\mathfrak{R}^+(\tau)$ and $R_1^- \subset \mathfrak{R}^-(\tau)$. Consider a simple closed curve $\lambda \subset R_1^+ \cup R_1^- \cup \tau_1 \cup \tau_2$ which cuts each one of τ_1 and τ_2 in a single transverse point and remains disjoint from the rest of the sutures. We further assume that $\tau_1.\lambda = 1$. Let ϕ denote the right-handed Dehn twist along τ_1 and ψ denote the left handed Dehn twist along λ. In particular, note that $\psi(\tau_1)$ and $\phi(\lambda)$ are both homologous to $\tau_1 + \lambda$. Let us assume that the algebra $\mathbb{A} = \mathbb{A}_\tau$ is generated by $\mathfrak{u}_1, \ldots, \mathfrak{u}_\kappa$ as before, where $\mathfrak{u}_i$ corresponds to the suture τ_i, $i = 1, \ldots, \kappa$. Note that in the relations ideal $\mathcal{I}(\tau)$ in $\mathbb{Z}[\kappa]$ (which defines $\mathbb{A}$ as $\mathbb{Z}[\kappa]/\mathcal{I}(\tau)$) the generators either contain $\mathfrak{u}_1\mathfrak{u}_2$, or they contain none of $\mathfrak{u}_1$ and $\mathfrak{u}_2$. We may thus introduce a new algebra $\mathbb{B}$ as a quotient of $\langle \mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_\kappa \rangle_{\mathbb{Z}}$ by an ideal $\overline{\mathcal{I}}(\tau)$. The generators of $\overline{\mathcal{I}}(\tau)$ are constructed from the generators of $\mathcal{I}(\tau)$ by replacing $\mathfrak{u}_1\mathfrak{u}_2$ with $\mathfrak{u}_0\mathfrak{u}_1\mathfrak{u}_2$. For $i = 0, 1, 2$ we obtain the embeddings $\imath^i$ of $\mathbb{A}$ in $\mathbb{B}$:

$$\imath^i : \mathbb{A} \to \mathbb{B}, \quad \imath^i(\mathfrak{u}_j) = \begin{cases} \mathfrak{u}_i & \text{if } j = 1 \\ \dfrac{\mathfrak{u}_0\mathfrak{u}_1\mathfrak{u}_2}{\mathfrak{u}_i} & \text{if } j = 2 \\ \mathfrak{u}_j & \text{if } 3 \le j \le \kappa \end{cases}.$$

We write $\mathbb{A}_i$ in order to refer to $\mathbb{A}$ as the sub-ring $\imath^i(\mathbb{A}) \subset \mathbb{B}$. Consider the following quotient of $\mathrm{H}^2(X, \partial X; \mathbb{Z})$.

$$\mathbb{H} = \frac{\mathrm{H}^2(X, \partial X; \mathbb{Z})}{\langle \eta \rangle}, \quad \text{where} \quad \eta = \mathrm{PD}[\tau_1 + \tau_2] = \mathrm{PD}[\psi(\tau_1) + \psi(\tau_2)]$$
$$= \mathrm{PD}[\phi(\psi(\tau_1)) + \phi(\psi(\tau_2))].$$

Let us denote by $\chi_j \in \mathbb{H}$ the Poincaré dual of the suture τ_j, for $j = 3, \ldots, \kappa$. Furthermore, let χ_0, χ_1 and χ_2 denote the Poincaré duals of τ_1, $\psi(\tau_1)$ and $-\phi \circ \psi(\tau_1)$, respectively. Note that $\chi_0 + \chi_1 + \chi_2 = 0$. Define the filtration map by

$$\chi : G(\mathbb{B}) \longrightarrow \mathbb{H}, \qquad \chi\left(\prod_{j=0}^{\kappa} \mathfrak{u}_i^{a_i}\right) := \sum_{j=0}^{\kappa} a_i \chi_i.$$

Associated with any Spin^c class $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$ let $\mathrm{CF}_i(\mathfrak{s}; \mathbb{A}_i)$ denote the complex $\mathrm{CF}(X, \tau, \mathfrak{s}; \mathbb{A}_0)$, $\mathrm{CF}(X, \tau_\psi, \mathfrak{s}; \mathbb{A}_1)$, or $\mathrm{CF}(X, \tau_{\phi\circ\psi}, \mathfrak{s}; \mathbb{A}_2)$ depending on whether $i = 0, 1$ or 2. Let $\mathrm{CF}_i(\mathfrak{s}; \mathbb{B}) = \mathrm{CF}_i(\mathfrak{s}; \mathbb{A}_i) \otimes_{\mathbb{A}_i} \mathbb{B}$.

Theorem 1.3. *With the above notation fixed, we have a triangle*

$$\begin{array}{ccc} \mathrm{CF}_0(\mathfrak{s};\mathbb{B}) & \xrightarrow{\ \mathfrak{f}_2^{\mathfrak{s}}\ } & \mathrm{CF}_1(\mathfrak{s};\mathbb{B}) \\ & {}^{\mathfrak{f}_1^{\mathfrak{s}}}\nwarrow \qquad \swarrow{}^{\mathfrak{f}_0^{\mathfrak{s}}} & \\ & \mathrm{CF}_2(\mathfrak{s};\mathbb{B}) & \end{array}$$

of filtered $(\mathbb{B},\mathbb{H})$ *chain maps such that* $\mathfrak{f}_1^{\mathfrak{s}}\circ\mathfrak{f}_0^{\mathfrak{s}}$, $\mathfrak{f}_2^{\mathfrak{s}}\circ\mathfrak{f}_1^{\mathfrak{s}}$, *and* $\mathfrak{f}_0^{\mathfrak{s}}\circ\mathfrak{f}_2^{\mathfrak{s}}$ *are null homotopic. Moreover,* $\mathrm{CF}_i(\mathfrak{s};\mathbb{B})$ *is filtered* $(\mathbb{B},\mathbb{H})$ *chain homotopic to the mapping cone of* $\mathfrak{f}_i^{\mathfrak{s}}$*. In particular, if a ring* R *has the structure of a* $\mathbb{B}$*-module, taking the tensor product of the above triangle with* R *and computing the homology groups we obtain a long exact sequence in homology:*

$$\cdots \xrightarrow{\mathfrak{f}_1^{\mathfrak{s}}} \mathrm{HF}(X,\tau,\mathfrak{s};R) \xrightarrow{\mathfrak{f}_2^{\mathfrak{s}}} \mathrm{HF}(X,\tau_\psi,\mathfrak{s};R) \xrightarrow{\mathfrak{f}_0^{\mathfrak{s}}} \mathrm{HF}(X,\tau_{\phi\circ\psi},\mathfrak{s};R) \xrightarrow{\mathfrak{f}_1^{\mathfrak{s}}} \cdots .$$

If the action of $\mathbb{B}$ on R respects the filtration by $\mathbb{H}$, the above exact sequence refines to an exact sequence corresponding to any of the relative Spin^c structures $\underline{\mathfrak{s}}\in\mathrm{Spin}^c(X,\tau)/\langle\eta\rangle$.

We say that the sutured manifold $(X,\widehat{\tau}=\tau\cup\{\tau_{\kappa+1},\tau_{\kappa+2}\})$ is obtained by a *simple stabilization* of (X,τ) if $-\tau_{\kappa+1}$ and $\tau_{\kappa+2}$ are oriented simple closed curves parallel to $\tau_\kappa\in\tau$, and τ_κ belongs to the common boundary of two genus zero components $R_l^+\subset\mathfrak{R}^+(\tau)$ and $R_k^-\subset\mathfrak{R}^-(\tau)$. Moreover, τ_κ and $\tau_{\kappa+1}$ bound an annulus R^+ in $\partial X-\widehat{\tau}$, while $\tau_{\kappa+1}$ and $\tau_{\kappa+2}$ bound $R^-\subset\partial X-\widehat{\tau}$, with $R^+,R^-\subset R_l^+$. We will denote $R_l^+-\overline{R^+\cup R^-}$ by R_{l+1}^+. Let $\mathbb{A}_{\widehat{\tau}}$ denote the algebra associated with $(X,\widehat{\tau})$, which is a quotient of $\mathbb{Z}[\mathfrak{u}_1,\ldots,\mathfrak{u}_{\kappa+2}]$. The following theorem (proved in Section 7) generalizes the stabilization theorem of [OS9], while improving the ring of coefficients to $\mathbb{Z}$ (instead of $\mathbb{Z}/2\mathbb{Z}$).

Proposition 1.4. *With the above notation fixed, for any given* Spin^c *class*

$$\mathfrak{s}\in\mathrm{Spin}^c\left(\overline{X}^{\widehat{\tau}}\right)=\mathrm{Spin}^c(\overline{X}),$$

the filtered chain homotopy type of the complex $\mathrm{CF}(X,\widehat{\tau},\mathfrak{s})$ *is the same as the filtered chain homotopy type of the chain complex obtained by equipping the module*

$$\mathrm{CF}\left(X,\tau,\mathfrak{s};\mathbb{A}_{\widehat{\tau}}\right)\oplus\mathrm{CF}\left(X,\tau,\mathfrak{s};\mathbb{A}_{\widehat{\tau}}\right)$$

with the differential

$$\widehat{\partial} = \begin{pmatrix} \partial_\tau & \mathfrak{u}_{\kappa+1} - \mathfrak{u} \\ \mathfrak{u}_\kappa - \mathfrak{u}_{\kappa+2} & -\partial_\tau \end{pmatrix}, \quad \textit{where} \;\; \mathfrak{u} := \prod_{\tau_\kappa \neq \tau_i \in \partial R_l^+} \mathfrak{u}_i \in \mathbb{A}_{\widehat{\tau}}.$$

Since $\mathbb{A}_{\widehat{\tau}}$ does not admit the structure of a natural $\mathbb{A}_\tau$-module, the module $\mathrm{CF}\,(X, \tau, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}})$, defined from an admissible Heegaard diagram for (X, τ), does not have the structure of a chain complex and its *differential*

$$\partial_\tau : \mathrm{CF}\,(X, \tau, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}}) \longrightarrow \mathrm{CF}\,(X, \tau, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}})$$

satisfies

$$\partial_\tau \circ \partial_\tau = (\mathfrak{u}^+(\tau) - \mathfrak{u}^-(\tau)).Id.$$

However, this is enough for $\widehat{\partial}$ to be a differential.

Finally, the behaviour of the sutured Floer complex under particular forms of product disk decompositions is described in Section 9.

Theorem 1.5. *Suppose that (X^i, τ^i) is a weakly balanced sutured manifold with a distinguished suture σ_i for $i = 1, 2$, and let (X, τ) denote the weakly balanced sutured manifold obtained as the boundary connected sum of (X^i, τ^i) along σ_i, $i = 1, 2$. Fix the* Spin^c *structures $\mathfrak{s}^i \in \mathrm{Spin}^c(\overline{X^i})$ for $i = 1, 2$. Then the filtered chain homotopy type of the two filtered $(\mathbb{A}, \mathbb{H})$ chain complexes*

$$\mathrm{CF}(X, \tau, \mathfrak{s}^1 \# \mathfrak{s}^2; \mathbb{A}) \quad \textit{and} \quad \mathrm{CF}(X^1, \tau^1, \mathfrak{s}^1; \mathbb{A}) \otimes_{\mathbb{A}} \mathrm{CF}(X^2, \tau^2, \mathfrak{s}^2; \mathbb{A})$$

are the same, where $\mathbb{A} = \mathbb{A}(\tau^1, \tau^2; \sigma_1, \sigma_2)$.

The algebra $\mathbb{A}(\tau^1, \tau^2; \sigma_1, \sigma_2)$ is discussed in Section 9. If the boundary components in $\mathfrak{R}^\pm(\tau^i)$ adjacent to σ_i have positive genus, one is forced to include *unwanted* relations in the above ring. Thus in some sense, Section 9 will provide a partial understanding of the extent to which a surface decomposition formula for the sutured Floer complex may be hoped for.

1.3. Outline of the paper

The paper is organized as follows. In Section 2 we review some of the basic notions, including the sutured manifolds, the corresponding Heegaard diagrams, and the Spin^c structures on sutured manifolds. We will also review some of the main constructions studied in this paper, including the action of the mapping class group of the boundary and filling the sutures.

In Section 3 we develop the language of chain complexes filtered by a module, and make some simple algebraic observations. Moreover, we construct an algebra associated with the boundary of a balanced sutured manifold, as well as a filtration of its generators by classes in $\mathrm{H}^2(X, \partial X; \mathbb{Z})$. The algebra plays the role of the coefficient ring for the Ozsváth-Szabó chain complex constructed in this paper.

In Section 4 we develop a notion of admissibility for Heegaard diagrams, which makes it possible to construct an Ozsváth-Szabó complex using Heegaard Floer theory. The admissibility condition is slightly weaker, in a sense, than the strong admissibility of Ozsváth and Szabó in the context of closed three-manifolds. However, it is strong enough for the construction of Ozsváth-Szabó complex to work. We show that all weakly balanced sutured manifolds admit admissible Heegaard diagrams corresponding to any Spin^c class.

In Section 5 we study the orientability issues for the corresponding moduli spaces. In particular, an appropriate orientation for the moduli spaces of boundary degenerations is required so that the differential ∂ of the associated Ozsváth-Szabó chain complex satisfies $\partial^2 = 0$. Analyzing the analytic aspects of the theory thus requires some new techniques which are developed in Section 5.

In Section 6 we show that the filtered chain homotopy type of the chain complex constructed in Section 5 and associated with an admissible Heegaard diagram for the balanced sutured manifold (X, τ) is invariant under Heegaard moves, and is independent of the choice of the path of almost structures on the symmetric product of the Heegaard surface. The choice of the algebra associated with the boundary plays a very crucial role both in defining the chain complex and proving the invariance of the filtered chain homotopy type.

In Section 7 we study how the filtered chain homotopy type of the Ozsváth-Szabó complex associated with a balanced sutured manifold (X, τ) changes when we add two parallel copies of an existing suture to the boundary with appropriate orientation. The operation is called the *stabilization* of the sutured manifold (X, τ).

In Section 8 we introduce a generalization of the surgery triangle for balanced sutured manifolds. The freedom to choose many marked points on the Heegaard diagram allows us to understand the chain maps in a better way, and refine the existing triangles, and long exact sequences.

Finally, in Section 9 we discuss product disk decomposition, and prove a connected sum formula for the sutured Floer complex. A number of examples are also discussed.

2. Background on sutured manifolds

2.1. Sutured manifolds and relative Spin^c structures

In this paper, we only deal with sutured manifolds without toroidal sutures, so we will modify the standard definition of sutured manifolds, by throwing away the possibility of having a torus component among the sutures.

Definition 2.1. A *sutured manifold* (without toroidal sutures) (X, τ) is a compact oriented three-manifold X with boundary ∂X, together with a set of disjoint oriented simple closed curves $\tau = \{\tau_1, \dots, \tau_\kappa\}$ on ∂X. We will denote by $A(\tau_i)$ a tubular neighbourhood of τ_i in ∂X, which will be an annulus. We let $A(\tau) = A(\tau_1) \cup \cdots \cup A(\tau_\kappa)$. Every component of $\mathfrak{R}(\tau) = \partial X - A(\tau)^\circ$ is oriented (where $A(\tau)^\circ$ denotes the interior of $A(\tau)$). Furthermore, $\mathfrak{R}(\tau) = \mathfrak{R}^+(\tau) \cup \mathfrak{R}^-(\tau)$ where $\mathfrak{R}^+(\tau)$ denotes the union of components of $\mathfrak{R}(\tau)$ with the property that the orientation induced on τ as the boundary of $\mathfrak{R}^+(\tau)$ agrees with the orientation of τ, while $\mathfrak{R}^-(\tau)$ denotes the union of components of $\mathfrak{R}(\tau)$ with the property that the orientation induced on τ as the boundary of $\mathfrak{R}^-(\tau)$ is the opposite of the orientation of τ. We assume that the orientation on the components of $\mathfrak{R}(\tau) \subset \partial X$ is induced by the orientation of X. A sutured manifold (X, τ) is called *weakly balanced* if for every connected component X_0 of X both $X_0 \cap \mathfrak{R}^+(\tau)$ and $X_0 \cap \mathfrak{R}^-(\tau)$ are non-empty and $\chi(\mathfrak{R}^+(\tau)) = \chi(\mathfrak{R}^-(\tau))$. A sutured manifold (X, τ) is called *balanced* if X is weakly balanced and the induced map $\pi_0(\tau) \to \pi_0(\partial X)$ is surjective.

Definition 2.2. A *Heegaard diagram* is a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ such that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a balanced Heegaard diagram i.e. Σ is a closed oriented surface and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are sets of disjoint oriented simple closed curves on Σ with $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = \ell$, and

$$\mathbf{z} = \{z_1, \dots, z_\kappa\} \subset \Sigma - \bigcup \boldsymbol{\alpha} - \bigcup \boldsymbol{\beta}$$

is a set of marked points.

Every Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ uniquely defines a weakly balanced sutured manifold manifold as follows. Let $\Sigma^\circ = \Sigma - D_1 - \cdots - D_\kappa$ denote the complement of the small disks $D_1, \dots, D_\kappa$ around $z_1, \dots, z_\kappa$, where $\mathbf{z} = \{z_1, \dots, z_\kappa\}$. The three-manifold X is obtained from $\Sigma^\circ \times [-1, 1]$ by attaching 3-dimensional 2-handles along the curves $\alpha_i \times \{-1\}$ and $\beta_j \times \{1\}$

for $i, j = 1, \ldots, \ell$. We may define the set of sutures on the boundary of X by

$$\tau = \Big\{\tau_1, \ldots, \tau_\kappa\Big\}, \quad \tau_i = \partial D_i \times \{0\}.$$

In this situation, we say that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is associated with the sutured three-manifold (X, τ). If each connected component of $\Sigma - \boldsymbol{\alpha}$ and $\Sigma - \boldsymbol{\beta}$ contains at least one marked point, then the sutured manifold (X, τ) is balanced.

Proposition 2.3. *For every weakly balanced sutured manifold (X, τ), there exists a Heegaard diagram associated with it in the above sense.*

Proof. The argument of [Ju1] may be extended to construct sutured Heegaard diagrams for weakly balanced sutured manifolds. Let $(\Sigma_\tau, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a sutured Heegaard diagram for the weakly balanced sutured manifold (X, τ) in the sense of [Ju1]. If $\tau = \{\tau_1, \ldots, \tau_\kappa\}$ consists of κ sutures, take Σ to be the surface obtained from Σ_τ by gluing κ disks $D_1, D_2, \ldots, D_\kappa$ to it along the boundary components corresponding to $\tau_1, \ldots, \tau_\kappa$. Let z_i be the center of D_i, $i = 1, \ldots, \kappa$. Then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z} = \{z_1, \ldots, z_\kappa\})$ is a Heegaard diagram for (X, τ). □

Proposition 2.4. *If $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{z})$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w})$ are two Heegaard diagrams for a weakly balanced sutured manifold (X, τ), then they are diffeomorphic after a finite set of Heegaard moves, which are supported away from the marked points.*

Proof. This is Proposition 2.15 from [Ju1]. □

For the most part of this paper, we will identify $\mathfrak{R}(\tau) = \mathfrak{R}^+(\tau) \cup \mathfrak{R}^-(\tau)$ as the connected components of $\partial X - \tau$. Thus the boundary of each connected component $R \subset \mathfrak{R}(\tau)$ may be identified with a union of curves in τ. In the few situations where the annuli $A(\tau_i)$ are relevant, we will emphasize them in the notation.

Suppose that (X, τ) is a balanced sutured manifold. One may define a nowhere vanishing vector field on ∂X as follows. Let v_τ be a vector field (with values in $TX|_{\partial X}$) which points outward on $\mathfrak{R}^+(\tau) \subset \partial X - A(\tau) = \mathfrak{R}(\tau)$, and points inward on $\mathfrak{R}^-(\tau) \subset \mathfrak{R}(\tau)$. Furthermore, under the identification $A(\tau_i) = \tau_i \times [-1, 1]$, let $v_\tau|_{A(\tau_i)}$ be the vector field $\frac{\partial}{\partial t}$ determining the unit tangent vector of the second factor, i.e. the interval $[-1, 1]$. In fact, we have to perturb v_τ on a small neighborhood $\partial A(\tau)$ to make it continuous, but we typically drop this perturbation from our notation.

Definition 2.5. Suppose that the non-vanishing vector fields v and w on X agree with v_τ on ∂X. We say that v and w are *homologous* if there is a ball $B \subset X^\circ$ such that the restrictions of v and w to $X - B$ are homotopic relative the boundary of X. We define the space $\mathrm{Spin}^c(X,\tau)$ of *relative* Spin^c *structures* on the sutured manifold (X,τ) to be the space of homology classes of such nowhere vanishing vector fields on X which agree with v_τ on ∂X.

Note that $\mathrm{Spin}^c(X,\tau)$ is an affine space over $\mathrm{H}^2(X,\partial X,\mathbb{Z})$. Let us assume that the Spin^c structure $\underline{\mathfrak{s}} \in \mathrm{Spin}^c(X,\tau)$ is represented by a nowhere vanishing vector field v, so that $v|_{\partial X} = v_\tau$. Let us define the first Chern class of $\underline{\mathfrak{s}}$ to be the first Chern class of the oriented 2-plane field $v^\perp$ over X, which lives in $\mathrm{H}^2(X,\mathbb{Z})$. Let us denote the inclusion of ∂X in X by $i : \partial X \to X$. We thus get a map

$$i^* : \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(\partial X,\mathbb{Z}).$$

The first Chern class of the 2-plane field $v_\tau^\perp$ lives in $\mathrm{H}^2(\partial X,\mathbb{Z})$ and $c_1(\underline{\mathfrak{s}})$ is thus included in

$$(i^*)^{-1}\Big(c_1(v_\tau^\perp)\Big) \subset \mathrm{H}^2(X,\mathbb{Z}).$$

Let (X,τ) be a balanced sutured manifold as above. Let I denote a subset of $\{1,\dots,\kappa\}$. We may glue a solid cylinder $D^2 \times [-1,1]$ (i.e. a 3-dimensional 2-handle) to each component $A(\tau_i)$ of $A(\tau)$ along $S^1 \times [-1,1]$. We will refer to this operation as *filling out* the suture τ_i. Consider the sutured manifold $(X(I),\tau(I))$ obtained by filling out the sutures of (X,τ) corresponding to the subset I. In particular, we will denote $X(1,\dots,\kappa)$ by $\overline{X}$. In terms of the Heegaard diagrams, if $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z} = \{z_1,\dots,z_\kappa\})$ is a Heegaard diagram associated with (X,τ) so that z_i corresponds to the suture τ_i, a diagram for $(X(I),\tau(I))$ will be the pointed Heegaard diagram

$$\big(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z} - \big\{z_i \mid i \in I\big\}\big)\,.$$

Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ be a Heegaard diagram for the balanced sutured manifold (X,τ). complete $\mathbf{z}$ to a collection $\overline{\mathbf{z}}$ of marked points in $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ so that each connected component contains at least one marked point. Consider the symmetric product

$$\mathrm{Sym}^\ell(\Sigma) = \frac{\Sigma^{\times \ell}}{S_\ell} = \frac{\Sigma \times \cdots \times \Sigma}{S_\ell}$$

equipped with a path of complex structures $\{J_t\}_{t\in[0,1]}$ which is of the form $\mathrm{Sym}^\ell(j_\Sigma)$ in a fixed small neighbourhood V of $\overline{\mathbf{z}}$, where j_Σ denotes a complex structure over the surface Σ. Following Ozsváth and Szabó [OS5] we call such almost complex structures *nearly symmetric*, dropping the Kähler structure and the complex structure j_Σ on Σ and the neighbourhood V from the notation. By a *generic* path of almost complex structures $\{J_t\}_{t\in[0,1]}$ over $\mathrm{Sym}^\ell(\Sigma)$ we mean a generic choice of the above path within the aforementioned family of paths. The last condition on the path of almost complex structures will be suppressed from the notation. The tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_\ell$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_\ell$ are totally real sub-manifolds of $\mathrm{Sym}^\ell(\Sigma)$. We may define a map

$$\underline{\mathfrak{s}} = \underline{\mathfrak{s}}_{\mathbf{z}} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c(X, \tau),$$

which is defined by choosing a Morse function compatible with the Heegaard diagram for the sutured manifold (X, τ), viewing an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ as a set of flow lines joining index-1 critical points to index-2 critical points of the Morse function, and perturbing the gradient vector field of the corresponding Morse function in a neighbourhood of this set of flow lines associated with $\mathbf{x}$ in order to obtain a nowhere vanishing vector field on X with the desired properties.

There is a notion of extending Spin^c structures from (X, τ) to $X(I)$ as follows. If $\underline{\mathfrak{s}} \in \mathrm{Spin}^c(X, \tau)$ is represented by the nowhere vanishing vector field v (so that $v|_{\partial X} = v_\tau$), we may extend v over each one of the glued cylinders $D^2 \times [-1, 1]$. In fact, v may be extended over $D^2 \times [-1, 1]$ by setting it equal to $\frac{\partial}{\partial t}$, where t denotes the variable associated with the interval $[-1, 1]$. We will denote this extension of the vector field v by v_I. Denote the natural maps obtained by extending the relative Spin^c structures on sutured manifolds over the attached solid cylinders as above by

$$s_I = s_I^\tau : \mathrm{Spin}^c(X, \tau) \longrightarrow \mathrm{Spin}^c(X(I), \tau(I)), \quad \forall\, I \subset \{1, \ldots, \kappa\}.$$

In particular, the extension map $s^\tau_{\{1,\ldots,\kappa\}}$ is denoted by

$$[.] : \mathrm{Spin}^c(X, \tau) \to \mathrm{Spin}^c(\overline{X}).$$

Note that there is an exact sequence

$$0 \longrightarrow \big\langle \mathrm{PD}[\tau_i] \mid i \in I \big\rangle_{\mathbb{Z}} \longrightarrow \mathrm{Spin}^c(X, \tau) \xrightarrow{s_I} \mathrm{Spin}^c(X(I), \tau(I)) \longrightarrow 0.$$

This sequence should be interpreted as follows. If two relative Spin^c structures $\underline{\mathfrak{s}}, \underline{\mathfrak{t}} \in \mathrm{Spin}^c(X, \tau)$ satisfy $s_I(\underline{\mathfrak{s}}) = s_I(\underline{\mathfrak{t}})$, then the cohomology class $\underline{\mathfrak{s}} - \underline{\mathfrak{t}}$ is generated by the Poincaré duals of the sutures corresponding to I.

Let us denote the inclusion of X in $X(I)$ by $\imath_I : X \to X(I)$. This inclusion gives a map

$$\imath_I^* : \mathrm{H}^2(X(I), \mathbb{Z}) \longrightarrow \mathrm{H}^2(X, \mathbb{Z}).$$

We know that if $\underline{\mathfrak{s}}$ is represented by v, $s_I(\underline{\mathfrak{s}})$ is represented by v_I. From the definition of the first Chern class,

$$\imath_I^* \left(c_1\left(s_I(\underline{\mathfrak{s}})\right)\right) = c_1(\underline{\mathfrak{s}}).$$

Let us assume that $S = \partial X$ is a connected surface. Associated with any element ϕ in the mapping class group $\mathrm{MCG}(S)$ of the surface S we would have a new sutured manifold (X, τ_ϕ), with the same ambient manifold, and with $\tau_\phi = \phi(\tau)$. Clearly, $\mathfrak{R}^+(\tau_\phi) = \phi(\mathfrak{R}^+(\tau))$ and $\mathfrak{R}^-(\tau_\phi) = \phi(\mathfrak{R}^-(\tau))$.

Consider the three-manifold with boundary $Y = [0,1] \times S$, together with the sutures

$$\sigma_\phi = (\{0\} \times \tau) \bigcup (\{1\} \times \tau_\phi).$$

Correspondingly, we will have

$$\begin{aligned} \mathfrak{R}^+(\sigma_\phi) &= \left(\{1\} \times \mathfrak{R}^+(\tau_\phi)\right) \cup \left(\{0\} \times \mathfrak{R}^-(\tau)\right) \quad \text{and} \\ \mathfrak{R}^-(\sigma_\phi) &= \left(\{1\} \times \mathfrak{R}^-(\tau_\phi)\right) \cup \left(\{0\} \times \mathfrak{R}^+(\tau)\right). \end{aligned}$$

Then (Y, σ_ϕ) is a sutured manifold and $\mathrm{Spin}^c(Y, \sigma_\phi)$ is an affine space over

$$\mathrm{H}^2(Y, \partial Y; \mathbb{Z}) = \mathrm{H}^2([0,1] \times S, \{0,1\} \times S; \mathbb{Z}) = \mathrm{H}_1([0,1] \times S; \mathbb{Z}).$$

Any relative Spin^c class in $\mathrm{Spin}^c(Y, \sigma_\phi)$ may thus be used to define a map from $\mathrm{Spin}^c(X, \tau)$ to $\mathrm{Spin}^c(X, \tau_\phi)$. In other words, we have a natural gluing map

$$\mathrm{Spin}^c(X, \tau) \times \mathrm{Spin}^c([0,1] \times \partial X, \sigma_\phi) \longrightarrow \mathrm{Spin}^c(X, \tau_\phi).$$

2.2. Relative Spin^c-structures and Heegaard diagrams

Let the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for the balanced sutured manifold (X, τ), the symmetric product $\mathrm{Sym}^\ell(\Sigma)$, the totally real tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, and the path of complex structures $\{J_t\}_{t \in [0,1]}$ (which is nearly symmetric) be as before.

Definition 2.6. Let $D \subset \mathbb{C}$ be the unit disk in the complex plane, and $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. A *Whitney disk* is a continuous map $\phi : D \to \mathrm{Sym}^\ell(\Sigma)$ such that $\phi(-i)=\mathbf{x},\phi(i)=\mathbf{y}$ and

$$\phi\big\{z\in\partial D \mid \mathrm{Re}(z)\geq 0\big\}\subset\mathbb{T}_\alpha \quad \text{and}$$
$$\phi\big\{z\in\partial D \mid \mathrm{Re}(z)\leq 0\big\}\subset\mathbb{T}_\beta.$$

The set of homotopy classes of the Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$ is denoted by $\pi_2(\mathbf{x},\mathbf{y})$. For any homotopy class $\phi\in\pi_2(\mathbf{x},\mathbf{y})$, we denote the moduli space of $\{J_t\}_t$-holomorphic representatives of ϕ by $\mathcal{M}(\phi)$. There exists a translation action of $\mathbb{R}$ on $\mathcal{M}(\phi)$. The quotient of $\mathcal{M}(\phi)$ under this action will be denoted by $\widehat{\mathcal{M}}(\phi)$. The Maslov index of ϕ is denoted by $\mu(\phi)$. For $i\in\mathbb{Z}$, we will denote by $\pi_2^i(\mathbf{x},\mathbf{y})$ the subset of $\pi_2(\mathbf{x},\mathbf{y})$ which consists of all ϕ with $\mu(\phi)=i$.

It is known ([OS5], and [OS9]) that for any generic path $\{J_t\}_t$ of complex structures, $\mathcal{M}(\phi)$ is a smooth manifold of dimension $\mu(\phi)$, which is not necessarily compact. Although the proof is given for a special class of pointed Heegaard diagrams the proof in this more general form is identical. This moduli space may be compactified by adding the Gromov limits of pseudo-holomorphic curves. But the boundary strata which correspond to degenerations of the domain are not necessarily smooth, or of lower dimension. We will return to this issue in Section 5.

Definition 2.7. Let $D_1,\dots,D_m$ be the connected components of $\Sigma-\boldsymbol{\alpha}-\boldsymbol{\beta}$. Each element of the free abelian group generated by $\{D_1,\dots,D_m\}$ is called a *domain*. A domain $\mathcal{D}=a_1D_1+\cdots+a_mD_m$ is called *positive*, denoted $\mathcal{D}\geq 0$, if $a_i\geq 0$ for $1\leq i\leq m$. It is called *periodic* if its formal boundary (as a 2-chain) is a sum of α and β curves.

For every Whitney disk ϕ connecting the intersection points $\mathbf{x}$ and $\mathbf{y}$, the domain associated with ϕ is defined as follows:

$$\mathcal{D}(\phi)=\sum_{i=1}^{m} n_{p_i}(\phi)D_i$$

where $p_i\in D_i$ is a marked point. Here $n_p(\phi)$ for a point $p\in\Sigma-\boldsymbol{\alpha}-\boldsymbol{\beta}$ denotes the algebraic intersection number of ϕ with the subvariety

$$\Delta_p=\big\{(p_1,\dots,p_\ell)\in\mathrm{Sym}^\ell(\Sigma)\,\big|\,p_i=p,\ \text{for some } 1\leq i\leq\ell\big\}.$$

If the map ϕ is pseudo-holomorphic then $\mathcal{D}(\phi)$ is a positive domain by positivity of intersection. We will denote by $\pi_2^+(\mathbf{x},\mathbf{y})$ the subset of $\pi_2(\mathbf{x},\mathbf{y})$ which consists of all ϕ with $\mathcal{D}(\phi)\geq 0$.

If $\mathcal{P}$ is a periodic domain we can associate to it a homology class in $H_2(\overline{X}, \mathbb{Z})$. More precisely, let

$$\partial \mathcal{P} = \sum_{i=1}^{\ell} a_i \alpha_i + \sum_{i=1}^{\ell} b_i \beta_i$$

and let $D(\alpha_i)$ be the union of $\alpha_i \times [-1, 0]$ with the core of the two-handles attached to $\alpha_i \times \{-1\}$ in X. Similarly, let $D(\beta_i)$ be the union of $\beta_i \times [0, 1]$ with the core of the two-handle attached to $\beta_i \times \{1\}$. Define

$$H(\mathcal{P}) = \mathcal{P} + \sum_{i=1}^{\ell} a_i D(\alpha_i) + \sum_{i=1}^{\ell} b_i D(\beta_i).$$

If $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$ is a Whitney disk connecting $\mathbf{x}$ to itself, with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the domain $\mathcal{D}(\phi)$ will be a periodic domain. Conversely, any periodic domain $\mathcal{P}$ determines the class of a Whitney disk in $\pi_2(\mathbf{x}, \mathbf{x})$ for any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, provided that $\ell > 1$. Thus the space of periodic domains may be identified with $\pi_2(\mathbf{x}, \mathbf{x})$ if we assume that $\ell > 1$.

Fix a metric on X and let ∇ denote the corresponding covariant derivative. Fix a self-indexing Morse function $f : X \to [\epsilon, 3 - \epsilon]$ compatible with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$, i.e. such that $\nabla f|_{A(\tau)}$ is a section of $T(\partial X)|_{A(\tau)}$,

$$\mathfrak{R}^+(\tau) = f^{-1}(3 - \epsilon), \quad \mathfrak{R}^-(\tau) = f^{-1}(\epsilon) \quad \text{and} \quad \Sigma - \coprod_{i=1}^{\kappa} D_i = f^{-1}(3/2),$$

and such that the curves in $\boldsymbol{\alpha}$ are identified as the ascending manifolds of the critical points of index 1, while the curves in $\boldsymbol{\beta}$ are identified with the descending manifolds of the critical points of index 2. For each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ let $\gamma_{\mathbf{x}}$ be the union of the flow lines connecting the index-1 critical points to the index-2 critical points passing through the union $\mathbf{x}$ of the intersection points on $\Sigma = f^{-1}(3/2) \subset X$.

Lemma 2.8. *For* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *we have* $\underline{\mathfrak{s}}(\mathbf{x}) - \underline{\mathfrak{s}}(\mathbf{y}) = \mathrm{PD}(\epsilon(\mathbf{x}, \mathbf{y}))$ *where* $\epsilon(\mathbf{x}, \mathbf{y}) = \gamma_{\mathbf{x}} - \gamma_{\mathbf{y}} \in \mathrm{H}_1(X, \mathbb{Z})$.

Proof. This is Lemma 4.7 from [Ju1]. □

Corollary 2.9. *If $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ then we have*

$$\underline{\mathfrak{s}}(\mathbf{x}) - \underline{\mathfrak{s}}(\mathbf{y}) = \sum_{i=1}^{\kappa} n_{z_i}(\phi)\mathrm{PD}[\tau_i].$$

Proof. The disk ϕ gives a domain $\mathcal{D}(\phi)$, with the property that $\epsilon(\mathbf{x}, \mathbf{y})$ is represented by

$$\partial(\mathcal{D}(\phi)) \in \mathrm{H}_1(X, \mathbb{Z}) = \frac{\mathrm{H}_1\big(\Sigma - \{z_1, \ldots, z_\kappa\}, \mathbb{Z}\big)}{\big\langle \alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_\ell \big\rangle_{\mathbb{Z}}}.$$

If ϵ_i denotes a small loop around $z_i \in \Sigma$, the domain $\mathcal{D}(\phi)$ gives a 2-chain connecting $\epsilon(\mathbf{x}, \mathbf{y})$ and $n_{z_1}(\phi)\epsilon_1 + \cdots + n_{z_\kappa}(\phi)\epsilon_\kappa$. However, ϵ_i is homologous to τ_i, and we thus have

$$\underline{\mathfrak{s}}(\mathbf{x}) - \underline{\mathfrak{s}}(\mathbf{y}) = \mathrm{PD}[\epsilon(\mathbf{x}, \mathbf{y})] = \sum_{i=1}^{\kappa} n_{z_i}(\phi)\mathrm{PD}[\tau_i].$$

This completes the proof of the corollary. □

Let us finish this subsection with a pair of lemmas for computing the Maslov index of a periodic domain. Let

$$\Sigma - \boldsymbol{\alpha} = \bigcup_{i=1}^{k} A_i \quad \text{and} \quad \Sigma - \boldsymbol{\beta} = \bigcup_{i=1}^{l} B_i,$$

and assume we have $m = k + l - 1$ points $w_1, \ldots, w_m$ on Σ such that $w_i \in A_i \cap B_1$ for $1 \leq i \leq k$, and $w_{i+k} \in A_k \cap B_{i+1}$ for $1 \leq i < l$.

Lemma 2.10. *For any periodic domain $\mathcal{P} \in \pi_2(\mathbf{x}, \mathbf{x})$ such that $n_{w_i}(\mathcal{P}) = 0$ for $1 \leq i \leq m$ we have:*

$$\mu(\mathcal{P}) = \big\langle c_1([\underline{\mathfrak{s}}(\mathbf{x})]), H(\mathcal{P})\big\rangle.$$

Proof. Let $\Sigma_{\mathbf{w}} = \Sigma - \mathrm{nd}(\mathbf{w})$, with $\mathbf{w} = \{w_1, \ldots, w_m\}$. Now $(\Sigma_{\mathbf{w}}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a sutured Heegaard diagram for a sutured manifold $X_{\mathbf{w}}$ which is obtained from $\overline{X}$ by removing neighbourhoods of the flow lines passing through $\mathbf{w}$. If $i : X_{\mathbf{w}} \to \overline{X}$ is the embedding of $X_{\mathbf{w}}$ in $\overline{X}$, then $i^{-1}(\mathcal{P})$ is a periodic domain in $(\Sigma_{\mathbf{w}}, \boldsymbol{\alpha}, \boldsymbol{\beta})$, and by Theorem 5.2 from [Ju1] we have

$$\mu(i^{-1}\mathcal{P}) = \left\langle c_1\big(\underline{\mathfrak{s}}_{\mathbf{w}}(\mathbf{x})\big), H\big(i^{-1}(\mathcal{P})\big)\right\rangle.$$

The periodic domains $\mathcal{P}$ and $i^{-1}(\mathcal{P})$ correspond to the homotopy class of a sphere

$$S \subset \mathrm{Sym}^{\ell}(\Sigma_{\mathbf{w}}) \subset \mathrm{Sym}^{\ell}(\Sigma)$$

and their Maslov index is thus the same, i.e. $\mu(\mathcal{P}) = \mu(i^{-1}(\mathcal{P}))$. We have $i^*H(\mathcal{P}) = H\big(i^{-1}(\mathcal{P})\big)$. Thus it is enough to show that

$$c_1\big(\underline{\mathfrak{s}}_{\mathbf{w}}(\mathbf{x})\big) = i^* c_1\big([\underline{\mathfrak{s}}(\mathbf{x})]\big). \tag{1}$$

Let ν be the vector field defining $\underline{\mathfrak{s}}(\mathbf{x})$, and let $\overline{\nu}$ be the extension of ν to $\overline{X}$. Then $i^*\overline{\nu}$ is the vector field defining $\underline{\mathfrak{s}}_{\mathbf{w}}(\mathbf{x})$ and thus Equation 1 is satisfied. □

Lemma 2.11. *For any periodic domain $\mathcal{P} \in \pi_2(\mathbf{x}, \mathbf{x})$ we have:*

$$\mu(\mathcal{P}) = \left\langle c_1\left([\underline{\mathfrak{s}}(\mathbf{x})]\right), H(\mathcal{P})\right\rangle.$$

Proof. If we move the curves by an isotopy to create new intersection points between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the periodic domain $\mathcal{P}$ corresponds to a periodic domain in the new Heegaard diagram. The two sides of the above equality remain unchanged under this correspondence. We may thus assume that we have $m = k + l - 1$ points $w_1, \ldots, w_m$ on Σ such that $w_i \in A_i \cap B_1$ for $1 \le i \le k$, and $w_{i+k} \in A_k \cap B_{i+1}$ for $1 \le i < l$. Let us denote $n_{w_i}(\mathcal{P})$ by n_i, and set

$$\mathcal{Q} = \mathcal{P} - \left(\sum_{i=1}^{k} n_i A_i\right) - \left(\sum_{i=1}^{l-1} (n_{i+k} - n_k) B_{i+1}\right).$$

Clearly $n_{w_i}(\mathcal{Q}) = 0$ for $i = 1, \ldots, m$, and Lemma 2.10 implies (setting $\mathfrak{s} = [\underline{\mathfrak{s}}(\mathbf{x})]$, and regarding $\mathcal{Q}$ as an element in $\pi_2(\mathbf{x}, \mathbf{x})$)

$$
\begin{aligned}
(2)\quad \mu(\mathcal{Q}) &= \big\langle c_1(\mathfrak{s}), H(\mathcal{Q})\big\rangle \\
&= \big\langle c_1(\mathfrak{s}), H(\mathcal{P})\big\rangle - \left(\sum_{i=1}^{k} n_i \big\langle c_1(\mathfrak{s}), H(A_i)\big\rangle\right) \\
&\quad - \left(\sum_{i=1}^{l-1} (n_{i+k} - n_k)\big\langle c_1(\mathfrak{s}), H(B_{i+1})\big\rangle\right) \\
&= \big\langle c_1(\mathfrak{s}), H(\mathcal{P})\big\rangle - \left(\sum_{i=1}^{k} n_i \chi(A_i)\right) - \left(\sum_{i=1}^{l-1} (n_{i+k} - n_k)\chi(B_i)\right).
\end{aligned}
$$

In the last equation we denote by $\chi(A_i)$ and $\chi(B_i)$ the expressions $2 - 2g_{A_i}$ and $2 - 2g_{B_i}$, respectively, where g_{A_i} and g_{B_i} denote the genera of the components in $\mathfrak{R}^-(\tau)$ and $\mathfrak{R}^+(\tau)$ which correspond to A_i and B_i, respectively.

On the other hand, the formula of Lipshitz ([Lip]) may be used to compute $\mu(A_i)$ and $\mu(B_j)$ as periodic domains in $\pi_2(\mathbf{x}, \mathbf{x})$. As such, we will have

$$
(3)\quad \mu(A_i) = \chi(A_i), \quad i = 1, \dots, k, \quad \text{and} \quad \mu(B_j) = \chi(B_i), \quad j = 1, \dots, l.
$$

Combining Equations 3 and 2 we obtain

$$
\begin{aligned}
\mu(\mathcal{P}) &= \big\langle c_1(\mathfrak{s}), H(\mathcal{P})\big\rangle - \left(\sum_{i=1}^{k} n_i \chi(A_i)\right) - \left(\sum_{i=1}^{l-1} (n_{i+k} - n_k)\chi(B_i)\right) \\
&\quad + \left(\sum_{i=1}^{k} n_i \mu(A_i)\right) + \left(\sum_{i=1}^{l-1} (n_{i+k} - n_k)\mu(B_i)\right) \\
&= \big\langle c_1(\mathfrak{s}), H(\mathcal{P})\big\rangle.
\end{aligned}
$$

This completes the proof of the lemma. □

3. Algebra input

3.1. The $\mathbb{A}$ chain complexes

Most part of this subsection is borrowed from Ozsváth and Szabó's [OS7] (Subsection 4.1) with minor modifications.

Let us assume that $\mathbb{A}$ is a (commutative) finitely generated $\mathbb{Z}$-algebra.

Definition 3.1. If $\mathbb{B}$ is another (commutative) ring, which has the structure of an $\mathbb{A}$-module, we will call $\mathbb{B}$ a *test ring* for $\mathbb{A}$. A chain complex with *coefficient ring* $\mathbb{A}$, or simply an $\mathbb{A}$ *chain complex*, is an $\mathbb{A}$-module C, together with a homomorphism of $\mathbb{A}$-modules $d : C \to C$, such that $d \circ d = 0$.

Let us assume that (C, d) is an $\mathbb{A}$ chain complex. Choose a test ring $\mathbb{B}$ and let $C(\mathbb{B}) = C \otimes_{\mathbb{A}} \mathbb{B}$. The differential d of the complex (C, d) induces a differential $d^{\mathbb{B}} : C(\mathbb{B}) \to C(\mathbb{B})$. If (C_1, d_1) and (C_2, d_2) are $\mathbb{A}$ chain complexes and $f : C_1 \to C_2$ is an $\mathbb{A}$ chain map, then f induces a $\mathbb{B}$ chain map

$$f^{\mathbb{B}} : (C_1(\mathbb{B}), d_1^{\mathbb{B}}) \longrightarrow (C_2(\mathbb{B}), d_2^{\mathbb{B}}),$$

where $d_i^{\mathbb{B}}$ denotes the differential induced by d_i on $C_i(\mathbb{B})$, $i = 1, 2$. Associated with any $\mathbb{A}$ chain complex (C, d), and any test ring $\mathbb{B}$, we consider the homology group

$$H_*(C, d; \mathbb{B}) := H_*(C(\mathbb{B}), d^{\mathbb{B}}).$$

We may denote this homology group by $H_*(C; \mathbb{B})$, if there is no confusion. Associated with $f : C_1 \to C_2$ as above, we thus obtain a homomorphism

$$f_*^{\mathbb{B}} : H_*(C_1, d_1; \mathbb{B}) \longrightarrow H_*(C_2, d_2; \mathbb{B}).$$

Definition 3.2. An $\mathbb{A}$ chain map $f : (C_1, d_1) \to (C_2, d_2)$ between $\mathbb{A}$ chain complexes is called a *quasi-isomorphism* if the induced map

$$f_*^{\mathbb{B}} : H_*(C_1, d_1; \mathbb{B}) \longrightarrow H_*(C_2, d_2; \mathbb{B})$$

is an isomorphism for any test ring $\mathbb{B}$. More generally, if $\mathfrak{B}$ is a family of test rings for $\mathbb{A}$, the $\mathbb{A}$ chain map f is called a $\mathfrak{B}$-*isomorphism* if $f_*^{\mathbb{B}}$ is an isomorphism for any test ring $\mathbb{B} \in \mathfrak{B}$. Two $\mathbb{A}$ chain complexes (C_1, d_1) and (C_2, d_2) are *quasi-isomorphic* if there is a third $\mathbb{A}$ chain complex (C, d), together with quasi-isomorphisms $f_i : (C_i, d_i) \to (C, d)$, $i = 1, 2$. Similarly, we may define $\mathfrak{B}$-*isomorphic* $\mathbb{A}$ chain complexes.

If $f : (C_1, d_1) \to (C_2, d_2)$ is a homotopy equivalence of $\mathbb{A}$ chain complexes, then f is clearly a quasi-isomorphism.

If (A_1, d_1) and (A_2, d_2) are $\mathbb{A}$ chain complexes and $f : A_1 \to A_2$ is an $\mathbb{A}$ chain map, we can form the mapping cone $\mathbb{M}(f)$ of f, whose underlying complex is the direct sum $A_1 \oplus A_2$, which is equipped with the differential

$$d_{\mathbb{M}} = \begin{pmatrix} d_1 & 0 \\ f & -d_2 \end{pmatrix}. \tag{4}$$

The chain complex $\mathbb{M}(f)$ inherits the structure of an $\mathbb{A}$-module from A_1 and A_2, and its differential respects the $\mathbb{A}$-module structure, since d_1 and

d_2 do so and f is an $\mathbb{A}$ chain map. Moreover, for any test ring $\mathbb{B}$ for $\mathbb{A}$, $\mathbb{M}(f)(\mathbb{B}) = \mathbb{M}(f^{\mathbb{B}})$. There is a short exact sequence of chain complexes

$$0 \longrightarrow A_2(\mathbb{B}) \xrightarrow{\imath^{\mathbb{B}}} \mathbb{M}(f)(\mathbb{B}) \xrightarrow{\pi^{\mathbb{B}}} A_1(\mathbb{B}) \longrightarrow 0,$$

induced from the natural sequence

$$0 \longrightarrow A_2 \xrightarrow{\imath} \mathbb{M}(f) \xrightarrow{\pi} A_1 \longrightarrow 0.$$

For each test ring $\mathbb{B}$ for $\mathbb{A}$ we thus obtain a long exact sequence in homology, or in fact an exact triangle

$$\begin{array}{ccc} H_*(A_1, d_1; \mathbb{B}) & \xrightarrow{f_*^{\mathbb{B}}} & H_*(A_2, d_2; \mathbb{B}) \\ & \nwarrow \quad \swarrow & \\ & H_*(\mathbb{M}(f), d_{\mathbb{M}}; \mathbb{B}) & \end{array}$$

The construction of the mapping cone is natural in the sense that a commutative diagram of $\mathbb{A}$ chain maps

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

induces an $\mathbb{A}$ chain map $\mathfrak{m}(\phi_1, \phi_2) : \mathbb{M}(f) \to \mathbb{M}(g)$ such that there is a homotopy commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_2 & \xrightarrow{\imath_f} & \mathbb{M}(f) & \xrightarrow{\pi_f} & A_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \mathfrak{m}(\phi_1, \phi_2)} & & \downarrow{\scriptstyle \phi_1} & & \\ 0 & \longrightarrow & B_2 & \xrightarrow{\imath_g} & \mathbb{M}(g) & \xrightarrow{\pi_g} & B_1 & \longrightarrow & 0. \end{array}$$

The following lemma is the main algebraic ingredient in the study of holomorphic triangles in this paper.

Lemma 3.3. (c.f. Lemma 4.4 from [OS7]) *Let $\{(A_i, d_i)\}_{i=1}^{\infty}$ be a collection of $\mathbb{A}$ chain complexes and $\{f_i : A_i \to A_{i+1}\}$ be a collection of $\mathbb{A}$ chain maps between these complexes which satisfy the following two properties:*

(1) There are $\mathbb{A}$ homomorphisms $H_i : A_i \to A_{i+2}$ such that

$$f_{i+1} \circ f_i = H_i \circ d_i + d_{i+2} \circ H_i,$$

i.e. $f_{i+1} \circ f_i$ is null-homotopic via $\mathbb{A}$ chain homotopy maps H_i.

(2)The difference

$$f_{i+2} \circ H_i - H_{i+1} \circ f_i : A_i \to A_{i+3}$$

is a homotopy equivalence for $i = 1, 2, \ldots$.

Then $\mathbb{M}(f_i)$ is homotopy equivalent to A_{i+2} for $i \geq 2$. Moreover, if

$$f_{i+2} \circ H_i - H_{i+1} \circ f_i : A_i \to A_{i+3}$$

is a $\mathfrak{B}$-isomorphism for some family $\mathfrak{B}$ of test rings for $\mathbb{A}$ and for $i = 1, 2, \ldots$, then $\mathbb{M}(f_i)$ is $\mathfrak{B}$-isomorphic to A_{i+2} for $i \geq 2$.

Proof. The maps $\phi_i = (-1)^i \left(f_{i+2} \circ H_i - H_{i+1} \circ f_i\right) : A_i \to A_{i+3}$ are $\mathbb{A}$ chain maps, making the following diagram homotopy commutative

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & A_{i+1} \\ \Big\downarrow{\scriptstyle \phi_i} & & \Big\downarrow{\scriptstyle \phi_{i+1}} \\ A_{i+3} & \xrightarrow{f_{i+3}} & A_{i+4}. \end{array} \tag{5}$$

In fact, using the first property in the statement of the lemma we will have

$$\phi_{i+1} \circ f_i - f_{i+3} \circ \phi_i = (-1)^i \left((H_{i+2} \circ H_i) \circ d_i - d_{i+4} \circ (H_{i+2} \circ H_i)\right),$$

and $\phi_{i+1} \circ f_i - f_{i+3} \circ \phi_i$ is thus null-homotopic. Let us denote $H_{i+2} \circ H_i$ by $L_i : A_i \to A_{i+4}$. We then define $\alpha_i : \mathbb{M}(f_i) \to A_{i+2}$ and $\beta_i : A_i \to \mathbb{M}(f_{i+1})$ by

$$\alpha_i(a_i, a_{i+1}) = f_{i+1}(a_{i+1}) - H_i(a_i) \quad \text{and} \quad \beta_i(a_i) = (f_i(a_i), H_i(a_i))$$

respectively. Then $\alpha_{i+1} \circ \beta_i = (-1)^i \phi_i$ is a homotopy equivalence by the second property above. All the squares in the following diagram commute up to homotopy

$$\begin{array}{ccccccccc}
A_i & \xrightarrow{f_i} & A_{i+1} & \xrightarrow{\imath_{i+1}} & \mathbb{M}(f_i) & \xrightarrow{(-1)^{i+1}\pi_i} & A_i & \xrightarrow{f_i} & A_{i+1}\\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle \alpha_i}\downarrow & & {\scriptstyle \phi_i}\downarrow & & {\scriptstyle \phi_{i+1}}\downarrow\\
A_i & \xrightarrow{f_i} & A_{i+1} & \xrightarrow{f_{i+1}} & A_{i+2} & \xrightarrow{f_{i+2}} & A_{i+3} & \xrightarrow{f_{i+3}} & A_{i+4}\\
{\scriptstyle \phi_i}\downarrow & & {\scriptstyle \phi_{i+1}}\downarrow & & {\scriptstyle \beta_{i+2}}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow\\
A_{i+3} & \xrightarrow{f_{i+3}} & A_{i+4} & \xrightarrow{(-1)^i\imath_{i+4}} & \mathbb{M}(f_{i+3}) & \xrightarrow{\pi_{i+3}} & A_{i+3} & \xrightarrow{f_{i+3}} & A_{i+4}.
\end{array}\tag{6}$$

The commutativity of the two squares on the right and the two squares on the left already follows from the commutativity of the square in Equation 5. The definition of α_i and β_{i+2} imply the equalities

$$f_{i+2} = \pi_{i+3} \circ \beta_{i+2} \quad \text{and} \quad f_{i+1} = \alpha_i \circ \imath_{i+1}.$$

For the remaining two squares, let us define

$$\begin{aligned}
&K_i^1 : \mathbb{M}(f_i) \to A_{i+3}, \quad K_i^1(a_i, a_{i+1}) := H_{i+1}(a_{i+1}),\\
&K_i^2 : A_i \to \mathbb{M}(f_{i+2}), \quad K_i^2(a_i) = (H_i(a_i), 0)
\end{aligned}$$

We can then compute

$$\begin{aligned}
&(-1)^{i+1}\phi_i \circ \pi_i - f_{i+2} \circ \alpha_i = K_i^1 \circ d_{\mathbb{M}_i} - d_{i+3} \circ K_i^1 \quad \text{and}\\
&\beta_{i+2} \circ f_{i+1} - (-1)^i \imath_{i+4} \circ \phi_{i+1} = K_{i+1}^2 \circ d_{i+1} + d_{\mathbb{M}_{i+3}} \circ K_{i+1}^2,
\end{aligned}$$

where $d_{\mathbb{M}_i}$ denotes the differential of $\mathbb{M}_i = \mathbb{M}(f_i)$. We first claim that

$$F_i = \beta_{i+2} \circ \alpha_i : \mathbb{M}(f_i) \to \mathbb{M}(f_{i+3})$$

is a chain homotopy equivalence. In fact, note that

$$\begin{aligned}
F_i(a_i, a_{i+1}) &= \beta_{i+2}(f_{i+1}(a_{i+1}) - H_i(a_i))\\
&= \Big(f_{i+2}\big(H_i(a_i) - f_{i+1}(a_{i+1})\big), H_{i+2}\big(H_i(a_i) - f_{i+1}(a_{i+1})\big)\Big)\\
&= \mathfrak{m}(\phi_i, \phi_{i+1})(a_i, a_{i+1}) + \big(d_{\mathbb{M}_{i+3}} \circ H^i + H^i \circ d_{\mathbb{M}_i}\big)(a_i, a_{i+1})
\end{aligned}$$

$$\text{where } \begin{cases} H^i(a_i, a_{i+1}) = \big(-H_{i+1}(a_{i+1}), 0\big),\\ \mathfrak{m}(\phi_i, \phi_{i+1})(a_i, a_{i+1}) = (-1)^i\big(\phi_i(a_i), L_i(a_i) - \phi_{i+1}(a_{i+1})\big). \end{cases}$$

Since $\mathfrak{m}(\phi_i,\phi_{i+1})$ is a chain homotopy equivalence, it follows that the same is true for F_i. Since $\alpha_{i+3}\circ\beta_{i+2}=(-1)^i\phi_{i+2}$ is a chain homotopy equivalence as well, it follows that β_{i+2} is a chain homotopy equivalence (one needs to use the fact that α_{i+3},β_{i+2} and F_i are all chain maps).

For the $\mathfrak{B}$-isomorphism statement, note that for any test ring $\mathbb{B}\in\mathfrak{B}$ for the ring $\mathbb{A}$, we may replace the complexes A_i with $A_i(\mathbb{B})$ and $\mathbb{M}(f_i)$ with $\mathbb{M}(f_i^{\mathbb{B}})$ in the commutative diagram 6. Then the maps induced on homology associated with the first and the third row of the above diagram are exact. From the five lemma, it follows that the map induced on homology by $\beta_{i+2}^{\mathbb{B}}\circ\alpha_i^{\mathbb{B}}$ is an isomorphism. Since $\alpha_{i+3}\circ\beta_{i+2}=\phi_{i+2}$ is a $\mathfrak{B}$-isomorphism, we conclude that β_{i+2}, and hence α_i are $\mathfrak{B}$-isomorphisms as well. □

3.2. Filtration by a $\mathbb{Z}$-module

Let us assume that $\mathbb{A}$ is an algebra over $\mathbb{Z}$ which is generated, as a free module over $\mathbb{Z}$, by a set $G(\mathbb{A})$ of generators. We will assume that $1\in G(\mathbb{A})$. The choice of this basis for $\mathbb{A}$ as a free module over $\mathbb{Z}$ will be implicit in our notation. Furthermore, let $\mathbb{H}$ be a $\mathbb{Z}$-module.

Definition 3.4. By a *filtration* for $\mathbb{A}$ with values in $\mathbb{H}$ we mean a choice of the basis $G(\mathbb{A})$ for the free $\mathbb{Z}$-module $\mathbb{A}$ which is closed under multiplication, and a map

$$\chi : G(\mathbb{A}) \longrightarrow \mathbb{H}$$

which satisfies $\chi(1)=0$ and $\chi(ab)=\chi(a)+\chi(b)$ for all $a,b\in G(\mathbb{A})$. The pair $(\mathbb{A},\chi:G(\mathbb{A})\to\mathbb{H})$ is called a *coefficient ring filtered by* $\mathbb{H}$. We will typically drop χ and the choice of $G(\mathbb{A})$ from the notation, if there is no confusion, and will denote the filtered ring by the pair $(\mathbb{A},\mathbb{H})$.

Suppose that $\mathbb{B}$ is a test ring for $\mathbb{A}$ which is a free $\mathbb{Z}$-module on its own with basis $G(\mathbb{B})$, and that

$$\chi_{\mathbb{B}} : G(\mathbb{B}) \longrightarrow \mathbb{H}$$

is a filtration for $\mathbb{B}$. Furthermore, assume that the $\mathbb{A}$-module structure on $\mathbb{B}$ induces a map

$$G(\mathbb{A})\times G(\mathbb{B})\to G(\mathbb{B}).$$

Definition 3.5. In the above situation, we say that $\chi_{\mathbb{B}}$ is *compatible* with χ if

$$\chi_{\mathbb{B}}(ab) = \chi_{\mathbb{A}}(a) + \chi_{\mathbb{B}}(b), \quad \forall\, a \in G(\mathbb{A}),\ b \in G(\mathbb{B}).$$

If this is the case, we will call $(\mathbb{B}, \chi_{\mathbb{B}})$ a *filtered test ring* for $(\mathbb{A}, \mathbb{H})$. Again, when there is no confusion we will denote this pair by $(\mathbb{B}, \mathbb{H})$.

Let us assume that (C, d) is an $\mathbb{A}$ chain complex, that $\chi : G(\mathbb{A}) \to \mathbb{H}$ is a filtration for $\mathbb{A}$, and that C is freely generated over $\mathbb{A}$ by some subset I of C.

Definition 3.6. We say that the $\mathbb{A}$ chain complex (C, d) is a *filtered* $(\mathbb{A}, \mathbb{H})$ chain complex if there is a basis $I \subset C$ for C over $\mathbb{A}$ and a *filtration*

$$\chi : I \times I \to \mathbb{H},$$

which satisfies:

1) $\chi(c_1, c_2) = -\chi(c_2, c_1)$ for all $c_1, c_2 \in I$.
2) $\chi(c_1, c_2) + \chi(c_2, c_3) = \chi(c_1, c_3)$, for all $c_1, c_2, c_3 \in I$.
3) For any $c \in I$, $d(c) = \sum_{i=1}^{N} m_i a_i c_i$, with $m_1, \ldots, m_N \in \mathbb{Z}$, $c_1, \ldots, c_N \in I$ and $a_1, \ldots, a_N \in G(\mathbb{A})$, such that

$$\chi(c, c_i) = \chi(a_i), \quad \forall\, i \in \{1, \ldots, N\}.$$

We are of course abusing the notation by denoting both the filtration of C and the filtration of $\mathbb{A}$ by χ.

If (C, d) is a filtered $(\mathbb{A}, \mathbb{H})$ chain complex as above, one may think of $\chi(c_1, c_2)$ as the difference $\chi(c_1) - \chi(c_2)$, where $\chi : I \to \mathbb{S}$ and $\mathbb{S}$ is an affine space over the module $\mathbb{H}$. With this new notation, $\chi(a.c)$ for $a \in G(\mathbb{A})$ and $c \in I$ may be defined as $\chi(a) + \chi(c) \in \mathbb{S}$.

Clearly, taking the tensor product of (C, d) with any filtered test ring $(\mathbb{B}, \mathbb{H})$ results in a filtered $(\mathbb{B}, \mathbb{H})$ chain complex.

Definition 3.7. An $\mathbb{A}$ chain map $f : (C_1, d_1) \to (C_2, d_2)$ between $(\mathbb{A}, \mathbb{H})$ chain complexes (C_i, d_i) with basis I_i, $i = 1, 2$ and filtrations χ_1, χ_2 is called a *filtered* $(\mathbb{A},\mathbb{H})$ chain map if for all $c, c' \in I_1$ we may write

$$f(c) = \sum_{i=1}^{N} m_i a_i b_i, \quad f(c') = \sum_{i=j}^{M} m'_j a'_j b'_j,$$
$$m_i, m'_j \in \mathbb{Z},\ a_i, a'_j \in G(\mathbb{A}) \quad \text{and} \quad b_i, b'_j \in I_2,$$

such that for any $i = 1, \dots, N$ and $j = 1, \dots, M$ we have

$$\chi_1(c, c') = \chi_2(b_i, b'_j) + \chi(a_i) - \chi(a'_j).$$

In particular, if for some affine space $\mathbb{S}$ over $\mathbb{H}$, there are maps $\chi_i : I_i \to \mathbb{S}$ which satisfy $\chi_i(c_1, c_2) = \chi_i(c_1) - \chi_i(c_2)$ for $i = 1, 2$ and $c_1, c_2 \in I_i$, the above condition may be translated to $\chi_2(a_i b_i) = \chi_1(c)$ for $i = 1, \dots, N$, whenever $f(c) = \sum_{i=1}^{N} m_i a_i b_i$ with $m_i \in \mathbb{Z}$, $a_i \in G(\mathbb{A})$ and $b_i \in I_i$.

Similarly, we may define the notion of a chain homotopy respecting the filtrations (i.e. $(\mathbb{A},\mathbb{H})$ chain homotopy), and filtered $(\mathbb{A},\mathbb{H})$ chain homotopy equivalence. Mapping cones of filtered $(\mathbb{A},\mathbb{H})$ chain maps are filtered $(\mathbb{A},\mathbb{H})$ chain complexes. Moreover, the following refinement of Lemma 3.3 may be proved with a similar argument.

Lemma 3.8. *With the notation of Lemma 3.3, if the $\mathbb{A}$ chain complexes A_i are all filtered $(\mathbb{A},\mathbb{H})$ chain complexes, the $\mathbb{A}$ chain maps f_i, as well as the $\mathbb{A}$-homomorphisms H_i are all $(\mathbb{A},\mathbb{H})$ filtered, and $f_{i+2} \circ H_i - H_{i+1} \circ f_i$ are all filtered $(\mathbb{A},\mathbb{H})$ chain homotopy equivalences, $\mathbb{M}(f_i)$ is filtered $(\mathbb{A},\mathbb{H})$ chain homotopy equivalent to A_{i+2}.*

3.3. The algebra associated with the boundary of a sutured manifold

Let (X, τ) be a weakly balanced sutured manifold. We will assume that

$$\mathfrak{R}^-(\tau) = \bigcup_{i=1}^{k} R_i^- \quad \text{and} \quad \mathfrak{R}^+(\tau) = \bigcup_{j=1}^{l} R_j^+.$$

Here R_i^- and R_j^+ are the connected components of $\mathfrak{R}^-(\tau)$ and $\mathfrak{R}^+(\tau)$ respectively, for $i = 1, \dots, k$ and $j = 1, \dots, l$. Let g_i^- denote the genus of R_i^- and g_j^+ denote the genus of R_j^+. We will denote $2 - 2g_i^-$ by χ_i^- and $2 - 2g_j^+$ by χ_j^+. The set of sutures $\tau = \{\tau_1, \dots, \tau_\kappa\}$ determines an algebras over $\mathbb{Z}$ as follows. Consider the free $\mathbb{Z}$-algebra $\mathbb{Z}[\kappa] := \mathbb{Z}\big[\mathfrak{u}_1, \dots, \mathfrak{u}_\kappa\big]$, and consider the following elements in it

$$\begin{aligned}
\mathfrak{u}^- = \mathfrak{u}^-(\tau) &:= \sum_{i=1}^{k} \mathfrak{u}(R_i^-) \quad \text{and} \quad \mathfrak{u}^+ = \mathfrak{u}^+(\tau) := \sum_{i=1}^{l} \mathfrak{u}(R_i^+), \quad \text{where} \\
\mathfrak{u}_i^- = \mathfrak{u}(R_i^-) &:= \prod_{\tau_j \subset \partial R_i^-} \mathfrak{u}_j, \quad 1 \le i \le k \quad \text{and} \\
\mathfrak{u}_i^+ = \mathfrak{u}(R_i^+) &:= \prod_{\tau_j \subset \partial R_i^+} \mathfrak{u}_j, \quad 1 \le i \le l.
\end{aligned}$$

Consider the quotient $\mathbb{A}_\tau = \mathbb{Z}[\kappa]/\mathcal{I}(\tau)$ of $\mathbb{Z}[\kappa]$ where $\mathcal{I}(\tau)$ is the following ideal of $\mathbb{Z}[\kappa]$, called the *relations ideal* associated with τ:

$$\mathcal{I}(\tau) := \left\langle \mathfrak{u}^+(\tau) - \mathfrak{u}^-(\tau) \right\rangle_{\mathbb{Z}[\kappa]} + \left\langle \mathfrak{u}_i^- \mid g_i^- > 0 \right\rangle_{\mathbb{Z}[\kappa]} + \left\langle \mathfrak{u}_i^+ \mid g_i^+ > 0 \right\rangle_{\mathbb{Z}[\kappa]}.$$

The algebra $\mathbb{A}_\tau$ will be used as the ring of coefficients for the Ozsváth-Szabó complex associated with the weakly balanced sutured manifold (X, τ). There is a quotient of $\mathbb{A}_\tau$ which is of interest in this paper as well:

$$\widehat{\mathbb{A}}_\tau := \frac{\mathbb{Z}[\kappa]}{\left\langle \mathfrak{u}_i^- \mid i = 1, \dots, k \right\rangle_{\mathbb{Z}[\kappa]} + \left\langle \mathfrak{u}_i^+ \mid i = 1, \dots, l \right\rangle_{\mathbb{Z}[\kappa]}}.$$

It is clear that both $\mathbb{A} = \mathbb{A}_\tau$ and $\widehat{\mathbb{A}} = \widehat{\mathbb{A}}_\tau$ are finitely generated algebras over $\mathbb{Z}$, which are generated, as a module over $\mathbb{Z}$, by elements of the form $\prod_{i=1}^{\kappa} \mathfrak{u}_i^{a_i}$, where a_i are non-negative integers. We will denote the set of all such monomials by $G(\mathbb{A})$.

The algebra $\mathbb{A}_\tau$ may be trivial (i.e. equal to zero). In particular, this would be the case if $\mathfrak{R}(\tau)$ contains a closed component of positive genus. When (X, τ) is balanced, there is a quotient map

$$q_\tau : \mathbb{A}_\tau \longrightarrow \mathbb{Z}$$

which gives $\mathbb{Z}$ the structure of an $\mathbb{A}_\tau$-module. This homomorphism is defined by sending any non-trivial monomial $\mathfrak{u} \in G(\mathbb{A}_\tau) \setminus \{1\}$ to zero (and extending this map to a homomorphism). In particular, $\mathbb{A}_\tau$ is non-trivial in this case. If $\mathbb{A}_\tau = 0$, the corresponding chain complex would be automatically trivial. We thus choose not to exclude this case.

One may define a natural map, which we call the Poincaré duality character, from $G(\mathbb{A})$ to the $\mathbb{Z}$-module $\mathbb{H} := \mathrm{H}^2(X, \partial X, \mathbb{Z})$ by

$$\chi : G(\mathbb{A}) \longrightarrow \mathbb{H} = \mathrm{H}^2(X, \partial X; \mathbb{Z}),$$
$$\chi\left(\prod_{i=1}^{\kappa} \mathfrak{u}_i^{a_i}\right) := a_1\mathrm{PD}[\tau_1] + \cdots + a_\kappa \mathrm{PD}[\tau_\kappa], \quad \forall\ a_1, \ldots, a_\kappa \in \mathbb{Z}^{\geq 0}.$$

As defined, χ is just a map from the set of generators for $\mathbb{Z}[\kappa]$ to $\mathrm{H}^2(X, \partial X, \mathbb{Z})$. However, since $\chi(\mathfrak{u}(R_i^-)) = -\mathrm{PD}[\partial R_i^-] = 0$ and $\chi(\mathfrak{u}(R_j^+)) = \mathrm{PD}[\partial R_j^+] = 0$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, l$, the map is well-defined on $G(\mathbb{A})$. The Poincaré duality character gives the filtration of $\mathbb{A}$ by $\mathbb{H} = \mathrm{H}^2(X, \partial X; \mathbb{Z})$.

We may also define a map from the set of positive Whitney disks associated with a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for (X, τ) to $G(\mathbb{A})$ by computing the local multiplicities of the domain associated with each disk at the marked points in $\mathbf{z}$:

$$\mathfrak{u} = \mathfrak{u}_{\mathbf{z}} : \coprod_{\mathbf{x},\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \pi_2^+(\mathbf{x}, \mathbf{y}) \longrightarrow G(\mathbb{A})$$
$$\mathfrak{u}(\phi) := \prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_{z_i}(\phi)}, \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \ \text{ and } \ \forall\ \phi \in \pi_2^+(\mathbf{x}, \mathbf{y}).$$

The composition $\chi(\mathfrak{u}(\phi))$ in $\mathrm{H}^2(X, \partial X, \mathbb{Z})$ will be denoted by $\widehat{H}(\phi)$ for any $\phi \in \pi_2^+(\mathbf{x}, \mathbf{y})$. Of course, the definition of $\widehat{H}(\phi)$ may be extended to arbitrary $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ by setting

$$\widehat{H}(\phi) = \sum_{i=1}^{\kappa} n_{z_i}(\phi)\mathrm{PD}[\tau_i].$$

Thus, Corollary 2.9 may be re-stated as

$$\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \quad \Rightarrow \quad \underline{\mathfrak{s}}(\mathbf{x}) - \underline{\mathfrak{s}}(\mathbf{y}) = \widehat{H}(\phi).$$

The algebra introduced above depends on $(\partial X, \tau)$. Below, we will compute the algebra in some of the familiar cases, and a few other interesting instances.

Example 3.9. (a) If ∂X is a union of n standard spheres and

$$\tau = \{\tau_1, \ldots, \tau_n\},$$

where τ_i is the equator of the i-th sphere, we will have

$$\mathbb{A}_\tau = \mathbb{Z}[\mathfrak{u}_1, \ldots, \mathfrak{u}_n].$$

(b) If ∂X is a torus T, and if τ consists of $2n$ parallel simple closed curves $\tau_1, \ldots, \tau_{2n}$, we will have

$$\mathbb{A}_\tau = \frac{\mathbb{Z}\big[\mathfrak{u}_1, \ldots, \mathfrak{u}_{2n}\big]}{\big\langle \mathfrak{u}_1\mathfrak{u}_2 + \cdots + \mathfrak{u}_{2n-1}\mathfrak{u}_{2n} = \mathfrak{u}_2\mathfrak{u}_3 + \cdots + \mathfrak{u}_{2n}\mathfrak{u}_1 \big\rangle}$$

In particular, for $n = 1$ the sutured manifold (X, τ) determines the complement of a knot K inside the three-manifold Y so that $X = Y - \mathrm{nd}(K)$ such that τ_1 and τ_2 represent two meridians for K with opposite orientation. The above relation is trivial in this case and $\mathbb{A}_\tau = \mathbb{Z}[\mathfrak{u}_1, \mathfrak{u}_2]$ is the coefficient ring used by Ozsváth and Szabó in defining $\mathrm{CF}^-(Y, K)$. There is an obvious quotient map

$$\mathbb{A}_\tau \longrightarrow \frac{\mathbb{Z}[\mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_{2n}]}{\big\langle \mathfrak{u}_{2i-1} = \mathfrak{u}_0 \mid i = 1, \ldots, n \big\rangle} = \mathbb{Z}[\mathfrak{u}_0, \mathfrak{u}_2, \ldots, \mathfrak{u}_{2n}].$$

The algebra $\mathbb{Z}[n+1]$ may thus be used as the coefficient ring as well.

(c) If $p_1, \ldots, p_n$ denote n distinct points on the standard sphere S_1 and $q_1, \ldots, q_n$ denote n distinct points on a second copy S_2 of the standard sphere, one may obtain a surface of genus $n-1$ by connecting S_1 and S_2 via n one-handles, so that i-th one-handle is attached to a neighbourhood of $\{p_i, q_i\}$. The cores of these one-handles determine a set of sutures $\tau = \{\tau_1, \ldots, \tau_n\}$ on the resulting surface S. The corresponding algebra is $\mathbb{Z}[\mathfrak{u}_1, \ldots, \mathfrak{u}_n]$.

(d) Consider a bipartite (2-colorable) graph G, with a coloring of its vertices with $+$ and $-$. We thus have a partition of the vertices $V(G)$ of G as $V(G) = V^+(G) \cup V^-(G)$. Adding valence zero vertices we may assume that the number of $+$ and $-$ vertices are equal. Let $S = S(G)$ denote the surface obtained as the boundary of a neighbourhood of G in $\mathbb{R}^3$, and $\tau = \tau(G)$ denote the set of simple closed curves which correspond to meridians of the edges of G (thus, corresponding to any edge $e \in E(G)$ we have a suture

$\tau_e \in \tau$). Associated with any vertex $v \in V(G)$ we thus have a monomial

$$\mathfrak{u}(v) = \prod_{\substack{e\in E(G)\\ v\in e}} \mathfrak{u}_e \in \mathbb{Z}\big[G\big] := \mathbb{Z}\big[\mathfrak{u}_e \mid e \in E(G)\big].$$

The algebra $\mathbb{A}_G = \mathbb{A}_{\tau(G)}$ may then be described as

$$\mathbb{A}_G = \frac{\mathbb{Z}[G]}{\Big\langle \sum_{v\in V^-(G)} \mathfrak{u}(v) = \sum_{v\in V^+(G)} \mathfrak{u}(v)\Big\rangle_{\mathbb{Z}[G]}}.$$

All previous cases are special cases of this construction. In particular, if G is the graph with three vertices $1, 2$ and 3 and two edges e_1 and e_2 such that e_1 connects 1 and 2 and e_2 connects 1 and 3, we will have

$$\mathbb{A}_G = \frac{\mathbb{Z}[\mathfrak{u}_1, \mathfrak{u}_2]}{\big\langle (\mathfrak{u}_1 - 1)(\mathfrak{u}_2 - 1)\big\rangle},$$

which corresponds to Example 4 from the introduction.

Let (X, τ) be a weakly balanced sutured manifold,

$$H = \big(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z} = \{z_1, \dots, z_\kappa\}\big)$$

be a Heegaard diagram associated with it, and $\mathbb{A}_\tau$ be the corresponding algebra. Associated with the Heegaard diagram is a free $\mathbb{A}_\tau$-module generated by the intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. We will denote this free $\mathbb{A}_\tau$-module by $\mathrm{CF}(X, \tau; H)$. We thus have

$$\begin{aligned}\mathrm{CF}(X, \tau; H) :&= \big\langle \mathbf{x} \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\big\rangle_{\mathbb{A}_\tau}\\ &= \big\langle a.\mathbf{x} \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \ \text{ and } \ a \in G(\mathbb{A}_\tau)\big\rangle_{\mathbb{Z}}.\end{aligned}$$

The assignment of relative Spin^c structures in $\mathbb{S} = \mathbb{S}_\tau = \mathrm{Spin}^c(X, \tau)$ to the intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ by the map $\underline{\mathfrak{s}} = \underline{\mathfrak{s}}_{\mathbf{z}}$ may thus be regarded as a filtration on the Ozsváth-Szabó module $\mathrm{CF}(X, \tau; H)$.

4. Admissible Heegaard diagrams

4.1. The notion of $\mathfrak{s}$-admissibility

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z} = \{z_1, \ldots, z_\kappa\})$ be a Heegaard diagram for the weakly balanced sutured manifold $(X, \tau = \{\tau_1, \ldots, \tau_\kappa\})$. As before, we let

$$\Sigma - \boldsymbol{\alpha} = \coprod_{i=1}^{k} A_i \quad \text{and} \quad \Sigma - \boldsymbol{\beta} = \coprod_{i=1}^{l} B_i.$$

Definition 4.1. Let $\overline{X} = X(1, \ldots, \kappa)$ be the three manifold obtained by filling the sutures in τ. For $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$, a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ of (X, τ) is called $\mathfrak{s}$-admissible if for any nontrivial periodic domain $\mathcal{P}$ with the property $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 0$ one of the following happens,

(a) There is a point $w \in \Sigma$ such that $n_w(\mathcal{P}) < 0$.

(b) We have $\mathcal{P} \geq 0$ and $\mathfrak{u}(\mathcal{P}) = 0$ in $\mathbb{A}$.

Lemma 4.2. *For $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$, let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ be an $\mathfrak{s}$-admissible Heegaard diagram for the weakly balanced sutured manifold (X, τ). Then for any two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\underline{\mathfrak{s}}(\mathbf{x}), \underline{\mathfrak{s}}(\mathbf{y}) \in \mathfrak{s} \subset \mathrm{Spin}^c(X, \tau)$, and for any integer j there are only finitely many $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ such that $\mu(\phi) = j$, $\mathcal{D}(\phi) \geq 0$ and $\mathfrak{u}(\phi) \neq 0$.*

Proof. Suppose that, for an integer j, there are infinitely many $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ such that $\mu(\phi) = j$, $\mathcal{D}(\phi) \geq 0$ and $\mathfrak{u}(\phi) \neq 0$. Fix an element $\phi_0 \in \pi_2(\mathbf{x}, \mathbf{y})$ with these properties. Any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with these properties can then be written as $\phi = \phi_0 + \mathcal{P}$ where $\mathcal{P} \in \pi_2(\mathbf{x}, \mathbf{x})$ and $\mu(\mathcal{P}) = \langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 0$. Thus the set

$$Q = \left\{\mathcal{P} \in \pi_2(\mathbf{x}, \mathbf{x}) \;\middle|\; \mu(\mathcal{P}) = 0,\ \mathcal{P} + \mathcal{D}(\phi_0) \geq 0,\ \mathfrak{u}(\phi_0 + \mathcal{P}) \neq 0\right\}$$

is not finite. Let m denote the total number of domains in $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$, and D_i, for $i = 1, \ldots, m$, denote the corresponding domains. Consider Q as a subset of the set of all lattice points in the vector space

$$V = \left\langle \mathcal{P} \in \pi_2(\mathbf{x}, \mathbf{x}) \;\middle|\; \langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 0 \right\rangle_{\mathbb{R}} \subset \mathbb{R}^m.$$

Here, $\mathbb{R}^m$ is the vector space generated by the components $D_i, i = 1, \ldots, m$ over $\mathbb{R}$. If Q is not finite, there is a sequence $(\mathcal{P}_i)_{i=1}^{\infty}$ in Q, such that

$\|\mathcal{P}_i\| \to \infty$. By passing to a subsequence if necessary, we may assume that the sequence $\left\{\frac{\mathcal{P}_i}{\|\mathcal{P}_i\|}\right\}_{i=1}^{\infty}$ is convergent on the unit ball inside V. Since $\|\mathcal{P}_i\| \to \infty$, and $\mathcal{P}_i + \mathcal{D}(\phi_0) \geq 0$, the sequence converges to a real vector in $\mathbb{R}^m$ with non-negative entries. Denote the limit of $\left(\frac{\mathcal{P}_i}{\|\mathcal{P}_i\|}\right)$ by $\widetilde{\mathcal{P}}$, which is a periodic domain with non-negative real entries.

For a rational periodic domain $\mathcal{P}$ such that $N\mathcal{P}$ is integral define

$$\mu(\mathcal{P}) := \mu(N\mathcal{P})/N = \frac{\langle c_1(\mathfrak{s}), H(N\mathcal{P})\rangle}{N}.$$

There is a positive rational periodic domain $\mathcal{P}$, sufficiently closed to $\widetilde{\mathcal{P}}$, such that

- The Maslov index $\mu(\mathcal{P})$ is zero, i.e. $\mathcal{P} \in V$
- If the coefficient of $\widetilde{\mathcal{P}}$ in some domain D_i is zero, the coefficient of $\mathcal{P}$ in D_i is zero as well.

Thus for a sufficiently large number M, $M\widetilde{\mathcal{P}} - \mathcal{P}$ is a positive periodic domain. After multiplying $\mathcal{P}$ with an appropriate positive integer N, we obtain a positive periodic domain $N\mathcal{P}$ with integral coefficients, and with Maslov index zero i.e. $\langle c_1(\mathfrak{s}), H(N\mathcal{P})\rangle = 0$. The $\mathfrak{s}$-admissibility condition implies that $\mathfrak{u}(N\mathcal{P}) = 0$. Since

$$\lim_{i\to\infty} \frac{MN}{\|\mathcal{P}_i\|}(\mathcal{D}(\phi_0) + \mathcal{P}_i) = MN\widetilde{\mathcal{P}},$$

and $\mathcal{D}(\phi_0) + \mathcal{P}_i \geq 0$, there exists a sufficiently large integer $K > 0$ such that

$$\frac{MN}{\|\mathcal{P}_i\|}(\mathcal{D}(\phi_0) + \mathcal{P}_i) - N\mathcal{P} \geq 0, \quad \forall\, i > K.$$

Note that $\|\mathcal{P}_i\| \to \infty$, thus for a sufficiently large K we have

$$\frac{MN}{\|\mathcal{P}_i\|} \ll 1 \quad \text{and} \quad (\mathcal{D}(\phi_0) + \mathcal{P}_i) - N\mathcal{P} \geq 0, \quad \forall\, i > K.$$

The equality $\mathfrak{u}(N\mathcal{P}) = 0$ implies that $\mathfrak{u}(\phi_i) = \mathfrak{u}(\mathcal{D}(\phi_0) + \mathcal{P}_i) = 0$ for any $i > K$, which is in contradiction with the assumption that the map $\mathfrak{u}$ is nonzero over the classes ϕ_i. □

Remark 4.3. When we use a test ring $\mathbb{B}$ for $\mathbb{A}_\mathcal{T}$ (which comes together with a ring homomorphism $\rho_\mathbb{B} : \mathbb{A}_\mathcal{T} \to \mathbb{B}$) as the ring of coefficients for the chain complex, it suffices to assume that the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$

is admissible in the following weaker sense: If $\mathcal{P}$ is a periodic domain with $\mathcal{P} \geq 0$ and $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 0$, then $\rho_{\mathbb{B}}(\mathfrak{u}(\mathcal{P})) = 0$. In particular, when (X, τ) is balanced and $\mathbb{B} = \mathbb{Z}$ this gives us the notion of weak admissibility used by Juhász [Ju1]. More generally, define

$$\mathbb{B}_\tau = \frac{\mathbb{Z}[\mathfrak{u}_1, \ldots, \mathfrak{u}_\kappa]}{\big\langle \mathfrak{u} \in G(\mathbb{A}_\tau) \setminus \{1\} \mid \chi(\mathfrak{u}) \text{ is torsion}\big\rangle}.$$

Clearly $\mathbb{B}_\tau$ is a quotient of $\mathbb{A}_\tau$. Let us denote the quotient map by

$$\rho_\tau : \mathbb{A}_\tau \longrightarrow \mathbb{B}_\tau.$$

Any positive periodic domain $\mathcal{P}$ with $\mathfrak{u}_{\mathbf{z}}(\mathcal{P}) = \prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_i}$ determines a 2-chain in X with boundary equal to $\sum_{i=1}^{\kappa} n_i \tau_i$. This implies that $\rho_\tau(\mathfrak{u}_{\mathbf{z}}(\mathcal{P})) = 0$, unless $n_1 = n_2 = \cdots = n_\kappa = 0$. Thus, the notion of admissibility for the coefficient ring $\mathbb{B}_\tau$ is a direct consequence of weak admissibility in the sense of Juhász [Ju1].

4.2. Existence of $\mathfrak{s}$-admissible Heegaard diagrams

Performing special isotopies on the curves in $\boldsymbol{\alpha}$, as in [OS5], produces $\mathfrak{s}$-admissible Heegaard diagrams.

Definition 4.4. Let γ be an oriented simple closed curve in Σ. Consider the coordinate system $(t, \theta) \in (-\epsilon, \epsilon) \times S^1$ in a neighbourhood of $\gamma = \{0\} \times S^1$. The diffeomorphism of Σ obtained by integrating a vector field ζ supported in this neighbourhood of γ with the property $d\theta(\zeta) > 0$ is called *winding* along γ. Let α be a simple closed curve which intersects γ in one point and ϕ be a winding around γ. If $\phi(\alpha)$ transversely intersects α in $2n$ points then we say that ϕ is an isotopy which *winds* α n*-times around* γ.

Lemma 4.5. *Let (X, τ) be a weakly balanced sutured manifold as before, $\overline{X}$ be the three-manifold obtained from (X, τ) by filling the sutures, and $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$ be a Spin^c-structure. Then (X, τ) admits an $\mathfrak{s}$-admissible Heegaard diagram. Moreover, every Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for (X, τ) may be modified to an $\mathfrak{s}$-admissible Heegaard diagram by performing isotopies (supported away from the marked points) on the curves in $\boldsymbol{\alpha}$.*

Proof. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ be a Heegaard diagram for (X, τ). Let

$$\Sigma - \boldsymbol{\alpha} = \coprod_{i=1}^{k} A_i \quad \text{and} \quad \Sigma - \boldsymbol{\beta} = \coprod_{i=1}^{l} B_i,$$

be the connected components in the complement of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively. It suffices to arrange that $\alpha_i \cap \beta_j \neq \emptyset$ for every $i, j \in \{1, \dots, \ell\}$. For any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ we can find a curve γ such that γ connects ∂A_i to ∂B_j, and avoids A_i and B_j, provided that $A_i \cap B_j = \emptyset$. By finger moving those α curves which intersect the curve γ (simultaneously) along γ, we create a new Heegaard diagram with the property that $A_i \cap B_j \neq \emptyset$. Repeating this procedure for all $i = 1, \dots, k$ and $j = 1, \dots, l$, we may thus assume that

$$A_i \cap B_j \neq \emptyset, \quad \forall\, i = 1, \dots, k,\ j = 1, \dots, l.$$

Let $\boldsymbol{\alpha}_0$ be a set of disjoint simple closed curves on Σ, disjoint from $\boldsymbol{\alpha}$, such that $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\alpha}_0$ has the same number of connected components as $\Sigma - \boldsymbol{\alpha}$, and all of its connected components have genus zero. Furthermore, let $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$ where

$$\boldsymbol{\alpha}_2 = \big\{\alpha_i \in \boldsymbol{\alpha} \mid \exists j,\ \alpha_i \subset \partial A_j\big\}.$$

For $i = 0, 1, 2$, let us denote the number of elements in $\boldsymbol{\alpha}_i$ by ℓ_i. Thus, in particular, $\ell = \ell_1 + \ell_2$.

We define a graph G with k vertices corresponding to the components $A_1, \dots, A_k$. The edges of G correspond to the elements of $\boldsymbol{\alpha}_2$, i.e. if $\alpha \in \boldsymbol{\alpha}_2$ is a curve in $\partial A_i \cap \partial A_j$ for $i \neq j$ we put an edge in G connecting A_i to A_j associated with α. If $= \Sigma[\boldsymbol{\alpha}_1]$ is the surface obtained from Σ by surgering out the elements of $\boldsymbol{\alpha}_1$, each loop in G corresponds to a homologically nontrivial simple closed curve in $\Sigma[\boldsymbol{\alpha}_1]$ which is disjoint from $\boldsymbol{\alpha}_0$. In other words, each loop in G corresponds to a homologically nontrivial simple closed curve in $\Sigma[\boldsymbol{\alpha}_0 \cup \boldsymbol{\alpha}_1]$. Furthermore, $h = \dim(\mathrm{H}_1(G, \mathbb{Z}))$ is the genus of $\Sigma[\boldsymbol{\alpha}_0 \cup \boldsymbol{\alpha}_1]$. One may thus compute $h = \ell_2 - k + 1$.

Consider a set of pairwise disjoint simple closed curves

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}_1 \cup \boldsymbol{\gamma}_2 = \big\{\gamma^1, \dots, \gamma^{\ell_1}\big\} \bigcup \big\{\gamma^{\ell_1+1}, \dots, \gamma^{\ell-k+1}\big\}$$

on Σ with the following properties. First of all, we assume that $\boldsymbol{\gamma}_1$ is a dual set for $\boldsymbol{\alpha}_1$ i.e. each element of $\boldsymbol{\gamma}_1$ intersects exactly one element of $\boldsymbol{\alpha}_1$ with intersection number one, and for each element of $\boldsymbol{\alpha}_1$ there is one element of $\boldsymbol{\gamma}_1$ intersecting it (with intersection number one). Furthermore, we assume that

$$\boldsymbol{\gamma}_1 \cap \boldsymbol{\alpha}_0 = \boldsymbol{\gamma}_1 \cap \boldsymbol{\alpha}_2 = \emptyset.$$

The set $\boldsymbol{\gamma}_2$ corresponds to a basis for $\mathrm{H}_1(G, \mathbb{Z})$ which is a set of disjoint, oriented, and linearly independent simple closed curves on $\Sigma[\boldsymbol{\alpha}_0 \cup \boldsymbol{\alpha}_1]$. There is a one to one map $i : \boldsymbol{\gamma}_2 \to \boldsymbol{\alpha}_2$ with the property that for each $\gamma \in \boldsymbol{\gamma}_2$ the curve $i(\gamma)$ has nonempty intersection with γ. In fact if this is not true, by Hall's theorem there is a subset of $\boldsymbol{\gamma}_2$ with n elements, say $\{\gamma^{i_1}, \dots, \gamma^{i_n}\} \subset \boldsymbol{\gamma}_2$, such that for

$$A = \left\{\alpha \in \boldsymbol{\alpha}_2 \;\middle|\; \exists j \in \{1, \dots, n\} \;\; s.t. \;\; \alpha \cap \gamma^{i_j} \neq \emptyset\right\}$$

we have $|A| < n$. Since the sum of the genera of the connected components of $\Sigma[\boldsymbol{\alpha}]$ is ℓ_0, $\Sigma[\boldsymbol{\alpha} - A]$ is a surface whose genus is less than or equal to $|A| + \ell_0$. Furthermore, the curves in $\{\gamma^{i_1}, \dots, \gamma^{i_n}\} \cup \boldsymbol{\alpha}_0$ are linearly independent in

$$\mathrm{H}_1\big(\Sigma[\boldsymbol{\alpha} - A], \mathbb{Z}\big).$$

Thus the genus of $\Sigma[\boldsymbol{\alpha} - A]$ is at least $n + \ell_0$ which is in contradiction with the assumption $|A| < n$.

Choose a parallel copy of each curve γ^i for $i = 1, \dots, \ell - k + 1$, with the opposite orientation and denote it by $\overline{\gamma}^i$. We will assume that $\overline{\gamma}^i$ is drawn on Σ very close to γ^i. Let $v_i \in \gamma^i$ be points which are not contained in any of the $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ curves for any $1 \leq i \leq \ell + k - 1$ and denote the corresponding points on $\overline{\gamma}^i$ by $\overline{v}_i$. For any integer N, by *winding the α curves N times along the γ curves* we mean winding all the α-curves which cut γ^i (and hence $\overline{\gamma}^i$) N times along γ^i and N times along $\overline{\gamma}^i$, for any of the curves $\gamma^i, i = 1, \dots, \ell - k + 1$. The windings around either of γ^i and $\overline{\gamma}^i$ will be done simultaneously for all the α curves, so that the new α-curves remain disjoint from each other.

Let Q be the $\mathbb{Q}$-vector space generated by the periodic domains $\mathcal{P}$ such that

$$\mu(\mathcal{P}) = \langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 0.$$

One may write Q as a direct sum

$$Q = (Q \cap \langle A_1, \dots, A_k, B_1, \dots, B_l\rangle_{\mathbb{Q}}) \oplus P,$$

for some subspace P of Q which is generated by the periodic domains $\{\mathcal{P}_1, \dots, \mathcal{P}_b\}$. Thus any periodic domain $\mathcal{P}$ in the vector space Q is of the form

$$\mathcal{P} = \sum_{i=1}^{b} q_i\mathcal{P}_i + \sum_{i=1}^{k} a_iA_i + \sum_{i=1}^{l} b_iB_i,$$

where of course the coefficients a_i and b_j for $i = 1, \ldots, k$ and $j = 1, \ldots, l$ should satisfy the relation

$$\mu\left(\sum_{i=1}^{k} a_i A_i + \sum_{i=1}^{l} b_i B_i\right) = \sum_{i=1}^{k} a_i(2 - 2g(A_i)) + \sum_{i=1}^{l} b_i(2 - 2g(B_i)) = 0.$$

In the above expression $g(A_i)$ and $g(B_j)$ denote the genus of A_i and B_j respectively, for $i = 1, \ldots, k$ and $j = 1, \ldots, l$.

Corresponding to any curve $\gamma^i \in \boldsymbol{\gamma}$ we define a map p_{γ^i} from P to $\mathbb{Q}$. If $\mathcal{P} \in P$ is a periodic domain and

$$\partial\mathcal{P} = \sum_{i=1}^{\ell} p_i \alpha_i + \sum_{i=1}^{\ell} q_i \beta_i = \partial_\alpha \mathcal{P} + \partial_\beta \mathcal{P},$$

we may define the functions p_{γ^i} by

$$p_{\gamma^i}(\mathcal{P}) = \sum_{j=1}^{\ell} p_j.\#(\gamma^i.\alpha_j), \quad \forall\, i \in \{1, \ldots, \ell - k + 1\}.$$

Here $\#(\gamma^i.\alpha_j)$ denotes the intersection number of γ^i with α_j. If for some periodic domain $\mathcal{P} \in P$ we have $p_{\gamma^i}(\mathcal{P}) = 0$, for $1 \leq i \leq \ell - k + 1$, we conclude that

$$\begin{aligned}
&\#(\partial_\alpha \mathcal{P}.\gamma^i) = p_{\gamma^i}(\mathcal{P}) = 0, \quad \forall\, 1 \leq i \leq \ell - k + 1 \\
\Rightarrow\ & \partial_\alpha \mathcal{P} = \partial\big(\sum_{i=1}^{k} a_i A_i\big), \quad \text{for some } a_1, \ldots, a_k \in \mathbb{Q} \\
\Rightarrow\ & \partial\big(\mathcal{P} - \sum_{i=1}^{k} a_i A_i\big) \in \big\langle \beta_1, \ldots, \beta_\ell \big\rangle_{\mathbb{Q}} \\
\Rightarrow\ & \mathcal{P} = \sum_{i=1}^{k} a_i A_i + \sum_{i=1}^{l} b_i B_i, \quad \text{for some } b_1, \ldots, b_l \in \mathbb{Q}
\end{aligned}$$

From the assumption $\mathcal{P} \in P$ we have $\mathcal{P} = 0$. Thus the map

$$\begin{aligned}
&e : P \longrightarrow \mathbb{Q}^{\ell - k + 1} \\
&e(\mathcal{P}) := (p_{\gamma^1}(\mathcal{P}), p_{\gamma^2}(\mathcal{P}), \ldots, p_{\gamma^{\ell-k+1}}(\mathcal{P}))
\end{aligned}$$

is one to one. By a change of basis in P, and changing the order of curves in $\boldsymbol{\gamma}$ if necessary, we can assume that

$$\pi_i(e(\mathcal{P}_j)) = \delta_{ij}, \quad \forall\ 1 \le i, j \le b,$$

where $\pi_i : \mathbb{Q}^{\ell-k+1} \to \mathbb{Q}$ is the projection over the i-th factor.

We would first like to show that for any positive periodic domain $\mathcal{Q}$ in Q, which is not included in the vector space generated by A_i's and B_j's, there is an integer $N = N(\mathcal{Q})$ such that by winding α-curves N times along the curves in $\boldsymbol{\gamma}$ (in both positive and negative directions) the new periodic domain obtained from $\mathcal{Q}$ will have some negative coefficient. Let

$$\mathcal{Q} = \sum_{i=1}^{b} q_i \mathcal{P}_i + \sum_{i=1}^{k} a_i A_i + \sum_{i=1}^{l} b_i B_i$$

be a positive periodic domain in Q such that there is an index i so that $q_i \neq 0$. Then we may choose an integer N such that

$$|q_i| N > \max \left\{ n_{v_i}(\mathcal{Q}), n_{\overline{v}_i}(\mathcal{Q}) \right\}.$$

Wind the α curves N times along γ curves. In the new diagram (obtained after the above winding procedure) let

$$\left\{ \mathcal{P}'_1, \ldots, \mathcal{P}'_b, A'_1, \ldots, A'_k, B'_1, \ldots, B'_l \right\}$$

be the new set of periodic domains obtained from

$$\left\{ \mathcal{P}_1, \ldots, \mathcal{P}_b, A_1, \ldots, A_k, B_1, \ldots, B_l \right\}.$$

For an appropriate choice of the windings we may compute the coefficients of these new domains at v_i and $\overline{v}_i$ from the following equations

$$\begin{aligned} n_{v_i}(A'_j) &= n_{v_i}(A_j), \quad \forall\ j = 1, \ldots, k, \\ n_{v_i}(B'_j) &= n_{v_i}(B_j), \quad \forall\ j = 1, \ldots, l, \\ \text{and} \quad n_{v_i}(\mathcal{P}'_j) &= \begin{cases} n_{v_i}(\mathcal{P}_j) & \textit{if } i \neq j \\ n_{v_i}(\mathcal{P}_j) + N & \textit{if } i = j \end{cases}. \end{aligned}$$

Similar equations are satisfied for the local coefficients at $\overline{v}_i$. In fact, we will have $n_{\overline{v}_i}(\mathcal{P}'_i) = n_{\overline{v}_i}(\mathcal{P}_i) - N$, while the rest of local coefficients remain unchanged. If $q_i < 0$ we thus have

$$n_{v_i}(\mathcal{Q}') = n_{v_i}(\mathcal{Q}) + q_i N < 0,$$

and if $q_i > 0$ then $n_{\overline{v}_i}(\mathcal{Q}') < 0$.

To finish the proof, first suppose that there is an integer N such that, after winding the α curves N times along the curves in $\boldsymbol{\gamma}$, any periodic domain $\mathcal{Q} \in Q$ with integer coefficients either has some negative coefficient or $\mathfrak{u}(\mathcal{Q}) = 0$. Then we are clearly done with the proof of the lemma. So, let us assume otherwise, that for any integer n there exists a periodic domain $\mathcal{Q}_n$ with integer coefficients in Q such that after winding the α curves n times along the curves in $\boldsymbol{\gamma}$, the resulting domain $\mathcal{Q}_n'$ is positive and satisfies $\mathfrak{u}(\mathcal{Q}_n') \neq 0$. Let $\{\mathcal{Q}_n\}_{n=1}^{\infty}$ be the sequence constructed from these elements of Q. As in the proof of Lemma 4.2, after passing to a subsequence if necessary, we may assume that the sequence $\left\{\frac{\mathcal{Q}_n}{\|\mathcal{Q}_n\|}\right\}_{n=1}^{\infty}$ is convergent.

Let us assume that

$$\widetilde{\mathcal{Q}} = \lim_{i\to\infty} \frac{\mathcal{Q}_n}{\|\mathcal{Q}_n\|} \in Q \otimes_{\mathbb{Q}} \mathbb{R}.$$

If $\widetilde{\mathcal{Q}}$ is not in the real vector space generated by A_i's and B_j's, there is an integer N with the property that after winding the α-curves N times along all the curves in $\boldsymbol{\gamma}$, the resulting domain $\widetilde{\mathcal{Q}}'$ will have some negative coefficient. So there is an integer K such that for any $i > K$, $\mathcal{Q}_i'$ has some negative coefficient after winding the α-curves N times along $\boldsymbol{\gamma}$. This is in contradiction with the definition of $\mathcal{Q}_i$ if $i > N$. Thus $\widetilde{\mathcal{Q}}$ may be written as

$$\widetilde{\mathcal{Q}} = \sum_{i=1}^{k} a_i A_i + \sum_{i=1}^{l} b_i B_i \tag{7}$$

for some coefficients a_i and b_i in $\mathbb{R}$.

Note that $\widetilde{\mathcal{Q}} \geq 0$, which implies that for any $w \in \Sigma$, we have

$$\sum_{i=1}^{k} a_i n_w(A_i) + \sum_{i=1}^{l} b_i n_w(B_i) \geq 0.$$

Since $A_i \cap B_j \neq \emptyset$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$, we may pick $w = w_{ij}$ to be a point in this intersection. But for this choice of w, the above inequality reads as $a_i + b_j \geq 0$. If b_j is the smallest of all $b_1, \ldots, b_l$, the above consideration implies that $a_i + b_j \geq 0$ for all $i = 1, \ldots, k$. We may thus compute

$$\widetilde{\mathcal{Q}} = \sum_{i=1}^{k} (a_i + b_j) A_i + \sum_{i=1}^{l} (b_i - b_j) B_i,$$

since $\sum_i A_i = \sum_i B_i = \Sigma$. However, all the coefficients in the above expression are non-negative. As a result, after replacing these new coefficients, we may assume that the real numbers a_i and b_j in Equation 7 are positive.

As in the proof of Lemma 4.2, choose a positive rational periodic domain

$$\mathcal{Q} = \sum_{i=1}^{k} a'_i A_i + \sum_{j=1}^{l} b'_j B_j$$

with $a'_i, b'_j \geq 0$, which is sufficiently close to $\widetilde{\mathcal{Q}}$, and such that the coefficient of $\mathcal{Q} \in Q$ in the domains where $\widetilde{\mathcal{Q}}$ has zero coefficient is zero as well. As before, there are integers N and M such that $N\mathcal{Q}$ is a periodic domain with integer coefficients and $M\widetilde{\mathcal{Q}} - \mathcal{Q} > 0$. Moreover, we may choose N so that

$$Na'_1, \ldots, Na'_k, Nb'_1, \ldots, NB'_l \in \mathbb{Z}.$$

The positivity of the coefficients of $N\mathcal{Q}$ imply that

$$\mathfrak{u}(N\mathcal{Q}) = \left(\prod_{i=1}^{k} \mathfrak{u}(A_i)^{Na'_i} \right) \left(\prod_{j=1}^{l} \mathfrak{u}(B_j)^{Nb'_j} \right) = 0.$$

Moreover, there is some positive integer $K > 0$ such that for $i > K$ we have

$$\begin{aligned} & MN \frac{\mathcal{Q}_i}{\|\mathcal{Q}_i\|} - N\mathcal{Q} \geq 0 \quad \text{and} \quad \frac{MN}{\|\mathcal{Q}_i\|} \ll 1 \\ \Rightarrow \quad & \mathcal{Q}_i - N\mathcal{Q} \geq 0. \end{aligned}$$

This means that for $i > K$, we have

$$\mathfrak{u}(\mathcal{Q}_i) = \mathfrak{u}(\mathcal{Q}_i - N\mathcal{Q})\mathfrak{u}(N\mathcal{Q}) = 0.$$

This is in clear contradiction with our assumption on the integral periodic domains $\mathcal{Q}_i$.

The above argument shows that there is an integer N with the property that after winding the curves in $\boldsymbol{\alpha}$ a total of N times along the curves in $\boldsymbol{\gamma}$

we obtain an $\mathfrak{s}$-admissible Heegaard diagram. This completes the proof of the lemma. □

Remark 4.6. The argument of Lemma 4.5 may be extended to show that for any weakly balanced sutured manifold (X, τ) and any Spin^c class $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$ there is a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ which is admissible in the following stronger sense. If $\mathcal{P}$ is a periodic domain with $\mathcal{P} \geq 0$, and $\mathfrak{u}(\mathcal{P}) \neq 0$ in $\mathbb{A}_\tau$ then

$$\big\langle c_1(\mathfrak{s}), H(\mathcal{P})\big\rangle > 0.$$

When there are genus zero components in $\mathfrak{R}(\tau)$, the above criteria is the same as the $\mathfrak{s}$-admissibility condition. To see this, note that if a Heegaard diagram is $\mathfrak{s}$-admissible and $\mathcal{P}$ is a periodic domain with $\mathcal{P} \geq 0$ and $\mathfrak{u}(\mathcal{P}) \neq 0$ then $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle \neq 0$. If $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle < 0$ we may add a positive multiple $\mathcal{Q}$ of the periodic domain which corresponds to the genus zero component to $\mathcal{P}$ so that $\langle c_1(\mathfrak{s}), H(\mathcal{P}+\mathcal{Q})\rangle = 0$. This contradiction with the $\mathfrak{s}$-admissibility criteria implies that $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle > 0$. Nevertheless, in certain situations where all the connected components of $\mathfrak{R}(\tau)$ have positive genus, using such Heegaard diagrams may be useful. We face this situation in Section 7.

5. The chain complex associated with a balanced sutured manifold

5.1. Holomorphic disks and boundary degenerations; orientation issues

Let us assume that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is an $\mathfrak{s}$-admissible Heegaard diagram for a weakly balanced sutured manifold (X, τ) and $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X}^\tau)$. We assume that $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = \ell$ and that $\mathbf{z} = \{z_1, \ldots, z_\kappa\}$. We have already defined $\pi_2(\mathbf{x}, \mathbf{y})$ for any two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. In discussing the analytic aspects of a Floer theory, we need to consider boundary degenerations and sphere bubblings as well. We recall the following definitions from [OS9].

Definition 5.1. Suppose that $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is an arbitrary intersection point. A continuous map

$$\psi : \mathbb{R} \times [0, \infty) \longrightarrow \mathrm{Sym}^\ell(\Sigma)$$

satisfying the boundary conditions

$$\psi\Big(\mathbb{R} \times \{0\}\Big) \subset \mathbb{T}_\alpha,$$

$$\lim_{|s| \to \infty} \psi(s, t) = \mathbf{x} \quad \text{and} \quad \lim_{t \to \infty} \psi(s, t) = \mathbf{x}$$

is called an $\boldsymbol{\alpha}$-*boundary degeneration*. The space of homotopy classes of such maps is denoted by $\pi_2^\alpha(\mathbf{x})$. The space $\pi_2^\beta(\mathbf{x})$ of $\boldsymbol{\beta}$-*boundary degenerations* is defined similarly.

If $\{J_t = \mathrm{Sym}^\ell(j_t)\}_{t\in[0,1]}$ is a generic path of almost complex structure, associated with any $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ we may consider the moduli space $\mathcal{M}(\phi)$ of the representatives

$$u : [0,1] \times \mathbb{R} \to \mathrm{Sym}^\ell(\Sigma)$$

of ϕ which satisfy the time dependent Cauchy-Riemann equation

$$\frac{\partial u}{\partial t}(t,s) + J_t \frac{\partial u}{\partial s}(t,s) = 0, \quad \forall\ (t,s) \in [0,1] \times \mathbb{R}.$$

Similarly, for any $\psi \in \pi_2^\beta(\mathbf{x})$, $\mathcal{N}(\psi)$ consists of the representatives $u : [0,\infty) \times \mathbb{R} \to \mathrm{Sym}^\ell(\Sigma)$ of ψ which are J_0-holomorphic. Also, for any $\psi \in \pi_2^\alpha(\mathbf{x})$, $\mathcal{N}(\psi)$ consists of the representatives $u : [0,\infty) \times \mathbb{R} \to \mathrm{Sym}^\ell(\Sigma)$ of ψ which are J_1- holomorphic. Let $\widehat{\mathcal{N}}(\psi)$ denote the quotient of $\mathcal{N}(\psi)$ under the action of the subgroup

$$\mathbb{G} = \left\{ \left(\begin{array}{cc} a & b \\ 0 & \frac{1}{a} \end{array} \right) \middle|\, a \in \mathbb{R}^+,\ b \in \mathbb{R} \right\} < \mathrm{PSL}_2(\mathbb{R})$$

in either of the above two cases. We define $\mathfrak{n}(\psi) = \#\widehat{\mathcal{N}}(\psi)$ if $\mu(\psi) = 2$ and $\mathfrak{n}(\psi) = 0$ otherwise.

The determinant line bundle associated with the linearization of the (time dependent) Cauchy-Riemann operator over the moduli of representatives of any of the above homotopy classes is trivial. This makes it possible to equip the corresponding moduli space with an orientation. Following Ozsváth and Szabó's approach in [OS5], we may choose a *coherent system of orientations* as follows.

As in the previous sections, let us assume that

$$\Sigma - \boldsymbol{\alpha} = \coprod_{i=1}^{k} A_i \quad \text{and} \quad \Sigma - \boldsymbol{\beta} = \coprod_{j=1}^{l} B_j,$$

where A_i and B_j correspond to the components $R_i^- \subset \mathfrak{R}^-(\tau)$ and $R_j^+ \subset \mathfrak{R}^+(\tau)$ respectively. Thus, the genus of A_i is g_i^- and the genus of B_j is g_j^+. Without loosing on generality, let us assume that $l \geq k$. Let $\mathbf{x}_0, \ldots, \mathbf{x}_m$ be all the intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ which correspond to the Spin^c class

$\mathfrak{s}$. Choose a disk class $\phi_i \in \pi_2(\mathbf{x}_0, \mathbf{x}_i)$ for each i, and complete the classes of the boundary degenerations $A_1, \ldots, A_k$ and $B_1, \ldots, B_{l-1}$ to a basis for the space of periodic domains in $\pi_2(\mathbf{x}_0, \mathbf{x}_0)$. Note that

$$B_l = A_1 + \cdots + A_k - (B_1 + \cdots + B_{l-1})$$

is the only relation satisfied among $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$. Let us denote this basis by $\psi_1, \ldots, \psi_n$. The choice of an orientation (i.e. one of the two classes represented by a non-vanishing section) on the determinant line bundle associated with the classes $\phi_1, \ldots, \phi_m$ and $\psi_1, \ldots, \psi_n$ induces an orientation on the moduli space corresponding to any class $\phi \in \pi_2(\mathbf{x}_i, \mathbf{x}_j)$, $0 \leq i, j \leq m$. In fact, $\phi + \phi_i - \phi_j$ is a periodic domain in $\pi_2(\mathbf{x}_0, \mathbf{x}_0)$, and is thus a linear combination of the classes $\psi_1, \ldots, \psi_n$. As a result, ϕ is a juxtaposition of (possibly several copies of) classes in

$$\big\{\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n\big\},$$

and thus inherits a natural orientation in our system of coherent orientations.

Let us study the boundary degenerations and their assigned orientation more carefully. Any periodic domain $\psi \in \pi_2(\mathbf{x}, \mathbf{x})$ such that $\partial\mathcal{D}(\psi)$ is a union of α-curves determines the class of an α boundary degeneration. Thus, the domain of any α boundary degeneration $\psi \in \pi_2^\alpha(\mathbf{x})$ is a linear combination of $A_1, \ldots, A_k$:

$$\mathcal{D}(\psi) = a_1 A_1 + \cdots + a_k A_k.$$

We may use Lipshitz' index formula to compute the Maslov index of ψ:

$$\mu(\psi) = a_1\chi(A_1) + \cdots + a_k\chi(A_k).$$

If furthermore $\mathcal{D}(\psi)$ is a positive domain, e.g. if ψ is a holomorphic boundary degeneration, then all a_i are non-negative. We may then define the map

$$\mathfrak{u} : \coprod_{\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \Big(\pi_2^{\beta,+}(\mathbf{x}) \coprod \pi_2^{\alpha,+}(\mathbf{x})\Big) \longrightarrow \mathbb{A}_\tau, \qquad \mathfrak{u}(\psi) := \prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_{z_i}(\psi)}.$$

Here, we use $\pi_2^{\alpha,+}(\mathbf{x})$ (respectively, $\pi_2^{\beta,+}(\mathbf{x})$) to denote the subset of $\pi_2^\alpha(\mathbf{x})$ (respectively, $\pi_2^\beta(\mathbf{x})$) which consists of the classes ψ with $\mathcal{D}(\psi) \geq 0$.

If an α boundary degeneration ψ as above is positive and $\mathfrak{u}(\psi) \neq 0$, we may conclude that for $i = 1, \ldots, k$, either $a_i = 0$ or the genus of A_i is zero. Without loosing on generality, assume that the genus of $A_1, \ldots, A_{k_0}$ is zero, and that the rest of A_i have positive genus. Thus $\mathcal{D}(\psi) \geq 0$ and $\mathfrak{u}(\psi) \neq 0$ implies that

$$\mathcal{D}(\psi) = a_1 A_1 + \cdots + a_{k_0} A_{k_0}.$$

Consequently $\mu(\psi) = 2(a_1 + \cdots + a_{k_0})$. Similarly, we may assume that the genera of $B_1, \ldots, B_{l_0}$ are zero, and that the rest of B_i have positive genus. This would imply that for any $\psi \in \pi_2^\beta(\mathbf{x})$ with $\mathcal{D}(\psi) \geq 0$, we will either have $\mathfrak{u}(\psi) = 0$, or

$$\mathcal{D}(\psi) = b_1 B_1 + \cdots + b_{l_0} B_{l_0} \quad \text{and} \quad \mu(\psi) = 2(b_1 + \cdots + b_{l_0}).$$

In Theorem 5.5 from [OS9], Ozsváth and Szabó prove the following statement. In fact, the statement of their result is less general, but the exact same proof applies in the following more general.

Lemma 5.2. *Let ψ be the class of a boundary degeneration, and fix a coherent choice of orientation for the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$. If $\mathcal{D}(\psi) \geq 0$, $\mathfrak{u}(\psi) \neq 0$, and $\mu(\psi) \leq 2$ then $\mathcal{D}(\psi) = A_i$ or $\mathcal{D}(\psi) = B_j$ for some $1 \leq i \leq k_0$ or $1 \leq j \leq l_0$ (or ψ is the class of the constant map). In the first case (i.e. $\mathcal{D}(\psi) = A_i$) we have*

$$\mathfrak{n}(\psi) = \begin{cases} 0 & \text{if } k = 1 \\ \pm 1 & \text{if } k > 1. \end{cases}$$

Similarly, for $\mathcal{D}(\psi) = B_j$ we have $\mathfrak{n}(\psi) = 0$ if $l = 1$ and $\mathfrak{n}(\psi) = \pm 1$ if $l > 1$.

Proof. See [OS9] Theorem 5.5. Note that the moduli spaces are now equipped with an orientation, and we may thus count the points of the moduli spaces with sign, instead of working modulo 2. The choice of the plus or minus sign comes from the choice on the orientation associated with the homotopy classes A_i and B_j of α and β boundary degenerations respectively. □

The argument of Ozsváth and Szabó in fact implies that there is a natural choice of orientation for $A_1, \ldots, A_{k_0}$ and $B_1, \ldots, B_{l_0}$ which makes the value of $\mathfrak{n}(\psi)$ equal to $+1$. After a modification of the complex structure on the surface Σ by stretching the appropriate necks, the moduli space of boundary degenerations associated with any of $A_1, \ldots, A_{k_0}$ and $B_1, \ldots, B_{l_0}$ may be identified with the group $\mathbb{G}$ via the Riemann mapping theorem. This identification gives an orientation for the index bundle which corresponds to any of $A_1, \ldots, A_{k_0}$ and $B_1, \ldots, B_{l_0}$, which will be called the *preferred* orientation over that index bundle. When we pick our coherent system of orientations we would like to choose the system so that the induced orientation over the above index bundles is the preferred orientation. If $k = k_0$

and $l = l_0$, since $A_1 + \cdots + A_k = B_1 + \cdots + B_l$ choosing the system of orientations so that the preferred orientation is induced over the index bundles corresponding to $A_1, \ldots, A_k, B_1, \ldots, B_{l-1}$ dictates an orientation over the index bundle corresponding to B_l. This dictated orientation matches with the preferred orientation associated with B_l. If $l_0 < l$ (or if $k_0 < k$), since the periodic domains $A_1, \ldots, A_{k_0}, B_1, \ldots, B_{l_0}$ are linearly independent any choice of orientation over the index bundles corresponding to them, including the preferred orientations, may be completed to a coherent system of orientations. We thus present the following definition.

Definition 5.3. A *coherent system of orientations* associated with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for the weakly balanced sutured manifold (X, τ) and the Spin^c class $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X}^\tau)$ is an assignment $\mathfrak{o}$ of an orientation to the determinant line bundle of the linearized Cauchy-Riemann operator associated with all classes in $\pi_2(\mathbf{x}, \mathbf{y})$ (for all $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$) with the following properties:

- $\mathfrak{o}(\phi_1 \star \phi_2)$ is the orientation induced by $\mathfrak{o}(\phi_1)$ and $\mathfrak{o}(\phi_2)$ via juxtaposition, for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ representing $\mathfrak{s}$, and any $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\phi_2 \in \pi_2(\mathbf{y}, \mathbf{z})$.
- For any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ representing $\mathfrak{s}$, any $R_i^- \subset \mathfrak{R}^-(\tau)$ with $g_i^- = 0$, and any $R_j^+ \subset \mathfrak{R}^+(\tau)$ with $g_j^+ = 0$, let us denote by $\psi^- \in \pi_2^\alpha(\mathbf{x})$ and $\psi^+ \in \pi_2^\beta(\mathbf{x})$ the classes of boundary degenerations corresponding to R_i^- and R_j^+ respectively. Then $\mathfrak{o}(\psi^-)$ and $\mathfrak{o}(\psi^+)$ are the preferred orientation of Ozsváth and Szabó on $\mathcal{N}(\psi^-)$ and $\mathcal{N}(\psi^+)$ respectively, which give $\mathfrak{n}(\psi^-) = 1$ and $\mathfrak{n}(\psi^+) = 1$ (if $k > 1$ and $l > 1$ respectively).

The last assumption implies, in particular, that the orientation induced on the moduli space corresponding to the periodic domain determined by Σ is the natural orientation on it, as defined in Section 3.6 of [OS5].

Let us assume that $\psi \in \pi_2^\alpha(\mathbf{x}_0)$ satisfies $\mathcal{D}(\psi) \in \{A_1, \ldots, A_{k_0}\}$, say $\mathcal{D}(\psi) = A_1$. Furthermore, assume that a preferred orientation on $\mathcal{N}(\psi)$ is fixed as before. At the same time ψ may be regarded as a class in $\pi_2(\mathbf{x}_0, \mathbf{x}_0)$, and a moduli space $\mathcal{M}(\psi)$ may also be associated with ψ. This moduli space is smooth and two dimensional as well, and gives an open 1-manifold $\widehat{\mathcal{M}}(\psi)$ after we mod out by the translation action of $\mathbb{R}$. The choice of orientation on $\mathcal{N}(\psi)$ induces an orientation on $\mathcal{M}(\psi)$ as well. The reason is that the determinant line bundle of the (time dependent) Cauchy-Riemann operator on both these moduli spaces is pulled back from the same model, as discussed in Subsection 3.6 in [OS5].

According to the discussion of Section 5 from [OS9], $\widehat{\mathcal{N}}(\psi)$ will then appear as a boundary point of the smooth one dimensional manifold $\widehat{\mathcal{M}}(\psi)$. This induces a second orientation on $\widehat{\mathcal{N}}(\psi)$, as the boundary of the oriented moduli space $\widehat{\mathcal{M}}(\psi)$. Whether this second orientation agrees with the orientation of $\widehat{\mathcal{N}}(\psi)$ as the quotient of $\mathcal{N}(\psi)$ under the action of $\mathbb{G}$ or not depends on our convention for the embedding of the translation group $\mathbb{R}$ (which acts on $\mathcal{M}(\psi)$) in $\mathbb{G}$, as will be discussed in more detail below. The same discussion is valid for β boundary degenerations.

By the Riemann mapping theorem,

$$\mathbb{H}^+ \setminus \{0\} = \big(\mathbb{R} \times [0, +\infty)\big) \setminus \{0\} \subset \mathbb{C}$$

is conformal to the complement of $\{\pm i\}$ in the unit disk, or to the strip $[0, 1] \times \mathbb{R} \subset \mathbb{C}$. We may thus think of $\mathbb{H}^+$ as the domain of the class ψ, when considered as an element in $\pi_2(\mathbf{x}, \mathbf{x})$. We may then fix a real number $r \in \mathbb{R}$ and interpret ψ as a class with

$$\psi\big([r, +\infty) \times \{0\}\big) \subset \mathbb{T}_\alpha \quad \text{and} \quad \psi\big((-\infty, r] \times \{0\}\big) \subset \mathbb{T}_\beta.$$

Furthermore, we have to assume that $\psi(r, 0)$ and $\psi(\infty)$ are both the intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The group $\mathbb{G}$ consists of the maps $\rho_{a,b}$ for $a > 0$ and $b \in \mathbb{R}$ which are defined by

$$\rho_{a,b}(z) := az + b.$$

With this notation fixed, the re-parametrization group of the domain of $\psi \in \pi_2(\mathbf{x}, \mathbf{x})$ is then identified as

$$\mathbb{R}_r = \big\{\rho_{a,b} \,\big|\, a \in \mathbb{R}^+ \text{ and } b = r(1 - a)\big\} < \mathbb{G}.$$

If we identify a as the exponential of a real number, the subgroup $\mathbb{R}_r$ is identified with $\mathbb{R}$. This induces an orientation on the one dimensional subgroup $\mathbb{R}_r$ of $\mathbb{G}$.

As r approaches $-\infty$, the limit of the subgroups $\mathbb{R}_r$ determines the embedding of the translation group in the automorphism group of the domain of α boundary degenerations. In this case, assuming $r \ll 0$ we may write

$$a = 1 - \frac{c}{r}, \quad c \in (r, \infty) \quad \Rightarrow \quad b = c.$$

As c grows large, a grows large as well. Thus the above parametrization of $\mathbb{R}_r$ by the interval $(r, +\infty)$ is orientation preserving. With r converging to

$-\infty$, the sequence $\{\rho_{1-(c/r),c}\}_r$ converges to

$$\rho_{1,c} : \mathbb{H}^+ \to \mathbb{H}^+, \qquad \rho_{1,c}(z) = z + c.$$

The limit of $\mathbb{R}_r$, as $r \to -\infty$, is thus the translation subgroup

$$\mathbb{R}_\alpha = \{\rho_{1,c} \mid c \in \mathbb{R}\} < \mathbb{G},$$

and the above parametrization of $\mathbb{R}_\alpha$ is orientation preserving.

On the other hand, when r approaches $+\infty$, the limit of the subgroups $\mathbb{R}_r$ determines the embedding of the translation group in the automorphism group of the domain of β boundary degenerations. In the this later case, assuming $r >> 0$ we may write

$$a = 1 - \frac{c}{r}, \quad c \in (-\infty, r) \quad \Rightarrow \quad b = c.$$

This time, as c grows large, a becomes small. Thus the above parametrization of $\mathbb{R}_r$ by the interval $(-\infty, r)$ is orientation reversing. With r growing large, the sequence $\{\rho_{1-(c/r),c}\}_r$ converges to

$$\rho_{1,c} : \mathbb{H}^+ \to \mathbb{H}^+, \qquad \rho_{1,c}(z) = z + c.$$

The limit of $\mathbb{R}_r$, as $r \to +\infty$, is thus the translation subgroup

$$\mathbb{R}_\beta = \{\rho_{1,c} \mid c \in \mathbb{R}\} < \mathbb{G},$$

but this time, the above parametrization of $\mathbb{R}_\beta$ is orientation reversing.

With the above conventions for the orientations of $\mathbb{G}$, $\mathbb{R}_r$, $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ fixed, we have thus proved the following lemma.

Lemma 5.4. *Let $\phi \in \pi_2^\alpha(\mathbf{x})$ and $\psi \in \pi_2^\beta(\mathbf{x})$ be the classes of α and β boundary degenerations respectively. Furthermore, assume that $\mu(\phi) = \mu(\psi) = 2$, and that $\mathcal{N}(\phi)$ and $\mathcal{N}(\psi)$ are smooth manifolds. Then the orientation induced on $\widehat{\mathcal{N}}(\phi)$ agrees with the boundary orientation induced from $\widehat{\mathcal{M}}(\phi)$, while the orientation induced on $\widehat{\mathcal{N}}(\psi)$ is the opposite of the boundary orientation induced from $\widehat{\mathcal{M}}(\psi)$.*

5.2. Energy bounds and relative gradings

Recall that for a Riemannian manifold (M, g) and a domain $\Omega \subset \mathbb{C}$ the energy of a smooth map $u : \Omega \to X$ is defined by

$$E(u) = \frac{1}{2}\int_{\Omega} \|du\|_g^2.$$

Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is a Heegaard diagram for a balanced sutured manifold (X, τ). Let

$$\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta} = \coprod_{i=1}^{m} D_i$$

be the connected components in the complement of the curves, and η denotes a Kähler form on Σ. We denote the area of D_i with respect to η by $\text{Area}_\eta(D_i)$, and for a domain $\mathcal{D} = \sum_{i=1}^{m} a_i D_i$ we define

$$\text{Area}_\eta(\mathcal{D}) = \sum_{i=1}^{m} a_i \text{Area}_\eta(D_i).$$

The following lemma is basically Lemma 3.5 from [OS5] and Theorem 6.3 from [Ju1].

Lemma 5.5. *There is a constant C which depends only on the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ and the Kähler form η such that for any pseudo-holomorphic Whitney disk*

$$u : (\mathbb{D}, \partial\mathbb{D}) \longrightarrow (\text{Sym}^{\ell}(\Sigma), \mathbb{T}_\alpha \cup \mathbb{T}_\beta)$$

we have

$$E(u) \leq C.\text{Area}_\eta\left(\mathcal{D}(u)\right).$$

The existence of energy bounds is needed in Gromov compactness arguments.

Finally, note that

$$\pi_1\left(\text{Sym}^{\ell}(\Sigma)\right) = \text{H}_1\left(\text{Sym}^{\ell}(\Sigma); \mathbb{Z}\right)$$

provided that $\ell > 1$. Throughout the construction, we will assume that the requirement $\ell > 1$ is satisfied, by stabilizing Heegaard diagrams if necessary.

Definition 5.6. For $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$ let

$$\mathfrak{d}(\mathfrak{s}) = \mathrm{gcd}_{h\in \mathrm{H}_2(\overline{X},\mathbb{Z})}\big\langle c_1(\mathfrak{s}), h\big\rangle.$$

If $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is an $\mathfrak{s}$-admissible Heegaard diagram for (X,τ) for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\underline{\mathfrak{s}}(\mathbf{x}), \underline{\mathfrak{s}}(\mathbf{y}) \in \mathfrak{s}$, and for $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we define the relative grading of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) \pmod{\mathfrak{d}(\mathfrak{s})}.$$

Thus, $\mathrm{gr}(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_{\mathfrak{d}(\mathfrak{s})} = \frac{\mathbb{Z}}{\mathfrak{d}(\mathfrak{s})\mathbb{Z}}$.

The relative grading is independent of the choice of ϕ. It induces a relative grading on the module $\mathrm{CF}(X, \tau, \mathfrak{s}; H)$. For this purpose, we should determine the grading associated with the generators $\mathfrak{u}_1, \dots, \mathfrak{u}_\kappa \in G(\mathbb{A})$. Each $\mathfrak{u}_i$ corresponds to the class

$$[\tau_i] \in \mathrm{Ker}\left(\imath_* : \mathrm{H}_1(X; \mathbb{Z}) \to \mathrm{H}_1(\overline{X}; \mathbb{Z})\right),$$

where $\imath : X \to \overline{X}$ is the inclusion map. It is thus the boundary of an integral 2-chain $A_i = A_{[\tau_i]}$ in $\overline{X}$, which is well defined up to addition of 2-cycles. The evaluation $d_i = -\langle c_1(\mathfrak{s}), A_i\rangle$ is thus well-defined as an element of $\frac{\mathbb{Z}}{\mathfrak{d}(\mathfrak{s})\mathbb{Z}}$. We may then define the grading on $G(\mathbb{A})$ by setting

$$\mathrm{gr}\left(\prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_i}\right) := \sum_{i=1}^{\kappa} d_i n_i \in \frac{\mathbb{Z}}{\mathfrak{d}(\mathfrak{s})\mathbb{Z}}, \quad \forall \prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_i} \in G(\mathbb{A}).$$

If $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$ is a positive disk, it determines a periodic domain, and

$$\mu(\phi) = \big\langle c_1(\mathfrak{s}), H(\phi)\big\rangle = \mathrm{gr}\left(\mathfrak{u}_{\mathbf{z}}(\phi)\right) \pmod{\mathfrak{d}(\mathfrak{s})}.$$

The $\mathbb{A}$-module $\mathrm{CF}(X, \tau, \mathfrak{s}; H)$ is thus equipped with a relative homological grading by the elements in $\frac{\mathbb{Z}}{\mathfrak{d}(\mathfrak{s})\mathbb{Z}}$. The differential of the corresponding Ozsváth-Szabó complex $\mathrm{CF}(X, \tau, \mathfrak{s}; H)$ which will be defined in the following subsection lowers this relative grading by one. In particular, a relative grading is induced on the homology groups corresponding to any test ring $\mathbb{B}$ for $\mathbb{A}_\tau$.

5.3. The construction of the chain complex

Let (X,τ) be a weakly balanced sutured manifold. As discussed in Section 3 we associate a coefficient ring $\mathbb{A} = \mathbb{A}_\tau$ with τ, which is a $\mathbb{Z}$-algebra. Let us denote by $\overline{X}$ the three-manifold (with positive and negative boundary components) obtained from X by filling out the sutures in τ. Let $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$ be a Spin^c structure on $\overline{X}$. Consider an $\mathfrak{s}$-admissible Heegaard diagram $H=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ for (X,τ). Associated with this Heegaard diagram, let

$$\mathrm{CF}(X,\tau,\mathfrak{s};H)=\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})=\big\langle \mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta \mid \underline{\mathfrak{s}}_{\mathbf{z}}(\mathbf{x})\in\mathfrak{s}\big\rangle_{\mathbb{A}}$$

be a free $\mathbb{A}$-module generated by the intersection points in $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ which represent the Spin^c class $\mathfrak{s}$. For every subset $I\subset\{1,\ldots,\kappa\}$ let

$$s_I:\mathrm{Spin}^c(X,\tau)\to\mathrm{Spin}^c(X(I),\tau(I))$$

denote the map defined in Subsection 2.1. Note that the "exact sequence"

$$0\longrightarrow \big\langle \mathrm{PD}[\tau_i]\big\rangle_{i=1}^{\kappa}\longrightarrow \mathrm{Spin}^c(X,\tau)\xrightarrow{[.]=s_{\{1,\ldots,\kappa\}}}\mathrm{Spin}^c(\overline{X})\longrightarrow 0$$

implies that the assignment of relative Spin^c structures gives a filtration on the $\mathbb{A}$-module $\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})$, which is compatible with the filtration $\chi: G(\mathbb{A})\to\mathbb{H}$. This module may be decomposed using the filtration by relative Spin^c structures:

$$\begin{aligned}\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s}) &= \bigoplus_{\underline{\mathfrak{s}}\in\mathfrak{s}\subset\mathrm{Spin}^c(X,\tau)}\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\underline{\mathfrak{s}})\\ \mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\underline{\mathfrak{s}}) &= \left\langle \mathfrak{u}\mathbf{x}\ \middle|\ \begin{matrix}\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta, & \mathfrak{u}\in G(\mathbb{A})\\ \underline{\mathfrak{s}}_{\mathbf{z}}(\mathbf{x})+\chi(\mathfrak{u})=\underline{\mathfrak{s}}\end{matrix}\right\rangle_{\mathbb{Z}}.\end{aligned}\tag{8}$$

Furthermore, fix a coherent system $\mathfrak{o}$ of orientations on the determinant line bundles of the linearization of Cauchy-Riemann operators associated with the classes of the Whitney disks (corresponding to $\mathfrak{s}$). We will drop $\mathfrak{o}$ from the notation, unless an issue related to the orientation should be discussed.

Define an $\mathbb{A}$-module homomorphism by the following equation

$$\begin{aligned}&\partial:\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})\longrightarrow\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})\\ &\partial(\mathbf{x}):=\sum_{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\{\phi\in\pi_2^+(\mathbf{x},\mathbf{y})|\mu(\phi)=1\}}\big(\mathfrak{m}(\phi)\mathfrak{u}(\phi)\big).\mathbf{y}.\end{aligned}$$

Here $\mathfrak{m}(\phi)=\#\widehat{\mathcal{M}}(\phi)$ is the algebraic count (i.e. with the signs determined by the orientation) of the points in $\widehat{\mathcal{M}}(\phi)$ for any class $\phi\in\pi_2^+(\mathbf{x},\mathbf{y})$ such that $\mathfrak{u}(\phi)\neq 0$. For other disk classes, the contribution $\mathfrak{m}(\phi)\mathfrak{u}(\phi)$ is trivial by definition.

It is important to note that for any $\phi \in \pi_2^+(\mathbf{x},\mathbf{y})$ with $\mu(\phi)=1$ and $\mathfrak{u}(\phi)\neq 0$, the moduli space $\widehat{\mathcal{M}}(\phi)$ is smooth, zero dimensional, oriented, and compact. Smoothness and zero dimensionality of the moduli space follows from the generic choice of the path of complex structures on $\mathrm{Sym}^{\ell}(\Sigma)$ (see the general discussion of [OS5], Section 3). The compactness is however more critical, due to boundary degenerations. If u_i is a sequence of pseudo-holomorphic representatives of ϕ, the amount of energy $E(u_i)$ remains bounded by Lemma 5.5. We may thus use the Gromov compactness theorem to describe the possible limits of this sequence. In fact, any possible Gromov limit of the sequence is the juxtaposition of some pseudo-holomorphic representative u of a class $\phi' \in \pi_2^+(\mathbf{x},\mathbf{y})$ with boundary degenerations and sphere bubblings. Let us assume that $v_1,\dots,v_p$ are the classes of degenerations and bubbles. Then the domains of u and v_i, $i=1,\dots,p$ are positive and

$$\mathfrak{u}(\phi)=\mathfrak{u}(u)\mathfrak{u}(v_1)\cdots\mathfrak{u}(v_p)\neq 0.$$

This implies that the domain of each v_i is a linear combination of the domains $A_1,\dots,A_{k_0}$ or $B_1,\dots,B_{l_0}$ (with non-negative coefficients). Here $A_1,\dots,A_{k_0}$ are the genus zero components in $\Sigma-\boldsymbol{\alpha}$ and $B_1,\dots,B_{l_0}$ are the genus zero components in $\Sigma-\boldsymbol{\beta}$. The Maslov index of each v_i is thus a positive even number. Since the moduli spaces $\mathcal{M}(\phi')$ are non-empty, $\mu(\phi')$ is non-negative and p is thus forced to be 0. However, this means that the Gromov limit of u_i is in $\widehat{\mathcal{M}}(\phi)$, i.e. $\widehat{\mathcal{M}}(\phi)$ is compact, and thus finite. In other words, for any class $\phi \in \pi_2^+(\mathbf{x},\mathbf{y})$ with $\mu(\phi)=1$, either ϕ does not contribute to the coefficient of $\mathbf{y}$ in $\partial(\mathbf{x})$ (e.g. $\mathfrak{u}(\phi)=0$), or $\mathfrak{m}(\phi)$ is finite.

The Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})$ is $\mathfrak{s}$-admissible, so by Lemma 4.5, for any intersection point $\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ there are only finitely many $\phi\in\pi_2(\mathbf{x},\mathbf{y})$ such that $\mu(\phi)=1$, $\mathcal{D}(\phi)\geq 0$, and $\mathfrak{u}(\phi)\neq 0$. Thus there are only finitely many classes $\phi\in\pi_2^+(\mathbf{x},\mathbf{y})$ with $\mu(\phi)=1$ and $\mathfrak{u}(\phi)\neq 0$ which admit holomorphic representatives. This shows that the terms which contribute to the coefficient of $\mathbf{y}$ in $\partial(\mathbf{x})$ are finite, and that the map ∂ is thus well-defined.

The map ∂ is, by definition, a homomorphism of $\mathbb{A}$-modules. It is obvious from the definition and the discussion of Subsection 3.3 that ∂ preserves the decomposition of Equation 8, and we thus obtain a set of $\mathbb{Z}$-module homomorphisms

$$\partial:\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\underline{\mathfrak{s}})\longrightarrow\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\underline{\mathfrak{s}}),$$

for any relative Spin^c class $\underline{\mathfrak{s}}\in\mathrm{Spin}^c(X,\tau)$.

Theorem 5.7. *The filtered* $\mathbb{A}$*-module* $\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})$ *is a filtered* $(\mathbb{A},\mathbb{H})$ *chain complex, where* $\mathbb{A}=\mathbb{A}_\tau$ *is the coefficient ring associated with* τ *and the filtration by the elements of the* $\mathbb{Z}$*-module* $\mathbb{H}=\mathrm{H}^2(X,\partial X;\mathbb{Z})$ *is given by the assignment of the relative* Spin^c *classes in* $\mathrm{Spin}^c(X,\tau)$ *to the generators in* $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ *using the map* $\underline{\mathfrak{s}}_{\mathbf{z}}$*.*

Before we start proving the above theorem we re-phrase Lemma 5.2 in the presence of a coherent system of orientation.

Lemma 5.8. *With the notation of Lemma 5.2 fixed, let* $\mathfrak{o}$ *be a coherent system of orientations associated with the Heegaard diagram. Let* ψ *be the class of a boundary degeneration. If* $\mathcal{D}(\psi)\geq 0$*,* $\mathfrak{u}(\psi)\neq 0$*, and* $\mu(\psi)\leq 2$ *then* $\mathcal{D}(\psi)=A_i$ *or* $\mathcal{D}(\psi)=B_j$ *for some* $1\leq i\leq k_0$ *or* $1\leq j\leq l_0$ *(or* ψ *is the class of the constant map). In the first case (i.e.* $\mathcal{D}(\psi)=A_i$*) we have*

$$\mathfrak{n}(\psi)=\begin{cases}0 & \text{if } k=1\\ 1 & \text{if } k>1.\end{cases}$$

Similarly, for $\mathcal{D}(\psi)=B_j$ *we have* $\mathfrak{n}(\psi)=0$ *if* $l=1$ *and* $\mathfrak{n}(\psi)=1$ *if* $l>1$*.*

Now we can prove Theorem 5.7 using the above lemma. The proof is similar to the proof of Lemma 4.3 in [OS9].

Proof of Theorem 5.7. Clearly, for any Whitney disk $\phi\in\pi_2(\mathbf{x},\mathbf{y})$, which has a holomorphic representative, we have $\mathcal{D}(\phi)\geq 0$ and $\mathfrak{u}(\phi)$ is thus a well-defined element of $\mathbb{A}$. Thus we only need to prove $\partial\circ\partial=0$. Let $\mathbf{x}$ and $\mathbf{y}$ be two intersection points in $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$. Fix a class $\phi\in\pi_2(\mathbf{x},\mathbf{y})$ such that $\mu(\phi)=2$. Consider the ends of the moduli space $\widehat{\mathcal{M}}(\phi)$. This space has three types of ends, which are in correspondence with the broken flow-lines. More precisely, these are (respectively)

(1) the ends corresponding to a holomorphic Whitney disk ϕ_1 connecting $\mathbf{x}$ to an intersection point $\mathbf{w}$ juxtaposed with a holomorphic Whitney disk ϕ_2 connecting $\mathbf{w}$ to $\mathbf{y}$ such that $\mu(\phi_1)=\mu(\phi_2)=1$,

(2) the ends corresponding to a sphere bubbling off i.e. a holomorphic Whitney disk ϕ' connecting $\mathbf{x}$ to $\mathbf{y}$ juxtaposed with a holomorphic sphere S in $\mathrm{Sym}^\ell(\Sigma)$, and

(3) the ends corresponding to a boundary bubbling i.e. a holomorphic Whitney disk ϕ' connecting $\mathbf{x}$ to $\mathbf{y}$ juxtaposed with a holomorphic boundary degeneration.

If $\mathbf{x} \neq \mathbf{y}$ the space $\widehat{\mathcal{M}}(\phi)$ does not have any boundary of the second and the third types, since any holomorphic boundary degeneration or holomorphic sphere with the property that its associated monomial in $\mathbb{A}$ is non-trivial will have Maslov index at least 2. Thus the remaining Whitney disk should have Maslov index less than or equal to zero. This implies that the moduli space associated with the Whitney disk is empty, or consists of a constant function (which can not happen by the assumption $\mathbf{x} \neq \mathbf{y}$). When $\mathbf{x} \neq \mathbf{y}$ the Gromov ends of this moduli space thus consist of

$$\coprod_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \coprod_{\substack{\phi_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \phi_2\in\pi_2(\mathbf{w},\mathbf{y})\\ \phi_1*\phi_2=\phi}} \Big(\widehat{\mathcal{M}}(\phi_1) \times \widehat{\mathcal{M}}(\phi_2)\Big)$$

For any fixed $\mathfrak{u} \in G(\mathbb{A})$ the coefficient of $\mathfrak{u}\mathbf{y}$ in $\partial^2\mathbf{x}$ (assuming $\mathbf{x} \neq \mathbf{y}$) is equal to

$$\sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=2\\ \mathfrak{u}(\phi)=\mathfrak{u}}} \sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\phi_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \phi_2\in\pi_2(\mathbf{w},\mathbf{y})\\ \phi_1*\phi_2=\phi}} (\mathfrak{m}(\phi_1).\mathfrak{m}(\phi_2))\,.$$

For each $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ the amount of the two interior sums in the above formula is the total number, counted with the sign determined by the coherent system $\mathfrak{o}$ of orientations, of the ends of the moduli space $\widehat{\mathcal{M}}(\phi)$, which is zero. Consequently the total sum in the above formula is trivial, and the coefficient of $\mathfrak{u}\mathbf{y}$ in $\partial\big(\partial(\mathbf{x})\big)$ is thus equal to 0.

Let us now assume that $\mathbf{x} = \mathbf{y}$. Let us denote the class of the generator of holomorphic spheres by S. The domain $\mathcal{D}(S)$ associated with S is the surface Σ:

$$\begin{aligned} &\mathcal{D}(S) = A_1 + \cdots + A_k = B_1 + \cdots + B_l,\\ \Rightarrow\quad &\mathfrak{u}(S) = \mathfrak{u}(A_1)\cdots\mathfrak{u}(A_k) = \mathfrak{u}(B_1)\cdots\mathfrak{u}(B_l). \end{aligned}$$

Thus $\mathfrak{u}(S) = 0$ unless $k = k_0 = l_0 = l$. In this latter case, the Maslov index of S is $2k$, which is greater than 2 unless $k = 1$. Combining with Lemma 5.8, we may thus conclude that in all possible cases, the total contribution to $\partial^2(\mathbf{x})$ from sphere bubblings is trivial.

We may thus assume, without loosing on generality, that the ends of the moduli space $\widehat{\mathcal{M}}(\phi)$ do not contain any sphere bubblings. If the ends of this moduli space contain a boundary disk degeneration, then the degeneration would consist of the juxtaposition of a constant function and a holomorphic boundary degeneration with Maslov index 2. If we denote the boundary degeneration by ψ, Lemma 5.8 implies that $\mathcal{D}(\psi) = A_i$ or $\mathcal{D}(\psi) = B_j$.

In the above situation, if $\mathcal{D}(\phi)=\mathcal{D}(\psi)=A_i$ or B_j, the boundary disk degeneration among the ends of $\widehat{\mathcal{M}}(\psi)$ are described by Lemma 5.8. Suppose first that $k,l>1$. Let $\mathcal{D}(\phi)=B_j$, and let ψ be the corresponding boundary disk degeneration with the same domain. Then the ends of $\mathcal{M}(\phi)$ consist of

$$\widehat{\mathcal{N}}(\psi)\bigcup\Bigg(\coprod_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \coprod_{\substack{\phi_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \phi_2\in\pi_2(\mathbf{w},\mathbf{x})\\ \phi_1\star\phi_2=\phi}}\big(\widehat{\mathcal{M}}(\phi_1)\times\widehat{\mathcal{M}}(\phi_2)\big)\Bigg).$$

According to Lemma 5.4, the orientation of $\widehat{\mathcal{N}}(\psi)$ is the opposite of the orientation induced from $\widehat{\mathcal{M}}(\psi)$. Thus the total number of the ends for this moduli space is equal to

$$-\mathfrak{n}(\psi)+\sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \phi_2\in\pi_2(\mathbf{w},\mathbf{x})\\ \phi_1\star\phi_2=\phi}}\big(\mathfrak{m}(\phi_1).\mathfrak{m}(\phi_2)\big)=0.$$

By Lemma 5.8, we have $\mathfrak{n}(\psi)=1$, thus the total value of the second sum is equal to $+1$. Note that $\mathfrak{u}(\psi)=\mathfrak{u}(R_j^+)=\mathfrak{u}_j^+$, since the domain associated with ψ is B_j. Thus, such degenerations contribute to the coefficient of $\mathfrak{u}(B_j)\mathbf{x}$, i.e. the contribution of ψ to $\partial^2(\mathbf{x})$ is $\mathfrak{u}_j^+.\mathbf{x}$.

Similarly, for a α boundary degeneration ψ with $\mathcal{D}(\psi)=A_i$, we obtain the equality

$$\sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \phi_2\in\pi_2(\mathbf{w},\mathbf{x})\\ \phi_1\star\phi_2=\phi}}\big(\mathfrak{m}(\phi_1).\mathfrak{m}(\phi_2)\big)=-1.$$

Thus the contribution of ψ to $\partial^2(\mathbf{x})$ is $-\mathfrak{u}_i^-.\mathbf{x}$.

The coefficient of $\mathbf{x}$ in $\partial\big(\partial(\mathbf{x})\big)$ is equal to

$$\sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{x})\\ \mathcal{D}(\phi)=A_i,1\le i\le k}}\mathfrak{u}(A_i)\Bigg(\sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \phi_2\in\pi_2(\mathbf{w},\mathbf{y})\\ \phi_1\star\phi_2=\phi}}\big(\mathfrak{m}(\phi_1).\mathfrak{m}(\phi_2)\big)\Bigg)+$$
$$\sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{x})\\ \mathcal{D}(\phi)=B_j,1\le j\le l}}\mathfrak{u}(B_j)\Bigg(\sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \phi_2\in\pi_2(\mathbf{w},\mathbf{y})\\ \phi_1\star\phi_2=\phi}}\big(\mathfrak{m}(\phi_1).\mathfrak{m}(\phi_2)\big)\Bigg)+$$
$$\sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{x})\\ \mathcal{D}(\phi)\neq A_i\text{ or }B_j}}\mathfrak{u}(\phi)\Bigg(\sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi_1\in\pi_2(\mathbf{x},\mathbf{w})\\ \phi_2\in\pi_2(\mathbf{w},\mathbf{y})\\ \phi_1\star\phi_2=\phi}}\big(\mathfrak{m}(\phi_1).\mathfrak{m}(\phi_2)\big)\Bigg).$$

Our argument shows that in the above sum, the sums in the first and the second line combine to give the following expression:

$$-\sum_{1\le i\le k}\mathfrak{u}(A_i)+\sum_{1\le i\le l}\mathfrak{u}(B_i)=\sum_{1\le i\le l}\mathfrak{u}(R_i^+)-\sum_{1\le i\le k}\mathfrak{u}(R_i^-)=0.$$

Thus the sum of the contributions from the first two lines in the above expression is zero. The last line is a sum of zero terms, by an argument similar to the case $\mathbf{x}\neq\mathbf{y}$, so it is trivial. Consequently, the coefficient of $\mathbf{x}$ in $\partial\big(\partial(\mathbf{x})\big)$ is zero.

When $k=1$ or $l=1$ one should be cautious. If $k=l=1$, the contributions from both α and β boundary degenerations is zero by Lemma 5.8. However, if $k=1$ and $l>1$, since the Heegaard diagram is balanced, we may conclude that $l>l_0$. However,

$$\mathfrak{u}(A_1)=\mathfrak{u}(\Sigma)=\prod_{i=1}^{l}\mathfrak{u}(B_i)=0\quad\Rightarrow\quad\sum_{i=1}^{l}\mathfrak{u}_i^+=\mathfrak{u}_1^-=0.$$

The rest of the argument in this case if completely identical with the case $k,l>1$. This completes the proof of the theorem. ☐

Remark 5.9. The relations ideal $\mathcal{I}(\tau)<\mathbb{Z}[\kappa]$ could have been chosen slightly smaller. In order for $\mathfrak{s}$-admissibility arguments of Section 4 to work, the relations ideal should include the ideal $\mathcal{J}(\tau)$ defined as

$$\left\langle \prod_{i=1}^{k}\left(\mathfrak{u}_i^-\right)^{n_i}\prod_{i=1}^{l}\left(\mathfrak{u}_i^+\right)^{m_i}\;\middle|\;\begin{array}{c} n_1,\dots,n_k\in\mathbb{Z}^{\ge 0}\\ m_1,\dots,m_l\in\mathbb{Z}^{\ge 0}\\ \sum_{i=1}^{k}n_i(1-g_i^-)+\sum_{i=1}^{l}m_i(1-g_i^+)=0\end{array}\right\rangle_{\mathbb{Z}[\kappa]}$$

In order for Lemma 5.2 to be true, we need to exclude the cases where the class ψ of boundary degeneration has negative Maslov index, while the corresponding moduli space $\mathcal{N}(\psi)$ is non-empty. Since the complex structure on Σ remains fixed when the moduli space corresponds to a boundary degeneration, the only method known to the authors for achieving the above goal is a form of somewhere injectivity assumption, i.e that the the coefficient of $\mathcal{D}(\psi)$ in some region is equal to 1 (see [Lip]). We thus need to include the relations

$$\mathcal{G}(\tau):=\big\langle(\mathfrak{u}_i^-)^2\,\big|\,g_i^->0\big\rangle_{\mathbb{Z}[\kappa]}+\big\langle(\mathfrak{u}_i^+)^2\,\big|\,g_i^+>0\big\rangle_{\mathbb{Z}[\kappa]}.$$

Finally, in order to prove Theorem 5.7, the extra relation which is required gives the following.

$$\mathcal{D}(\tau) := \left\langle \sum_{g_i^-=0} \mathfrak{u}_i^- = \sum_{g_j^+=0} \mathfrak{u}_j^+ \right\rangle_{\mathbb{Z}[\kappa]}.$$

Instead of the relations ideal $\mathcal{I}(\tau)$ we may thus use the smaller ideal of relations

$$\widetilde{\mathcal{I}}(\tau) = \mathcal{J}(\tau) + \mathcal{G}(\tau) + \mathcal{D}(\tau).$$

Correspondingly, the ring of coefficients may be improved to $\widetilde{\mathbb{A}}_\tau = \mathbb{Z}[\kappa]/\widetilde{\mathcal{I}}(\tau)$. It is clear that $\mathbb{A}_\tau$ is a quotient of $\widetilde{\mathbb{A}}_\tau$. However, in order to keep the exposition simpler, we choose to use the algebra $\mathbb{A}_\tau$ as the ring of coefficients. The only relations which may be redundant are the ones in $\mathcal{G}(\tau)$. However, the authors do not know how to remove these relations.

Remark 5.10. In the module $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \tilde{\mathbb{A}})$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ over the algebra

$$\tilde{\mathbb{A}} = \tilde{\mathbb{A}}_\tau = \frac{\mathbb{Z}[\mathfrak{u}_1, \ldots, \mathfrak{u}_\kappa]}{\langle \mathfrak{u}_i^+ \mid g_i^+ > 0 \rangle + \langle \mathfrak{u}_i^- \mid g_i^- > 0 \rangle}$$

Theorem 5.7 implies that the differential ∂ satisfies

$$\partial \circ \partial = \big(\mathfrak{u}^+(\tau) - \mathfrak{u}^-(\tau)\big)\,\mathrm{Id}.$$

This stronger form will be needed in Section 7.

In the following section we will prove the following theorem.

Theorem 5.11. *The filtered* $(\mathbb{A}, \mathbb{H})$ *chain homotopy type of the filtered* $(\mathbb{A}, \mathbb{H})$ *chain complex* $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s})$ *is an invariant of the weakly balanced sutured manifold* (X, τ) *and the* Spin^c *class* $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$*. In particular, for any filtered test ring* $\mathbb{B}$ *for* $(\mathbb{A}, \mathbb{H})$ *and for any* $\underline{\mathfrak{s}} \in \mathrm{Spin}^c(X, \tau)$*, the chain homotopy type of*

$$\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \underline{\mathfrak{s}}; \mathbb{B}) \subset \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}) \otimes_{\mathbb{A}} \mathbb{B}$$

is also an invariant of $(X, \tau, \underline{\mathfrak{s}})$*.*

Definition 5.12. We may thus denote the filtered $(\mathbb{A}, \mathbb{H})$ chain homotopy type of the filtered $(\mathbb{A}, \mathbb{H})$ chain complex $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s})$ and its invariant decomposition into chain complexes $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \underline{\mathfrak{s}})$ by

$$\mathrm{CF}(X, \tau; \mathfrak{s}) = \bigoplus_{\underline{\mathfrak{s}} \in \mathfrak{s} \subset \mathrm{Spin}^c(X,\tau)} \mathrm{CF}(X, \tau; \underline{\mathfrak{s}}).$$

5.4. Additional algebraic structure

From the definitions, it is clear that the multiplication by a generator $\mathfrak{u} \in G(\mathbb{A})$ gives a map

$$m_{\mathfrak{u}} : \mathrm{CF}(X, \tau; \underline{\mathfrak{s}}) \longrightarrow \mathrm{CF}(X, \tau; \underline{\mathfrak{s}} + \chi(\mathfrak{u})).$$

This map shifts the homological grading by $\mathrm{gr}(\mathfrak{u})$. In particular, if $\mathfrak{u} = \prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_i}$ and $\sum_i n_i \tau_i$ is homologically trivial, multiplication by $\mathfrak{u}$ preserves the relative Spin^c class and the homological degree. This map generalizes the U-action in the original construction of Ozsváth and Szabó [OS5] and [OS9].

With $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ as before, let $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ denote the space of paths connecting $\mathbb{T}_\alpha$ to $\mathbb{T}_\beta$ in $\mathrm{Sym}^\ell(\Sigma)$. Any intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, viewed as a constant path, is a point in $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$, and for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and any Whitney disk u representing a class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, u may be viewed as a path connecting $\mathbf{x}$ to $\mathbf{y}$ in $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. The homotopy class of this path depends only on ϕ. As in Section 4 of [OS5] for any one-cocycle

$$\zeta \in Z^1\left(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z}\right)$$

the evaluation $\zeta(\phi)$ is well-defined. Correspondingly, we may define the map

$$\begin{aligned} &A_\zeta : \mathrm{CF}(X, \tau; \mathfrak{s}) \longrightarrow \mathrm{CF}(X, \tau; \mathfrak{s}) \\ &A_\zeta(\mathbf{x}) := \sum_{\substack{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \underline{\mathfrak{s}}(\mathbf{y}) \in \mathfrak{s}}} \sum_{\substack{\phi \in \pi_2^+(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1}} \left(\zeta(\phi).\mathfrak{u}_{\mathbf{z}}(\phi).\mathfrak{m}(\phi)\right).\mathbf{y}, \quad \begin{array}{l} \forall\, \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ s.t.\ \ \underline{\mathfrak{s}}(\mathbf{x}) \in \mathfrak{s} \end{array}. \end{aligned}$$

The map A_ζ is then extended as a homomorphism of $\mathbb{A}$-modules to $\mathrm{CF}(X, \tau; \mathfrak{s})$. It respects the decomposition according to relative Spin^c structures in $\mathfrak{s} \subset \mathrm{Spin}^c(X, \tau)$. As in Lemmas 4.18 and 4.19 from [OS5] one may prove that the map A_ζ satisfies

$$(9) \qquad \begin{array}{rl} (i) & \partial \circ A_\zeta + A_\zeta \circ \partial = 0 \quad \text{and} \\ (ii) & A_\zeta = \partial \circ H_\zeta - H_\zeta \circ \partial, \quad \text{if } \zeta \text{ is a coboundary,} \end{array}$$

for some $\mathbb{A}$-module homomorphism H_ζ which respects the filtration by $\mathbb{H}$. As a result, the following proposition may be proved in this generalized setup.

Proposition 5.13. *There is a natural action of* $\mathrm{H}^1(\Omega^1(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z})$ *on the complex* $\mathrm{CF}(X, \tau; \mathfrak{s})$ *lowering degree by one, which is well-defined up to filtered chain homotopy equivalence. Furthermore, this induces an action of the exterior algebra*

$$\wedge^* \left(\mathrm{H}_1(\overline{X}; \mathbb{Z})/\mathrm{Tors}\right) \subset \wedge^* \left(\mathrm{H}^1\left(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z}\right)\right)$$

on the module $\mathrm{CF}(X, \tau; \mathfrak{s})$*, which is well-defined up to chain homotopy equivalence.*

Proof. One should simply repeat the proof of Proposition 4.17 from [OS5]. From the properties stated in Equation 9 and the isomorphisms

$$\begin{aligned}\mathrm{H}^1\left(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z}\right) &\cong \mathrm{Hom}\left(\pi_1\left(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)\right), \mathbb{Z}\right)\\ &\cong \pi_2\left(\mathrm{Sym}^\ell(\Sigma)\right) \oplus \mathrm{Hom}\left(\mathrm{H}^1(\overline{X}, \mathbb{Z}), \mathbb{Z}\right),\end{aligned}$$

the proof of the above proposition is reduced to showing $A_\zeta \circ A_\zeta = 0$. For this purpose, let $f : \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to S^1$ denote a representative of ζ. For a generic point $p \in S^1$ we set $V_p = f^{-1}(p)$ and observe that for any generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ representing the Spin^c class $\mathfrak{s}$

$$A_\zeta(\mathbf{x}) = \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\\ \underline{\mathfrak{s}}(\mathbf{y})\in\mathfrak{s}}} \sum_{\substack{\phi\in\pi_2^+(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \left(a(\zeta, \phi).\mathfrak{u}_{\mathbf{z}}(\phi)\right).\mathbf{y},$$
$$\text{where} \quad a(\zeta, \phi) = \#\left\{u \in \mathcal{M}(\phi) \,\middle|\, u([0,1] \times \{0\}) \in V_p\right\}.$$

Let us now consider a positive homotopy class $\phi \in \pi_2^+(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) = 2$. Associated with ϕ, and for generic points $p, q \in S^1$, we consider the one-dimensional moduli space

$$\Xi_{p,q}(\phi) := \left\{(s, u) \in [0, \infty) \times \mathcal{M}(\phi) \,\middle|\, \begin{array}{l} u([0,1] \times \{s\}) \subset V_p \\ u([0,1] \times \{-s\}) \subset V_q \end{array}\right\}.$$

This one-manifold does not have any boundary at $s = 0$. Furthermore, if we set $I_0 = [0, 1] \times \{0\}$, the boundary at infinity (i.e. the structure of the moduli space as $s \to \infty$) is modeled on

$$\coprod_{\substack{\phi_1 \star \phi_2 = \phi \\ \mu(\phi_1)=\mu(\phi_2)=1}} \left(\{u_1 \in \mathcal{M}(\phi_1) \mid u_1(I_0) \subset V_p\} \times \{u_2 \in \mathcal{M}(\phi_2) \mid u_2(I_0) \subset V_q\}\right).$$

Other possible boundary points correspond to boundary disk degenerations and sphere bubblings. Let us first assume that $\mathbf{x} \neq \mathbf{y}$. If we furthermore assume that $\mathfrak{u}_{\mathbf{z}}(\phi) \neq 0$, any boundary disk degeneration or sphere bubbling will reduce the Maslov index at least by 2. Thus the moduli space corresponding to such degenerations would be empty, if we choose a generic path of almost complex structures.

When $\mathbf{x} = \mathbf{y}$, everything is as before except that boundary degenerations or sphere bubblings of Maslov index 2 are now possible. The total contribution of such degenerations is $\mathfrak{u}^+(\tau) - \mathfrak{u}^-(\tau) = 0$, c.f. the proof of Theorem 5.7.

The number of points in the boundary of $\Xi_{p,q}(\phi)$, counted with sign, would vanish. On the other hand, this total count corresponds to the contribution of the pairs (ϕ_1, ϕ_2) with $\phi = \phi_1 \star \phi_2$ and $\mu(\phi_1) = \mu(\phi_2) = 1$ to the coefficient of $\mathfrak{u}_{\mathbf{z}}(\phi).\mathbf{y}$ in $A_\zeta^2(\mathbf{x})$. Thus $A_\zeta^2 = 0$ for all $\zeta \in \mathrm{H}^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z})$. Thus the action descends to an action of the exterior algebra

$$\wedge^* \left(\mathrm{H}^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z})\right).$$

This completes the proof of the proposition. □

5.5. Taut sutured manifolds

Theorem 5.11, together with Remark 4.3 imply that for computing $\mathrm{CF}(X, \tau; \mathfrak{s}; \mathbb{B})$, given a test ring $\mathbb{B}$ for $\mathbb{A}_\tau$, one may use any Heegaard diagram which is weakly $\mathfrak{s}$-admissible in the sense of Remark 4.3. In particular, we can easily prove the following proposition.

Proposition 5.14. *The irreducible balanced sutured manifold* (X, τ) *is taut if and only if the filtered* $(\mathbb{B}_\tau, \mathbb{H}_\tau)$ *chain homotopy type of the complex* $\mathrm{CF}(X, \tau; \mathfrak{s}; \mathbb{B}_\tau)$ *is not trivial for some* Spin^c *structure* $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$.

Proof. Suppose that (X, τ) is an irreducible balanced sutured manifold which is not taut. As in the proof of Proposition 9.18 from [Ju1], there is a weakly admissible Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for (X, τ) such that $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is empty. This Heegaard diagram is $\mathfrak{s}$-admissible for the test ring $\rho_\tau : \mathbb{A}_\tau \to \mathbb{B}_\tau$ and for any $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$ by Remark 4.3, and may thus be used to compute

$$\mathrm{CF}(X, \tau; \mathfrak{s}; \mathbb{B}_\tau) \cong 0, \quad \forall\, \mathfrak{s} \in \mathrm{Spin}^c(\overline{X}),$$

where $\cong$ denotes the equivalence of filtered chain homotopy types.

Conversely, if (X, τ) is taut, Theorem 1.4 from [Ju2] implies that $\mathrm{HF}(X, \tau; \mathbb{Z}) \neq 0$. Since $\mathbb{Z}$ is a test ring for $\mathbb{B}_\tau$ this implies, in particular, that the filtered chain homotopy type of $\mathrm{CF}(X, \tau; \mathbb{B}_\tau)$ is non-trivial. □

In fact, the proof of Proposition 5.14 implies the following corollary.

Corollary 5.15. *For an irreducible balanced sutured manifold (X, τ), the filtered $(\mathbb{B}_\tau, \mathbb{H}_\tau)$ chain homotopy type of the Ozsváth-Szabó complex $\mathrm{CF}(X, \tau; \mathbb{B}_\tau)$ is trivial if and only if $\mathrm{SFH}(X, \tau) = 0$.*

6. Invariance of the filtered chain homotopy type

6.1. Pseudo-holomorphic m-gons

Let us assume that the Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \ldots, \boldsymbol{\alpha}^m, \mathbf{z})$ is given, so that Σ is a closed Riemann surface of genus g, each $\boldsymbol{\alpha}^i$ is a set of ℓ disjoint simple closed curves on Σ, and $\mathbf{z} = \{z_1, \ldots, z_\kappa\}$ is a set of marked points in

$$\Sigma - \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^2 - \cdots - \boldsymbol{\alpha}^m.$$

The Heegaard diagram $(\Sigma, \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j, \mathbf{z})$ determines a weakly balanced sutured manifold (X_{ij}, τ_{ij}). Let $\overline{X_{ij}}$ denote the three-manifold obtained from X_{ij} by filling out the sutures in τ_{ij}, and fix the Spin^c classes $\mathfrak{s}_{ij} \in \mathrm{Spin}^c(\overline{X_{ij}})$. Assume that for any pair of indices $i < j$, $(\Sigma, \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j, \mathbf{z})$ is an $\mathfrak{s}_{ij}$-admissible Heegaard diagram for the weakly balanced sutured manifold (X_{ij}, τ_{ij}). Furthermore, let $\mathfrak{o}_{ij}$ be a coherent system of orientations on $(\Sigma, \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j, \mathbf{z})$ associated with $\mathfrak{s}_{ij}$. Finally, suppose that

$$\mathrm{CF}\big(\Sigma, \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j, \mathbf{z}; \mathfrak{s}_{ij}\big) = \bigoplus_{\underline{\mathfrak{s}}_{ij} \in \mathfrak{s}_{ij}} \mathrm{CF}\big(\Sigma, \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j, \mathbf{z}; \underline{\mathfrak{s}}_{ij}\big)$$

is the corresponding chain complex, and its decomposition into relative Spin^c classes.

Let us assume that

$$\Sigma - \boldsymbol{\alpha}^i = \coprod_{j=1}^{k_i} A^i_j, \quad i = 1, 2, \ldots, m,$$

are the connected components in the complements of the curves in $\boldsymbol{\alpha}^i$. We will denote the genus of A^i_j by $g^i_j \in \mathbb{Z}^{\geq 0}$. We will also denote

$$\sum_{p=1}^{k_i} \mathfrak{u}\big(A^i_p\big) \in \mathbb{Z}[\kappa]$$

by $\mathfrak{u}(\boldsymbol{\alpha}^i)$. For any subset I of the set of indices $\{1, \ldots, m\}$ introduce the $\mathbb{Z}$-algebra

$$\mathbb{A}_I = \frac{\mathbb{Z}[\mathfrak{u}_1, \ldots, \mathfrak{u}_\kappa]}{\big\langle \mathfrak{u}(\boldsymbol{\alpha}^i) = \mathfrak{u}(\boldsymbol{\alpha}^j) \mid \forall\, i, j \in I \big\rangle \oplus \big\langle \mathfrak{u}(A^i_j) \mid i \in I,\ g^i_j > 0 \big\rangle}.$$

If for two subsets $I, J \subset \{1, \ldots, m\}$ we have $I \subset J$, then $\mathbb{A}_J$ would be a quotient of $\mathbb{A}_I$, and we have a natural homomorphism

$$\rho_{IJ} : \mathbb{A}_I \longrightarrow \mathbb{A}_J.$$

This homomorphism may be used to give $\mathbb{A}_J$ the structure of an $\mathbb{A}_I$-module. As a result, from any $\mathbb{A}_I$ chain complex (C, d), we obtain a natural $\mathbb{A}_J$ chain complex $C \otimes_{\mathbb{A}_I} \mathbb{A}_J$. In particular, for any index set I which contains i, j, we may consider the $\mathbb{A}_I$ chain complex

$$C_{ij}(I) = \mathrm{CF}(\Sigma, \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j, \mathbf{z}; \mathfrak{s}_{ij}) \otimes_{\mathbb{A}_{ij}} \mathbb{A}_I.$$

We will denote $C_{ij}\big(\{1, \ldots, m\}\big)$ by C_{ij} for simplicity.

Associated with each set of curves $\boldsymbol{\alpha}^i$ is a torus $\mathbb{T}_{\alpha^i} \subset \mathrm{Sym}^\ell(\Sigma)$. A Whitney m-gon is a continuous map u from the standard m-gon $\mathbb{D}_m$ into $\mathrm{Sym}^\ell(\Sigma)$ which maps the i-th edge of the m-gon to $\mathbb{T}_{\alpha^i}$. If we fix

$$\mathbf{x}_i \in \mathbb{T}_{\alpha^i} \cap \mathbb{T}_{\alpha^{i+1}}, \quad i = 1, \ldots, m-1 \quad \text{and} \quad \mathbf{x}_m \in \mathbb{T}_{\alpha^m} \cap \mathbb{T}_{\alpha^1},$$

we may let $\pi_2(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ denote the set of homotopy classes of the Whitney m-gons which map the vertex v_i between the i-th edge and the $(i+1)$-th edge to $\mathbf{x}_i$ (for $i = 1, \ldots, m-1$), and the vertex v_m between the m-th edge and the first edge to $\mathbf{x}_m$.

Let us fix a generic continuous family $\{J_p\}_{p\in\mathbb{D}_m}$ of (nearly symmetric) almost complex structures on $\mathrm{Sym}^\ell(\Sigma)$. Thus the family is of the form $\mathrm{Sym}^\ell(j_\Sigma)$ in a neighbourhood of a collection $\overline{\mathbf{z}}$ of marked points, where $\mathbf{z}\subset\overline{\mathbf{z}}$ and the intersection of every connected component A of

$$\Sigma-\boldsymbol{\alpha}^1-\boldsymbol{\alpha}^2-\cdots-\boldsymbol{\alpha}^m$$

with $\overline{\mathbf{z}}$ is non-empty. Furthermore, we will assume that under a fixed identification of a neighbourhood of the i-th vertex v_i of $\mathbb{D}_m$ with $[0,1]\times(0,\infty)$ the family is translation invariant, i.e.

$$J_{(s,t)}=J_{(s,t+R)},\quad \forall\ (s,t)\in[0,1]\times(0,\infty),\quad R\in\mathbb{R}^+.$$

We will drop this generic family $\{J_p\}_{p\in\mathbb{D}_m}$ from our notation. For $\phi\in\pi_2(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ we let $\mathcal{M}(\phi)$ denote the set of pseudo-holomorphic representatives of ϕ.

Fix a subset $I=\{i_1<i_2<\cdots<i_p\}\subset\{1,\ldots,m\}$. This subset determines a sub-diagram

$$H_I=(\Sigma,\boldsymbol{\alpha}^{i_1},\ldots,\boldsymbol{\alpha}^{i_p},\mathbf{z})$$

of H. Correspondingly, we may consider the p-gons associated with H_I. We will say that two p-gons $\phi\in\pi_2(\mathbf{x}_{i_1},\ldots,\mathbf{x}_{i_p})$ and $\phi'\in\pi_2(\mathbf{y}_{i_1},\ldots,\mathbf{y}_{i_p})$, with

$$\mathbf{x}_{i_j},\mathbf{y}_{i_j}\in\mathbb{T}_{\alpha^{i_j}}\cap\mathbb{T}_{\alpha^{i_{j+1}}},\quad j=1,\ldots,p-1\quad\text{and}\quad \mathbf{x}_{i_p},\mathbf{y}_{i_p}\in\mathbb{T}_{\alpha^{i_p}}\cap\mathbb{T}_{\alpha^{i_1}},$$

are equivalent if and only if there exist Whitney disk classes $\psi_{i_j}\in\pi_2(\mathbf{x}_{i_j},\mathbf{y}_{i_j})$ for $j=1,\ldots,p$ such that ϕ is obtained from ϕ' by juxtaposition of the disk ψ_{i_j} at the vertices $\mathbf{y}_{i_j}$ for $j=1,\ldots,p$. The set of equivalence classes of such p-gons will be denoted by $\mathrm{Spin}^c(H,I)$. It is important to note that $\mathrm{Spin}^c(H,\{i,j\})$ determines a subset of the set of Spin^c structures on the three-manifold $\overline{X_{ij}}$, which are realized by the Heegaard diagram.

Definition 6.1. Suppose that we have a pair of index sets $I,J\subset\{1,\ldots,m\}$ such that $I=\{i_1<i_2<\cdots<i_p\}$ and $J=\{i_r=j_1<j_2<\cdots<j_q=i_{r+1}\}$. We will call the pair I,J *attachable*, and define

$$I\star J:=\big\{i_1<\cdots<i_r=j_1<j_2<\cdots<j_q=i_{r+1}<i_{r+2}<\cdots<i_p\big\}.$$

We will denote r by $r(I,J)$ for future reference. Suppose that I and J are attachable index sets as above, and we are given a p-gon ϕ and a q-gon ψ

$$\phi\in\pi_2(\mathbf{x}_{i_1},\ldots,\mathbf{x}_{i_p})\quad\text{and}\quad\psi\in\pi_2(\mathbf{y}_{j_1},\ldots,\mathbf{y}_{j_q}),$$

where $\mathbf{x}_{i_s} \in \mathbb{T}_{\alpha^{i_s}} \cap \mathbb{T}_{\alpha^{i_{s+1}}}$ and $\mathbf{y}_{j_s} \in \mathbb{T}_{\alpha^{j_s}} \cap \mathbb{T}_{\alpha^{j_{s+1}}}$. Furthermore, assume that $\mathbf{x}_{i_r} = \mathbf{y}_{j_q}$. Then we may *juxtapose* ϕ and ψ to obtain the class of some $(p+q-2)$-gon, which will be denoted by $\phi \star \psi$, see Figure 1.

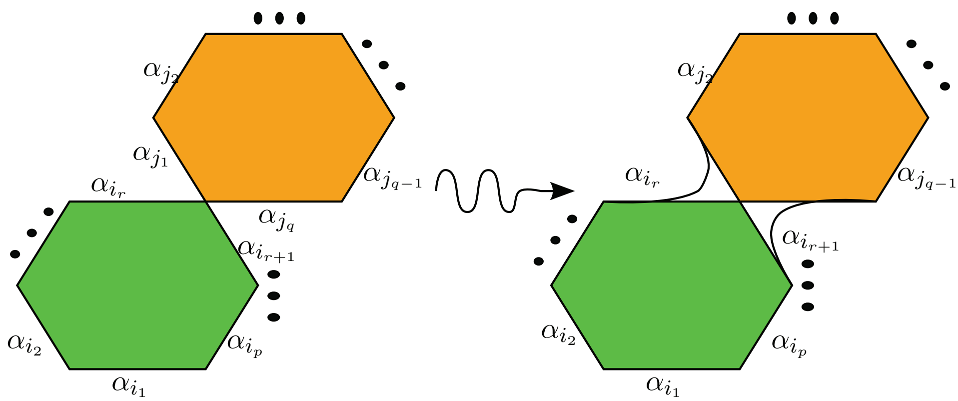


Figure 1: The juxtaposition of a p-gon and a q-gon.

Let us now restrict ourselves to the polygon classes whose vertices correspond to the fixed set $\mathfrak{S} = \{\mathfrak{s}_{ij}\}_{i<j}$ of Spin^c structures. Let us denote the subset of Spin$^c(H, I)$ which consists of polygons such that the Spin^c structures associated with the vertices are in $\mathfrak{S}$ by Spin$^c(H, I; \mathfrak{S})$.

Definition 6.2. With the above notation fixed, a *coherent system of* Spin^c *structures on polygons* for the Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^m, \mathbf{z})$, and compatible with $\mathfrak{S}$ is a choice of classes

$$\mathfrak{T} = \big\{[\phi_I] \in \mathrm{Spin}^c(H, I; \mathfrak{S}) \,\big|\, I \subset \{1, \ldots, m\},\ |I| \geq 3\big\},$$

represented by the polygon classes $\{\phi_I \mid I \subset \{1, \ldots, m\}, |I| \geq 3\}$, such that the following is satisfied. If I and J are attachable index sets, then we have

$$[\phi_{I \star J}] = [\phi_I \star \phi_J]\,.$$

Lemma 6.3. *Let us assume that a coherent system* $\{[\phi_I]\}_I$ *of* Spin^c *structures is fixed for the Heegaard diagram* H*. If* $K = I \star J$ *and a polygon* ψ_K

in the same class as

$$\phi_K \in [\phi_I \star \phi_J] \in \mathrm{Spin}^c(H, K; \mathfrak{S})$$

is decomposed as $\psi_K = \psi_I \star \psi_J$*, then the the class of* ψ_I *in* $\mathrm{Spin}^c(H, I; \mathfrak{S})$ *is equal to the class of* ϕ_I *and the class of* ψ_J *in* $\mathrm{Spin}^c(H, J; \mathfrak{S})$ *is equal to the class of* ϕ_J*.*

Proof. Fix the above notation and let $K = I \star J$. After addition of disk classes we may assume that the corners of ψ_K are the same as the corners of ϕ_K (i.e. both are chosen from $\{\mathbf{x}_{ij}\}_{i<j}$). This means that

$$\begin{aligned}
&\phi_I \in \pi_2\big(\mathbf{x}_{i_1 i_2}, \ldots, \mathbf{x}_{i_{p-1} i_p}, \mathbf{x}_{i_1 i_p}\big),\\
&\psi_I \in \pi_2\big(\mathbf{x}_{i_1 i_2}, \ldots, \mathbf{x}_{i_{r-1} i_r}, \mathbf{y}, \mathbf{x}_{i_{r+1} i_{r+2}}, \ldots, \mathbf{x}_{i_{p-1} i_p}, \mathbf{x}_{i_1 i_p}\big),\\
&\phi_J \in \pi_2\big(\mathbf{x}_{j_1 j_2}, \ldots, \mathbf{x}_{j_{q-1} j_q}, \mathbf{x}_{j_1 j_q}\big) \quad \text{and} \quad \psi_J \in \pi_2\big(\mathbf{x}_{j_1 j_2}, \ldots, \mathbf{x}_{j_{q-1} j_q}, \mathbf{y}\big)
\end{aligned}$$

and we have $\phi_I \star \phi_J = \psi_I \star \psi_J$. We thus have the following relation among the associated domains:

$$\mathcal{D}(\phi_I) - \mathcal{D}(\psi_I) = \mathcal{D}(\psi_J) - \mathcal{D}(\phi_J) = \mathcal{D}.$$

The coefficients of the domains in the expression appearing on the left hand side of the above equality on both sides of any curve in $\boldsymbol{\alpha}^i$, with $i \notin I$, are equal. Similarly, the coefficients of the domains in the expression appearing as the middle term in the above equality on the two sides of any curve in $\boldsymbol{\alpha}^j$, with $j \notin J$, are equal. This implies that $\partial(\mathcal{D})$ is included in

$$\coprod_{i \in I \cap J} \boldsymbol{\alpha}^i = \boldsymbol{\alpha}^{i_r} \coprod \boldsymbol{\alpha}^{i_{r+1}}.$$

Thus, $\mathcal{D}$ is the domain associated with a disk in $\pi_2(\mathbf{x}_{i_r i_{r+1}}, \mathbf{y})$, and the Spin^c class of ψ_I is the same as that of ϕ_I. Similarly, the Spin^c class of ψ_J is the same as that of ϕ_J. This completes the proof of the lemma. □

The above lemma implies that a coherent system of Spin^c structures on polygons for H is completely determined by the choice of triangle classes

$$\big\{[\phi_{ijk}] \in \mathrm{Spin}^c(H, \{i, j, k\}; \mathfrak{S}) \,\big|\, 1 \le i < j < k \le m\big\},$$

which satisfy the following compatibility relation

$$\phi_{ikl} \star \phi_{ijk} = \phi_{ijl} \star \phi_{jkl}, \quad \forall\, 1 \le i < j < k < l \le m. \tag{10}$$

Furthermore, the above lemma implies that for ϕ_{ijk}, ϕ_{ikl} and ϕ_{ijl} as above, there exists at most one class ϕ_{jkl} such that Equation 10 is satisfied. This observation implies that a coherent system of Spin^c classes of polygons for the Heegaard diagram H is determined by the family of triangle classes

$$\{\phi_{1ij} \mid 1 < i < j \leq m\}.$$

However, this family should have the property that for any triple $1 < i < j < k \leq m$ of indices, there is a triangle class ψ such that

$$\phi_{1jk} \star \phi_{1ij} = \phi_{1ik} \star \psi. \tag{11}$$

If this is the case, we will write

$$\mathfrak{T} = \{\phi_I\}_I = \langle \phi_{1ij} \,\big|\, 1 < i < j \leq m \rangle.$$

Let us fix a system $\mathfrak{T}$ of compatible Spin^c structures for the Heegaard diagram H as above, which is generated by the triangle classes ϕ_{1ij}. The set of periodic domains for polygons in $\mathfrak{T}$ is generated by periodic domains for each pair $(\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j)$. To be more precise, let us denote by $\mathfrak{P}_{ij}$ the set of periodic domains for the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j)$. Then any periodic domain which appears as the difference of two q-gons with the same set of vertices

$$\mathbf{y}_j \in \mathbb{T}_{\alpha^{i_j}} \cap \mathbb{T}_{\alpha^{i_{j+1}}}, \quad j = 1, \ldots, q, \; i_{q+1} := i_1 \;\text{ and }\; i_1 < i_2 < \cdots < i_q,$$

and representing the same Spin^c class may be written as a sum of periodic domains in $\mathfrak{P}_{i_1 i_2}, \mathfrak{P}_{i_2 i_3}, \ldots, \mathfrak{P}_{i_{q-1} i_q}$, and $\mathfrak{P}_{i_1 i_q}$.

Definition 6.4. Let the Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \ldots, \boldsymbol{\alpha}^m, \mathbf{z})$ and $\mathfrak{S}$, $\mathfrak{T}$, and $\mathfrak{P}_{ij}$ be as above. The Heegaard diagram H is called $\mathfrak{S}$-*admissible* if for any index set $I = \{i_1 < \cdots < i_q\}$, and any periodic domain

$$\mathcal{P} = \mathcal{P}_{i_1 i_2} + \mathcal{P}_{i_2 i_3} + \cdots + \mathcal{P}_{i_{q-1} i_q} + \mathcal{P}_{i_1 i_q}$$

with $\mathcal{P}_{ij} \in \mathfrak{P}_{ij}$, the following is true. If

$$\sum_{j=1}^{q} \langle c_1(\mathfrak{s}_{i_j i_{j+1}}), H(\mathcal{P}_{i_j i_{j+1}}) \rangle = 0$$

then either the coefficient of the domain $\mathcal{P}$ at some point w is negative, or $\mathfrak{u}(\mathcal{P}) = 0$ in $\mathbb{A}_I$.

The existence of $\mathfrak{S}$-admissible Heegaard diagrams, and the possibility of modifying H to an admissible Heegaard diagram using finger moves, follows with an argument completely similar to the arguments of Section 4. Furthermore, the $\mathfrak{S}$-admissibility of the Heegaard diagram H implies that for any index set

$$I = \{i_1 < \cdots < i_q\} \subset \{1, \ldots, m\},$$

any integer N, and any set of corners

$$\mathbf{y}_j \in \mathbb{T}_{\alpha^{i_j}} \cap \mathbb{T}_{\alpha^{i_{j+1}}}, \quad j = 1, \ldots, q, \; i_{q+1} := i_1 \text{ and } i_1 < i_2 < \cdots < i_q,$$

such that $\underline{\mathfrak{s}}_{\mathbf{z}}(\mathbf{y}_j) \in \mathfrak{s}_{i_j i_{j+1}}$, there are at most finitely many classes $\phi \in \pi_2(\mathbf{y}_1, \ldots, \mathbf{y}_q)$ satisfying the following three conditions.

- $\phi = \phi_I \in \mathrm{Spin}^c(H, I; \mathfrak{S})$.
- $\mu(\phi) = N$.
- $\mathcal{D}(\phi) \geq 0$ and $\mathfrak{u}_{\mathbf{z}}(\phi; I) \neq 0$, where $\mathfrak{u}_{\mathbf{z}}(\phi; I)$ is defined by

$$\mathfrak{u}_{\mathbf{z}}(\phi; I) := \prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_{z_i}(\phi)} \in \mathbb{A}_I.$$

The construction of Ozsváth and Szabó in Subsection 8.2 from [OS5] may be extended to this more general context without any major modification. Namely, for any index set $I \subset \{1, \ldots, m\}$, and any polygon class $[\phi] = [\phi_I]$, the determinant line bundle of the Cauchy-Riemann operator over $\mathcal{M}(\phi)$ is trivial, and one may thus choose an orientation, i.e. one of the two classes of nowhere vanishing sections of this determinant line bundle, associated with ϕ.

Definition 6.5. A *coherent system of orientations* associated with the Heegaard diagram H and the coherent system $\mathfrak{T}$ of Spin^c classes of polygons of H is a choice of orientation $\mathfrak{o}_I(\phi)$ for any polygon class ϕ with $\phi \in [\phi_I] \in \mathrm{Spin}^c(H, I; \mathfrak{S})$, such that the following are satisfied.

- For any $1 \leq i < j \leq m$, $\mathfrak{o}_{ij}$ is a coherent system of orientations associated with the Spin^c class $\mathfrak{s}_{ij}$ for the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j, \mathbf{z})$.
- For any pair I, J of attachable index sets and any attachable polygon classes ϕ and ψ, with

$$\phi \in [\phi_I] \in \mathrm{Spin}^c(H, I; \mathfrak{S}) \quad \text{and} \quad \psi \in [\phi_J] \in \mathrm{Spin}^c(H, J; \mathfrak{S}),$$

we have

$$(-1)^{r(I,J)|J|}\mathfrak{o}_I(\phi) \wedge \mathfrak{o}_J(\psi) = \mathfrak{o}_{I\star J}(\phi \star \psi).$$

Lemma 6.3 implies that in order for us to obtain a coherent system of orientations associated with the Heegaard diagram H and the coherent system of Spin^c classes of polygons $\mathfrak{T}$, it suffices to determine $\mathfrak{o}_{ij}$ and $\mathfrak{o}_{1ij}$ for any pair of indices $1 \leq i < j \leq m$. This observation implies that the following lemma, which was proved in [OS5] as Lemma 8.7, is valid in our setup. Although the Heegaard diagrams are more general, the proof carries over without any major modification.

Lemma 6.6. *Suppose that the Heegaard diagram H, and the coherent system of* Spin^c *classes of polygons $\mathfrak{T}$ are as above. Then for any choice of coherent systems of orientations $\mathfrak{o}_{1i}$ corresponding to the* Spin^c *classes $\mathfrak{s}_{1i}$ (with $1 < i \leq m$), and any choice of $\mathfrak{o}_{1ij}(\phi_{1ij})$ for $1 < i < j \leq m$, there always exists a coherent system of orientations $\mathfrak{o} = \{\mathfrak{o}_I\}_I$ such that $\mathfrak{o}_{1i}$ is the initial choice of the coherent system of orientations corresponding to $\mathfrak{s}_{1i}$ and $\mathfrak{o}_{1ij}(\phi_{1ij})$ is the prescribed orientation.*

Proof. For an index set $I = \{i_1 < \cdots < i_q\}$ let $i(I) = i_1$ and $j(I) = i_q$ denote the smallest and largest element of I respectively. Let us assume that $\phi \in \pi_2(\mathbf{y}_1, \ldots, \mathbf{y}_q)$ is a q-gon class in the same Spin^c class as ϕ_I.

If $1 \in I$, then we may assume $|I| \geq 3$, since otherwise, we already have a choice of orientation. In this case, we may write, in a unique way,

$$\begin{aligned}\phi &= \phi_I \star \phi_1 \star \cdots \star \phi_q, \quad \phi_j \in \pi_2(\mathbf{y}_j, \mathbf{x}_{i_j i_{j+1}})\\ \phi_I &= \phi_{1i_2i_3} \star \phi_{1i_3i_4} \star \cdots \star \phi_{1i_{q-1}i_q}.\end{aligned}$$

Thus $\mathfrak{o}_I(\phi)$ is determined if we determine all the maps $\mathfrak{o}_{ij}$ for $1 < i < j \leq m$ in a compatible way. Note that $\mathfrak{o}_{1ij}(\phi_{1ij})$ is already defined. If otherwise $1 \notin I$, we may write, again in a unique way

$$\begin{aligned}\phi &= \phi_I \star \phi_1 \star \cdots \star \phi_q, \quad \phi_j \in \pi_2(\mathbf{y}_j, \mathbf{x}_{i_j i_{j+1}})\\ \phi_{1i(I)j(I)} \star \phi_I &= \phi_{1i_1i_2} \star \phi_{1i_2i_3} \star \cdots \star \phi_{1i_{q-1}i_q}.\end{aligned}$$

Thus, in order to determine the orientation $\mathfrak{o}_I(\phi)$, it suffices to determine all maps $\mathfrak{o}_{ij}$ for $1 < i < j \leq m$. In order to determine the $\mathfrak{o}_{ij}$ from $\mathfrak{o}_{1i}$, $\mathfrak{o}_{1j}$ and $\mathfrak{o}_{1ij}(\phi_{1ij})$, one may then use the argument of Lemma 8.7 from [OS5]. □

Remark 6.7. Note that the choice of $\mathfrak{o}_{1i}$ for $1 < i \leq m$ determines the orientation for all boundary degenerations in a unique way. In fact, suppose that ψ is the class of some $\boldsymbol{\alpha}^i$ boundary degeneration corresponding to the corner $\mathbf{y}\in\mathbb{T}_{\alpha^i}\cap\mathbb{T}_{\alpha^j}$, say for some $j>i$, and that ϕ is a Whitney disk in $\pi_2(\mathbf{x}_{ij},\mathbf{y})$. Furthermore, let ψ' denote the class in $\pi_2^{\alpha^i}(\mathbf{x}_{1i})$ which has the same domain as ψ. We may then write

$$\phi_{1ij}\star\phi\star\psi=\phi_{1ij}\star\psi'\star\phi,$$

implying that $\mathfrak{o}_{ij}(\psi)$ is uniquely determined by $\mathfrak{o}_{1i}(\psi')$, and is equal to it as the class of an α^i boundary degeneration.

Let H be an $\mathfrak{S}$-admissible Heegaard diagram, and $\mathfrak{T}$ be a system of compatible Spin^c structures as before. Correspondingly, assume that

$$\mathfrak{o}=\big\{\mathfrak{o}_I\mid I\subset\{1,\dots,m\},\ |I|\geq 2\big\}$$

is a coherent system of orientations associated with $\mathfrak{T}$. Associated with any subset $I=\{i_1,\dots,i_q\}\subset\{1,\dots,m\}$ of indices, we may define a holomorphic polygon map

$$\begin{aligned}f_I:\bigotimes_{j=1}^{q-1}&\mathrm{CF}(\Sigma,\boldsymbol{\alpha}^{i_j},\boldsymbol{\alpha}^{i_{j+1}},\mathbf{z};\mathfrak{s}_{i_ji_{j+1}})\otimes_{\mathbb{A}_{\{i_j,i_{j+1}\}}}\mathbb{A}_I\\&\longrightarrow\mathrm{CF}(\Sigma,\boldsymbol{\alpha}^{i_1},\boldsymbol{\alpha}^{i_q},\mathbf{z};\mathfrak{s}_{i_1i_q})\otimes_{\mathbb{A}_{\{i_1,i_q\}}}\mathbb{A}_I.\end{aligned}$$

In other words, if $\{i<j\}\lhd I$ denotes that i and j are consecutive elements in I, and $i(I),j(I)$ denote the smallest and largest elements of I respectively, we will have a map

$$\begin{aligned}&f_I:\bigotimes_{\{i<j\}\lhd I}C_{ij}(I)=\bigotimes_{j=1}^{q-1}C_{i_ji_{j+1}}(I)\longrightarrow C_{i_1i_q}(I)=C_{i(I),j(I)}(I)\\&f_I\big(\mathbf{y}_1\otimes\mathbf{y}_2\otimes\cdots\otimes\mathbf{y}_{q-1}\big):=\sum_{\substack{\mathbf{y}_q\in\mathbb{T}_{\beta^1}\cap\mathbb{T}_{\beta^q}\\ [\underline{\mathfrak{s}}(\mathbf{y}_q)]=\mathfrak{t}_{1q}}}\ \sum_{\substack{\phi\in\pi_2(\mathbf{y}_1,\mathbf{y}_2,\dots,\mathbf{y}_q)\\ \mu(\phi)=3-q\\ \phi\in[\phi_I]}}\big(\mathfrak{m}(\phi)\mathfrak{u}_{\mathbf{z}}(\phi;I)\big).\mathbf{y}_q,\end{aligned}$$

where $\boldsymbol{\beta}^j=\boldsymbol{\alpha}^{i_j}$ and $\mathfrak{t}_{1q}=\mathfrak{s}_{i_1i_q}$.

Since H is admissible, it follows that only finitely many terms would contribute to the above sum, and f_I is thus well-defined.

These maps satisfy a generalized associativity property, which may be stated in our setup as follows (we will only state the associativity corresponding to the full index set $\{1,\dots,m\}$).

Theorem 6.8. *With the above notation fixed, if we set* $[m]=\{1,\dots,m\}$ *to be the full index set, the map*

$$\begin{aligned} &F_{[m]}: C_{12}\big([m]\big)\otimes C_{23}\big([m]\big)\otimes\cdots\otimes C_{m-1,m}\big([m]\big)\longrightarrow C_{1m}\big([m]\big),\\ (12)\qquad &F_{[m]}:=\sum_{1\le i<j\le m}(-1)^{ij} f_{\{1,2,\dots,i,j,j+1,\dots,m\}}\circ f_{\{i,i+1,\dots,j\}}\end{aligned}$$

is trivial.

Proof. Let us denote the set $\{1,2,\dots,i,j,j+1,\dots,m\}$ of indices by $I(i,j)$, and $\{i,i+1,\dots,j\}$ by $J(i,j)$. We have to show that for any set $\mathbf{y}_1,\dots,\mathbf{y}_m$ of intersection points with $\mathbf{y}_i\in\mathbb{T}_{\alpha^i}\cap\mathbb{T}_{\alpha^{i+1}}$, and such that $\underline{\mathfrak{s}}(\mathbf{y}_i)\in\mathfrak{s}_{i,(i+1)}$, the coefficient of $\mathbf{y}_m$ in

$$\begin{aligned}&\sum_{1\le i<j\le m}(-1)^{ij} f_{I(i,j)}\mathbf{y}_{I(i,j)},\quad \text{with}\\ &\mathbf{y}_{I(i,j)}:=\mathbf{y}_1\otimes\cdots\otimes\mathbf{y}_{i-1}\otimes f_{J(i,j)}\big(\mathbf{y}_i\otimes\cdots\otimes\mathbf{y}_{j-1}\big)\otimes\mathbf{y}_j\otimes\cdots\otimes\mathbf{y}_{m-1},\end{aligned}$$

is zero. Let us consider a Whitney polygon class $\psi\in\pi_2(\mathbf{y}_1,\dots,\mathbf{y}_m)$ with Maslov index $4-m$ and in the same class as $\psi_{[m]}$, and consider the ends of $\mathcal{M}(\psi)$. The ends of this moduli space do not contain any boundary disk degenerations or sphere bubblings. The reason is that the Maslov index of the holomorphic boundary disk degenerations and holomorphic spheres are greater than or equal to 2 if the corresponding element of the coefficient ring is non-trivial. This would imply that the remaining component should have Maslov index at most $2-m$. As a result, the moduli space associated with the remaining part would be empty.

Thus all degenerations of this moduli space (for dimensional reasons) are degenerations along an arc which connects two different edges of the m-gon. The ends corresponding to a degeneration along an arc connecting the i-th edge to the j-th edge, with $i<j$, correspond to a degeneration of ψ into the juxtaposition of a holomorphic Whitney $(j-i+1)$-gon connecting $\mathbf{y}_i,\dots,\mathbf{y}_{j-1}$ to an intersection point $\mathbf{x}\in\mathbb{T}_{\alpha^i}\cap\mathbb{T}_{\alpha^j}$ with Maslov index $2-j+i$, with a holomorphic $(m-j+i+1)$-gon connecting $\mathbf{y}_1,\dots,\mathbf{y}_{i-1},\mathbf{x},\mathbf{y}_j,\dots,\mathbf{y}_{m-1}$ to $\mathbf{y}_m$ with Maslov index $2-m+j-i$. Thus, the ends of $\mathcal{M}(\psi)$ will have the following form.

$$\partial\mathcal{M}(\psi)=\coprod_{\substack{1\le i<j\le m\\ \mathbf{x}\in\mathbb{T}_{\alpha^i}\cap\mathbb{T}_{\alpha^j}}}\ \coprod_{\substack{\phi_{ij}\in\pi_2^{2-j+i}(\mathbf{y}_i,\dots,\mathbf{y}_{j-1},\mathbf{x})\\ \psi_{ij}\in\pi_2^{2-m+j-i}(\mathbf{y}_1,\dots,\mathbf{y}_{i-1},\mathbf{x},\mathbf{y}_j,\dots,\mathbf{y}_m)\\ \psi_{ij}\star\phi_{ij}=\psi}}\big(\mathcal{M}(\psi_{ij})\times\mathcal{M}(\phi_{ij})\big).$$

In the above decomposition, we are dropping the condition that the polygons represent the Spin^c class determined by $\mathfrak{T}$. The sign difference between the orientation we assign to the component $\mathcal{M}(\psi_{ij}) \times \mathcal{M}(\phi_{ij})$, and its orientation as a boundary component of $\partial\mathcal{M}(\psi)$ is computed as

$$\epsilon\big(\mathcal{M}(\psi_{ij}) \times \mathcal{M}(\phi_{ij})\big) = (-1)^{r(I(i,j),J(i,j))|J(i,j)|} = (-1)^{i(j-i+1)} = (-1)^{ij}.$$

Note that the total number of points in the moduli space on the right hand side of the above equation, when counted with the above induced signs, will be zero. We should of course mod out by possible automorphisms of the domain, when necessary. Fix a generator $\mathfrak{u} \in G(\mathbb{A}_{[m]})$. The coefficient of $\mathfrak{u}.\mathbf{y}_m$ in the expression

$$\sum_{1\le i<j\le m} (-1)^{ij} f_{I(i,j)}\mathbf{y}_{I(i,j)},$$
$$\mathbf{y}_{I(i,j)} = \mathbf{y}_1 \otimes \cdots \otimes \mathbf{y}_{i-1} \otimes f_{J(i,j)}\big(\mathbf{y}_i \otimes \cdots \otimes \mathbf{y}_{j-1}\big) \otimes \mathbf{y}_j \otimes \cdots \otimes \mathbf{y}_{m-1},$$

is equal to

$$\sum_{\substack{1\le i<j\le m\\ \mathbf{x}\in\mathbb{T}_{\alpha^i}\cap\mathbb{T}_{\alpha^j}}} \sum_{\substack{\phi_{ij}\in\pi_2^{2-j+i}(\mathbf{y}_i,\dots,\mathbf{y}_{j-1},\mathbf{x})\\ \psi_{ij}\in\pi_2^{2-m+j-i}(\mathbf{y}_1,\dots,\mathbf{y}_{i-1},\mathbf{x},\mathbf{y}_j,\dots,\mathbf{y}_m)\\ \mathfrak{u}(\phi_{ij})\mathfrak{u}(\psi_{ij})=\mathfrak{u}}} (-1)^{ij}\big(\mathfrak{m}(\psi_{ij})\mathfrak{m}(\phi_{ij})\big)$$
$$= \sum_{\substack{\psi\in\pi_2^{4-m}(\mathbf{y}_1,\dots,\mathbf{y}_m)\\ \mathfrak{u}(\psi)=\mathfrak{u}}} \#\Big(\partial\big(\mathcal{M}(\psi)\big)\Big) = 0.$$

The above computation thus completes the proof of the theorem. □

Remark 6.9. The maps

$$f_I : \bigotimes_{\{i<j\}\lhd I} C_{ij}(I) \to C_{i(I)j(I)}(I)$$

will sometimes refine to the maps respecting the relative Spin^c structures. We will face this situation in the upcoming sections several times. Each time we will give a separate argument for such an splitting, to avoid the complexity of a general treatment.

In order to prove the above associativity, we do not need to use the full system $\mathfrak{P}$ of compatible Spin^c structures. In fact, a subsystem containing the classes of polygons associated with the index sets $I(i,j)$ and $J(i,j)$ suffices for this purpose. In other words, we only make use of the Spin^c classes in the subset

$$\mathfrak{T}_1 = \left\{[\phi_{I(i,j)}] \,\middle|\, 1 \leq i < j \leq m\right\} \cup \left\{[\phi_{J(i,j)}] \,\middle|\, 1 \leq i < j \leq m\right\} \subset \mathfrak{T}$$

for defining the maps appearing on the left-hand-side of Equation 12.

Definition 6.10. The set

$$\mathfrak{T}_1 = \left\{[\phi_{I(i,j)}] \in \mathrm{Spin}^c(H, I(i,j)) \,\middle|\, 1 \leq i < j \leq m\right\}$$
$$\bigcup \left\{[\phi_{J(i,j)}] \in \mathrm{Spin}^c(H, J(i,j)) \,\middle|\, 1 \leq i < j \leq m\right\}$$

is called a *system of first degenerations* for $\phi_{[m]} \in \mathrm{Spin}^c(H, [m]; \mathfrak{S})$ if

$$[\phi_{I(i,j)}] \star [\phi_{J(i,j)}] = [\phi_{[m]}], \quad \forall\, 1 \leq i < j \leq m.$$

Thus, instead of $\mathfrak{T}$, we may fix a system of first degenerations for a class $\phi_{[m]} \in \mathrm{Spin}^c(H, [m]; \mathfrak{S})$, together with a compatible system of coherent orientations associated with them. Then Theorem 6.8 would still remain true.

6.2. Special Heegaard diagrams corresponding to handle slides

Let us assume that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is a Heegaard diagram, which corresponds to a weakly balanced sutured manifold (X, τ). Let us assume that

$$\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_\ell\} \quad \text{and} \quad \boldsymbol{\beta} = \{\beta_1, \ldots, \beta_\ell\},$$

and that β_i is the image of α_i under a small Hamiltonian isotopy for $i = 2, \ldots, \ell$ so that β_i is disjoint from α_j for $j \neq i$ and cuts α_i in a pair of canceling intersection points. The area bounded between α_i and β_i is thus of the form $\mathcal{P}_i = D_i^+ - D_i^-$, such that D_i^+ and D_i^- are two of the connected components in

$$\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta},$$

and $\partial\mathcal{P}_i = \alpha_i - \beta_i$. Furthermore, assume that β_1 is obtained from α_1 by first moving it by a small Hamiltonian isotopy, and then doing a handle slide along α_2. Thus, the only curve in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ that intersects β_1 is α_1, which cuts β_1 in a pair of intersection points. These two intersection points are connected by a bi-gon, which we will denote by D_1^+. There is a domain with small area which is bounded between $\alpha_1, \beta_1, \alpha_2$ and β_2, denoted by D_1^-, so that $\mathcal{P}_1 = D_1^+ - D_1^- - D_2^-$ is a periodic domain satisfying

$$\partial\mathcal{P}_1 = \alpha_1 + \alpha_2 - \beta_1.$$

We will assume that none of the marked points $\mathbf{z} = \{z_1, \ldots, z_\kappa\}$ are in any of $D_1^+, \ldots, D_\ell^+$ or $D_1^-, \ldots, D_\ell^-$. Let us assume

$$\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta} = \left(\coprod_{i=1}^{\ell} D_i^+\right) \bigcup \left(\coprod_{i=1}^{\ell} D_i^-\right) \bigcup \left(\coprod_{i=1}^{m} E_i\right),$$

and that $\mathbf{z}^i = \{z_1^i, \ldots, z_{j_i}^i\}$ are the marked points in E_i for $i = 1, \ldots, m$. Thus, $\mathbf{z} = \mathbf{z}^1 \cup \cdots \cup \mathbf{z}^m$. In the ring $\mathbb{A}_\tau$, let $\mathfrak{u}(z_j^i)$ denote the element associated with $z_j^i \in \mathbf{z}$. Furthermore, define

$$\mu_i = \prod_{j=1}^{j_i} \mathfrak{u}(z_j^i) \in \mathbb{A}_\tau, \quad i = 1, \ldots, m.$$

If $\Sigma - \boldsymbol{\alpha} = \coprod_{i=1}^{m_a} A_i$ and $\Sigma - \boldsymbol{\beta} = \coprod_{i=1}^{m_b} B_i$, we will have $m_a = m_b = m$, and after renaming the indices if necessary, we may assume $\mathbf{z} \cap A_i = \mathbf{z} \cap B_i$, and $\mathfrak{u}(A_i) = \mathfrak{u}(B_i) = \mu_i$. Let us denote by $\mathbb{A}_\mu$ the sub-ring of $\mathbb{A}_\tau$ generated by $\mu_1, \ldots, \mu_m$.

Any pair of curves (α_i, β_i) intersect in a pair of points x_i^+, x_i^-, so that the bi-gon D_i^+ connects x_i^+ to x_i^-. Any map $\epsilon : \{1, \ldots, \ell\} \to \{+, -\}$ thus corresponds to an intersection point

$$\mathbf{x}^\epsilon = \left\{x_1^{\epsilon(1)}, x_2^{\epsilon(2)}, \ldots, x_\ell^{\epsilon(\ell)}\right\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta.$$

These all correspond to the same Spin^c class in $\mathrm{Spin}^c(\overline{X}^\tau)$, which will be denoted by $\mathfrak{s}_0$. For $\epsilon : \{1, \ldots, \ell\} \to \{+, -\}$ let $|\epsilon|$ denote the number of elements in $\epsilon^{-1}\{+\}$. We may refine the homological grading of the generators of $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}_0)$ into a relative $\mathbb{Z}$-grading by setting

$$\mathrm{gr}(\epsilon, \delta) = |\epsilon| - |\delta|, \quad \forall\, \epsilon, \delta : \{1, \ldots, \ell\} \longrightarrow \{+, -\}.$$

We will show below that this gives a well-defined relative grading in an appropriate sense.

The periodic domains corresponding to the above Heegaard diagram are generated, as a free abelian group, by $\mathcal{P}_1, \ldots, \mathcal{P}_\ell, A_1, \ldots, A_m$. Note that

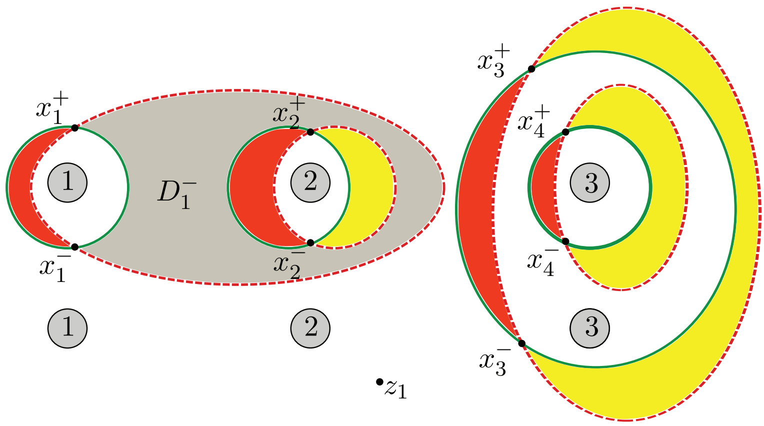


Figure 2: The green curves denote the elements of $\boldsymbol{\alpha}$ and the dashed red curves denote the elements of $\boldsymbol{\beta}$. The domains D_i^+ are shaded red, while the domains D_i^- are shaded yellow. The domain D_1^- is shaded gray.

$\mathfrak{u}(\mathcal{P}_i) = 1$. If

$$\mathcal{P} = q_1\mathcal{P}_1 + \cdots + q_\ell\mathcal{P}_\ell + a_1A_1 + \cdots + a_mA_m \geq 0$$

is a positive periodic domain with $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 0$ for some $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X}^\tau)$, we will have $a_1, \ldots, a_m \geq 0$ and

$$0 = a_1(2 - 2g(A_1)) + a_2(2 - 2g(A_2)) + \cdots + a_m(2 - 2g(A_m)).$$

Here $g(A_i)$ denotes the genus of A_i.

The reason for the equality $\langle c_1(\mathfrak{s}), H(A_i)\rangle = 2 - 2g(A_i)$ is that $H(A_i)$ is represented by one of the boundary components in $\overline{X}^\tau$, where $\mathfrak{s}$ is represented the vector field which is normal to the tangent space of the corresponding component, i.e. $\langle c_1(\mathfrak{s}), H(A_i)\rangle$ is the Euler characteristic of the corresponding component. If moreover we know that $\mathfrak{u}(\mathcal{P}) \neq 0$, we may conclude that if $a_i \neq 0$ then $g(A_i) = 0$. Let us assume that $A_1, \ldots, A_k$ are the components of genus zero, and the rest of A_i have positive genus. This implies that

$$a_1, \ldots, a_m \geq 0, \quad 0 = a_1 + \cdots + a_k \quad \text{and} \quad a_{k+1} = \cdots = a_m = 0.$$

Thus all a_i are zero, and $\mathcal{P} = q_1\mathcal{P}_1 + \cdots + q_\ell\mathcal{P}_\ell$.

Since $\mathcal{P}_2, \ldots, \mathcal{P}_\ell$ are disjoint, and all $\mathcal{P}_i$ have both positive and negative coefficients, one can easily conclude that $\mathcal{P}$ has both positive and negative coefficients. Thus the constructed Heegaard diagram is admissible for all Spin^c classes. Note that D_i^+ and D_i^- are both domains of Whitney disks, for $i = 2, \dots, \ell$, and the number of points in

$$\widehat{\mathcal{M}}(D_i^+) \bigcup \widehat{\mathcal{M}}(D_i^-)$$

is even.

In the applications we face in this paper the orientations induced over the above two moduli spaces from a system $\mathfrak{o}$ of coherent orientations is always the opposite of one another. More precisely, such Heegaard diagrams appear as a part of triples

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$$

where a system of coherent orientations on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z})$ is given. The aforementioned choice of orientation for the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$, together with the choice of orientation for the triangle classes, determines a corresponding system of coherent orientations on $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$. With this choice of $\mathfrak{o}$, the sign associated with the moduli spaces of holomorphic representatives of D_i^+ and D_i^- are different. Thus the signed count of the number of points in the moduli space

$$\widehat{\mathcal{M}}(D_i^+) \bigcup \widehat{\mathcal{M}}(D_i^-)$$

for this particular system of orientations $\mathfrak{o}$ is zero.

The complex $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s})$ is thus trivial for $\mathfrak{s} \neq \mathfrak{s}_0$ and is equal to

$$\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}; \mathfrak{s}_0) \otimes_{\mathbb{A}_\mu} \mathbb{A}_\tau$$

for $\mathfrak{s} = \mathfrak{s}_0$ where $\mathbf{w} = \{z_1^1, z_1^2, \dots, z_1^m\}$. For the rest of the computation, we may thus assume that $\mathbf{w} = \mathbf{z}$, i.e. that there is a single marked point in each connected component of $\Sigma - \boldsymbol{\alpha}$. Each μ_i (or under the assumption $\mathbf{w} = \mathbf{z}$, each $\mathfrak{u}_i$) corresponds to some component A_i. Thus $\mu_i = 0$ if the genus of A_i is positive. With our previous notation, this means that $\mu_{k+1} = \cdots = \mu_m = 0$. We set the degree associated with μ_i, $i = 1, \dots, k$, equal to $-2 = \langle c_1(\mathfrak{s}_0), H(A_i) \rangle$. This gives a grading on the complex $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}; \mathfrak{s}_0)$.

Let us assume that $\epsilon, \delta : \{1, \dots, \ell\} \to \{-, +\}$ are a pair of indices. After re-naming the elements of $\{1, \dots, \ell\}$ we may assume that

$$\begin{cases} \epsilon(i) = + \text{ and } \delta(i) = - & \text{if } 1 \leq i \leq \ell_1 \\ \epsilon(i) = - \text{ and } \delta(i) = + & \text{if } \ell_1 < i \leq \ell_2 \\ \epsilon(i) = \delta(i) & \text{if } \ell_2 < i \leq \ell, \end{cases}$$

for some $1 \leq \ell_1 \leq \ell_2 \leq \ell$. Then $\mathcal{D} = D_1^+ + \cdots + D_{\ell_1}^+ - D_{\ell_1+1}^+ - \cdots - D_{\ell_2}^+$ is the domain of a disk connecting $\mathbf{x}^\epsilon$ to $\mathbf{x}^\delta$. If $\mathcal{P}$ is a periodic domain and $\mathcal{D} + \mathcal{P}$ is the domain of a positive disk ϕ with $\mathfrak{u}(\phi) \neq 0$, the same argument as before implies that

$$\mathcal{P} = a_1 A_1 + \cdots + a_k A_k + q_1 \mathcal{P}_1 + \cdots + q_\ell \mathcal{P}_\ell.$$

Furthermore, the assumption $\mathcal{D}(\phi) \geq 0$ implies that all a_i are non-negative. In order to prove that the above grading assignment is well-defined, one only needs to check the following easy equality

$$\mu(\phi) = 2\left(\sum_{i=1}^{k} a_i\right) + 2\ell_1 - \ell_2. \tag{13}$$

If the coefficient ring $\widehat{\mathbb{A}}_\tau$ is used instead of $\mathbb{A}_\tau$, the corresponding quotient $\widehat{\mathbb{A}}_\mu$ of $\mathbb{A}_\mu$ will be equal to $\mathbb{Z}$. One would then quickly conclude from the above presentation of the domain $\mathcal{P}$ that if $\mathfrak{u}(\phi) \neq 0$ (as an element in the quotient $\widehat{\mathbb{A}}_\tau$), then $a_1 = \cdots = a_k = 0$ and $\ell_2 = \ell_1$. If $\mu(\phi) = 1$ then $\ell_1 = \ell_2 = 1$. We can then carry out the rest of the argument for any choice of indices ϵ and δ, if the coefficient ring is replaced with $\widehat{\mathbb{A}}_\tau$.

With coefficients in $\mathbb{A}$, however, in order to complete our investigation we need to assume $\epsilon(i) = \{+\}$ for $i = 1, \ldots, \ell$. The corresponding generator is often called the *top generator*. In this case the equality $\ell_1 = \ell_2$ is automatic. Replacing $\mu(\phi) = 1$ in Equation 13 we obtain $a_1 = \cdots = a_k = 0$ and $\ell_1 = 1$. From here we will have (from positivity of the domain) that $q_2 = \cdots = q_\ell = 0$. This means that if the top intersection point $\mathbf{x}^\epsilon$ is connected to an intersection point $\mathbf{x}^\delta$ by a positive domain ϕ of index 1, such that $\mathfrak{u}(\phi) \neq 0$, δ differs from ϵ only over one element of $\{1, \ldots, \ell\}$, where ϵ gives $+$ and δ gives $-$. Let us assume that this element is $i \in \{1, \ldots, \ell\}$.

If $i \neq 1$, the possible domains one may obtain as $\mathcal{D} + \mathcal{P}$ are D_i^+ and D_i^-, and the total contribution of $\mathbf{x}^\delta$ to $\partial(\mathbf{x}^\epsilon)$ is zero. However, if $i = 1$, the possible domains are $D_1^+, D_1^- + D_2^-$ and $D_1^- + D_2^+$. Again, the total contribution of these three domains is zero by the argument of [OS5] (Lemma 9.4). The above discussion implies that the top generator $\mathbf{x}^\epsilon$ is closed and represents a non-trivial element of the homology groups corresponding to either of the chain complexes

$$\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}; \mathfrak{s}_0) \quad \text{and} \quad \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}_0) = \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}_0) \otimes_{\mathbb{A}_\mu} \mathbb{A}_\tau.$$

Moreover, the same argument implies that all the generators of the form $\mathbf{x}^\delta$ are closed, when the coefficient ring $\widehat{\mathbb{A}}_\tau$ is used instead of $\mathbb{A}_\tau$, giving rise to an identification of the chain complexes:

$$\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s}_0;\widehat{\mathbb{A}}) = \widehat{\mathrm{HF}}(\#^{\ell}S^1\times S^2,\mathfrak{t}_0)\otimes_{\mathbb{Z}}\widehat{\mathbb{A}}_{\tau}.$$

The above equality means that the differential on the right hand side of the above equality is trivial. Any module isomorphism of the right hand side which respects the filtration by relative Spin^c structures is thus a filtered chain homotopy equivalence. The top generator $\mathbf{x}^{\epsilon}$ of $\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s}_0)$ is usually denoted by Θ, or $\Theta_{\alpha\beta}$.

The above example illustrates how the arguments of Ozsváth and Szabó for the study of the special Heegaard diagrams, i.e. Heegaard diagrams where most of β_i are Hamiltonian isotopes of some corresponding curves in $\boldsymbol{\alpha}$, may be generalized to the present situation. The above type of Heegaard diagrams appear in the arguments for the invariance under handle-slide. We will face similar Heegaard diagrams again. In particular this happens when we study the exact triangles. Each time, a separate argument should be presented for computing the contribution of holomorphic disks and polygons. However, the argument is always a straight forward modification of the corresponding argument for Heegaard diagrams arising from closed three-manifolds.

6.3. The triangle map and the invariance

Fix a Heegaard triple

$$H = (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{z})$$

and assume that $\mathbf{z}=\{z_1,\dots,z_\kappa\}$. We will denote the weakly balanced sutured manifold associated with $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ by $(X,\tau)=(X_{\alpha\beta},\tau_{\alpha\beta})$, and the corresponding coefficient ring by $\mathbb{A}=\mathbb{A}_\tau$. Similarly, let $(X_{\alpha\gamma},\tau_{\alpha\gamma})$ and $(X_{\beta\gamma},\tau_{\beta\gamma})$ be the weakly balanced sutured manifolds associated with the Heegaard diagrams $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma},\mathbf{z})$, and $(\Sigma,\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{z})$ respectively. Suppose that

$$\Sigma-\boldsymbol{\alpha}=\coprod_{i=1}^{k}A_i,\quad \Sigma-\boldsymbol{\beta}=\coprod_{i=1}^{l}B_i\quad\text{and}\quad \Sigma-\boldsymbol{\gamma}=\coprod_{i=1}^{m}C_i,$$

where A_i, B_i and C_i are the connected components of the curve complements. We will furthermore assume that $m=l$, and that these components are labeled so that for each $i=1,\dots,l$ we have $C_i\cap\mathbf{z}=B_i\cap\mathbf{z}$, and $g(C_i)=g(B_i)$. This implies that $\mathfrak{u}(\boldsymbol{\beta})=\mathfrak{u}(\boldsymbol{\gamma})$ in $\langle\mathfrak{u}_1,\dots,\mathfrak{u}_\kappa\rangle_{\mathbb{Z}}$, and more importantly, $\mathfrak{u}(C_i)=\mathfrak{u}(B_i)$ for $i=1,\dots,l$.

Assume that the coefficient rings $\mathbb{A}_{\beta\gamma}$ and $\mathbb{A}_{\alpha\gamma}$ are associated with the Heegaard diagrams $(\Sigma,\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{z})$ and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma},\mathbf{z})$ respectively. Then the above observation implies that

$$\mathbb{A}_{\beta\gamma} = \frac{\mathbb{Z}[\mathfrak{u}_1, \dots, \mathfrak{u}_\kappa]}{\big\langle \mathfrak{u}(B_i) \,\big|\, g(B_i) = g(C_i) > 0 \big\rangle_{\mathbb{Z}[\kappa]}}$$

is naturally mapped by a quotient homomorphism

$$\rho_{\beta\gamma} : \mathbb{A}_{\beta\gamma} \longrightarrow \mathbb{A}_{\alpha\beta} = \mathbb{A}$$

to $\mathbb{A}$. We may thus consider the $\mathbb{A}$-module

$$\mathrm{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z}) \otimes_{\mathbb{A}_{\beta\gamma}} \mathbb{A},$$

which will have the structure of a filtered $(\mathbb{A}, \mathbb{H})$ chain complex, where $\mathbb{H}$ is the $\mathbb{Z}$-module $\mathrm{H}^2(X, \partial X; \mathbb{Z})$ as before.

Any triangle class $\psi_H \in \mathrm{Spin}^c\big(H, \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}\big)$ determines a set of three Spin^c structures

$$\begin{aligned} &\mathfrak{s}_{\alpha\beta} \in \mathrm{Spin}^c\big(\overline{X_{\alpha\beta}}(\tau_{\alpha\beta})\big), \quad \mathfrak{s}_{\alpha\gamma} \in \mathrm{Spin}^c\big(\overline{X_{\alpha\gamma}}(\tau_{\alpha\gamma})\big) \\ \text{and} \quad &\mathfrak{s}_{\beta\gamma} \in \mathrm{Spin}^c\big(\overline{X_{\beta\gamma}}(\tau_{\beta\gamma})\big). \end{aligned}$$

These three Spin^c classes, together with the triangle class of ψ_H give a coherent system of Spin^c structures for H which will be denoted by $\mathfrak{T}$. We will assume that $\mathfrak{T}$, or equivalently the triangle class ψ_H, is fixed, and will drop them from the notation when there is no confusion. In particular, by the admissibility of a Heegaard diagram we would mean $\mathfrak{T}$-admissibility.

Any choice of coherent systems of orientations $\mathfrak{o}_{\alpha\beta}$ and $\mathfrak{o}_{\alpha\gamma}$ associated with the Spin^c classes $\mathfrak{s}_{\alpha\beta}$ and $\mathfrak{s}_{\alpha\gamma}$ may be completed to a coherent system of orientations for $\mathfrak{T}$ by Lemma 6.6. Furthermore, we are free to choose the orientation associated with a fixed representative of ψ_H. Let us fix such a coherent system $\mathfrak{o}$ of orientations. Once again, we will drop this choice of orientation from the notation.

Assuming that the Heegaard triple H is admissible, the triangle map corresponding to H (and the triangle class ψ_H) is defined via the construction of Subsection 6.1.

$$\begin{aligned} &f_{\alpha\beta\gamma} : \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}_{\alpha\beta}) \otimes (\mathrm{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z}; \mathfrak{s}_{\beta\gamma}) \otimes \mathbb{A}) \to \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z}; \mathfrak{s}_{\alpha\gamma}) \\ &f_{\alpha\beta\gamma}(\mathbf{x} \otimes \mathbf{q}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\substack{\psi \in \pi_2^0(\mathbf{x}, \mathbf{q}, \mathbf{y}) \\ (\psi) = (\psi_H)}} \big(\mathfrak{m}(\psi) \mathfrak{u}_{\mathbf{z}}(\psi)\big).\mathbf{y} \end{aligned}$$

As usual, $\mathfrak{u}_{\mathbf{z}}$ is the map

$$\mathfrak{u}_{\mathbf{z}}:\coprod_{\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \coprod_{\mathbf{q}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma}\ \coprod_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma}\pi_2^+(\mathbf{x},\mathbf{q},\mathbf{y})\longrightarrow G(\mathbb{A})$$
$$\mathfrak{u}_{\mathbf{z}}(\psi):=\prod_{i=1}^{\kappa}\mathfrak{u}_i^{n_{z_i}(\psi)}\in G(\mathbb{A}).$$

The admissibility of the Heegaard diagram implies that $f_{\alpha\beta\gamma}$ is well-defined.

Lemma 6.11. *The map $f_{\alpha\beta\gamma}$ is an $\mathbb{A}$ chain map. More precisely*

$$f_{\alpha\beta\gamma}\big(\partial(\mathbf{x})\otimes\mathbf{q}\big)+f_{\alpha\beta\gamma}\big(\mathbf{x}\otimes\partial(\mathbf{q})\big)=\partial\big(f_{\alpha\beta\gamma}(\mathbf{x}\otimes\mathbf{q})\big)$$

for all $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ and $\mathbf{q}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma$ corresponding to the Spin^c *classes $\mathfrak{s}_{\alpha\beta}$ and $\mathfrak{s}_{\beta\gamma}$ respectively.*

Proof. The equality

$$f_{\alpha\beta\gamma}\big(\partial(\mathbf{x})\otimes\mathbf{q}\big)+f_{\alpha\beta\gamma}\big(\mathbf{x}\otimes\partial(\mathbf{q})\big)=\partial\big(f_{\alpha\beta\gamma}(\mathbf{x}\otimes\mathbf{q})\big)$$

in the above lemma is nothing but the following special case (i.e. the case $m=3$) of Theorem 6.8:

$$f_{\alpha\beta\gamma}\big(f_{\alpha\beta}(\mathbf{x})\otimes\mathbf{q}\big)+f_{\alpha\beta\gamma}\big(\mathbf{x}\otimes f_{\beta\gamma}(\mathbf{q})\big)-f_{\alpha\gamma}\big(f_{\alpha\beta\gamma}(\mathbf{x}\otimes\mathbf{q})\big)=0.\qquad\square$$

As in [OS5], holomorphic triangle maps satisfy an associativity law, which comes from considering Heegaard quadruples. Let $K=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta},\mathbf{z})$ be an admissible Heegaard quadruple. This means that we have a coherent system $\mathfrak{T}$ of Spin^c classes of polygons, which consists of a square class

$$\psi_K\in\mathrm{Spin}^c\big(K,\{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}\}\big)$$

and triangle classes

$$\psi_\alpha\in\mathrm{Spin}^c\big(K,\{\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}\}\big),\quad \psi_\beta\in\mathrm{Spin}^c\big(K,\{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta}\}\big)$$
$$\psi_\gamma\in\mathrm{Spin}^c\big(K,\{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta}\}\big)\quad\text{and}\quad\psi_\delta\in\mathrm{Spin}^c\big(K,\{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\}\big).$$

We implicitly assume that the set of corners of these representatives of the triangle classes is a fixed set of 6 intersection points between the pairs of tori from $\{\mathbb{T}_\alpha, \mathbb{T}_\beta, \mathbb{T}_\gamma, \mathbb{T}_\delta\}$. These classes have to satisfy the following compatibility criteria

$$\psi_K = \psi_\alpha \star \psi_\gamma = \psi_\beta \star \psi_\delta.$$

The triangle classes also determine Spin^c structures on the Heegaard diagrams determined by any pair of curve collections. These Spin^c structures will be denoted by $\mathfrak{s}_{\alpha\beta} \in \mathrm{Spin}^c(\overline{X_{\alpha\beta}}(\tau_{\alpha\beta}))$, etc. Moreover, we will assume that

$$\Sigma - \boldsymbol{\alpha} = \coprod_{i=1}^{k} A_i, \quad \Sigma - \boldsymbol{\beta} = \coprod_{i=1}^{l} B_i,$$
$$\Sigma - \boldsymbol{\gamma} = \coprod_{i=1}^{l} C_i \quad \text{and} \quad \Sigma - \boldsymbol{\delta} = \coprod_{i=1}^{l} D_i$$

are labelled so that $B_i \cap \mathbf{z} = C_i \cap \mathbf{z} = D_i \cap \mathbf{z}$ for $i = 1, \dots, l$. Furthermore, we will assume that $g(B_i) = g(C_i) = g(D_i)$ for $i = 1, \dots, l$. Then we will have $\mathfrak{u}(\boldsymbol{\beta}) = \mathfrak{u}(\boldsymbol{\gamma}) = \mathfrak{u}(\boldsymbol{\delta})$ in $\langle \mathfrak{u}_1, \dots, \mathfrak{u}_\kappa \rangle$, and $\mathfrak{u}(B_i) = \mathfrak{u}(C_i) = \mathfrak{u}(D_i)$ for $i = 1, \dots, l$.

One may also choose a coherent system of orientations associated with $\mathfrak{T}$. In fact, we are free to choose $\mathfrak{o}_{\alpha\beta}, \mathfrak{o}_{\alpha\gamma}, \mathfrak{o}_{\alpha\delta}$, and the orientation of the triangle classes ψ_β, ψ_γ and ψ_δ. Once again, we keep such a coherent system of orientations implicit in our notation.

We may thus consider the following filtered $(\mathbb{A}, \mathbb{H})$ chain complexes, which are relevant for the associativity:

$$\begin{aligned}
C_{\alpha\beta} &= \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}_{\alpha\beta}), \quad C_{\beta\gamma} = \mathrm{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z}; \mathfrak{s}_{\beta\gamma}) \otimes \mathbb{A}\\
C_{\alpha\gamma} &= \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z}; \mathfrak{s}_{\alpha\gamma}), \quad C_{\beta\delta} = \mathrm{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\delta}, \mathbf{z}; \mathfrak{s}_{\beta\delta}) \otimes \mathbb{A}\\
C_{\alpha\delta} &= \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{z}; \mathfrak{s}_{\alpha\delta}) \quad \text{and} \quad C_{\gamma\delta} = \mathrm{CF}(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, \mathbf{z}; \mathfrak{s}_{\gamma\delta}) \otimes \mathbb{A}.
\end{aligned}$$

Following the construction of subsection 6.1, we define a rectangle map as in [OS5]:

$$\begin{aligned}
&h_{\alpha\beta\gamma\delta} : C_{\alpha\beta} \otimes C_{\beta\gamma} \otimes C_{\gamma\delta} \longrightarrow C_{\alpha\delta}\\
&h_{\alpha\beta\gamma\delta}(\mathbf{x} \otimes \mathbf{p} \otimes \mathbf{q}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta} \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \mathbf{p}, \mathbf{q}, \mathbf{y}) \\ \mu(\psi) = -1}} \big(\mathfrak{m}(\psi)\mathfrak{u}(\psi)\big)\mathbf{y}.
\end{aligned}$$

Lemma 6.12. *The rectangle map $h_{\alpha\beta\gamma\delta}$ gives a chain homotopy between the chain maps $f_{\alpha\gamma\delta}(f_{\alpha\beta\gamma}(.\otimes .)\otimes .)$ and $f_{\alpha\beta\delta}(.\otimes f_{\beta\gamma\delta}(.\otimes .))$ in the sense* *that*

$$\begin{aligned}&f_{\alpha\gamma\delta}\big(f_{\alpha\beta\gamma}(\mathbf{x}\otimes\mathbf{p})\otimes\mathbf{q}\big)-f_{\alpha\beta\delta}\big(\mathbf{x}\otimes f_{\beta\gamma\delta}(\mathbf{p}\otimes\mathbf{q})\big)\\ =&\,\partial\big(h_{\alpha\beta\gamma\delta}(\mathbf{x}\otimes\mathbf{p}\otimes\mathbf{q})\big)+h_{\alpha\beta\gamma\delta}\big(\partial(\mathbf{x}\otimes\mathbf{p}\otimes\mathbf{q})\big)\end{aligned}$$

for any $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta, \mathbf{p}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma$, *and* $\mathbf{q}\in\mathbb{T}_\gamma\cap\mathbb{T}_\delta$.

Proof. Once again, this is a special case of Theorem 6.8, where we put $m=4$ and use the above data. □

Proof of Theorem 5.11. The proof of the independence from the choice of the path of almost complex structures, as well as the proof of the isotopy invariance of the filtered $(\mathbb{A},\mathbb{H})$ chain homotopy type of $\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})$ is the same as the proof of the special case discussed in [OS5]. We only need to keep track of the marked points, and that the constructed chain homotopy equivalence respects the decomposition into relative Spin^c classes in $\mathrm{Spin}^c(X,\tau)$. The same is almost true for the handle slides supported away from the marked points. We will present the proof in this case, to give an illustration of the procedure, which involves the use of holomorphic triangles and squares introduced above.

Fix a Spin^c class $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$ and let $\underline{\mathfrak{s}}\in\mathfrak{s}\subset\mathrm{Spin}^c(X,\tau)$ be a fixed relative Spin^c class in $\mathfrak{s}$. To prove the handle slide invariance consider the Heegaard quadruple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta},\mathbf{z})$ where $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are obtained from $\boldsymbol{\beta}$ as follows. Let $\boldsymbol{\beta}=\{\beta_1,\dots,\beta_\ell\}$. Then we let δ_i to be a small Hamiltonian isotope of β_i for $i=1,\dots,\ell$ which cuts it in a pair of transverse cancelling intersection points. Similarly, for $i=2,\dots,\ell$, we let γ_i be a small Hamiltonian isotope of β_i which cuts either of the curves β_i and δ_i in a pair of transverse cancelling intersection points. Finally, we let γ_1 be the simple closed curve obtained by first moving β_1 by a small Hamiltonian isotopy, and then taking its handle slide over β_2. We may assume that γ_1 cuts either of β_1 and δ_1 in a pair of canceling intersection points, while it is disjoint from the rest of the curves β_i,γ_i and δ_i. We let

$$\boldsymbol{\gamma}=\big\{\gamma_1,\dots,\gamma_\ell\big\}\quad\text{and}\quad\boldsymbol{\delta}=\big\{\delta_1,\dots,\delta_\ell\big\}.$$

Consider the (admissible) Heegaard diagram $(\Sigma,\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{z})$, which is a standard Heegaard diagram of the type studied in Subsection 6.2. Note that all marked points which are in the same connected component of $\Sigma-\boldsymbol{\beta}$ or $\Sigma-\boldsymbol{\gamma}$ are in the same connected component of $\Sigma-\boldsymbol{\beta}-\boldsymbol{\gamma}$. Let $\Theta_{\beta\gamma}$ be the top generator of the complex $\mathrm{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{z};\mathfrak{s}_0)$ corresponding to its canonical Spin^c structure. This generator is represented by the intersection point in $\mathbb{T}_\beta \cap \mathbb{T}_\gamma$ which contains positive intersection points between the corresponding curves β_i and γ_i. Similarly, associated with the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\delta}, \mathbf{z})$ the top generator of the homology is denoted by a $\Theta_{\beta\delta}$, and is represented by the positive intersection points between the corresponding curves β_i and δ_i. Finally, $\Theta_{\gamma\delta}$ is defined in a similar way. We may consider $\Theta_{\beta\gamma}, \Theta_{\gamma\delta}$ and $\Theta_{\beta\delta}$ as generators of the complexes $C_{\beta\gamma}, C_{\gamma\delta}$ and $C_{\beta\delta}$ respectively. Here, we assume $\mathfrak{s}_{\alpha\bullet} = \mathfrak{s}$ for $\bullet \in \{\beta, \gamma, \delta\}$, and that

$$\mathfrak{s}_{\beta\gamma} = \mathfrak{s}_{\beta\delta} = \mathfrak{s}_{\gamma\delta} = \mathfrak{s}_0$$

is the canonical Spin^c structure on $\overline{X_{\beta\gamma}} = \overline{X_{\gamma\delta}} = \overline{X_{\beta\delta}}$.

Note that $\Theta_{\beta\gamma}, \Theta_{\gamma\delta}$ and $\Theta_{\beta\delta}$ are connected to each other by a natural triangle class Δ_α of small area. Moreover, for any fixed $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\underline{\mathfrak{s}}(\mathbf{x}) \in \mathfrak{s}$, we have a generator $I(\mathbf{x}) \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$, determined by the closest intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\delta$ to $\mathbf{x}$. Similarly, there is a generator $J(\mathbf{x})$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ determined as the closest intersection points between $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ to $\mathbf{x}$. There is a triangle class Δ_γ connecting $\Theta_{\beta\delta}$, $\mathbf{x}$ and $I(\mathbf{x})$ with very small area. Similarly, there is a triangle class Δ_δ connecting $\Theta_{\beta\gamma}$, $\mathbf{x}$ and $J(\mathbf{x})$ with very small area. Finally, there is a triangle class Δ_β which connects $I(\mathbf{x}), J(\mathbf{x})$ and $\Theta_{\gamma\delta}$. Let $\square$ be the square class $\Delta_\gamma \star \Delta_\alpha$. Then $\square$ may also be degenerated as $\square = \Delta_\delta \star \Delta_\beta$. The data

$$\mathfrak{P} = \left\{\square, \Delta_\alpha, \Delta_\beta, \Delta_\gamma, \Delta_\delta\right\}$$

thus gives a coherent system $\mathfrak{T}$ of Spin^c classes of polygons for the Heegaard quadruple, which will be implicit for the rest of the construction.

Lemma 6.13. *If the Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is $\mathfrak{s}$-admissible then the Heegaard quadruple $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \mathbf{z})$ is $\mathfrak{T}$-admissible.*

Proof. We will prove the lemma for the class of $\square$. The rest of the admissibility claims are similar, and in fact simpler. Let us denote the small periodic domains constructed as the domain bounded between β_i and δ_i by $\mathcal{Q}_i$ for $i = 1, \dots, \ell$. Thus $\mathcal{Q}_i$ is the difference of two bi-gons. Similarly, let $\mathcal{Q}_{i+\ell}$ be the domain bounded between β_i and γ_i for $i = 2, \dots, \ell$, and $\mathcal{Q}_{\ell+1}$ be the domain bounded between β_1, γ_1 and β_2, as in the previous subsection. We will thus have

$$\begin{cases} \partial\mathcal{Q}_i = \beta_i - \delta_i, & \text{for } i = 1, \dots, \ell, \\ \partial\mathcal{Q}_{i+\ell} = \beta_i - \gamma_i, & \text{for } i = 2, \dots, \ell, \quad \text{and} \\ \partial\mathcal{Q}_{\ell+1} = \beta_1 + \beta_2 - \gamma_1. & \end{cases}$$

Finally, let $A_1,\dots,A_k,B_1,\dots,B_l,\mathcal{P}_1,\dots,\mathcal{P}_m$ be the periodic domains corresponding to $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$. As before $\Sigma-\boldsymbol{\alpha}=\coprod A_i$ and $\Sigma-\boldsymbol{\beta}=\coprod B_i$. It may be checked then that the space of periodic domains for the Heegaard diagrams $(\Sigma,\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{z})$, $(\Sigma,\boldsymbol{\beta},\boldsymbol{\delta},\mathbf{z})$ and $(\Sigma,\boldsymbol{\gamma},\boldsymbol{\delta},\mathbf{z})$ is generated by the following periodic domains respectively

$$\begin{aligned}&\big\langle B_1,\dots,B_l,\mathcal{Q}_{\ell+1},\dots,\mathcal{Q}_{2\ell}\big\rangle,\quad \big\langle B_1,\dots,B_l,\mathcal{Q}_1,\dots,\mathcal{Q}_\ell\big\rangle\quad\text{and}\\ &\big\langle \widehat{B}_1,\dots,\widehat{B}_l,\mathcal{Q}_1+\mathcal{Q}_2-\mathcal{Q}_{\ell+1},\mathcal{Q}_2-\mathcal{Q}_{2+\ell},\dots,\mathcal{Q}_\ell-\mathcal{Q}_{2\ell}\big\rangle.\end{aligned}$$

Here, $\widehat{B}_i$ is the domain obtained from B_i by adding an appropriated combination of $\mathcal{Q}_j,\ j=1,\dots,\ell$, so that its boundary is supported on the curves in $\boldsymbol{\delta}$.

Let us now assume that we have a periodic domain $\mathcal{P}$ with $\mathfrak{u}(\mathcal{P})\neq 0$ and

$$\begin{aligned}\mathcal{P}=\mathcal{P}_{\alpha\beta}+\mathcal{P}_{\beta\gamma}+\mathcal{P}_{\gamma\delta}+\mathcal{P}_{\alpha\delta}\geq 0,\\ \big\langle c_1(\mathfrak{s}),H(\mathcal{P}_{\alpha\beta})\big\rangle+\big\langle c_1(\mathfrak{s}_0),H(\mathcal{P}_{\beta\gamma})\big\rangle\\ +\big\langle c_1(\mathfrak{s}_0),H(\mathcal{P}_{\gamma\delta})\big\rangle+\big\langle c_1(\mathfrak{s}),H(\mathcal{P}_{\alpha\delta})\big\rangle=0.\end{aligned}$$

Then $\mathcal{P}$ may then be written as

$$\mathcal{P}=\sum_{i=1}^{k}a_iA_i+\sum_{i=1}^{l}b_iB_i+\sum_{i=1}^{m}p_i\mathcal{P}_i+\sum_{i=1}^{2\ell}q_i\mathcal{Q}_i.$$

With the above notation fixed, computing the evaluation of Spin^c classes over the periodic domains (i.e. re-writing the last equation above) we obtain

$$0=\sum_{i=1}^{k}a_i(2-2g(A_i))+\sum_{i=1}^{l}b_i(2-2g(B_i))+\sum_{i=1}^{m}p_i\big\langle c_1(\mathfrak{s}),H(\mathcal{P}_i)\big\rangle, \tag{14}$$

since the Maslov index of all $\mathcal{Q}_i$ are zero for all Spin^c structures, according to Lipshitz's index formula [Lip].

Let us set

$$\mathcal{Q}=\sum_{i=1}^{k}a_iA_i+\sum_{i=1}^{l}b_iB_i+\sum_{i=1}^{m}p_i\mathcal{P}_i.$$

Then $\mathcal{Q}$ is a periodic domain for the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$, with $\mathcal{P}-\mathcal{Q}$ only consisting of the domains with very small area. The assumption $\mathcal{P}\geq 0$ thus implies that $\mathcal{Q}\geq 0$. Furthermore $\mathfrak{u}(\mathcal{Q})=\mathfrak{u}(\mathcal{P})$, since no marked point lives in the small domains. Equation 14 implies that $\langle c_1(\mathfrak{s}),H(\mathcal{Q})\rangle=0$.

The $\mathfrak{s}$-admissibility of of the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ thus implies that $\mathcal{Q} = 0$. As a result,

$$\mathcal{P} = \sum_{i=1}^{2\ell} q_i \mathcal{Q}_i \geq 0.$$

It is then an easy combinatorial exercise to check from this last equality that all q_i need to vanish. We have thus shown that $\mathcal{P} = 0$. This completes the proof of the admissibility claim. □

Finally, the last step towards defining the holomorphic triangle map and the holomorphic square map using the Heegaard diagram H is choosing the orientation. Note that the choice of orientation over the Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z})$, and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{z})$ may be done without any restriction, and we may thus choose the system of orientations corresponding to the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ (and correspondingly, the induced orientation corresponding to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{z})$), as well as the system of orientations corresponding to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z})$ to be our preferred choice of orientation. Orienting the triangles and the square in $\mathfrak{T}$ will then provide us with a coherent system $\mathfrak{o}$ of orientations for the Heegaard diagram H.

We may thus define the triangle and the square maps associated with this Heegaard diagram and $\mathfrak{T}$. The argument of Ozsváth and Szabó from [OS5] (Lemma 9.7) applies here to give

$$f_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta}) = \Theta_{\beta\delta}.$$

We may define a map

$$F = F_{\alpha\beta\gamma} : \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}) \longrightarrow \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z}; \mathfrak{s})$$

by setting $F(\mathbf{x}) := f_{\alpha\beta\gamma}\big(\mathbf{x} \otimes \Theta_{\beta\gamma}\big)$. Since $\Theta_{\beta\gamma}$ is closed, F is a chain map. More importantly, F respects the decomposition into relative Spin^c structures, and the image of $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \underline{\mathfrak{s}})$, for the fixed relative Spin^c structure

$$\underline{\mathfrak{s}} \in \mathfrak{s} \subset \mathrm{Spin}^c(X_{\alpha\beta}, \tau_{\alpha\beta}) = \mathrm{Spin}^c(X, \tau)$$

is in $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z}; \underline{\mathfrak{s}})$. Let us denote by G the similar filtered $(\mathbb{A}, \mathbb{H})$ chain map

$$G = F_{\alpha\gamma\delta} : \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z}; \mathfrak{s}) \longrightarrow \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{z}; \mathfrak{s})$$

defined by $G(\mathbf{y}) := f_{\alpha\gamma\delta}\big(\mathbf{y} \otimes \Theta_{\gamma\delta}\big)$. Also, define the map

$$H = H_{\alpha\beta\gamma\delta} : \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}) \longrightarrow \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{z}; \mathfrak{s})$$

by $H(\mathbf{x}) := h_{\alpha\beta\gamma\delta}\big(\mathbf{x}\otimes\Theta_{\beta\gamma}\otimes\Theta_{\gamma\delta}\big)$. Checking that all the above maps respect the relative Spin^c structures is straight forward. Using Lemma 6.12, and the fact that $f_{\beta\gamma\delta}(\Theta_{\beta\gamma}\otimes\Theta_{\gamma\delta})=\Theta_{\beta\delta}$ we have

$$G\big(F(\mathbf{x})\big)-f_{\alpha\beta\delta}\big(\mathbf{x}\otimes\Theta_{\beta\delta}\big)=\partial\big(H(\mathbf{x})\big)+H\big(\partial(\mathbf{x})\big).$$

The small triangles which contribute to $f_{\alpha\beta\delta}(\mathbf{x}\otimes\Theta_{\beta\delta})$ may be used to show that in terms of an appropriate energy filtration we have

$$f_{\alpha\beta\delta}\big(\mathbf{x}\otimes\Theta_{\beta\delta}\big)=I(x)+\epsilon(\mathbf{x})$$

where $\epsilon(\mathbf{x})$ consists of a combination of generators with smaller energy than $\mathbf{x}$. This implies that there is a filtered $(\mathbb{A},\mathbb{H})$ chain equivalence

$$K:\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\delta},\mathbf{z};\mathfrak{s})\longrightarrow\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s}),$$

respecting the decomposition according to relative Spin^c structures, such that $K\big(f_{\alpha\beta\delta}(\mathbf{x}\otimes\Theta_{\beta\delta})\big)=\mathbf{x}$. Thus setting $G'=K\circ G$ and $H'=K\circ H$ we have

$$G'\circ F-Id=H'\circ\partial+\partial\circ H',$$

and $G'\circ F$ is chain homotopic to the identity. The other composition is similarly chain homotopic to the identity. This completes the proof of the handle slide invariance.

The invariance under isotopy and stabilization-destabilization is completely similar to the proofs presented in Sections 7 and 10 of [OS5]. Thus the filtered $(\mathbb{A},\mathbb{H})$ chain homotopy type of $\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})$ is an invariant of $(X,\tau,\mathfrak{s})$, and will be denoted by

$$\mathrm{CF}(X,\tau;\mathfrak{s})=\bigoplus_{\underline{\mathfrak{s}}\in\mathfrak{s}}\mathrm{CF}(X,\tau;\underline{\mathfrak{s}}).\qquad\square$$

7. Stabilization of sutured manifolds

7.1. The analytic input

Before we start proving the main result of this section, which is a generalization of the stabilization theorem of [OS9], we need to rephrase the statements of Theorem 5.1, Lemma 6.3 and Lemma 6.4 from [OS9] for weakly balanced sutured manifolds and the corresponding Heegaard diagrams.

Let (X,τ) be a sutured manifold with the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ and consider a point v on Σ. Let $\phi\in\pi_2(\mathbf{x},\mathbf{y})$ be the homotopy class of a

Whitney disk connecting the intersection points $\mathbf{x}$ and $\mathbf{y}$, and assume that $n_v(\phi) = k \in \mathbb{Z}^{\geq 0}$. We may define a map

$$\rho^v : \mathcal{M}(\phi) \longrightarrow \mathrm{Sym}^k(\mathbb{D})$$
$$\rho^v(u) = u^{-1}\big(v \times \mathrm{Sym}^{\ell-1}(\Sigma)\big)$$

Correspondingly, we may define the moduli spaces $\mathcal{M}(\phi, t)$ and $\mathcal{M}(\phi, \Delta)$ by

$$\mathcal{M}(\phi, t) = \mathcal{M}(\phi, t; v) := \big\{u \in \mathcal{M}(\phi) \,\big|\, (t, 0) \in \rho^v(u)\big\} \quad \text{and}$$
$$\mathcal{M}(\phi, \Delta) = \mathcal{M}(\phi, \Delta; v) := \big\{u \in \mathcal{M}(\phi) \,\big|\, \rho^v(u) = \Delta\big\}$$

where $t \in [0, 1]$ and $\Delta \in \mathrm{Sym}^k(\mathbb{D})$.

Let (X_1, τ_1) and (X_2, τ_2) be weakly balanced sutured manifolds with the corresponding Heegaard diagrams $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{z}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{z}_2)$. We can form their connected sum along the points w and v on Σ_1 and Σ_2 to obtain a new sutured manifold (X, τ) with the corresponding Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ where

$$\Sigma = \Sigma_1 \# \Sigma_2, \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2, \quad \boldsymbol{\beta} = \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2 \quad \text{and} \quad \mathbf{z} = \mathbf{z}_1 \cup \mathbf{z}_2.$$

Note that $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = (\mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}) \times (\mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2})$. Consider two pairs of intersection points $\mathbf{x}_1, \mathbf{y}_1 \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}$ and $\mathbf{x}_2, \mathbf{y}_2 \in \mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2}$. Any homology class

$$\phi \in \pi_2(\mathbf{x}_1 \times \mathbf{x}_2, \mathbf{y}_1 \times \mathbf{y}_2)$$

can be uniquely decomposed as $\phi = \phi_1 \# \phi_2$ where

$$\phi_1 \in \pi_2(\mathbf{x}_1, \mathbf{y}_1), \quad \phi_2 \in \pi_2(\mathbf{x}_2, \mathbf{y}_2) \quad \text{and} \quad n_w(\phi_1) = n_v(\phi_2).$$

Conversely, any pair of homology classes $\phi_1 \in \pi_2(\mathbf{x}_1, \mathbf{y}_1)$ and $\phi_2 \in \pi_2(\mathbf{x}_2, \mathbf{y}_2)$ such that $n_w(\phi_1) = n_v(\phi_2)$ can be combined to give a homology class

$$\phi = \phi_1 \# \phi_2 \in \pi_2(\mathbf{x}_1 \times \mathbf{x}_2, \mathbf{y}_1 \times \mathbf{y}_2).$$

Theorem 7.1. *Let (X_1, τ_1) and (X_2, τ_2) be weakly balanced sutured manifolds with the corresponding Heegaard diagrams*

$$(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{z}_1) \quad \textit{and} \quad (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{z}_2)$$

respectively. Consider the weakly balanced sutured manifold (X, τ) obtained by taking the connected sum of the two Heegaard diagrams along w and v as *described above. For any homotopy class*

$$\phi = \phi_1 \# \phi_2 \in \pi_2(\mathbf{x}_1 \times \mathbf{x}_2, \mathbf{y}_1 \times \mathbf{y}_2)$$

we then have

$$\mu(\phi) = \mu(\phi_1) + \mu(\phi_2) - 2k$$

where $\mathbf{x}_1, \mathbf{y}_1 \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}$, $\mathbf{x}_2, \mathbf{y}_2 \in \mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2}$ *and* $k = n_w(\phi_1) = n_v(\phi_2)$.

Suppose furthermore that $\mu(\phi_1) = 1$, $\mu(\phi_2) = 2k$, *that* w *(respectively,* v*) is in a genus zero connected component of either of* $\Sigma_1 - \boldsymbol{\alpha}_1$ *and* $\Sigma_1 - \boldsymbol{\beta}_1$ *(respectively,* $\Sigma_2 - \boldsymbol{\alpha}_2$ *and* $\Sigma_2 - \boldsymbol{\beta}_2$*), and that one of the following is true:*

- *At least one component of* $\mathfrak{R}(\tau_2)$ *has nonzero genus and* $\mathfrak{u}(\phi) \neq 0$.
- *All the components of* $\mathfrak{R}(\tau_2)$ *are genus zero components, and*

$$\ell_2 = |\boldsymbol{\alpha}_2| = |\boldsymbol{\beta}_2| > g(\Sigma_2).$$

Then assuming the fibered product

$$\mathcal{M}(\phi_1) \times_{\mathrm{Sym}^k(\mathbb{D})} \mathcal{M}(\phi_2) = \left\{ u_1 \times u_2 \in \mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2) \,\middle|\, \rho^w(u_1) = \rho^v(u_2) \right\}$$

of $\mathcal{M}(\phi_1)$ *and* $\mathcal{M}(\phi_2)$ *is a smooth manifold, and taking the length of the connected sum tube sufficiently large, there is an identification of this moduli space with* $\mathcal{M}(\phi)$.

Proof. The proof is similar to the proof of Theorem 5.1 in [OS9]. As in that proof we use Lipshitz's cylindrical formulation. However, we keep the same notation for the moduli spaces and the corresponding maps for the sake of simplicity.

The formula for the Maslov index follows from the excision principle for the linearized $\overline{\partial}$ operator, using the cylindrical formulation [Lip]. For the second part of the theorem, if all components of $\mathfrak{R}(\tau)$ are genus zero components and $\ell_2 > g(\Sigma_2)$, the proof of Theorem 5.1 from [OS9] applies word by word. In the other case, the proof requires some modification, as follows. We drop the details and only highlight the differences. For more details, we refer the reader to [OS9].

Suppose that $\mathfrak{R}(\tau_2)$ has a component with nonzero genus. Consider a sequence of paths of almost complex structures $\{J_t(s)\}_{s\in[1\infty)}$ on $\Sigma_1 \# \Sigma_2$, where for $s \in [1, \infty)$ $\{J_t(s)\}_{t\in[0,1]}$ denotes the path of almost complex structures determined by a pair of generic paths of complex structures $\{j_t^1\}_t$ and $\{j_t^2\}_t$ on Σ_1 and Σ_2, and by setting the neck-length equal to s. Let us assume that $\widehat{\mathcal{M}}_{J_t(s)}(\phi) \neq \emptyset$ as $s \longrightarrow \infty$. Consider a sequence of pseudo-holomorphic curves $\{u_s\}_{s\in\mathbb{Z}^+}$ such that $u_s \in \mathcal{M}_{J_t(s)}(\phi)$. Under the assumptions $\mu(\phi_1) = 1$ and $\mu(\phi_2) = 2k$, and using the Gromov compactness theorem, a subsequence of this sequence is weakly convergent to a pseudo holomorphic representative u_1 of ϕ_1 and a broken flow-line representative of ϕ_2. This broken flow-line can not contain any sphere bubblings, since otherwise our assumption on $\mathfrak{R}(\tau_2)$ implies that $\mathfrak{u}(\phi_2) = 0$, and thus $\mathfrak{u}(\phi) = 0$. Hence we may follow the argument of Ozsváth and Szabó from here, and conclude that there is a component u_2 of this broken flow line such that u_1 and u_2 represents a pre-glued Whitney disk, i.e. that

$$\rho^w(u_1) = \rho^v(u_2).$$

Let ϕ_2' be the homotopy class represented by u_2. If $\phi_2' \neq \phi_2$, the above Gromov limit contains boundary degenerations or other flow lines. The assumption $\mathfrak{u}(\phi_2) \neq 0$ then implies that $\mu(\phi_2') < \mu(\phi_2) = 2k$. Let us consider the map

$$\rho^v : \mathcal{M}(\phi_2') \longrightarrow \mathrm{Sym}^k(\mathbb{D}).$$

For any point $\Delta \in \mathrm{Sym}^k(\mathbb{D})$ the moduli space $(\rho^v)^{-1}(\Delta)$ will have the expected dimension equal to $\mu(\phi') - 2k < 0$. Thus for a generic choice of $\Delta \in \mathrm{Sym}^k(\mathbb{D})$, this moduli space is empty. This observation implies that $\phi_2' = \phi_2$, as in the proof of Theorem 5.1 from [OS9].

Thus the Gromov limit of a sequence of holomorphic representatives of ϕ, as we stretch the neck, is a pre-glued flow line representing ϕ_1 and ϕ_2. Conversely, given a pre-glued flow line, one obtains a pseudo-holomorphic representative of ϕ in $\mathcal{M}(\phi)$ by the gluing theorem of Lipshitz [Lip], as in the proof of Theorem 5.1 from [OS9]. This completes the proof of Theorem 7.1. □

Lemma 7.2. *Let (X, τ) be a weakly balanced sutured manifold represented by the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$, and let $v \in \mathbf{z}$ be one of the marked points. Let $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ be the homotopy class of a Whitney disk connecting the intersection point $\mathbf{x}$ to $\mathbf{y}$. Assume furthermore that $\mathcal{D}(\phi) \geq 0$ and that $\mathfrak{u}(\phi) \neq 0$. If $\mu(\phi) = 2$ then $\mathcal{M}(\phi, t)$ is generically a zero dimensional moduli space. Furthermore, there is a number $\epsilon > 0$ such that for all $t \leq \epsilon$ the only non-empty such moduli spaces $\mathcal{M}(\phi, t; v)$ are the moduli spaces corresponding to $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$ where ϕ is obtained by splicing a boundary degeneration with Maslov index 2 corresponding to one of the genus zero components of $\mathfrak{R}^+(\tau)$*

and a constant flow line. For any such moduli space we have

$$\#\mathcal{M}(\phi, t; v) = \begin{cases} 0 & \textit{if } l = 1 \\ 1 & \textit{if } l > 1 \end{cases},$$

where l is the number of components in $\mathfrak{R}^+(\tau)$.

Proof. Given Lemma 5.8, the proof is exactly the same as the proof of Lemma 6.3 in [OS9]. □

Consider the Heegaard diagram $(S, \alpha, \beta, \mathbf{z})$, where $S = S^2$ is the Riemann sphere, and α and β are simple closed curves on S intersecting transversely in two points $\{x, y\}$, and $\mathbf{z} = \{z_1, z_2, z_3, z_4\}$ where there is one marked point in each one of the four connected components of $S - \alpha - \beta$. Assume furthermore, that the marked points are labelled so that the bigons corresponding to z_4 and z_2 do not have any edges in common, and that the edge belonging to the common boundary of the pairs of the bigons corresponding to z_1 and z_2 is on β.

Lemma 7.3. *Consider the Heegaard diagram $(S, \alpha, \beta, \mathbf{z})$ as above. For a generic point $\Delta \in \mathrm{Sym}^k(\mathbb{D})$ (for any positive integer k), we have*

$$\sum_{\substack{\phi \in \pi_2^{2k}(a,a) \\ n_{z_4}(\phi)=0}} \#\mathcal{M}(\phi, \Delta; z_2) = 1,$$

for $a \in \{x, y\}$. Moreover, if the generic set $\Delta = \{p_1, \dots, p_k\}$ of points is chosen so that the points $p_i = (x_i, y_i) \in \mathbb{D}$, $i = 1, \dots, k$ are sufficiently close to $\{0\} \times \mathbb{R}$ (i.e. x_i is sufficiently small) we will have

$$\sum_{\substack{\phi \in \pi_2^{2k}(a,a) \\ n_{z_4}(\phi)=0}} \#\mathcal{M}(\phi, \Delta; z_2)\mathfrak{u}_1^{n_{z_1}(\phi)} = 1 \tag{15}$$

Proof. The first claim is precisely Lemma 6.4 from [OS9]. In fact, the proof of the second claim is almost included in the proof of Lemmas 6.3 and 6.4 in [OS9], as outlined below.

The homotopy classes $\phi \in \pi_2^{2k}(a, a)$ with $n_{z_4}(\phi) = 0$ (and $n_{z_2}(\phi) = k$) are determined by $j = n_{z_1}(\phi) \in \{0, \dots, k\}$. Denote the corresponding homotopy class by ϕ_j, and let $\mathcal{M}_j(\Delta)$ denote the moduli space $\mathcal{M}(\phi_j, \Delta; z_2)$. Let us assume that for a sequence of sets $\Delta_t = \{p_1(t), \dots, p_k(t)\}$, with $p_i(t) =$

$(x_i(t), y_i(t)) \in [0,1] \times \mathbb{R}$ and $x_i(t)$ going to zero as t goes to infinity, the moduli space $\mathcal{M}_j(\Delta_t)$ is non-empty. Applying Gromov's compactness theorem to a sequence $\{u_t\}_t$ with $u_t \in \mathcal{M}_j(\Delta_t)$ we obtain a broken flow line u in the limit. As in the proof of Lemma 6.3 from [OS9], since z_2 is disjoint from the curves α and β, while the set $u_t^{-1}(z_2)$ contains k points which converge to $\{0\} \times \mathbb{R}$, we may conclude that the domain of u includes some β boundary degenerations $u^1, \ldots, u^p$ with

$$n_{z_2}(u^1) + n_{z_2}(u^2) + \cdots + n_{z_2}(u^p) = k.$$

This already implies, since $\mu(u^1) + \cdots + \mu(u^p) = 2k = \mu(u)$, that

$$\mathcal{D}(\phi_j) = \mathcal{D}(u) = \mathcal{D}(u^1) + \cdots + \mathcal{D}(u^p) \quad \Rightarrow \quad j = 0.$$

The only class contributing to Equation 15 is thus ϕ_0 provided that the set Δ consists of the points sufficiently close to $\{0\} \times \mathbb{R}$. The second part of the lemma then follows from the first part. □

7.2. Simple stabilization of a weakly balanced sutured manifold

Let us fix a weakly balanced sutured manifold $(X, \tau = \{\tau_1, \ldots, \tau_\kappa\})$ and let

$$\mathfrak{R}^-(\tau) = \bigcup_{i=1}^{k} R_i^- \quad \text{and} \quad \mathfrak{R}^+(\tau) = \bigcup_{j=1}^{l} R_j^+,$$

as before.

Definition 7.4. We say that a sutured manifold $(X, \widehat{\tau})$ is obtained by a *simple stabilization* of (X, τ) if $\widehat{\tau} = \tau \cup \{\tau_{\kappa+1}, \tau_{\kappa+2}\}$ and $\tau_{\kappa+1}$ and $\tau_{\kappa+2}$ are oriented simple closed curves so that $-\tau_{\kappa+1}$ and $\tau_{\kappa+2}$ are both parallel to an oriented suture $\tau_i \in \tau$, where τ_i is in the common boundary of two genus zero components of $\mathfrak{R}(\tau)$. Moreover, τ_i and $\tau_{\kappa+1}$ bound an annulus in $\partial X - \widehat{\tau}$.

Without loss of generality, assume that $i = \kappa$, $\tau_i = \tau_\kappa \in \partial R_k^- \cap \partial R_l^+$ and $\tau_{\kappa+1}, \tau_{\kappa+2} \subset R_l^+$. Let us denote the connected components of $R_l^+ - (\tau_{\kappa+1} \cup \tau_{\kappa+2})$ by $R^+ \coprod R^- \coprod R_{l+1}^+$, where R^+ and R^- are the annulus components with the boundary sets $\{\tau_\kappa, \tau_{\kappa+1}\}$ and $\{\tau_{\kappa+1}, \tau_{\kappa+2}\}$ respectively, as illustrated in Figure 3.

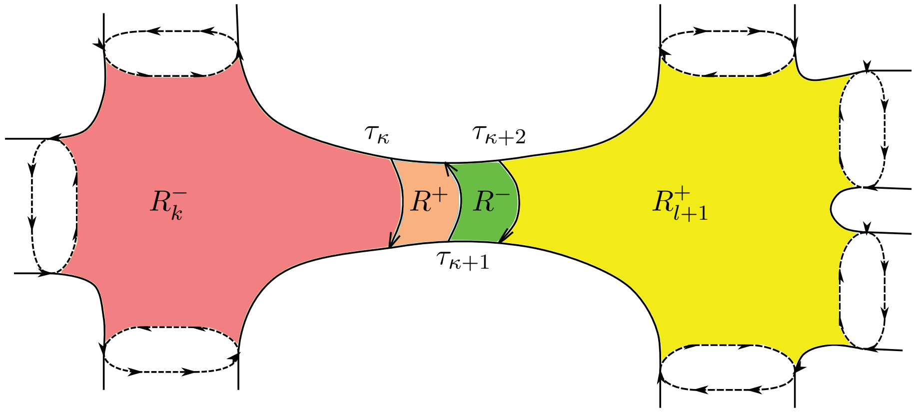


Figure 3: The simple stabilization of a sutured manifold. Here we have $R_l^+ = R_{l+1}^+ \coprod R^- \coprod R^+$.

Note that

$$\begin{aligned} &\mathfrak{R}^-(\widehat{\tau}) = \mathfrak{R}^-(\tau)\coprod R^-, \quad \mathfrak{R}^+(\widehat{\tau}) = (\mathfrak{R}^+(\tau) - R_l^+)\coprod R^+\coprod R_{l+1}^+,\\ &\mathfrak{u}^-(\widehat{\tau}) = \mathfrak{u}^-(\tau) + \mathfrak{u}_{\kappa+1}\mathfrak{u}_{\kappa+2}, \quad \mathfrak{u}^+(\widehat{\tau}) = \mathfrak{u}^+(\tau) + \mathfrak{u}_\kappa\mathfrak{u}_{\kappa+1} + \mathfrak{u}_{l+1}^+ - \mathfrak{u}_l^+ \end{aligned} \tag{16}$$

where $\mathfrak{u}_{l+1}^+ = \mathfrak{u}(R_{l+1}^+)$. The algebra associated with $(X,\widehat{\tau})$ is defined by

$$\begin{aligned} \mathbb{A}_{\widehat{\tau}} &= \frac{\widetilde{\mathbb{A}}_{\widehat{\tau}}}{\langle \mathfrak{u}^+(\widehat{\tau}) - \mathfrak{u}^-(\widehat{\tau})\rangle_{\widetilde{\mathbb{A}}_{\widehat{\tau}}}}, \quad \text{where}\\ \widetilde{\mathbb{A}}_{\widehat{\tau}} &= \frac{\langle \mathfrak{u}_1,\dots,\mathfrak{u}_\kappa,\mathfrak{u}_{\kappa+1},\mathfrak{u}_{\kappa+2}\rangle_{\mathbb{Z}}}{\langle \mathfrak{u}_i^+ \mid g_i^+>0\rangle_{\mathbb{Z}[\kappa+2]} + \langle \mathfrak{u}_j^- \mid g_j^->0\rangle_{\mathbb{Z}[\kappa+2]}} = \widetilde{\mathbb{A}}_\tau[\mathfrak{u}_{\kappa+1},\mathfrak{u}_{\kappa+2}]. \end{aligned}$$

Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ be a Heegaard diagram for (X,τ). A Heegaard diagram $(\Sigma,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}},\widehat{\mathbf{z}})$ for $(X,\widehat{\tau})$ may then be constructed from $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ as follows. We set

$$\widehat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\cup\{\alpha_{\ell+1}\}, \quad \widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}\cup\{\beta_{\ell+1}\} \quad \text{and} \quad \widehat{\mathbf{z}} = \mathbf{z}\cup\{z_{\kappa+1},z_{\kappa+2}\},$$

where the additional curves $\alpha_{\ell+1}$ and $\beta_{\ell+1}$ are isotopic simple closed curves on Σ in the the connected component of $\Sigma - \boldsymbol{\alpha}\cup\boldsymbol{\beta}$ containing the marked point z_κ with the following properties. We assume that $\#\alpha_{\ell+1}\cap\beta_{\ell+1} = 2$ and that $\alpha_{\ell+1}$ and $\beta_{\ell+1}$ bound the disks A_{k+1} and B_{l+1} in $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ respectively. Furthermore, we assume that

$$z_\kappa \in B_{l+1} - A_{k+1}, \quad z_{\kappa+1}\in A_{k+1}\cap B_{l+1} \quad \text{and} \quad z_{\kappa+2}\in A_{k+1} - B_{l+1}.$$

The picture around the marked point z_κ is illustrated in Figure 4.

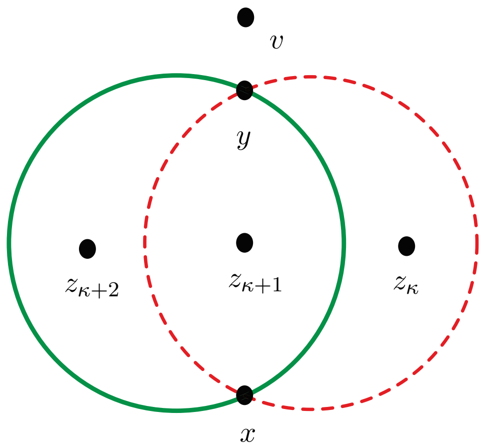


Figure 4: For simple stabilization, a pair of intersecting null-homotopic simple closed curves $\alpha_{\ell+1}$ and $\beta_{\ell+1}$ are added to the Heegaard diagram close to the marked point z_κ. The locations of the new marked points $z_\kappa, z_{\kappa+1}$ and $z_{\kappa+2}$ are illustrated in the figure. The marked point v is used as the connected sum point of the current diagram (on a Riemann sphere) with the old Heegaard diagram.

Note that $H(A_{k+1}) = [S^-]$ and $H(B_{l+1}) = [S^+]$ in $H_2(\overline{X}_{\widehat{\tau}} = \overline{X}^{\widehat{\tau}})$ where S^+ and S^- are the sphere boundary components of $\overline{X}_{\widehat{\tau}}$ corresponding to R^+ and R^- in $\mathfrak{R}(\widehat{\tau})$ respectively. In the above situation, we say that the Heegaard diagram $(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}})$ is obtained from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ by a *simple stabilization*.

We may define a map

$$\widehat{\imath} : \mathrm{Spin}^c(X, \tau) \longrightarrow \mathrm{Spin}^c(X, \widehat{\tau})$$

as follows. Fix $\widehat{\underline{\mathfrak{s}}} \in \mathrm{Spin}^c(X, \widehat{\tau})$ and let $\widehat{v}$ be a nowhere vanishing vector field on X representing $\widehat{\underline{\mathfrak{s}}}$ such that $\widehat{v}|_{\partial X} = v_{\widehat{\tau}}$. Consider a neighbourhood N in X of the annulus

$$R = \overline{R^+ \cup R^-} \subset \partial X$$

together with a diffeomorphism

$$\psi : N \longrightarrow S^1 \times I \times I, \quad \text{s.t.}$$
$$\psi(R) = S^1 \times \{0\} \times I \quad \text{and} \quad \psi_* \left(\widehat{v}|_N\right)|_{S^1 \times I \times \{0,1\}} = \frac{\partial}{\partial s},$$

where $I=[0,1]$ is the unit interval and s denotes the standard parameter on the third component of the product $S^1\times I\times I$. The vector field $\psi_*(\widehat{v})$ may be changed through an isotopy to a new vector field $\psi_*(v)$ on $S^1\times I\times I$ with the property $\psi_*(v)|_{S^1\times\{0\}\times I}=\frac{\partial}{\partial s}$, where the vector field remains fixed through the isotopy on

$$\left(S^1\times\{1\}\times I\right)\bigcup\left(S^1\times I\times\{0\}\right)\bigcup\left(S^1\times I\times\{1\}\right).$$

The vector field v on N may be glued to $\widehat{v}|_{X-N}$ to give a vector field on X, still denote by v, which represents an element $\underline{\mathfrak{s}}$ in $\mathrm{Spin}^c(X,\tau)$. It is not hard to see that the above construction gives an isomorphism between $\mathrm{Spin}^c(X,\widehat{\tau})$ and $\mathrm{Spin}^c(X,\tau)$, and we may thus define $\widehat{\imath}(\underline{\mathfrak{s}}):=\widehat{\underline{\mathfrak{s}}}$.

We may also picture $\overline{X}_{\widehat{\tau}}$ as the three-manifold obtained from $\overline{X}$ by removing a pair of spheres. This gives an embedding $i_{\overline{X}}:\overline{X}_{\widehat{\tau}}\to\overline{X}$. The suture $\tau_{\kappa+1}$ gives an arc connecting the above two balls in $\overline{X}_{\widehat{\tau}}$. Abusing the notation, let

$$i^*_{\overline{X}}:\mathrm{Spin}^c(\overline{X})\to\mathrm{Spin}^c(\overline{X}_{\widehat{\tau}})$$

denote the isomorphism obtained by first pulling back a non-vanishing vector field from $\overline{X}$ to $\overline{X}_{\widehat{\tau}}$ using the embedding $i_{\overline{X}}$, and then modifying the resulting vector field in a neighbourhood of the two removed balls and the arc joining them, so that the corresponding boundary conditions are satisfied. It is then easy to verify that the following diagram is commutative

$$\begin{array}{ccc}\mathrm{Spin}^c(X,\tau) & \xrightarrow{\widehat{\imath}} & \mathrm{Spin}^c(X,\widehat{\tau})\\ \Big\downarrow[.] & & \Big\downarrow[.]\\ \mathrm{Spin}^c(\overline{X}_{\tau}) & \xrightarrow{i^*_{\overline{X}}} & \mathrm{Spin}^c(\overline{X}_{\widehat{\tau}})\end{array}.$$

Let (X,τ) be a weakly balanced sutured manifold, $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$ be a Spin^c class on $\overline{X}$, and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ be an $\mathfrak{s}$-admissible Heegaard diagram for (X,τ). Let $(X,\widehat{\tau})$ be the sutured manifold obtained by a simple stabilization on (X,τ) and $(\Sigma,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}},\widehat{\mathbf{z}})$ be the Heegaard diagram for $(X,\widehat{\tau})$ obtained by the simple stabilization of the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z})$ as above. Abusing the notation, let

$$\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathfrak{s};\mathbb{A}_{\widehat{\tau}})$$

denote the free module generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ over $\mathbb{A}_{\widehat{\tau}}$. Counting holomorphic disks then gives a homomorphism of $\mathbb{A}_{\widehat{\tau}}$ modules

$$\partial : \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}}) \longrightarrow \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}}),$$

which satisfies

$$(17) \qquad (\partial \circ \partial)(\mathbf{x}) = (\mathfrak{u}^+(\tau) - \mathfrak{u}^-(\tau)).\mathbf{x}, \quad \forall\, \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \quad \text{s.t. } \underline{\mathfrak{s}}_{\mathbf{z}}(\mathbf{x}) \in \mathfrak{s}.$$

Since $\mathfrak{u}^+(\tau) - \mathfrak{u}^-(\tau)$ is not trivial in $\mathbb{A}_{\widehat{\tau}}$, ∂ is not a differential. However, we will prove the following proposition in the upcoming subsection.

Proposition 7.5. *With the above notation fixed, for any given* Spin^c *class*

$$\mathfrak{s} \in \mathrm{Spin}^c(\overline{X}_{\widehat{\tau}}) = \mathrm{Spin}^c(\overline{X}),$$

the filtered chain homotopy type of the complex $\mathrm{CF}(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}}, \mathfrak{s})$ *is the same as the filtered chain homotopy type of the chain complex obtained by equipping the module*

$$\mathrm{CF}\left(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, i^*_{\overline{X}}(\mathfrak{s}); \mathbb{A}_{\widehat{\tau}}\right) \oplus \mathrm{CF}\left(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, i^*_{\overline{X}}(\mathfrak{s}); \mathbb{A}_{\widehat{\tau}}\right)$$

with the differential

$$\widehat{\partial} = \begin{pmatrix} \partial & \mathfrak{u}_{\kappa+1} - \mathfrak{u} \\ \mathfrak{u}_\kappa - \mathfrak{u}_{\kappa+2} & -\partial \end{pmatrix}, \quad \textit{where } \mathfrak{u} := \prod_{i:\tau_i \in \partial R_l^+, i \neq \kappa} \mathfrak{u}_i \in \mathbb{A}_{\widehat{\tau}}.$$

It is important to note that the relation $\widehat{\partial}^2 = 0$ follows from Equation 17, the fact that $\mathfrak{u}.\mathfrak{u}_\kappa = \mathfrak{u}(R_l^+)$ while $\mathfrak{u}.\mathfrak{u}_{\kappa+2} = \mathfrak{u}(R_{l+1}^+)$, and the relations in the second line of Equation 16.

7.3. Proof of the stabilization formula

In this subsection we prove Proposition 7.5.

Proof of Proposition 7.5. Let $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X}_{\widehat{\tau}})$ be a Spin^c structure on $\overline{X}_{\widehat{\tau}}$. Consider a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for (X, τ), which is $i^*_{\overline{X}}(\mathfrak{s})$-admissible. Furthermore, assume that for any positive periodic domain $\mathcal{P}$ we have the following implication:

$$\left\langle c_1\left(i_{\overline{X}}^*(\mathfrak{s})\right), H(\mathcal{P})\right\rangle \leq 0 \quad \Rightarrow \quad \mathfrak{u}(\mathcal{P}) = 0,$$

where the last vanishing takes place in $\widetilde{\mathbb{A}}_\tau$, rather than $\mathbb{A}_\tau$. The existence of such a Heegaard diagram is guaranteed by Remark 4.6.

Let $\widehat{H} = (\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}})$ be the Heegaard diagram for $(X, \widehat{\tau})$ obtained by a simple stabilization on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$. We claim that this Heegaard diagram is $\mathfrak{s}$-admissible. Suppose that $\mathcal{P}$ is a positive periodic domain corresponding to $\widehat{H}$ such that $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 0$. Then there are integers a and b, and a positive periodic domain $\mathcal{P}_0$ for H such that

$$\mathcal{P} = \mathcal{P}_0 + aA_{k+1} + bB_{l+1} \quad \text{and} \quad n_{z_\kappa}(\mathcal{P}_0) = n_{z_{\kappa+1}}(\mathcal{P}_0) = n_{z_{\kappa+2}}(\mathcal{P}_0).$$

Thus $\mathcal{P}_0$ may be viewed as a periodic domain associated with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$. If $n_{z_\kappa}(\mathcal{P}_0) = d$ then

$$\left\langle c_1(\mathfrak{s}), (i_{\overline{X}})_* H(\mathcal{P}_0)\right\rangle = \left\langle c_1(i_{\overline{X}}^*(\mathfrak{s})), H(\mathcal{P}_0)\right\rangle + 2d.$$

From here we may conclude

$$\begin{aligned} &\left\langle c_1(\mathfrak{s}), (i_{\overline{X}})_* H(\mathcal{P}_0)\right\rangle = -2(a+b) \\ \Rightarrow\ &\left\langle c_1(i_{\overline{X}}^*(\mathfrak{s})), H(\mathcal{P}_0)\right\rangle = -2(a+b+d) = -n_{z_{\kappa+1}}(\mathcal{P}_0) \leq 0. \end{aligned}$$

Since the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is $i_{\overline{X}}^*(\mathfrak{s})$-admissible in the stronger sense of Remark 4.6, we conclude that $\mathfrak{u}(\mathcal{P}_0) = 0$ in $\widetilde{\mathbb{A}}_\tau$. The condition that z_κ is in the genus zero components of $\Sigma - \boldsymbol{\alpha}$ and $\Sigma - \boldsymbol{\beta}$ then implies that $\mathfrak{u}(\mathcal{P}) = 0$ in $\widetilde{\mathbb{A}}_{\widehat{\tau}}$. This proves the $\mathfrak{s}$-admissibility of the Heegaard diagram $\widehat{H}$.

Let us consider the Heegaard diagram $(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}})$ as the connected sum

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z} \cup \{w\} - \{z_\kappa\}) \# (S, \alpha_{\ell+1}, \beta_{\ell+1}, \{v, z_\kappa, z_{\kappa+1}, z_{\kappa+2}\}),$$

where S is a sphere, w and v are the corresponding connected sum points such that w is in the same domain as z_κ in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$. Choose w so that it is sufficiently close to one of the β curves and is sufficiently far from the curves in $\boldsymbol{\alpha}$. If $\alpha_{\ell+1} \cap \beta_{\ell+1} = \{x, y\}$ then $\mathbb{T}_{\widehat{\alpha}} \cap \mathbb{T}_{\widehat{\beta}} = (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \{x, y\}$, and for any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we have

$$\widehat{\imath}(\underline{\mathfrak{s}}(\mathbf{x})) = \underline{\mathfrak{s}}(\mathbf{x} \times \{x\}) = \underline{\mathfrak{s}}(\mathbf{x} \times \{y\}) + \mathrm{PD}[\tau_{\kappa+1}].$$

Let C_x and C_y be the submodules of $\mathrm{CF}\big(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}}, \mathfrak{s}\big)$ generated by the intersection points containing x and y respectively. Thus we have a module splitting

$$\mathrm{CF}\big(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}}, \mathfrak{s}\big) = C_x \oplus C_y.$$

The Heegaard diagram $(S, \alpha, \beta, \{v, z_\kappa, z_{\kappa+1}, z_{\kappa+2}\})$ then corresponds to a sutured manifold and the corresponding ring of coefficients is

$$\mathbb{A}_S = \frac{\mathbb{Z}[\mathfrak{u}_\kappa, \mathfrak{u}_{\kappa+1}, \mathfrak{u}_{\kappa+2}, \mathfrak{u}_v]}{\langle (\mathfrak{u}_{\kappa+1} - \mathfrak{u}_v)(\mathfrak{u}_\kappa - \mathfrak{u}_{\kappa+2}) = 0 \rangle}.$$

The chain complex is generated by x and y. The homotopy classes of disks with positive domains and Maslov index 1 are the four bi-gons containing the markings. Since $\partial \circ \partial = 0$ we should have

$$\partial(y) = \pm(\mathfrak{u}_\kappa - \mathfrak{u}_{\kappa+2})x \quad \text{and} \quad \partial(x) = \pm(\mathfrak{u}_{\kappa+1} - \mathfrak{u}_v)y.$$

A more detailed consideration implies that the unique holomorphic representatives of the bi-gons containing v and $z_{\kappa+2}$ have the same sign, and this same sign is opposite to the signs of the unique holomorphic representatives of the bi-gons containing z_κ and $z_{\kappa+1}$. We may thus assume that the former two representatives get a negative sign and the latter two get a positive sign. The alternative choice on the system of orientations results in the same conclusion in our proof. The system of coherent orientations associated with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, \mathfrak{s})$, together with this (or the alternative) system of coherent orientations for $(S, \alpha, \beta, \{v, z_\kappa, z_{\kappa+1}, z_{\kappa+2}\})$ gives a system of coherent orientations for $(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}}, \mathfrak{s})$.

First we consider the C_x-components of the differential of the complex on the generators of C_x. Let $\mathbf{x} \times \{x\}$ be a generator of C_x and $\phi \in \pi_2(\mathbf{x} \times \{x\}, \mathbf{y} \times \{x\})$ be the homology class of a Whitney disk with $\mu(\phi) = 1$. We may thus decompose ϕ as $\phi = \phi_1 \# \phi_2$ where $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\phi_2 \in \pi_2(x, x)$. Theorem 7.1 then implies that

$$\mu(\phi) = \mu(\phi_1) + \mu(\phi_2) - 2k = \mu(\phi_1) + 2n_{z_{\kappa+1}}(\phi_2),$$

where $k = n_w(\phi_1) = n_v(\phi_2)$. If $\mathcal{M}(\phi) \neq \emptyset$ for long enough neck-length, then ϕ_2 admits holomorphic representatives and $\mathcal{D}(\phi_2) \geq 0$. This implies that

$$\mu(\phi_2) - 2n_v(\phi_2) = 2n_{z_{\kappa+1}}(\phi) \geq 0,$$

and that the equality happens if and only if $n_{z_{\kappa+1}}(\phi) = 0$. If $\mu(\phi_2) - 2n_v(\phi_2) > 0$ then $\mu(\phi_1) \leq -1$ and $\mathcal{M}(\phi_1)$ is generically empty. Thus $n_{z_{\kappa+1}}(\phi_2)$ should be zero and $\mu(\phi_2) = 2n_v(\phi_2) = 2k$. Theorem 7.1 then guarantees that for a sufficiently large connected sum length, we have an identification of $\mathcal{M}(\phi)$ as follows.

$$\begin{aligned}
\mathcal{M}(\phi) &= \mathcal{M}(\phi_1) \times_{\mathrm{Sym}^k(\mathbb{D})} \mathcal{M}(\phi_2) \\
&= \big\{u_1 \times u_2 \in \mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2) \,\big|\, \rho^w(u_1) = \rho^v(u_2)\big\} \\
\Rightarrow \quad \#\widehat{\mathcal{M}}(\phi) &= \sum_{u_1 \in \widehat{\mathcal{M}}(\phi_1)} \#\big\{u_2 \in \mathcal{M}(\phi_2) \,\big|\, \rho^w(u_1) = \rho^v(u_2)\big\}.
\end{aligned}$$

The coefficient of $\mathbf{y} \times \{x\}$ in the expression $\widehat{\partial}(\mathbf{x} \times \{x\})$ in $\mathrm{CF}\big(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}}, \mathfrak{s}\big)$ is thus equal to

$$\sum_{\substack{\phi_1 \in \pi_2^1(\mathbf{x},\mathbf{y}) \\ \phi_2 \in \pi_2(x,x) \\ n_{\kappa+1}(\phi_2)=0 \\ n_w(\phi_1)=n_v(\phi_2) \\ u_1 \in \widehat{\mathcal{M}}(\phi_1)}} \epsilon(u_1) \prod_{i=1}^{\kappa-1} \mathfrak{u}_i^{n_i(\phi_1)} \left(\mathfrak{u}_\kappa^{n_\kappa(\phi_2)} \mathfrak{u}_{\kappa+2}^{n_{\kappa+2}(\phi_2)} \# \left\{ u_2 \,\middle|\, \begin{array}{c} u_2 \in \mathcal{M}(\phi_2) \\ \rho^w(u_1) = \rho^v(u_2) \end{array} \right\} \right),$$

where $\epsilon(u_1)$ denotes the sign associated with $u_1 \in \widehat{\mathcal{M}}(\phi_1)$ via a coherent system of orientations for the Heegaard diagram (which is suppressed from the notation). Suppose now that the marked point w is moved sufficiently close to one of the β curves, as stated before. Consequently, $\rho^w(u_1)$ would be a collection of $k = n_w(\phi_1)$ points in $\mathbb{D}$ which are sufficiently close to $\{0\} \times \mathbb{R}$.

Lemma 7.3 may then be used to compute the interior count of holomorphic curves. Since $n_v(\phi_2) = k$, the total value of the above sum is thus equal to

$$\sum_{\phi_1 \in \pi_2^1(\mathbf{x},\mathbf{y})} \sum_{u_1 \in \widehat{\mathcal{M}}(\phi_1)} \epsilon(u_1) \left(\prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_i(\phi_1)} \right),$$

which is the coefficient of $\mathbf{y}$ in $\partial\mathbf{x}$ in $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}})$. With the same argument, the C_y-component of the differential of the generators in C_y is identified with the differential of $C_y = \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}})$.

We now consider the C_x-component of the differential of a generator in C_y. For any $\phi \in \pi_2(\mathbf{x} \times \{y\}, \mathbf{y} \times \{x\})$ we can write $\phi = \phi_1 \# \phi_2$. By Theorem 7.1 if $\mu(\phi) = 1$ then $\mu(\phi_1) + \mu(\phi_2) - 2n_v(\phi_2) = 1$. By Lipshitz' Index formula we have

$$\mu(\phi_2) = 2n_v(\phi_2) + 2n_{z_{\kappa+1}}(\phi_2) + 1.$$

This implies that $\mu(\phi_2) - 2n_v(\phi_2) \geq 1$, and that the equality holds if and only if $n_{z_{\kappa+1}}(\phi_2) = 0$. This last equality should thus be satisfied and $\mu(\phi_1) = 0$. Hence ϕ_1 is constant, $\mu(\phi_2) = 1$ and $n_v(\phi_2) = 0$. These conditions imply that the possible domains for ϕ_2 are two different bi-gons in S connecting y to x, which contain z_κ and $z_{\kappa+2}$ respectively. For either of these bi-gons $\widehat{\mathcal{M}}(\phi_2)$ consists of one element, while the orientation assignment for these two bi-gons are different, and determined by our choice of the system of orientations above. The coefficient of $\mathbf{y} \times \{x\}$ in $\widehat{\partial}(\mathbf{x} \times \{y\})$ is thus $\mathfrak{u}_\kappa - \mathfrak{u}_{\kappa+2}$, i.e. the C_x-component of the differential of the generators in C_y corresponds to scalar multiplication by $\mathfrak{u}_\kappa - \mathfrak{u}_{\kappa+2}$.

Finally, we consider the C_y-component of the differential of a generator in C_x. Again, degenerate $\phi \in \pi_2(\mathbf{x} \times \{x\}, \mathbf{y} \times \{y\})$ with $\mu(\phi) = 1$ and $\mathfrak{u}(\phi) \neq 0$ as the connected sum $\phi = \phi_1 \# \phi_2$. We thus have $\mu(\phi_1) + \mu(\phi_2) - 2n_v(\phi_2) = 1$, implying $\mu(\phi_2) - 2n_v(\phi_2) \leq 1$. By Lipshitz' index formula we have

$$\mu(\phi_2) = 2n_v(\phi_2) + 2n_{z_{\kappa+1}}(\phi_2) - 1,$$

which implies that $\mu(\phi_2)$ is an odd number and $\mu(\phi_2) - 2n_v(\phi_2) \geq -1$. Thus $\mu(\phi_2) - 2n_v(\phi_2)$ is equal to 1 or -1.

If $\mu(\phi_2) - 2n_v(\phi_2) = 1$ then $\mu(\phi_1) = 0$. Thus ϕ_1 is constant and $\mathcal{D}(\phi)$ is the bi-gon containing $z_{\kappa+1}$. In this case $\#\widehat{\mathcal{M}}(\phi) = 1$. Thus the corresponding component of the differential, as a map from C_x to C_y, is given by

$$\begin{aligned} &\partial^1_{xy} : C_x \longrightarrow C_y \\ &\partial^1_{xy}(\mathbf{x} \times \{x\}) := \mathfrak{u}_{\kappa+1}(\mathbf{x} \times \{y\}). \end{aligned}$$

The second possibility is the case where $\mu(\phi_2) - 2n_v(\phi_2) = -1$. If $\mu(\phi_2) = n_v(\phi_2) = 1$ then $\mu(\phi_1) = 2$ and $n_w(\phi_1) = 1$. If furthermore $\mathfrak{R}(\tau)$ has at least one component with nonzero genus then by Theorem 7.1 for sufficiently large connected sum length $\mathcal{M}(\phi)$ is identified with

$$\begin{aligned} \mathcal{M}(\phi_1) \times_{\mathbb{D}} \mathcal{M}(\phi_2) &= \left\{ u_1 \times u_2 \in \mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2) \,\middle|\, \rho^w(u_1) = \rho^v(u_2) \right\} \\ &= \left\{ u_1 \times u_2 \in \mathcal{M}(\phi_1) \times \mathcal{M}(\phi_2) \,\middle|\, \rho^w(u_1) = u_2^{-1}(v) \right\}. \end{aligned}$$

Now $\mu(\phi_2) = n_v(\phi_2) = 1$ implies that the domain of ϕ_2 is the bi-gon in S containing v and thus it has a unique holomorphic representative up to translation. We can fix a holomorphic representative u_2 such that $u_2^{-1}(v) = (t, 0)$.

Since our system of coherent orientations is induced from a coherent system of orientations on $(S, \alpha, \beta, \{v, z_\kappa, z_{\kappa+1}, z_{\kappa+2}\})$ where the sign associated with the unique (upto translation) holomorphic representative of the bi-gon containing v is -1 we find

$$\begin{aligned}\#\widehat{\mathcal{M}}(\phi) &= -\#\big\{u_1 \in \mathcal{M}(\phi_1)\,\big|\,\rho^w(u_1) = (t,0)\big\}\\ &= -\#\mathcal{M}(\phi_1, t).\end{aligned}$$

Let us now assume that the point v is chosen very close to the curve β. By Lemma 7.2, for t sufficiently small $\mathcal{M}(\phi_1, t)$ is nonempty if and only if $\phi_1 \in \pi_2^\beta(\mathbf{x})$ is the class of a β boundary degeneration. If furthermore $l > 1$ then $\#\mathcal{M}(\phi_1, t) = 1$. Thus this case contributes to the C_y-component of the restriction of the differential to C_x via a map

$$\begin{aligned}&\partial^2_{xy} : C_x \longrightarrow C_y\\ &\partial^2_{xy}\left(\mathbf{x} \times \{x\}\right) = -\left(\prod_{\tau_\kappa \neq \tau_i \in \partial R_l^+} \mathfrak{u}_i\right) . \left(\mathbf{x} \times \{y\}\right).\end{aligned}$$

Similarly, if $l = 1$ then $\partial^2_{xy}(\mathbf{x} \times \{x\}) = 0$.

To deal with the other terms corresponding to the homotopy classes ϕ with $n_v(\phi_2) > 1$ we define a one parameter family of connected sum points $v(r)$ on S such that when r goes to infinity, $v(r)$ tends towards a point v_∞ on the curve β.

Let $\mathcal{M}_r(\phi)$ be the moduli space of holomorphic representations of ϕ when we used the connected sum point $v(r)$ in S. Assume that for a sequence $\{r_i\}$ converging to infinity, the moduli space $\mathcal{M}_{r_i}(\phi) \neq \emptyset$ for all choice of connected sum length. For sufficiently large connected sum length the moduli space $\mathcal{M}_{r_i}(\phi)$ is identified with the fibered product $\mathcal{M}(\phi_1) \times_{\mathrm{Sym}^k(\mathbb{D})} \mathcal{M}(\phi_2)$. Consider a sequence $u_1^i \times u_2^i$ in the fibered product. Let $\overline{u}_1^\infty$ and $\overline{u}_2^\infty$ be Gromov limits of $\{u_1^i\}$ and $\{u_2^i\}$. The assumption $\mu(\phi_1) = 2$ implies that there are three possible types for the limit $\overline{u}_1^\infty$. The limit can be a holomorphic disk or a singly broken flow line or it can contain a boundary degeneration. If it contain a boundary degeneration, $\mathfrak{u}(\phi) \neq 0$ implies that the remaining component has Maslov index zero and it should be constant. Thus $k = 1$ and this situation is already considered in the previous case.

If $\overline{u}_1^\infty$ is not a broken flow line and it is a holomorphic disk, $\overline{u}_2^\infty$ has a component u_2^∞ such that $\rho^w(\overline{u}_1^\infty) = \rho^v(u_2^\infty)$. Since $v(r_i)$ tends toward v_∞ on β, $\rho^v(u_2^\infty)$ includes some points on $\{0\} \times \mathbb{R}$. Thus for large i, $\rho^w(u_1^i)$ contains points sufficiently close to $\{0\} \times \mathbb{R}$. By Lemma 7.2 the holomorphic curve u_1^i

should be a boundary degeneration for i sufficiently large. This implies that $k = 1$ and again, we are within the cases considered earlier, and there is no new contribution to the C_y-component of the restriction of the differential to C_x from this case.

Finally, if $\overline{u}_1^\infty$ is a broken flow line (i.e. it is of the form $\overline{u}_1^\infty = a \star b$ and $\mu(a) = \mu(b) = 1$) then $\overline{u}_2^\infty$ degenerates, correspondingly, as $\overline{u}_2^\infty = a' \star b'$. The Maslov index of ϕ_2 is odd, thus one of a' and b' has odd Maslov index. Let us assume that $\mu(a')$ is odd. Then $(a')^{-1}(v^\infty)$ contains some points on $\{0\} \times \mathbb{R}$. If a is the holomorphic representative of a homology class ϕ_1' of a Whitney disk, then for r sufficiently large $\mathcal{M}(\phi_1')$ includes holomorphic representatives a^r such that $\rho^w(a^r)$ contains points of distance less than $1/r$ to $\{0\} \times \mathbb{R}$. Since $\mu(\phi_1') = 1$, ϕ_1' has finitely many holomorphic representative up to translation. Thus for any holomorphic representative u of ϕ_1', $\rho^w(u)$ does not include points arbitrary close to $\{0\} \times \mathbb{R}$, since w is not on $\boldsymbol{\beta}$.

Gathering the above considerations, we observe that if either of the two assumptions in the second part of Theorem 7.1 is satisfied the C_y component of the restriction of the differential to C_x is given by the map

$$\begin{aligned}
&\partial_{xy} : C_x \longrightarrow C_y \\
&\partial_{xy}\left(\mathbf{x} \times \{x\}\right) = \partial_{xy}^1\left(\mathbf{x} \times \{x\}\right) + \partial_{xy}^2\left(\mathbf{x} \times \{x\}\right) \\
&\qquad = \mathfrak{u}_{\kappa+1}(\mathbf{x} \times \{y\}) - \left(\prod_{\gamma_\kappa \neq \gamma_i \in \partial R_l^+} \mathfrak{u}_i\right) . \left(\mathbf{x} \times \{y\}\right) \\
&\qquad = \left(\mathfrak{u}_{\kappa+1} - \mathfrak{u}\right)\left(\mathbf{x} \times \{y\}\right),
\end{aligned}$$

provided that the path of almost complex structures is chosen by setting the connected sum length equal to a sufficiently large real number. The proof in the case where all the components of $\mathfrak{R}(\tau)$ have genus zero and $\ell_2 = g(\Sigma)$ is exactly the same as the last part of the proof of Proposition 6.5 in [OS9]. This completes the proof of Proposition 7.5. □

8. A triangle for sutured Floer complex

8.1. The triangle associated with the surgery Heegaard quadruple

Let us assume that Σ is a closed Riemann surface of genus g and that

$$\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_\ell\} \quad \text{and} \quad \boldsymbol{\beta}_0 = \{\beta_1, \ldots, \beta_{\ell-1}\}$$

are collections of disjoint simple closed curves on Σ. Let $\mu_i,\ i=0,1,2$ be three oriented simple closed curves which are disjoint from $\boldsymbol{\beta}_0$, and mutually intersect each other in single transverse points, so that the intersection numbers $\mu_0.\mu_1, \mu_1.\mu_2$ and $\mu_2.\mu_0$ are equal to 1. We will assume that

$$\mu_1 \cap \mu_2 = \{p_0\}, \quad \mu_2 \cap \mu_0 = \{p_1\} \quad \text{and} \quad \mu_0 \cap \mu_1 = \{p_2\}.$$

Furthermore, suppose that the three intersection points p_0, p_1 and p_2 are the vertices of a small triangle Δ, which is one of the connected components in

$$S = \Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}_0 - \{\mu_0, \mu_1, \mu_2\}.$$

Let $\mathbf{z} = \{z_0, \dots, z_\kappa\}$ be a collection of marked points in S. We choose the marked points z_0, z_1 and z_2 outside Δ and very close to its edges, so that z_0 is close to the edge e_0 connecting p_1 to p_2, z_1 is close to the edge e_1 connecting p_2 to p_0, and z_2 is close to the edge e_2 connecting p_0 to p_1. Finally, we will fix a marked point p in Δ for further reference. We will denote by $\mathbf{w}$ the set of marked points $\mathbf{w} = \{z_3, \dots, z_\kappa\}$. We will assume that the component A of $\Sigma - \boldsymbol{\alpha}$ which contains the marked point p (and thus the marked points z_0, z_1 and z_2) is a genus zero component. Furthermore, we will assume that $\Sigma - \boldsymbol{\beta}_0 - \{\mu_0, \mu_1, \mu_2\}$ consists of a pair of triangles bounded between the curves μ_0, μ_1 and μ_2 which will be denoted by Δ and Δ', a component B of genus zero containing the marked points $z_0, z_1, \dots, z_m$ (with $m \le \kappa$), and a union of periodic domains $B_2, \dots, B_l$ with boundary in $\boldsymbol{\beta}_0$. Moreover, Δ is the connected component of S containing the marked point p which was considered earlier, and Δ' does not contain any marked points. The notation is illustrated in Figure 5.

Consider the Heegaard diagrams

$$H_i = \big(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^i = \{\beta_1^i, \dots, \beta_{\ell-1}^i, \mu_i\},\ \mathbf{z}^i = \mathbf{w} \cup \{z_i, p\}\big), \quad i \in \{0,1,2\},$$

where we assume that β_j^i are small Hamiltonian isotopes of the curve β_j, for $i = 0,1,2$, so that any pair of curves in $\{\beta_j^0, \beta_j^1, \beta_j^2\}$ intersect each-other in a pair of transverse canceling intersection points for $j = 1, \dots, \ell-1$. This Heegaard diagram determines a sutured manifold which will be denoted by (X, τ^i). Similarly, suppose that (Y, ς^i) is the sutured manifold obtained by extending the set of marked points to $\{p\} \cup \mathbf{z}$ in H_i. The three-manifolds Y and X do not depend on i and would be the same for $i = 0,1,2$. In fact, instead of gluing a disk to μ_i, one may fill out the suture τ_p corresponding to the marked point p. The identification is illustrated in Figure 6 for Y.

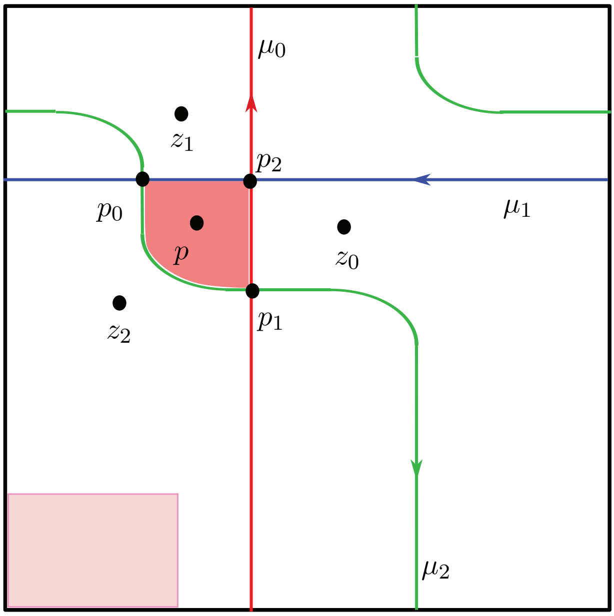


Figure 5: A neighbourhood of the curves μ_0, μ_1 and μ_2 is illustrated. One should take the connected sum of the torus obtained by identifying the opposite edges of the rectangle, with another Riemann surface to obtain the Heegaard surface Σ. The attaching circle of the connected sum tube lands in the shaded area in the lower left corner of the figure. The α curves live close to the boundary of the rectangle, or on the Riemann surface which is attached to this torus. The marked points $\{p, z_0, z_1, z_2\}$ and the intersection points p_0, p_1 and p_2 are illustrated.

Furthermore, the orientation of the the curve μ_i may be used to identify the spaces of relative Spin^c structures associated with the sutured manifolds (X, τ^i), and we may thus fix the identifications

$$\text{Spin}^c(X, \tau^0) = \text{Spin}^c(X, \tau^1) = \text{Spin}^c(X, \tau^2). \tag{18}$$

The identification in Equation 18 is done as follows. The assumptions on the Heegaard diagrams imply that, associated with the sutured manifolds (X, τ^i), $i = 0, 1, 2$, the marked points p and z_i correspond to a pair of parallel sutures τ_p and τ_i on the boundary of the three-manifold X. Denote by (X, τ) the unbalanced sutured manifold obtained by removing these two sutures from the boundary of X. Clearly τ does not depend on $i \in \{0, 1, 2\}$. The vector fields v_{τ^i} on $\partial X \times \{0\}$ and v_τ on $\partial X \times \{1\}$ give a vector field defined on the boundary of $\partial X \times [0,1]$ with values in the tangent space of this three-manifold. This vector field may be extended, in a natural way, to the interior of $\partial X \times [0,1]$. In fact, let $R \simeq (-1,1) \times S^1$ denote a neighbourhood of the annulus bounded by the sutures τ_p and τ_i on ∂X. The vector fields v_τ and v_{τ^i} agree on $\partial X - R$. It is thus enough to do the extension on $R \times [0,1]$. Denote by v the vector field $\partial/\partial\theta$ on

$$R \times \left\{\frac{1}{2}\right\} = (-1,1) \times S^1 \times \left\{\frac{1}{2}\right\},$$

where θ denotes the coordinate on S^1 which is compatible with the orientation of the sutures τ_p and τ_i, as determined by the orientation of μ_i. In the standard Euclidean model, the angle between the vector fields v and v_{τ^i} is less than π (and is in fact always equal to $\pi/2$). We may thus smoothly isotope v_{τ^i} to v on $R \times [0,1/2]$. Similarly, the angle between the vector fields v and v_τ is less than π (and is again equal to $\pi/2$). We may thus smoothly isotope v to v_τ on $R \times [1/2,1]$. The two isotopies may be kept constant close to $(\partial R) \times [0,1]$ to produce a natural vector field w on $(\partial X) \times [0,1]$ which connects v_{τ^i} and v_τ. Using this vector field we obtain the identifications of Equation 18.

The algebra $\mathbb{A}_{\varsigma^i}$ is independent of i. We will denote by $\mathbb{A}$ the quotient

$$\mathbb{A} = \frac{\mathbb{A}_{\varsigma^i}}{\langle \mathfrak{u}_p = 1 \rangle}, \qquad i = 0,1,2,$$

where $\mathfrak{u}_p$ denotes the variable corresponding to p. In fact, passing to this quotient means that we are forgetting the marked point p in the Heegaard diagram. Let us denote the generator corresponding to the marked point z_j by $\mathfrak{u}_j$, for $j = 0,1,\dots,\kappa$. Denote the generator associated with the marked point z_j in $\mathbb{A}_{\tau^i}$ by $\mathfrak{v}_j$ for $j = 3,\dots,\kappa$, and denote the generator associated with z_i and p by $\mathfrak{v}_1$ and $\mathfrak{v}_2$ respectively. Each Heegaard diagram H_i determines an embedding of $\mathbb{A}_i = \mathbb{A}_{\tau^i}$ in $\mathbb{A}$. More precisely, we may define

$$\imath^i : \mathbb{A}_i \hookrightarrow \mathbb{A}, \ \imath^i(\mathfrak{v}_j) = \begin{cases} \mathfrak{u}_i & \text{if } j = 1 \\ \frac{\mathfrak{u}_0\mathfrak{u}_1\mathfrak{u}_2}{\mathfrak{u}_i} & \text{if } j = 2 \\ \mathfrak{u}_j & \text{if } 3 \le j \le \kappa \end{cases}.$$

Note that $\mathbb{A}_0, \mathbb{A}_1$ and $\mathbb{A}_2$ are isomorphic. However, the index is used to distinguish them as sub-rings of $\mathbb{A}$, using the embeddings $\imath^i : \mathbb{A}_i \hookrightarrow \mathbb{A}$, $i = 0,1,2$.

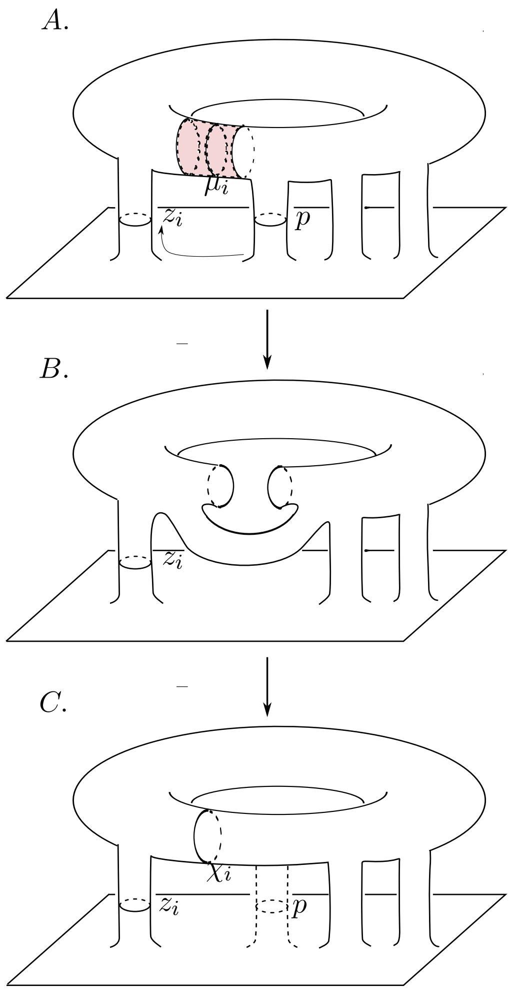


Figure 6: Instead of attaching a 2-handle along μ_i, the suture which corresponds to the marked point p may be filled. In part A, a 2-handle is attached to μ_i (think of Y as the three manifold outside the torus and above the plane illustrated in this picture). Then we slide the 1-handle corresponding to p over the 1-handle corresponding to z_i, as illustrated in part B. The result, after smoothing the appropriate corners, is the picture illustrated in part C, in which the suture corresponding to p is filled out and instead, no 2-handle is attached to μ_i. The picture corresponds to the case $m = 2$.

Let $\mathbb{D}_\alpha$ denote a set of ℓ copies of $D^2 \times [-\epsilon, \epsilon]$ (for some small positive real number ϵ) corresponding to the curves in $\boldsymbol{\alpha}$, and $\mathbb{D}_{\beta_0}$ denote a set of $\ell - 1$

copies of $D^2 \times [-\epsilon, \epsilon]$ corresponding to curves in $\boldsymbol{\beta}_0$. Denote small tubular neighbourhoods of the curves in $\boldsymbol{\alpha}$ and the curves in $\boldsymbol{\beta}_0$ by $\mathrm{nd}(\boldsymbol{\alpha})$ and $\mathrm{nd}(\boldsymbol{\beta}_0)$, respectively. These neighbourhoods may be identified with subsets of $\partial\mathbb{D}_\alpha$ and $\partial\mathbb{D}_{\beta_0}$ respectively. Under the identification of Y with the three-manifold

$$\big([0,1] \times \big(\Sigma \setminus \mathrm{nd}(\mathbf{z})\big)\big) \bigcup_{\mathrm{nd}(\alpha)\times\{0\}} \mathbb{D}_\alpha \bigcup_{\mathrm{nd}(\beta_0)\times\{1\}} \mathbb{D}_{\beta_0},$$

each marked point z_j determines an oriented simple closed curve on the boundary of Y. The Poincaré dual of this curve determines an element $\chi_j \in \mathrm{H}^2(Y, \partial Y; \mathbb{Z})$ for $3 \le j \le \ell$. For $i \in \{0,1,2\}$ we will denote the element of $\mathrm{H}^2(Y, \partial Y; \mathbb{Z})$ corresponding to the marked point z_i by η_i. The assumptions on the Heegaard diagram imply that

$$(\eta_0 + \eta_1 + \eta_2) + (\chi_3 + \cdots + \chi_m) = 0.$$

The particular case where $m = 3$, or equivalently $\mathbf{w} \cap B = \emptyset$, is of particular interest. We will denote $\eta_0 + \eta_1 + \eta_2$ by η. The Poincaré duals of the curves corresponding to the marked point z_j in (Y, ς^i) will be denoted by $\chi(i,j)$, for $i = 0,1,2$ and $0 \le j \le \ell$. Furthermore, let χ_i denote the Poincaré dual $\mathrm{PD}[\mu_i]$ of the simple closed curve $\mu_i \subset \partial Y$ for $i = 0,1,2$. One may check that

$$\chi(i,j) = \begin{cases} \chi_j & \text{if } j \in \{3, \dots, \kappa\} \\ \eta_j & \text{if } j \ne i \text{ and } j \in \{0,1,2\} \\ \chi_i + \eta_i & \text{if } j = i \end{cases}$$

Associated with any of the Heegaard diagrams $H_i,\ i = 0,1,2$ (and independent of i) we define a filtration map

$$\chi : G(\mathbb{A}) \longrightarrow \mathrm{H}^2(Y, \partial Y; \mathbb{Z}), \quad \chi(\mathfrak{u}_j) := \chi_j \quad \text{for } j \in \{0, 1, \dots, \kappa\}.$$

Note that with this assignment we may compute

$$\chi \circ \imath^i \left(\prod_{j=1}^{\kappa} \mathfrak{v}_j^{i_j} \right) = (i_1 - i_2)\chi_i + \sum_{j=3}^{\kappa} i_j \chi_j = \chi_{\tau^i} \left(\prod_{j=1}^{\kappa} \mathfrak{v}_j^{i_j} \right) - i_1 \eta_i,$$

for all $i \in \{0,1,2\}$. Consider the following $\mathbb{Z}$ module associated with the three-manifold Y:

$$\mathbb{H} := \frac{\mathrm{H}^2(Y, \partial Y; \mathbb{Z})}{\langle \eta_0, \eta_1, \eta_2 \rangle_{\mathbb{Z}}}$$

which acts by translation on either of

$$\frac{\mathrm{Spin}^c(X,\tau^i)}{\langle\eta\rangle}=\frac{\mathrm{Spin}^c(Y,\varsigma^i)}{\langle\eta_0,\eta_1,\eta_2,\rangle}.$$

Abusing the notation, we will denote this later quotient by $\mathrm{Spin}^c(X,\tau)$. When $\eta=0$ in $H^2(X,\partial X;\mathbb{Z})$ (e.g. when $m=2$) the quotient maps

$$\mathrm{Spin}^c(X,\tau^i)\to\mathrm{Spin}^c(X,\tau),\quad i=0,1,2$$

are bijections. The above observation implies that the composition of $\chi\circ\imath^i$ with the projection over $\mathbb{H}$ is the same as the composition of χ_{τ^i} with the projection map from $\mathrm{H}^2(X,\partial X;\mathbb{Z})$ to $\mathbb{H}$.

We continue to denote the image of $\chi_i\in\mathrm{H}^2(Y,\partial Y;\mathbb{Z})$ in $\mathbb{H}$ by χ_i. The filtration map $\chi:G(\mathbb{A})\to\mathrm{H}^2(Y,\partial Y;\mathbb{Z})$ may be composed with the quotient map from $\mathrm{H}^2(Y,\partial Y;\mathbb{Z})$ to $\mathbb{H}$ to define a new filtration map, yet denoted by χ. Clearly, $\mathbb{H}$ acts on $\mathfrak{S}=\mathrm{Spin}^c(X,\tau)$ by translation. We may abuse the notation and define

$$\mathrm{Spin}^c(\overline{X})=\frac{\mathrm{Spin}^c(X,\tau)}{\langle\chi_0,\chi_1,\dots,\chi_\kappa\rangle}.$$

Thus, $\mathrm{Spin}^c(\overline{X})$ is a natural quotient of any of $\mathrm{Spin}^c(\overline{X}^{\tau^i})$. We fix a class $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$ for the rest of this section. We assume that the Heegaard diagrams $H_i,\ i=0,1,2$ are $\mathfrak{s}$-admissible. In fact, we will drop the admissibility issues, as well as orientation issues, from our discussion in the remainder of this section. Taking care of these issues is completely straight forward, and follows the lines of the arguments given in the earlier sections.

For $i=0,1,2$, consider the filtered $(\mathbb{A},\mathbb{H})$ chain complex

$$\mathrm{CF}_i(\mathfrak{s}):=\mathrm{CF}(X,\tau^i,\mathfrak{s})\otimes_{\mathbb{A}_i}\mathbb{A}=\big\langle\mathbf{x}\,\big|\,\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta^i}\text{ and }\underline{\mathfrak{s}}_{\mathbf{z}^i}(\mathbf{x})\in\mathfrak{s}\big\rangle_{\mathbb{A}}.$$

The set of marked points $\mathbf{z}=\{z_0,\dots,z_\kappa\}$ defines a map

$$\mathfrak{u}_{\mathbf{z}}:\coprod_{i=0}^{3}\ \coprod_{\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta^i}}\pi_2(\mathbf{x},\mathbf{y})\longrightarrow G(\mathbb{A}),\quad \mathfrak{u}_{\mathbf{z}}(\phi):=\prod_{i=0}^{\kappa}\mathfrak{u}_i^{n_i(\phi)}.$$

The differential ∂_i of the complex $\mathrm{CF}_i(\mathfrak{s})$, as an $\mathbb{A}$-module homomorphism, is then defined by

$$\partial_i(\mathbf{x}):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta^i}}\ \sum_{\phi\in\pi_2^1(\mathbf{x},\mathbf{y})}\big(\mathfrak{m}(\phi)\mathfrak{u}_{\mathbf{z}}(\phi)\big).\mathbf{y}.$$

We define a map from the set of generators of $\mathrm{CF}_i(\mathfrak{s})$ to $\mathfrak{S}$ by setting

$$\begin{aligned}&\underline{\mathfrak{s}}^i : G(\mathbb{A}) \times (\mathbb{T}_\alpha \cap \mathbb{T}_{\beta^i}) \longrightarrow \mathfrak{S}\\ &\underline{\mathfrak{s}}^i(\mathfrak{u}.\mathbf{x}) := \underline{\mathfrak{s}}_{\mathbf{z}^i}(\mathbf{x}) + \chi(\mathfrak{u}).\end{aligned}$$

Abusing the notation, we will sometimes denote $\underline{\mathfrak{s}}^i(\mathfrak{u}.\mathbf{x})$ by $\underline{\mathfrak{s}}(\mathfrak{u}\mathbf{x}) = \underline{\mathfrak{s}}(\mathbf{x}) + \chi(\mathfrak{u})$, dropping the index i from the notation.

Lemma 8.1. *If a generator* $\mathfrak{u}.\mathbf{y}$*, with* $\mathfrak{u} \in G(\mathbb{A})$*, appears with non-zero coefficient in* $\partial_i\big(\mathbf{x}\big)$*, we will have* $\underline{\mathfrak{s}}^i(\mathbf{x}) = \underline{\mathfrak{s}}^i(\mathfrak{u}.\mathbf{y})$ *in* $\mathfrak{S}$*.*

Proof. Without loosing on generality, let us assume that $i = 0$. Suppose that $\underline{\mathfrak{s}}(\mathbf{x}), \underline{\mathfrak{s}}(\mathbf{y}) \in \mathfrak{s}$, and that there is a Whitney disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ contributing to $\partial_0(\mathbf{x})$ with $\mathfrak{u} = \mathfrak{u}_{\mathbf{z}}(\phi)$. Then we will have $n_1(\phi) = n_2(\phi) = n_p(\phi)$. The existence of this disk implies that

$$\begin{aligned}&\underline{\mathfrak{s}}(\mathbf{x}) = \underline{\mathfrak{s}}(\mathbf{y}) + \big(n_0(\phi) - n_p(\phi)\big)\chi_0 + \sum_{j=3}^{\kappa} n_j(\phi).\chi_j\\ \Rightarrow\quad &\underline{\mathfrak{s}}^0(\mathbf{x}) = \underline{\mathfrak{s}}^0(\mathbf{y}) + \chi\left(\prod_{j=0}^{\kappa} \mathfrak{u}_j^{n_j(\phi)}\right) = \underline{\mathfrak{s}}^0(\mathfrak{u}.\mathbf{y}).\end{aligned}$$

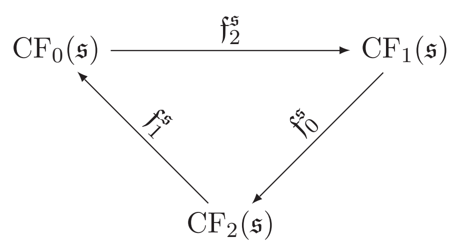
For the equality in the second line, we use the relation $\chi_0 + \chi_1 + \chi_2 = 0$. □

The above assignment of relative Spin^c structures is thus respected by the differential ∂_i of $\mathrm{CF}_i(\mathfrak{s})$, and $\mathrm{CF}_i(\mathfrak{s})$ is thus decomposed as

$$\mathrm{CF}_i(\mathfrak{s}) = \bigoplus_{\underline{\mathfrak{s}} \in \mathfrak{s} \subset \mathfrak{S}} \mathrm{CF}_i(\underline{\mathfrak{s}}).$$

Associated with the Spin^c class $\mathfrak{s}$, we will describe a triangle of chain maps

(19)



such that the compositions $\mathfrak{f}^{\mathfrak{s}}_{i+1} \circ \mathfrak{f}^{\mathfrak{s}}_i$, $i \in \frac{\mathbb{Z}}{3\mathbb{Z}} = \{0, 1, 2\}$ are chain homotopic to zero.

To define $\mathfrak{f}^{\mathfrak{s}}_{i-1}$, note that the special Heegaard diagram $(\Sigma,\boldsymbol{\beta}^i,\boldsymbol{\beta}^{i+1},\mathbf{z})$ is admissible for all the corresponding Spin^c classes (c.f. the arguments of Subsection 6.2). We may thus compute

$$\mathrm{CF}\left(L_{i-1},\nu_{i-1};\mathbb{A}\right)=\mathrm{CF}(\Sigma,\boldsymbol{\beta}^i,\boldsymbol{\beta}^{i+1},\mathbf{z})\otimes\mathbb{A}$$

where (L_{i-1},ν_{i-1}) is the sutured manifold corresponding to the special Heegaard diagram $(\Sigma,\boldsymbol{\beta}^i,\boldsymbol{\beta}^{i+1},\mathbf{z})$. There is a unique Spin^c class

$$\mathfrak{s}_{i-1}\in\mathrm{Spin}^c\left(\overline{L_{i-1}}\right),\quad c_1(\mathfrak{s}_{i-1})=0,$$

as well as a top generator Θ_{i-1} corresponding to $\mathfrak{s}_{i-1}$ (which is a closed element) in the above Heegaard Floer complex. The generator Θ_{i-1} is obtained as the union of p_{i-1} and the positive intersection points of β^i_j and β^{i+1}_j for $j=1,\dots,\ell-1$. The generator Θ_{i-1} corresponds to a relative Spin^c class which will be denoted by $\underline{\mathfrak{s}}_{i-1}\in\mathfrak{s}_{i-1}$. Consider the holomorphic triangle map

$$f^{\mathfrak{s}}_{i-1}:\mathrm{CF}_i(\mathfrak{s})\otimes_{\mathbb{A}}\mathrm{CF}(\Sigma,\boldsymbol{\beta}^i,\boldsymbol{\beta}^{i+1},\mathbf{z};\mathfrak{s}_{i-1};\mathbb{A})\longrightarrow\mathrm{CF}_{i+1}(\mathfrak{s}).$$

On a generator $\mathbf{x}\otimes\mathbf{q}$ of the left hand side, with $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta^i}$ and $\mathbf{q}$ a generator corresponding to the Spin^c class $\mathfrak{s}_{i-1}$, $f^{\mathfrak{s}}_{i-1}(\mathbf{x}\otimes\mathbf{q})$ is defined by

$$f^{\mathfrak{s}}_{i-1}\big(\mathbf{x}\otimes\mathbf{q}\big):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\Delta\in\pi_2^0(\mathbf{x},\mathbf{q},\mathbf{y})}\Big(\mathfrak{m}(\Delta)\mathfrak{u}_{\mathbf{z}}(\Delta)\Big)\mathbf{y},\tag{20}$$

where $\pi_2^0(\mathbf{x},\mathbf{q},\mathbf{y})$ denotes the subset of $\pi_2(\mathbf{x},\mathbf{q},\mathbf{y})$ consisting of the triangle classes Δ such that $\mu(\Delta)=0$. The map $f^{\mathfrak{s}}_{i-1}$ is then extended, as an $\mathbb{A}$-module homomorphism, to all of $\mathrm{CF}_i(\mathfrak{s})\otimes_{\mathbb{A}}\mathrm{CF}(\Sigma,\boldsymbol{\beta}^i,\boldsymbol{\beta}^{i+1},\mathbf{z};\mathfrak{s}_{i-1};\mathbb{A})$. One should also fix the Spin^c class of the triangles contributing to the sum in Equation 20. Let us assume that the intersection points $\mathbf{x}^i\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta^i}$ for $i=0,1,2$ are fixed so that $\underline{\mathfrak{s}}^i(\mathbf{x}^i)\in\mathfrak{s}\subset\mathfrak{S}$. Furthermore, assume that, after possible re-labelling of the curves in $\boldsymbol{\alpha}$, we have

$$\mathbf{x}^i=\{x^i_1,\dots,x^i_\ell\},\quad x^i_j\in\begin{cases}\alpha_j\cap\beta^i_j & \text{if } 1\le j<\ell\\ \alpha_\ell\cap\mu_i & \text{if } j=\ell\end{cases}.$$

Also, for $j=1,\dots,\ell-1$, we will assume that x^0_j,x^1_j and x^2_j are very close to each other, and correspond to one another by the Hamiltonian isotopies considered above.

We may always change the α curves in the Heegaard diagram by isotopy so that the above condition is satisfied. In order to specify the class of triangles used in Equation 20, we need to specify triangle classes $\Delta_i \in \pi_2(\mathbf{x}^{i+1}, \Theta_i, \mathbf{x}^{i-1})$ for any $i \in \frac{\mathbb{Z}}{3\mathbb{Z}} = \{0,1,2\}$. The domain $\mathcal{D}(\Delta_i)$ consists of a union of ℓ triangles. The first $\ell - 1$ triangles are small triangles determined by the small Hamiltonian isotopy changing the simple closed curves in $\boldsymbol{\beta}^{i+1} - \{\mu_{i+1}\}$ to those in $\boldsymbol{\beta}^{i-1} - \{\mu_{i-1}\}$. Two of the vertices of the j-th triangle are the intersection points x_j^{i+1} and x_j^{i-1}, while the last vertex belongs to the top generator Θ_i. The ℓ-th triangle connects three intersection points between μ_i, μ_{i+1} and $\alpha_\ell \in \boldsymbol{\alpha}$. With this notation fixed, let

$$\mathcal{D} = \mathcal{D}(\Delta_0) + \mathcal{D}(\Delta_1) + \mathcal{D}(\Delta_2).$$

We assume that no α curve appears in $\partial\mathcal{D}$. Furthermore, we may assume that $n_p(\mathcal{D}) = -1$ while $n_j(\mathcal{D}) = 0$ for $j = 0, 1, \dots, \kappa$. The 2-chain $\mathcal{D}$ is then the domain of a triangle class $\widehat{\Delta} \in \pi_2(\Theta_0, \Theta_1, \Theta_2)$ with small area. Note that achieving all these properties may be done through a correct choice of the last triangle among the ℓ triangles chosen above.

The choice of this last triangle class (with the above properties) determines how the map $f_{i-1}^{\mathfrak{s}}$ changes the relative Spin^c classes. We will specify this last choice after the following lemma.

Lemma 8.2. *There exists a cohomology class $h_i \in \mathbb{H}$ for $i = 0, 1, 2$ with the following property. If for a generator $\mathbf{x}$ of $\mathrm{CF}_i(\mathfrak{s})$ we have*

$$\underline{\mathfrak{s}}(\mathbf{x}) = \underline{\mathfrak{s}} \in \mathfrak{s} \subset \mathfrak{S},$$

and for the intersection point $\mathbf{q} \in \mathbb{T}_{\beta^i} \cap \mathbb{T}_{\beta^{i+1}}$ we have $\underline{\mathfrak{s}}(\mathbf{q}) = \underline{\mathfrak{s}}_{i-1}$, then

$$f_{i-1}^{\mathfrak{s}}\big(\mathbf{x} \otimes \mathbf{q}\big) \in \mathrm{CF}_{i+1}(\underline{\mathfrak{s}} + h_{i-1}).$$

Furthermore, the cohomology classes satisfy $h_0 + h_1 + h_2 = 0$.

Proof. Once again, it suffices to prove the lemma for $i = 0$. The cyclic symmetry of all definitions then implies the lemma in general. Let $\mathbf{q} \in \mathbb{T}_{\beta^0} \cap \mathbb{T}_{\beta^1}$ be an intersection point corresponding to the relative Spin^c class $\underline{\mathfrak{s}}_2$. Suppose that $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^1}$ is a generator and $\Delta \in \pi_2(\mathbf{x}, \mathbf{q}, \mathbf{y})$. Using the fact that $n_2(\Delta)=n_p(\Delta)$ we then have

$$\begin{aligned}\underline{\mathfrak{s}}^0(\mathbf{x}) &= \big(\underline{\mathfrak{s}}(\mathbf{y})+h_2\big)+\big(n_0(\Delta)-n_p(\Delta)\big)\chi_0\\ &\quad+\big(n_1(\Delta)-n_p(\Delta)\big)\chi_1+\sum_{j=3}^{\kappa}n_j(\Delta)\chi_j\\ &=\big(\underline{\mathfrak{s}}^1(\mathbf{y})+h_2\big)+\sum_{j=0}^{\kappa}n_j(\Delta)\chi_j=\underline{\mathfrak{s}}^1(\mathfrak{u}_{\mathbf{z}}(\Delta).\mathbf{y})+h_2.\end{aligned}$$

Here h_2 is a cohomology class in $\mathbb{H}$ which depends on the choice of the triangle classes

$$\Delta_{i-1}\in\pi_2(\mathbf{x}_i,\Theta_{i-1},\mathbf{x}_{i+1}),\quad i\in\{0,1,3\}=\frac{\mathbb{Z}}{3\mathbb{Z}}$$

corresponding to the Heegaard diagrams $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^i,\boldsymbol{\beta}^{i+1},\mathbf{z})$ defined earlier. We have assumed that the triangle classes are chosen so that

$$\mathcal{D}=\mathcal{D}(\Delta_0)+\mathcal{D}(\Delta_1)+\mathcal{D}(\Delta_2)=-\mathcal{D}(\widehat{\Delta})$$

is the domain of the triangle class $\widehat{\Delta}\in\pi_2(\Theta_0,\Theta_1,\Theta_2)$ so that $n_p(\mathcal{D})=-1$ and $n_j(\mathcal{D})=0$ for $j=0,\dots,\kappa$. The above computation then implies that

$$\begin{aligned}\underline{\mathfrak{s}}^0(\mathbf{x}_0)&=\underline{\mathfrak{s}}^1\left(\mathfrak{u}_{\mathbf{z}}(\Delta_2).\mathbf{x}_1\right)+h_2\\ &=\underline{\mathfrak{s}}^2\left(\mathfrak{u}_{\mathbf{z}}(\Delta_0)\mathfrak{u}_{\mathbf{z}}(\Delta_2).\mathbf{x}_2\right)+h_0+h_2\\ &=\underline{\mathfrak{s}}^0\left(\mathfrak{u}_{\mathbf{z}}(\Delta_0)\mathfrak{u}_{\mathbf{z}}(\Delta_1)\mathfrak{u}_{\mathbf{z}}(\Delta_2).\mathbf{x}_0\right)+h_0+h_1+h_2.\\ &=\underline{\mathfrak{s}}^0\left(\mathfrak{u}_{\mathbf{z}}(\widehat{\Delta}).\mathbf{x}_0\right)+h_0+h_1+h_2.\\ \Rightarrow\quad 0&=h_0+h_1+h_2.\end{aligned}$$

This completes the proof of the lemma. □

In fact, the choices of the triangle classes Δ_0 and Δ_1 (which forces the choice of triangle class Δ_2 via the relation $\widehat{\Delta}\star\Delta_1=\Delta_0\star\Delta_2$), may be made so that with the notation of the above lemma we have

$$f_i^{\mathfrak{s}}\big(\mathbf{x}\otimes\mathbf{q}\big)\in\mathrm{CF}_{i-1}(\underline{\mathfrak{s}}),\quad\forall\, i\in\frac{\mathbb{Z}}{3\mathbb{Z}},$$

or equivalently, $h_i=0$. This last condition determines the triangle classes in a unique way. The closed top generator

$$\Theta_{i-1}\in\mathrm{CF}\left(L_{i-1},\nu_{i-1};\mathfrak{s}_{i-1}\right)\otimes\mathbb{A}$$

may then be used to define the map $\mathfrak{f}_{i-1}^{\mathfrak{s}}$ by

$$\mathfrak{f}_{i-1}^{\mathfrak{s}} : \mathrm{CF}_i(\mathfrak{s}) \longrightarrow \mathrm{CF}_{i+1}(\mathfrak{s}), \quad \mathfrak{f}_{i-1}^{\mathfrak{s}}(\mathbf{x}) := f_{i-1}^{\mathfrak{s}}(\mathbf{x} \otimes \Theta_{i-1}).$$

For a relative Spin^c class $\underline{\mathfrak{s}} \in \mathfrak{s} \subset \mathfrak{S}$, the restriction of $\mathfrak{f}_i^{\mathfrak{s}}$ to $\mathrm{CF}_{i+1}(\underline{\mathfrak{s}}) \subset \mathrm{CF}_{i+1}(\mathfrak{s})$ will be denoted by $\mathfrak{f}_i^{\underline{\mathfrak{s}}}$. Lemma 8.2 implies that the image of $\mathfrak{f}_i^{\underline{\mathfrak{s}}}$ is in $\mathrm{CF}_{i+2}(\underline{\mathfrak{s}})$.

Straight forward arguments in Heegaard Floer homology (c.f. Section 7 of [OS5]) may be used to show the following proposition, using the closedness of the generators Θ_0, Θ_1 and Θ_2:

Proposition 8.3. *The maps $f_i^{\underline{\mathfrak{s}}}$ for $\underline{\mathfrak{s}} \in \mathfrak{s} \subset \mathfrak{S}$, as defined above, are all chain maps, which are induced by the $(\mathbb{A}, \mathbb{H})$ chain maps*

$$\mathfrak{f}_i^{\mathfrak{s}} : \mathrm{CF}_{i+1}(\mathfrak{s}) \longrightarrow \mathrm{CF}_{i+2}(\mathfrak{s}), \quad i \in \frac{\mathbb{Z}}{3\mathbb{Z}} = \{0, 1, 2\}.$$

8.2. Compositions in the triangle are null-homotopic

The maps defined in the previous subsection give a triangle of filtered $(\mathbb{A}, \mathbb{H})$ chain maps between filtered $(\mathbb{A}, \mathbb{H})$ chain complexes:

$$\begin{array}{ccc} \mathrm{CF}_0(\mathfrak{s}) = \bigoplus_{\underline{\mathfrak{s}} \in \mathfrak{s}} \mathrm{CF}_0(\underline{\mathfrak{s}}) & \xrightarrow{\quad \mathfrak{f}_2^{\mathfrak{s}} \quad} & \mathrm{CF}_1(\mathfrak{s}) = \bigoplus_{\underline{\mathfrak{s}} \in \mathfrak{s}} \mathrm{CF}_1(\underline{\mathfrak{s}}) \\ \nwarrow f_1^{\mathfrak{s}} & & \swarrow \mathfrak{f}_0^{\mathfrak{s}} \\ & \mathrm{CF}_2(\mathfrak{s}) = \bigoplus_{\underline{\mathfrak{s}} \in \mathfrak{s}} \mathrm{CF}_2(\underline{\mathfrak{s}}) & \end{array} \qquad (21)$$

The maps in this triangle preserve the associated relative Spin^c decompositions, as described in Lemma 8.2. Our first observation is the following theorem.

Theorem 8.4. *With the notation of the previous subsection, the compositions $\mathfrak{f}_{i+1}^{\mathfrak{s}} \circ \mathfrak{f}_i^{\mathfrak{s}}$, $i \in \frac{\mathbb{Z}}{3\mathbb{Z}} = \{0, 1, 2\}$ from the triangle in Equation 21 are $(\mathbb{A}, \mathbb{H})$ chain homotopic to zero for each Spin^c class $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$. More precisely, there are $(\mathbb{A}, \mathbb{H})$ homotopy maps*

$$H_i^{\mathfrak{s}} : \mathrm{CF}_{i-1}(\mathfrak{s}) \to \mathrm{CF}_{i+1}(\mathfrak{s}), \quad i \in \frac{\mathbb{Z}}{3\mathbb{Z}} = \{0, 1, 2\}, \quad s.t.$$

$$H_i^{\mathfrak{s}} \circ \partial_{i-1} + \partial_{i+1} \circ H_i^{\mathfrak{s}} = \mathfrak{f}_{i-1}^{\mathfrak{s}} \circ \mathfrak{f}_{i+1}^{\mathfrak{s}} \quad \forall\, i \in \frac{\mathbb{Z}}{3\mathbb{Z}}.$$

Proof. Let $\Theta_i \in \mathbb{T}_{\beta^{i+1}} \cap \mathbb{T}_{\beta^{i-1}}$ denote the top intersection point of the corresponding tori. Define the homotopy map $H_i^{\mathfrak{s}}$ using the Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^{i-1}, \boldsymbol{\beta}^i, \boldsymbol{\beta}^{i+1}, \mathbf{z})$ by

$$\begin{aligned} &H_i^{\mathfrak{s}} : \mathrm{CF}_{i-1}(\mathfrak{s}) \longrightarrow \mathrm{CF}_{i+1}(\mathfrak{s}) \\ &H_i^{\mathfrak{s}}(\mathbf{x}) := \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta^{i+1}} \\ \square\in\pi_2^{-1}(\mathbf{x},\Theta_{i+1},\Theta_{i-1},\mathbf{y})}} (\mathfrak{m}(\square)\mathfrak{u}_{\mathbf{z}}(\square)) .\mathbf{y}. \end{aligned} \tag{22}$$

Here $\pi_2^j(\mathbf{x}, \Theta_{i+1}, \Theta_{i-1}, \mathbf{y})$ denotes the subset of $\pi_2(\mathbf{x}, \Theta_{i+1}, \Theta_{i-1}, \mathbf{y})$ consisting of the squares $\square$ with $\mu(\square) = j$, and $\mathfrak{m}(\square)$ denotes the number of points in the moduli space $\mathcal{M}(\square)$, counted with sign. In Equation 22, we only count square classes which may be represented as the juxtaposition of the small triangle class $\widehat{\Delta}$ in $\pi_2(\Theta_{i+1}, \Theta_{i-1}, \Theta_i)$ with the triangle class Δ_i in $\pi_2(\mathbf{x}, \Theta_i, \mathbf{y})$. We will drop this condition from the notation for the sake of simplicity.

Lemma 8.5. *With the above notation fixed, for any relative* Spin^c *class* $\underline{\mathfrak{s}} \in \mathfrak{s} \subset \mathfrak{S}$ *the image of*

$$H_i^{\underline{\mathfrak{s}}} = H_i^{\mathfrak{s}}|_{\mathrm{CF}_{i-1}(\underline{\mathfrak{s}})} : \mathrm{CF}_{i-1}(\underline{\mathfrak{s}}) \longrightarrow \mathrm{CF}_{i+1}(\mathfrak{s})$$

is in the sub-complex $\mathrm{CF}_{i+1}(\underline{\mathfrak{s}}) \subset \mathrm{CF}_{i+1}(\mathfrak{s})$*.*

Proof. Without loosing on generality, we may assume that $i = 0$. Let $\square \in \pi_2^{-1}(\mathbf{x}, \Theta_1, \Theta_2, \mathbf{y})$ be a square connecting $\mathbf{x}$ to $\mathbf{y}$. We can thus find an element $h \in \mathbb{H}$ such that for all such generators and square classes we have

$$\begin{aligned} \underline{\mathfrak{s}}^2(\mathbf{x}) &= \underline{\mathfrak{s}}^1(\mathbf{y}) + h + \sum_{i=1}^{2}(n_i(\square) - n_p(\square))\chi_i + \sum_{j=3}^{\kappa} n_j(\square)\chi_j \\ &= \underline{\mathfrak{s}}^1(\mathbf{y}) + h + \chi\left(\prod_{j=0}^{\kappa} \mathfrak{u}_j^{n_j(\square)}\right) = \underline{\mathfrak{s}}^1\left(\mathfrak{u}_{\mathbf{z}}(\square).\mathbf{y}\right) + h. \end{aligned}$$

Considering the square classes which are obtained as the juxtaposition of triangles corresponding to $f_{i-1}^{\mathfrak{s}}$ and $f_{i+1}^{\mathfrak{s}}$, and using the coherence of the system of Spin^c classes, we may compute $h = h_2 + h_1 = 0$. This completes the proof. □

If $\mathbf{y}$ is an intersection point in $\mathbb{T}_\alpha \cap \mathbb{T}_{\beta^{i+1}}$ and if $\square \in \pi_2(\mathbf{x}, \Theta_{i+1}, \Theta_{i-1}, \mathbf{y})$ is a square with $\mu(\square) = 0$, we may consider the moduli space $\mathcal{M}(\square)$, which is a smooth, oriented 1-dimensional manifold with boundary. The boundary points of this moduli space correspond to different types of degenerations of $\square$. Four types of these degenerations, are degenerations of $\square$ to a bi-gon and a square. Since Θ_{i-1} and Θ_{i+1} are closed elements in their corresponding chain complexes, counting such degenerations contribute to the coefficient of $\mathfrak{u}_{\mathbf{z}}(\square).\mathbf{y}$ in the expression

$$\left(H_i^{\mathfrak{s}} \circ \partial_{i-1} + \partial_{i+1} \circ H_i^{\mathfrak{s}}\right)(\mathbf{x}).$$

Then we have the possibility of a degeneration of $\square$ as $\Delta \star \Delta'$ with $\Delta \in \pi_2(\mathbf{x}, \mathbf{q}, \mathbf{y})$ and $\Delta' \in \pi_2(\Theta_{i+1}, \Theta_{i-1}, \mathbf{q})$ for some $\mathbf{q} \in \mathbb{T}_{\beta^{i-1}} \cap \mathbb{T}_{\beta^{i+1}}$ satisfying $\mu(\Delta) = \mu(\Delta') = 0$. Such degenerations correspond to the appearance of $\mathbf{y}$ in the expression

$$\Psi_i\big(\mathbf{x} \otimes \Phi_i(\Theta_{i+1} \otimes \Theta_{i-1})\big),$$

where the holomorphic triangle maps Ψ_i and Φ_i are defined by

$$\begin{aligned}
&\Psi_i : \mathrm{CF}_{i-1}(\mathfrak{s}) \otimes \mathrm{CF}(\Sigma, \boldsymbol{\beta}^{i-1}, \boldsymbol{\beta}^{i+1}, \mathbf{z}; \mathfrak{s}_i; \mathbb{A}) \longrightarrow \mathrm{CF}_{i+1}(\mathfrak{s})\\
&\Psi_i\big(\mathbf{x} \otimes \mathbf{p}\big) := \sum_{\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^{i+1}}} \sum_{\Delta \in \pi_2^0(\mathbf{x}, \mathbf{p}, \mathbf{w})} \left(\mathfrak{m}(\Delta)\mathfrak{u}_{\mathbf{z}}(\Delta)\right)\mathbf{w}\\
&\Phi_i(\Theta_{i+1} \otimes \Theta_{i-1}) := \sum_{\substack{\mathbf{p} \in \mathbb{T}_{\beta^{i-1}} \cap \mathbb{T}_{\beta^{i+1}}\\ \Delta \in \pi_2^0(\Theta_{i+1}, \Theta_{i-1}, \mathbf{p})}} \left(\mathfrak{m}(\Delta)\mathfrak{u}_{\mathbf{z}}(\Delta)\right)\mathbf{p}.
\end{aligned}$$

Since the Heegaard diagram $(\Sigma, \boldsymbol{\beta}^{i-1}, \boldsymbol{\beta}^{i}, \boldsymbol{\beta}^{i+1}, \mathbf{z})$ is a standard diagram, one may easily observe that $\Phi_i(\Theta_{i+1} \otimes \Theta_{i-1}) = 0$. The reason for this vanishing is that holomorphic triangles which contribute to the above sum come in pairs. This is in fact the same phenomena as what happens in the surgery exact sequence of Ozsváth and Szabó [OS3]. Moreover, the element of $\mathbb{A}$ associated with either of the two triangle classes in a pair is the same by the assumptions on the Heegaard triple. We thus have $\Psi_i\big(\mathbf{x} \otimes \Phi_i(\Theta_{i+1} \otimes \Theta_{i-1})\big) = 0$.

Finally, the last type of degeneration for the domain $\square$ is a degeneration of $\square$ as $\square = \Delta \star \Delta'$, where $\Delta \in \pi_2^0(\mathbf{x}, \Theta_{i+1}, \mathbf{w})$ and $\Delta' \in \pi_2^0(\mathbf{w}, \Theta_{i-1}, \mathbf{y})$ for some $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^i}$. The assumption on the class of the square then imply that Δ represents the same class as Δ_{i+1} and that Δ' represents the same class as Δ_{i-1}, compare with the argument of Lemma 6.3. Counting the end points of $\mathcal{M}(\square)$ corresponding to such degenerations gives the coefficient of $\mathfrak{u}_{\mathbf{z}}(\square).\mathbf{y}$ in $\big(\mathfrak{f}_{i-1}^{\mathfrak{s}} \circ \mathfrak{f}_{i+1}^{\mathfrak{s}}\big)(\mathbf{x})$.

Gathering all this data we conclude that the following relations are satisfied

$$H_i^{\mathfrak{s}} \circ \partial_{i+1} + \partial_{i-1} \circ H_i^{\mathfrak{s}} = \mathfrak{f}_{i-1}^{\mathfrak{s}} \circ \mathfrak{f}_{i+1}^{\mathfrak{s}}, \quad \forall\, i \in \frac{\mathbb{Z}}{3\mathbb{Z}} = \big\{0,1,2\big\},$$

implying that $\mathfrak{f}_{i-1}^{\mathfrak{s}} \circ \mathfrak{f}_{i+1}^{\mathfrak{s}}$ is $(\mathbb{A},\mathbb{H})$ chain homotopic to zero for $i \in \mathbb{Z}/3\mathbb{Z}$. In particular, the decomposition into relative Spin^c classes in $\mathfrak{S}$ is respected by the maps by Lemmas 8.5 and 8.2. □

8.3. Exactness and computation of chain homotopy type

We would like to apply Lemma 3.3 to the triangle of Equation 21. From Lemma 8.2 we know that $\mathfrak{f}_i^{\mathfrak{s}}$ is a filtered $(\mathbb{A},\mathbb{H})$ map between filtered $(\mathbb{A},\mathbb{H})$ chain complexes $\mathrm{CF}_{i+1}(\mathfrak{s})$ and $\mathrm{CF}_{i-1}(\mathfrak{s})$ which decomposes as a sum of maps

$$\mathfrak{f}_i^{\underline{\mathfrak{s}}} : \mathrm{CF}_{i+1}(\underline{\mathfrak{s}}) \longrightarrow \mathrm{CF}_{i-1}(\underline{\mathfrak{s}}), \quad \forall\, \underline{\mathfrak{s}} \in \mathfrak{s} \subset \mathfrak{S}.$$

We may also decompose the maps $H_i^{\mathfrak{s}}$ as $H_i^{\mathfrak{s}} = \oplus_{\underline{\mathfrak{s}}\in\mathfrak{s}} H_i^{\underline{\mathfrak{s}}}$ in a similar way.

Theorem 8.6. *With our previous notation fixed and for any*

$$i \in \frac{\mathbb{Z}}{3\mathbb{Z}} = \{0,1,2\},$$

the map from $\mathrm{CF}_i(\mathfrak{s})$ *to the mapping cone of* $\mathfrak{f}_i^{\mathfrak{s}}$ *defined by* $\mathcal{I}_i^{\mathfrak{s}} = \oplus_{\underline{\mathfrak{s}}\in\mathfrak{s}} \mathcal{I}_i^{\underline{\mathfrak{s}}}$, *and*

$$\begin{aligned} &\mathcal{I}_i^{\underline{\mathfrak{s}}} : \mathrm{CF}_i(\underline{\mathfrak{s}}) \longrightarrow \mathrm{CF}_{i+1}(\underline{\mathfrak{s}}) \oplus \mathrm{CF}_{i-1}(\underline{\mathfrak{s}}) \\ &\mathcal{I}_i^{\underline{\mathfrak{s}}}(\mathfrak{u}.\mathbf{z}) := \big(\mathfrak{f}_{i-1}^{\underline{\mathfrak{s}}}(\mathfrak{u}.\mathbf{z}), H_{i+1}^{\underline{\mathfrak{s}}}(\mathfrak{u}.\mathbf{z})\big), \end{aligned}$$

is a filtered chain homotopy equivalence of filtered $(\mathbb{A},\mathbb{H})$ *chain complexes.*

Proof. For any integer $j \in \mathbb{Z}$ let us define

$$A_j := \begin{cases} \mathrm{CF}_0(\mathfrak{s}) & \text{if } j = 0 \pmod 3 \\ \mathrm{CF}_1(\mathfrak{s}) & \text{if } j = 1 \pmod 3 \\ \mathrm{CF}_2(\mathfrak{s}) & \text{if } j = 2 \pmod 3 \end{cases}$$

Denote the differential of A_j by d_j. Furthermore, define $f_j : A_j \to A_{j+1}$ to be $\mathfrak{f}_2^{\mathfrak{s}}$, $\mathfrak{f}_0^{\mathfrak{s}}$ or $\mathfrak{f}_1^{\mathfrak{s}}$ for $j = 0,1$ or 2 modulo 3, respectively. Let $H_j : A_j \to A_{j+2}$, depending on whether $j = 0,1$ or 2 modulo 3, be the maps $H_1^{\mathfrak{s}}, H_2^{\mathfrak{s}}$ and $H_0^{\mathfrak{s}}$, respectively. By Lemma 3.3, in order to show that the map $\mathcal{I}_i^{\mathfrak{s}}$ is a chain homotopy equivalence of filtered $(\mathbb{A},\mathbb{H})$ chain complexes we have to show that the differences $\phi_i = f_{i+2}\circ H_i - H_{i+1}\circ f_i : A_i \to A_{i+3}$ are chain homotopy equivalences. Checking that all the constructions respect the decomposition into relative Spin^c classes in $\mathfrak{S}$ is straight-forward from the Lemmas 8.2 and 8.5.

As in [OS7] and [OS3], checking the above claim is done by considering holomorphic pentagons associated with Heegaard diagrams of the form

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^j, \boldsymbol{\beta}^{j+1}, \boldsymbol{\beta}^{j+2}, \boldsymbol{\beta}^{j+3}, \mathbf{z}),$$

where $\boldsymbol{\beta}^j$ denotes a set of ℓ simple closed curves which are Hamiltonian isotopes of the curves in $\boldsymbol{\beta}^i$, where $i \in \frac{\mathbb{Z}}{3\mathbb{Z}}$ is equal to $0, 1$ or 2 and j is congruent to i modulo 3. Let us denote the top generator of the Heegaard Floer homology group associated with $(\Sigma, \boldsymbol{\beta}^j, \boldsymbol{\beta}^{j+1}, \mathbf{z})$ by Θ_{j-1}, by little abuse of notation. More generally, the top generator associated with $(\Sigma, \boldsymbol{\beta}^i, \boldsymbol{\beta}^j, \mathbf{z})$ will be denoted by Θ_{ij}. For any three indices $i < j < k$, there is a triangle, with small area (assuming that the Hamiltonian isotopies changing the curve collection to each other are small) which connects Θ_{ij}, Θ_{jk} and Θ_{ik}. Denote this triangle class by Δ_{ijk}.

Without loosing on generality, we may assume that $j = 0$ modulo 3. Choose a generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^j}$ so that $\underline{\mathfrak{s}}(\mathbf{x}) = \underline{\mathfrak{s}} \in \mathfrak{s}$. The curves in $\boldsymbol{\beta}^{j+3}$ are Hamiltonian isotopes of those in $\boldsymbol{\beta}^j$. Thus there is a natural *closest point* map

$$I : \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^j} \to \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^{j+3}}.$$

There is a natural triangle class connecting $\Theta_{j,j+3}, \mathbf{x}$ and $I(\mathbf{x})$ which will be denoted by $\Delta_{\mathbf{x}}$.

Let us denote the complex associated with $(\Sigma, \boldsymbol{\beta}_j, \boldsymbol{\beta}_{j+1}, \mathbf{z})$ and the coefficient ring $\mathbb{A}$ with B_j, and the complex associated with $(\Sigma, \boldsymbol{\beta}_j, \boldsymbol{\beta}_{j+2}, \mathbf{z})$ (again with coefficient ring $\mathbb{A}$) by C_j, and finally the complex associated with $(\Sigma, \boldsymbol{\beta}_j, \boldsymbol{\beta}_{j+3}, \mathbf{z})$ by D_j. We omit the straight forward details of the definitions.

Define a map $\mathcal{P}_j : A_j \to A_{j+3} \cong A_j$ by

$$\mathcal{P}_j(\mathbf{x}) = \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_{j+3}} \\ \pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_{j-1},\Theta_j,\Theta_{j+1},\mathbf{y})}} (\mathfrak{m}(\pentagon)\mathfrak{u}_{\mathbf{z}}(\pentagon))\,\mathbf{y}.$$

The class of the pentagons counted in the above sum is determined by juxtaposing a triangle class $\Delta_{\mathbf{x}} \in \pi_2(\mathbf{x}, \Theta_{j,j+3}, I(\mathbf{x}))$ with an standard square class $\widehat{\square} \in \pi_2(\Theta_{j-1}, \Theta_j, \Theta_{j+1}, \Theta_{j,j+3})$ with small area. As usual, we will drop this class from the notation.

Let us assume that $\pentagon \in \pi_2^{-1}(\mathbf{x}, \Theta_{j-1}, \Theta_j, \Theta_{j+1}, \mathbf{y})$ is a pentagon class which has Maslov index -1. Consider the ends of the smooth orientable one dimensional moduli space $\mathcal{M}(\pentagon)$, which correspond to the degenerations discussed in Theorem 6.8.

Considering the possible degenerations at the boundary of $\mathcal{M}(\pentagon)$, Theorem 6.8 implies

$$\begin{aligned}\phi_j(a_j) &= (\mathcal{P}_j \circ d_j - d_j \circ \mathcal{P}_j)(a_j) \\ &\quad + I_j\big(a_j \otimes K_j(\Theta_{j-1} \otimes \Theta_j \otimes \Theta_{j+1})\big), \quad \forall\, a_j \in A_j,\end{aligned} \tag{23}$$

where the maps $I_j : A_j \otimes D_j \to A_{j+3}$ and $K_j(\Theta_{j-1} \otimes \Theta_j \otimes \Theta_{j+1})$ are defined as follows.

$$\begin{aligned} I_j\big(\mathbf{x} \otimes \mathbf{q}\big) &:= \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^{j+3}}} \sum_{\Delta \in \pi_2^0(\mathbf{x},\mathbf{q},\mathbf{y})} (\mathfrak{m}(\Delta)\mathfrak{u}_{\mathbf{z}}(\Delta))\,\mathbf{y} \\ K_j(\Theta_{j-1} \otimes \Theta_j \otimes \Theta_{j+1}) &:= \sum_{\substack{\mathbf{q} \in \mathbb{T}_{\beta^j} \cap \mathbb{T}_{\beta^{j+3}} \\ \square \in \pi_2^{-1}(\Theta_{j-1}\, \Theta_j, \Theta_{j+1}, \mathbf{q})}} (\mathfrak{m}(\square)\mathfrak{u}_{\mathbf{z}}(\square))\,\mathbf{q}. \end{aligned} \tag{24}$$

Two of the terms appearing in Theorem 6.8 vanish and are not present in the Equation 23. These are the terms that correspond to degenerations containing a triangle in $\pi_2(\Theta_{j-1}, \Theta_j, \mathbf{q})$ for some $\mathbf{q} \in \mathbb{T}_{\beta^j} \cap \mathbb{T}_{\beta^{j+2}}$, or a triangle in $\pi_2(\Theta_j, \Theta_{j+1}, \mathbf{q})$ for some $\mathbf{q} \in \mathbb{T}_{\beta^{j+1}} \cap \mathbb{T}_{\beta^{j+3}}$. The total contribution of such triangles vanishes, since they come in canceling pairs. Thus the terms containing such degenerations would vanish as well.

Note that the map $\mathbf{x} \mapsto I_j(\mathbf{x} \otimes \Theta_{j,j+3})$ is a perturbation of the isomorphism I with a map $\epsilon : A_j \to A_{j+3}$ which takes a generator $\mathbf{x}$ to generators with smaller energy than $I(\mathbf{x})$, when we equip A_{j+3} with an appropriate energy filtration. This follows since the contributions from triangle classes other than $\Delta_{\mathbf{x}}$ will contribute more than the small energy associated with $\Delta_{\mathbf{x}}$. Standard arguments in Heegaard Floer theory (c.f. Ozsváth and Szabó's original paper [OS5]) may then be applied to construct an explicit inverse for this map up to filtered $(\mathbb{A}, \mathbb{H})$ chain homotopy.

In order to complete the proof of the theorem, it is thus enough to show that

$$K_j(\Theta_{j-1} \otimes \Theta_j \otimes \Theta_{j+1}) = \Theta_{j,j+3}.$$

This can be proved directly, since the Heegaard quadruple

$$(\Sigma, \boldsymbol{\beta}^j, \boldsymbol{\beta}^{j+1}, \boldsymbol{\beta}^{j+2}, \boldsymbol{\beta}^{j+3}, \mathbf{z})$$

is a special Heegaard diagram, which may be analysed without too much difficulty, following earlier considerations of Ozsváth and Szabó (e.g. in [OS7], Subsection 4.2). Here is a quick review of the proof. There is a preferred square class which contributes to the second sum of Equation 24. This square class has small total area, and multiplicity 1 at p. The contribution of this class would give $\Theta_{j,j+3}$. The rest of contributing square classes come in pairs and the elements of $\mathbb{A}$ associated with both elements in each pair are the same (with opposite sign). Thus the two square classes in each pair cancel each other. □

8.4. Special cases of the surgery exact sequence

Suppose that

$$(X, \tau = \{\tau_1, \dots, \tau_\kappa\})$$

is a weakly balanced sutured manifold. Furthermore, assume that τ_1 and τ_2 belong to the common boundary of the genus zero connected components $R_1^+ \in \mathfrak{R}^+(\tau)$ and $R_1^- \in \mathfrak{R}^-(\tau)$. Let λ denote a simple closed curve in $\partial X - \cup_{i=3}^{\kappa}\tau_i$ which cuts either of τ_1 and τ_2 in a single transverse point. Choose the orientation on λ so that the intersection number $\tau_1.\lambda$ is 1. Let ϕ denote the right-handed (positive) Dehn twist along τ_1 and ψ denote the left-handed (negative) Dehn twist along λ. Thus in particular $\phi(\lambda)$ and $\psi(\tau_1)$ are both homologous to $\tau_1 + \lambda$. We set $\tau^0 = \tau$, $\tau^1 = \phi(\tau)$ and $\tau^2 = \psi(\phi(\tau))$. It is then easy to construct a Heegaard quadruple associated with the three sutured manifolds (X, τ^i) $i = 0, 1, 2$, which is of the form described in the first subsection of this section.

One will thus have a triangle of chain complexes connecting

$$\mathrm{CF}(X, \tau^i, \mathfrak{s}), \quad i = 0, 1, 2.$$

The Spin^c class $\mathfrak{s}$ belongs to a subset of the set of Spin^c structures on any of the three 3-manifold obtained by filling out the sutures in τ^i, $i = 0, 1, 2$. The filtration of the chain complexes in the exact triangle is, however, by the relative Spin^c classes in

$$\frac{\mathrm{Spin}^c(X, \tau)}{\langle \mathrm{PD}[\tau_1 + \tau_2] \rangle},$$

rather than $\mathrm{Spin}^c(X, \tau)$.

Many of the surgery exact sequences in Heegaard Floer theory arise as special cases of the above situation. Namely, suppose that Y is a three-manifold and that K is a null-homologous knot inside Y. Let us assume that X is the three-manifold $Y - \mathrm{nd}(K)$ obtained by removing a tubular neighbourhood of K from Y. Let $\tau_{p/q}$ denote a set of two parallel sutures on the torus boundary of X, such that the sutures correspond to the framing giving the p/q surgery on K. For $i \in \{0,1,2\} = \mathbb{Z}/3\mathbb{Z}$, let the rational number $p_i/q_i \in \mathbb{Q}$ be chosen so that the three equations $p_iq_{i+1} - q_ip_{i+1} = 1$ are satisfied. Let $X^i = Y_{p_i/q_i}(K)$ denote the three-manifold obtained from Y by performing p_i/q_i surgery on K, and $K^i = K_{p_i/q_i}$ denote the corresponding knot inside X^i. Associated with any relative Spin^c class $\underline{\mathfrak{s}} \in \mathrm{Spin}^c(Y,K) = \mathrm{Spin}^c(X^i,K^i)$ we thus obtain the exact triangle

$$
\begin{array}{ccc}
\mathrm{CF}(X^0,K^0,\underline{\mathfrak{s}};\mathbb{A}) & \xrightarrow{\ \mathfrak{f}_2^{\underline{\mathfrak{s}}}\ } & \mathrm{CF}(X^1,K^1,\underline{\mathfrak{s}};\mathbb{A}) \\
{\scriptstyle \mathfrak{f}_1^{\underline{\mathfrak{s}}}}\nwarrow & & \swarrow{\scriptstyle \mathfrak{f}_0^{\underline{\mathfrak{s}}}} \\
 & \mathrm{CF}(X^2,K^2,\underline{\mathfrak{s}};\mathbb{A}) &
\end{array}
\qquad (25)
$$

If we set the variables $\mathfrak{u}_0, \mathfrak{u}_1$ and $\mathfrak{u}_2$ equal to zero, we arrive at the test ring $\mathbb{Z}$ for $\mathbb{A}$. Correspondingly, we have the following short exact sequence in homology

$$
\begin{array}{ccc}
\widehat{\mathrm{HFK}}(X^0,K^0,\underline{\mathfrak{s}}) & \xrightarrow{\ \mathfrak{f}_2^{\underline{\mathfrak{s}}}\ } & \widehat{\mathrm{HFK}}(X^1,K^1,\underline{\mathfrak{s}}) \\
{\scriptstyle \mathfrak{f}_1^{\underline{\mathfrak{s}}}}\nwarrow & & \swarrow{\scriptstyle \mathfrak{f}_0^{\underline{\mathfrak{s}}}} \\
 & \widehat{\mathrm{HFK}}(X^2,K^2,\underline{\mathfrak{s}}) &
\end{array}
\qquad (26)
$$

In particular, let us assume that $p_0/q_0 = 1/0, p_1/q_1 = 0/1$ and $p_2/q_2 = (-1)/(-1)$. Then the exact triangle of Equation 26 we obtain the following exact triangle

$$
\begin{array}{ccc}
\widehat{\mathrm{HFK}}(Y,K,\underline{\mathfrak{s}}) & \xrightarrow{\ \mathfrak{f}_2^{\underline{\mathfrak{s}}}\ } & \widehat{\mathrm{HFK}}(Y_0(K),K_0,\underline{\mathfrak{s}}) \\
{\scriptstyle \mathfrak{f}_1^{\underline{\mathfrak{s}}}}\nwarrow & & \swarrow{\scriptstyle \mathfrak{f}_0^{\underline{\mathfrak{s}}}} \\
 & \widehat{\mathrm{HFK}}(Y_1(K),K_1,\underline{\mathfrak{s}}) &
\end{array}
\qquad (27)
$$

which is the same as the exact triangle used in [Ef1, Ef5] and [Ef2].

In fact, Theorem 8.6 may be generalized to the situation where $\mu_i.\mu_{i+1} = m_{i-1},\ i \in \{0, 1, 2\} = \mathbb{Z}/3\mathbb{Z}$ are integers such that the relation

$$m_0\mu_0 + m_1\mu_1 + m_2\mu_2 = 0 \in \mathrm{H}_1(X;\mathbb{Z})$$

is satisfied. This stronger form appeared in the earlier version of this paper (see [AE], Theorem 9.6), but in the interest of much simpler exposition we prefer to restrict our attention to the simple case discussed above. We refer the interested reader to the 9th section of the initial version of this paper. This stronger form then generalizes the exact sequence in homology, which appears as Theorem 1.7 in [OS6]. Also, Theorem 4.7 from [OS7] is a special case of Theorem 9.6 in its stronger form. Furthermore, Theorem 8.2 from [OS1] is also a corollary in this situation. When $m_1 = m_2 = 1$, and $m_0 = m$ is an arbitrary integer, the main result of [Ef4] is obtained as a special case as well. The surgery exact sequence of Theorem 3.1 in [OS3] is in turn a consequence of this last result. In a similar way, Theorem 6.2 from [OS4] follows from this last consideration.

9. Properties and Examples

9.1. Product disk decomposition

Product disk decomposition is a special case of surface decomposition defined by Gabai. We refer the interested reader to [Gab1] for the complete definition of surface decomposition, and will only recall the following definition.

Definition 9.1. We say that the sutured manifold (X', τ') is obtained from (X, τ) by a *product disk decomposition* if there is a properly embedded disk D in (X, τ) such that $|D \cap \tau| = 2$ and such that decomposing (X, τ) along the embedded disk D results in (X', τ').

Lemma 9.13 from [Ju1] shows that product disk decomposition preserves the sutured Floer homology, i.e. we have

$$\mathrm{SFH}(X, \tau) = \mathrm{SFH}(X', \tau').$$

The purpose of this section is to give a description of the behaviour of the sutured Floer complex under product disk decomposition. It will be observed meanwhile, that there are serious obstructions in the way towards a surface decomposition formula for the full sutured Floer complex.

Suppose that (X', τ') is obtained from $(X, \tau = \{\tau_0, \tau_1, \dots, \tau_\kappa\})$ by a product disk decomposition along the disk D as above. The boundary of the disk D intersects the sutures transversely in two points. The easier case to understand is the case where the two intersection points in $D \cap \tau$ belong to different sutures, say τ_0 and τ_1. In this case, $\overline{X} = \overline{X'}$. The sutures τ_0 and τ_1 belong to the boundary of the same component in $\mathfrak{R}^+(\tau)$ (respectively, $\mathfrak{R}^-(\tau)$). The set of sutures τ' will be the union of $\{\tau_2, \dots, \tau_\kappa\}$ with a suture σ. The algebra $\mathbb{A}_{\tau'}$ associated with the weakly balanced sutured manifold (X', τ') is then a quotient of $\mathbb{Z}[\mathfrak{u}, \mathfrak{u}_2, \dots, \mathfrak{u}_\kappa]$, where $\mathfrak{u}$ denotes the variable associated with σ. There is an embedding

$$\imath : \mathbb{A}_{\tau'} \hookrightarrow \mathbb{A}_\tau, \quad \imath(\mathfrak{u}) := \mathfrak{u}_0\mathfrak{u}_1 \quad \text{and} \quad \imath(\mathfrak{u}_j) := \mathfrak{u}_j, \quad j = 2, \dots, \kappa,$$

and one may easily prove that

$$\mathrm{CF}(X, \tau, \mathfrak{s}) = \mathrm{CF}(X', \tau', \mathfrak{s}) \otimes_{\mathbb{A}_{\tau'}} \mathbb{A}_\tau, \quad \forall\, \mathfrak{s} \in \mathrm{Spin}^c(\overline{X}) = \mathrm{Spin}^c(\overline{X'}).$$

In fact, if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}' = \{z_1, \dots, z_\kappa\})$ is an $\mathfrak{s}$-admissible Heegaard diagram for (X', τ') with z_1 corresponding to $\sigma \in \tau'$, and if z_0 is a marked point in the same component of $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ as z_1, then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z} = \{z_0, z_1, \dots, z_\kappa\})$ is an $\mathfrak{s}$-admissible Heegaard diagram for (X, τ), making the above conclusion straight forward.

The more interesting situation is the product disk decomposition where the two points in $D \cap \tau$ belong to the same suture $\tau_0 \in \tau$. We may thus find a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z} = \{z_0, \dots, z_\kappa\})$ for the weakly balanced sutured manifold (X, τ), as discussed in the proof of Lemma 9.13 from [Ju1], where the disk D corresponds to an arc δ, starting and ending at z_0, which stays disjoint from the curves in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We may thus cut Σ open along the simple closed curve δ, and glue a pair of disks to the resulting boundary components of Σ. Denote the resulting surface by Σ' and the centers of the aforementioned disks by w_1 and w_2. The weakly balanced sutured manifold $(X', \tau' = \{\sigma_1, \sigma_2, \tau_1, \dots, \tau_\kappa\})$ then corresponds to the Heegaard diagram

$$\left(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}' = \{w_1, w_2, z_1, \dots, z_\kappa\}\right).$$

Let us further assume that the simple closed curve δ is separating, and thus $(X', \tau') = (X^1, \tau^1) \coprod (X^2, \tau^2)$ is a disjoint union of two other sutured manifolds. This case corresponds to the situation in the following definition. Let $\kappa = m + n$ and suppose that the weakly balanced sutured manifolds $(X^1, \tau^1 = \{\sigma_1, \tau_1, \dots, \tau_m\})$ and $(X^2, \tau^2 = \{\sigma_2, \tau_{m+1}, \dots, \tau_\kappa\})$, with distinguished sutures $\sigma_1 \in \tau^1$ and $\sigma_2 \in \tau^2$ are given.

Definition 9.2. Choose the embedded disk $D_i \subset \partial X^i$ for $i = 1, 2$ so that $D_i \cap \sigma_i$ is a closed connected segment on σ_i and the intersection of D_i with the other sutures in τ^i is empty. We set X to be the 3-manifold obtained from X^1 and X^2 by the identification of D_1 and D_2, such that $D_1 \cap \mathfrak{R}^\bullet(\tau^1)$ is identified with $D_2 \cap \mathfrak{R}^\bullet(\tau^2)$ for $\bullet \in \{+, -\}$, and set τ_0 to be the simple closed curve obtained by concatenation of $\sigma_1 - (\sigma_1 \cap D_1)$ and $\sigma_2 - (\sigma_2 \cap D_2)$. We define the *boundary connected sum* of (X^1, τ^1) with (X^2, τ^2) along the sutures σ_1 and σ_2 to be the sutured manifold $(X, \tau = \{\tau_0, \dots, \tau_\kappa\})$.

In the situation of Definition 9.2, if we decompose the sutured manifold (X, τ) along the disk D obtained by identifying D_1 and D_2 we recover $(X^1, \tau^1) \coprod (X^2, \tau^2)$. Since $D \cap \tau = D \cap \tau_0$ consists of precisely two points, $(X^1, \tau^1) \coprod (X^2, \tau^2)$ is obtained from (X, τ) by a product disk decomposition. Let us further assume that τ_0 belongs to the genus zero components of $\mathfrak{R}^+(\tau)$ and $\mathfrak{R}^-(\tau)$. It is thus implied that each σ_i belongs to the boundary of genus zero components of $\mathfrak{R}^+(\tau^i)$ and $\mathfrak{R}^-(\tau^i)$, for $i = 1, 2$.

Consider the Heegaard diagrams

$$\begin{aligned} H_1 &= (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{z}_1 = \{w_1, z_1, \dots, z_m\}) \quad \text{and} \\ H_2 &= (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{z}_2 = \{w_2, z_{m+1}, \dots, z_\kappa\}) \end{aligned}$$

associated with (X^1, τ^1) and (X^2, τ^2), respectively. We assume that the marked point w_i corresponds to the suture σ_i, and that the marked point z_j corresponds to the suture τ_j, for $i = 1, 2$ and $j = 1, \dots, \kappa$. One may construct a Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for the sutured manifold (X, τ) by taking the connecting sum of H_1 and H_2 along w_1 and w_2. Thus $\Sigma = \Sigma_1 \# \Sigma_2$ and

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2, \quad \boldsymbol{\beta} = \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2 \quad \text{and} \quad \mathbf{z} = \{z_0\} \cup \mathbf{z}_1 \cup \mathbf{z}_2 - \{w_1, w_2\},$$

where the marked point z_0 is placed on the connected sum tube.

Let us assume that

$$\mathfrak{R}^-(\tau^j) = \coprod_{i=0}^{k_j} R_i^-(j) \quad \text{and} \quad \mathfrak{R}^+(\tau^j) = \coprod_{i=0}^{l_j} R_i^+(j), \quad j = 1, 2,$$

and suppose that $R_0^\pm(j)$ for $j = 1, 2$ are the genus zero components which have σ_j as a boundary component. We will assume that $g_i^\bullet(j)$ denotes the genus of $R_i^\bullet(j)$ for $\bullet \in \{-, +\}$. Denote the generator associated with τ^i, $i = 0, 1, \dots, \kappa$ by $\mathfrak{u}_i$, and the generator corresponding to σ_j, $j = 1, 2$ by $\mathfrak{v}_j$. Let

$\mathfrak{u}_i^\bullet(j)$ denote $\mathfrak{u}(R_i^\bullet(j))$ in $\mathbb{Z}[\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{u}_0,\dots,\mathfrak{u}_\kappa]$. We also set $\mathfrak{u}^\bullet(j) := \mathfrak{u}^\bullet(\tau^j)$. One may thus compute

$$\mathfrak{u}^+(\tau) = \mathfrak{u}^+(1) + \mathfrak{u}^+(2) - \mathfrak{u}_0^+(1) - \mathfrak{u}_0^+(2) + \frac{\mathfrak{u}_0\mathfrak{u}_0^+(1)\mathfrak{u}_0^+(2)}{\mathfrak{v}_1\mathfrak{v}_2} \quad \text{and}$$
$$\mathfrak{u}^-(\tau) = \mathfrak{u}^-(1) + \mathfrak{u}^-(2) - \mathfrak{u}_0^-(1) - \mathfrak{u}_0^-(2) + \frac{\mathfrak{u}_0\mathfrak{u}_0^-(1)\mathfrak{u}_0^-(2)}{\mathfrak{v}_1\mathfrak{v}_2}.$$

From here, one finds that $\mathbb{A}_1 = \mathbb{A}_{\tau^1}$ and $\mathbb{A}_2 = \mathbb{A}_{\tau^2}$ may both be embedded in the quotient $\mathbb{A} = \mathbb{A}(\tau^1,\tau^2)$ of $\mathbb{A}_\tau$ defined by

$$\mathbb{A} = \frac{(\mathbb{A}_1 \otimes_{\mathbb{Z}} \mathbb{A}_2)[\mathfrak{u}_0]}{\left\langle \mathfrak{v}_1 - \frac{\mathfrak{u}_0\mathfrak{u}_0^-(2)}{\mathfrak{v}_2}\right\rangle + \left\langle \mathfrak{v}_2 - \frac{\mathfrak{u}_0\mathfrak{u}_0^+(1)}{\mathfrak{v}_1}\right\rangle}. \tag{28}$$

The embeddings of $\mathbb{A}_i$ in $\mathbb{A}$ and the quotient map $\mathbb{A}_\tau \to \mathbb{A}$ give $\mathbb{A}$ the structure of a test ring for $\mathbb{A}_i$ and $\mathbb{A}_\tau$. Moreover, there are natural gluing maps

$$\# : \mathrm{Spin}^c(\overline{X^1}) \times \mathrm{Spin}^c(\overline{X^2}) \to \mathrm{Spin}^c(\overline{X}) \quad \text{and}$$
$$\# : \mathrm{Spin}^c(X^1,\tau^1) \times \mathrm{Spin}^c(X^2,\tau^2) \to \mathrm{Spin}^c(X,\tau).$$

Let $\mathbb{H}^1 = H^2(X^1,\partial X^1,\mathbb{Z})$, $\mathbb{H}^2 = H^2(X^2,\partial X^2,\mathbb{Z})$ and $\mathbb{H} = H^2(X,\partial X,\mathbb{Z})$. Let χ_i denote the filtration map corresponding to $\mathbb{A}_i$ with values in $\mathbb{H}^i$ for $i=1,2$ and χ_τ denote the filtration map of $\mathbb{A}_\tau$ with values in $\mathbb{H}$. The inclusion of X^i in X gives a map induces a map from $H_1(X_i;\mathbb{Z})$ to $H_1(X;\mathbb{Z})$ and by Lefshetz duality we get a map $\imath_i : \mathbb{H}_i \to \mathbb{H}$ for $i=1,2$. We may extend the filtration map χ_i to a filtration by $\mathbb{H}$ using $\imath_i$. Correspondingly, $\mathrm{CF}(X^i,\tau^i,\mathfrak{s}^i)$ is a filtered $(\mathbb{A}_i,\mathbb{H})$ chain complex for any $\mathfrak{s}^i \in \mathrm{Spin}^c(\overline{X^i})$.

With the above notation fixed, we prove the following proposition in the following two subsections.

Proposition 9.3. *Fix the* Spin^c *structures* $\mathfrak{s}^i \in \mathrm{Spin}^c(\overline{X^i})$ *for* $i=1,2$. *The filtered chain homotopy type of the two filtered* $(\mathbb{A},\mathbb{H})$ *chain complexes*

$$\mathrm{CF}(X,\tau,\mathfrak{s}^1\#\mathfrak{s}^2;\mathbb{A}) \quad \textit{and} \quad \mathrm{CF}(X^1,\tau^1,\mathfrak{s}^1;\mathbb{A}) \otimes_{\mathbb{A}} \mathrm{CF}(X^2,\tau^2,\mathfrak{s}^2;\mathbb{A})$$

are the same.

9.2. A special case

Let us start with the following special case. Suppose that (X^2,τ^2) is the sutured manifold which corresponds to a special Heegaard diagram $H_2 =$

$(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{z}_2)$ of the following type. We assume that $\boldsymbol{\alpha}_2 = \{\alpha_{\ell_1+1}, \dots, \alpha_{\ell_1+\ell_2}\}$ and $\boldsymbol{\beta}_2 = \{\beta_{\ell_1+1}, \dots, \beta_{\ell_1+\ell_2}\}$ where β_i is the image of α_i under a small Hamiltonian isotopy for $i = \ell_1 + 1, \dots, \ell_1 + \ell_2$. We thus assume that β_i is disjoint from α_j for $j \neq i \in \{\ell_1 + 1, \dots, \ell_1 + \ell_2\}$, and that it cuts α_i is a pair of cancelling intersection points. Let $\Theta_{\alpha_2\beta_2}$ denote the top generator of the complex $\mathrm{CF}(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{z}_2)$. For $\mathfrak{s}^1 \in \mathrm{Spin}^c(\overline{X^1})$ let us denote the Spin^c structure on $\overline{X}$ obtained by pairing $\mathfrak{s}^1$ with the canonical Spin^c class corresponding to H_2 by $\imath^*(\mathfrak{s}^1)$.

Lemma 9.4. *With the above notation, for any* $\mathfrak{s}^1 \in \mathrm{Spin}^c(\overline{X^1})$ *the map*

$$\Phi_1 : \mathrm{CF}(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{z}_1, \mathfrak{s}^1; \mathbb{A}) \longrightarrow \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, \imath^*(\mathfrak{s}^1); \mathbb{A})$$
$$\Phi_1(\mathbf{x}) = \mathbf{x} \times \Theta_{\alpha_2\beta_2}$$

is a filtered $(\mathbb{A}, \mathbb{H})$ *chain map.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}$, $\Theta \in \mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2}$ and $\phi \in \pi_2(\mathbf{x} \times \Theta_{\alpha_2\beta_2}, \mathbf{y} \times \Theta)$ be a Whitney disk with Maslov index one and $\mathfrak{u}_{\mathbf{z}}(\phi) \neq 0$. We may degenerate ϕ as the connected sum $\phi = \phi_1 \# \phi_2$ where $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\phi_2 \in \pi_2(\Theta_{\alpha_2\beta_2}, \Theta)$. By Theorem 7.1 we have

$$\mu(\phi) = \mu(\phi_1) + \mu(\phi_2) - 2n_{z_0}(\phi) = 1.$$

Similar to the discussions in the Subsection 6.2, we have

$$\mathcal{D}(\phi_2) = \mathcal{D}(\phi_2') + a_0 A_0 + \dots + a_{k_2} A_{k_2} + q_1 \mathcal{P}_1 + \dots + q_{\ell_2} \mathcal{P}_{\ell_2},$$

where ϕ_2' is a positive Whitney disk with $n_{\mathbf{z}_2}(\phi_2') = 0$, $\mathcal{P}_i$ is the periodic domains bounded by $\alpha_{i+\ell_1}$ and $\beta_{i+\ell_1}$, and A_i is the domain in $\Sigma_2 - \boldsymbol{\alpha}_2$ which corresponds to $R_i^-(2)$. Furthermore, we have $a_i \geq 0$ for $i = 0, \dots, k_2$. Note that $\mathfrak{u}_{\mathbf{z}_2}(\phi_2) \neq 0$, and we thus know that if $a_i > 0$ then the genus $g_i^-(2) = g_i^+(2)$ is zero. Note also, that $a_0 = n_{z_0}(\phi)$. From here we have

$$\mu(\phi_2) = \mu(\phi_2') + 2n_{z_0}(\phi) + 2(a_1 + \dots + a_{k_2}).$$

Each pair of curves (α_i, β_i) for $i = \ell_1 + 1, \dots, \ell_1 + \ell_2$ intersect in a pair of points; the top one which will be denoted by x_i and the bottom one which will be denoted by y_i. For $\Theta \in \mathbb{T}_{\alpha^2} \cap \mathbb{T}_{\beta^2}$ let $|\Theta|$ denote the number of bottom intersection points in Θ (i.e. the number of y_i's in Θ). It is then easy to check that

$$\begin{aligned} & \mu(\phi_2') = |\Theta| - |\Theta_{\alpha_2,\beta_2}| = |\Theta| \\ \Rightarrow \quad & \mu(\phi_1) = 1 + 2n_{z_0}(\phi) - \mu(\phi_2) \\ & \qquad = 1 - |\Theta| - 2(a_1 + \cdots + a_{k_2}). \end{aligned}$$

Since $\mathcal{M}(\phi_1)$ is non-empty $\mu(\phi_1) \geq 0$ and we have $a_1 = a_2 = \cdots = a_{k_2} = 0$, and $\mu(\phi_2') = 0$ or 1. If $\mu(\phi_2') = 1$ then $\mu(\phi_2) = \mu(\phi_2') = 1$ and $\mu(\phi_1) = 0$. Thus ϕ_1 should be the class of the constant disk and $n_{z_0}(\phi) = n_{w_1}(\phi_1) = 0$, implying $\phi_2 = \phi_2'$. Therefore, the coefficient of $\mathbf{x} \times \Theta$ in $\partial(\mathbf{x} \times \Theta_{\alpha_2\beta_2})$ is equal to

$$\sum_{\phi_2 \in \pi_2^1(\Theta_{\alpha_2\beta_2},\Theta)} \mathfrak{m}(\phi_2) \prod_{i=1}^{\kappa_2} (\mathfrak{u}_{i+m})^{n_{i+m}(\phi_2)},$$

which is equal to zero because $\Theta_{\alpha_2\beta_2}$ is closed.

The second possibility is the case $\mu(\phi_2') = 0$. In this case $\mu(\phi_1) = 1$ and $|\Theta| = 0$. Thus $\Theta = \Theta_{\alpha_2,\beta_2}$ and ϕ_2' is the domain of the constant map, or equivalently $\mathcal{D}(\phi_2) = n_{z_0}(\phi)A_0$, or $\mathcal{D}(\phi_2) = n_{z_0}(\phi)B_0$. Here, B_0 denotes the connected component of $\Sigma_2 - \boldsymbol{\beta}_2$ which contains the marked point w_2. By Theorem 7.1, for a sufficiently large connected sum tube length we have

$$\mathfrak{m}(\phi) = \sum_{u_1 \in \widehat{\mathcal{M}}(\phi_1)} \epsilon(u_1).\#\big\{u_2 \in \mathcal{M}(\phi_2) \,\big|\, \rho^{w_1}(u_1) = \rho^{w_2}(u_2)\big\}$$

Suppose now that the marked point w_1 is moved sufficiently close to one of the β curves. Lemma 7.2 implies that the above sum is equal to $\mathfrak{m}(\phi_1)$. Thus the coefficient of $\mathbf{y} \times \Theta_{\alpha_2\beta_2}$ in $\partial(\mathbf{x} \times \Theta_{\alpha_2\beta_2})$ is equal to

$$\sum_{\phi_1 \in \pi_2^1(\mathbf{x},\mathbf{y})} \mathfrak{m}(\phi_1) \left(\mathfrak{u}_{\mathbf{z}_1}(\phi_1) \left(\frac{\mathfrak{u}_0 \mathfrak{u}_0^-(2)}{\mathfrak{v}_1 \mathfrak{v}_2} \right)^{n_{w_1}(\phi_1)} \right),$$

Which is equal to the coefficient of $\mathbf{y}$ in $\partial\mathbf{x}$ in $\mathrm{CF}(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{z}_1, \mathfrak{s}^1; \mathbb{A})$. This completes the proof of the lemma. □

9.3. Proof of the proposition

We are now ready to prove Proposition 9.3.

Proof of Proposition 9.3. Assume that the Heegaard diagram H_i considered above is an $\mathfrak{s}^i$-admissible Heegaard diagram for (X^i, τ^i) in the stronger sense of Remark 4.6 for $i = 1, 2$. This implies that the connected sum Heegaard diagram $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is $\mathfrak{s}^1\#\mathfrak{s}^2$- admissible. To prove the connected sum formula, consider the Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\beta}, \mathbf{z})$ where $\boldsymbol{\delta} = \boldsymbol{\delta}_1 \cup \boldsymbol{\delta}_2$, $\boldsymbol{\delta}_1 = \{\delta_1, \ldots, \delta_{\ell_1}\}$, $\boldsymbol{\delta}_2 = \{\delta_{\ell_1+1}, \ldots, \delta_{\ell_1+\ell_2}\}$ are constructed as follows. For any $i = 1, \ldots, \ell_1$, δ_i is small Hamiltonian isotope of β_i such that they intersect in two cancelling intersection points. Similarly, for $j = \ell_1 + 1, \ldots, \ell_1 + \ell_2$, δ_j is small Hamiltonian isotope of α_j which intersects it in pair of cancelling intersection points.

Consider the Heegaard diagrams $(\Sigma_1, \boldsymbol{\delta}_1, \boldsymbol{\beta}_1, \mathbf{z}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\delta}_2, \mathbf{z}_2)$ which are special, in the sense that they are of the type studies in the previous subsection. Let $\Theta_{\delta_1\beta_1}$ and $\Theta_{\alpha_2\delta_2}$ denote the top generators of the chain complexes $\mathrm{CF}(\Sigma_1, \boldsymbol{\delta}_1, \boldsymbol{\beta}_1, \mathbf{z}_1)$ and $\mathrm{CF}(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\delta}_2, \mathbf{z}_2)$ corresponding to the canonical Spin^c structure for either Heegaard diagram. For any fixed intersection point $\mathbf{x} \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\delta_1}$ with $\underline{\mathfrak{s}}(\mathbf{x}) \in \mathfrak{s}^1$ we have a generator $I_1(\mathbf{x}) \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}$ determined as the closest intersection point in $\mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}$ to $\mathbf{x}$. Similarly, for any fixed intersection point $\mathbf{y} \in \mathbb{T}_{\delta_2} \cap \mathbb{T}_{\beta_2}$ with $\underline{\mathfrak{s}}(\mathbf{y}) \in \mathfrak{s}^2$ there is an intersection point $I_2(\mathbf{y}) \in \mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2}$ determined as the closest intersection point in $\mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2}$ to $\mathbf{y}$.

In the Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\beta}, \mathbf{z})$ there is a triangle class Δ connecting $\Theta_{\delta_1\beta_1} \times \mathbf{y}$, $\mathbf{x} \times \Theta_{\alpha_2\delta_2}$ and $I_1(\mathbf{x}) \times I_2(\mathbf{y})$. The triangle class Δ represents a Spin^c class for the Heegaard triple. One may easily check that the $\mathfrak{s}^1$ and $\mathfrak{s}^2$ admissibility of H_1 and H_2 in the stronger sense implies the admissibility of the Heegaard triple with respect to this Spin^c structure. Moreover, the $\mathbb{Z}$-algebra $\mathbb{A}$ is a test ring for the $\mathbb{Z}$-algebra associated to the Heegaard triple.

The triangle class Δ determines a Spin^c class $\imath^*(\mathfrak{s}^1)$ corresponding to the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{z})$, and a Spin^c class $\jmath^*(\mathfrak{s}^2)$ corresponding to the Heegaard diagram $(\Sigma, \boldsymbol{\delta}, \boldsymbol{\beta}, \mathbf{z})$. There is a holomorphic triangle map

$$\begin{aligned}\Phi : \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{z}, \imath^*(\mathfrak{s}_1); \mathbb{A}) \otimes \mathrm{CF}(\Sigma, \boldsymbol{\delta}, \boldsymbol{\beta}, \mathbf{z}, \jmath^*(\mathfrak{s}_2); \mathbb{A})\\ \longrightarrow \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}; \mathfrak{s}_1\#\mathfrak{s}_2; \mathbb{A}).\end{aligned}$$

Moreover, using the top generators $\Theta_{\alpha_2\delta_2}$ and $\Theta_{\delta_1\beta_1}$ we may define the homomorphisms

$$\begin{aligned}&\Phi_1 : \mathrm{CF}(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\delta}_1, \mathbf{z}_1, \mathfrak{s}^1; \mathbb{A}) \to \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{z}, \imath^*(\mathfrak{s}^1); \mathbb{A}),\\ &\Phi_2 : \mathrm{CF}(\Sigma_2, \boldsymbol{\delta}_2, \boldsymbol{\beta}_2, \mathbf{z}_2, \mathfrak{s}^2; \mathbb{A}) \to \mathrm{CF}(\Sigma, \boldsymbol{\delta}, \boldsymbol{\beta}, \mathbf{z}, \jmath^*(\mathfrak{s}^2); \mathbb{A}),\\ &\Phi_1(\mathbf{x}) := \mathbf{x} \times \Theta_{\alpha_2\delta_2} \quad \text{and} \quad \Phi_2(\mathbf{y}) := \Theta_{\delta_1\beta_1} \times \mathbf{y}.\end{aligned}$$

Using Lemma 9.4 we know that both Φ_1 and Φ_2 are chain maps. The chain complexes $\mathrm{CF}(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\delta}_1, \mathfrak{s}^1; \mathbb{A})$ and $\mathrm{CF}(\Sigma_2, \boldsymbol{\delta}_2, \boldsymbol{\beta}_2, \mathfrak{s}^2; \mathbb{A})$ may be identified as

$$\mathrm{CF}(X^1, \tau^1, \mathfrak{s}^1; \mathbb{A}) \quad \text{and} \quad \mathrm{CF}(X^2, \tau^2, \mathfrak{s}^2; \mathbb{A}),$$

respectively. We may thus define a a chain map

$$\begin{aligned}&\Gamma : \mathrm{CF}(X^1, \tau^1, \mathfrak{s}^1; \mathbb{A}) \otimes \mathrm{CF}(X^2, \tau^2, \mathfrak{s}^2; \mathbb{A}) \to \mathrm{CF}(X, \tau, \mathfrak{s}^1 \# \mathfrak{s}^2; \mathbb{A})\\ &\Gamma = \Phi \circ (\Phi_1 \otimes \Phi_2)\end{aligned}$$

In fact, Γ is an $(\mathbb{A}, \mathbb{H})$ filtered chain map. We may assume that the total unsigned area in the region between the curves in $\boldsymbol{\delta}_1$ and $\boldsymbol{\beta}_1$ and the total unsigned area in the region between the curves in δ_2 and α_2 are sufficiently small. Using appropriate energy filtration we then have

$$\Gamma(\mathbf{x} \otimes \mathbf{y}) = I_1(\mathbf{x}) \times I_2(\mathbf{y}) + \epsilon(\mathbf{x} \otimes \mathbf{y})$$

where $\epsilon(\mathbf{x} \otimes \mathbf{y})$ consists of terms of smaller energy than $I_1(\mathbf{x}) \times I_2(\mathbf{y})$. This implies that Γ is a homotopy equivalence of filtered chain complexes. $\square$

Returning to Lemma 9.4, let us drop the assumption that the component of $\mathfrak{R}^-(\tau^2)$ containing the suture σ_2 as a boundary curve has non-trivial genus. One may then encounter Whitney disks $\phi \in \pi_2(\mathbf{x} \times \Theta_{\alpha_2\beta_2}, \mathbf{y} \times \Theta)$ with $\mathbf{x} \neq \mathbf{y}$ and $\Theta \neq \Theta_{\alpha_2\beta_2}$ which have non-trivial contribution to $\partial(\mathbf{x} \times \Theta_{\alpha_2\beta_2})$. In fact, in the decomposition $\phi = \phi_1 \# \phi_2$, if the coefficient of ϕ at z_0 is large, no restriction on $\mu(\phi_1)$ and $\mu(\phi_2)$ may be made. In order to prevent the contribution of such disks, one would then need to add the relation $\mathfrak{u}_0 . \frac{\mathfrak{u}_0^-(2)}{\mathfrak{v}_2} = 0$ to the ring of coefficients. This would then force the relation $\mathfrak{v}_1 = 0$. Similarly, if the connected component of $\mathfrak{R}^+(\tau^1)$ containing the suture σ_1 as a boundary curve has positive genus, we are forced to include the relation $\mathfrak{v}_2 = 0$ in the ring of coefficients. Define

$$\delta : \mathbb{Z} \to \{0, 1\} \qquad \delta(i) := \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i \neq 0 \end{cases}.$$

Motivated by the above observation, in the situation of definition 9.2 we may thus set the algebra $\mathbb{A} = \mathbb{A}(\tau^1, \tau^2; \sigma_1, \sigma_2)$ to be

$$\mathbb{A} := \frac{(\mathbb{A}_1 \otimes \mathbb{A}_2)[\mathfrak{u}_0]}{\left\langle \mathfrak{v}_1 - \frac{\mathfrak{u}_0 \mathfrak{u}_0^-(2)}{\mathfrak{v}_2},\ \delta\left(g_0^-(2)\right) \mathfrak{v}_1 \right\rangle + \left\langle \mathfrak{v}_2 - \frac{\mathfrak{u}_0 \mathfrak{u}_0^+(1)}{\mathfrak{v}_1},\ \delta\left(g_0^+(1)\right) \mathfrak{v}_2 \right\rangle}$$

The rest of the argument of Lemma 9.4 and Proposition 9.3 may then be extended to prove the following generalization.

Theorem 9.5. *Suppose that (X^i,τ^i) is a weakly balanced sutured manifold with a distinguished suture σ_i and let (X,τ) denote the weakly balanced sutured manifold obtained as the connected sum of (X^i,τ^i) along σ_i, $i=1,2$. Fix the* Spin^c *structures $\mathfrak{s}^i\in\mathrm{Spin}^c(\overline{X^i})$ for $i=1,2$. The filtered chain homotopy type of the two filtered $(\mathbb{A},\mathbb{H})$ chain complexes*

$$\mathrm{CF}(X,\tau,\mathfrak{s}^1\#\mathfrak{s}^2;\mathbb{A})\quad \textit{and}\quad \mathrm{CF}(X^1,\tau^1,\mathfrak{s}^1;\mathbb{A})\otimes_{\mathbb{A}}\mathrm{CF}(X^2,\tau^2,\mathfrak{s}^2;\mathbb{A})$$

are the same, where $\mathbb{A}=\mathbb{A}(\tau^1,\tau^2;\sigma_1,\sigma_2)$.

9.4. Examples

Below, we will discuss a category of examples using Proposition 9.3.

Example 9.6. Consider a bipartite (2-colorable) graph G, and fix a coloring of vertices $V(G)$ of G by $+$ and $-$, i.e. $V(G)=V^-(G)\coprod V^+(G)$. Adding empty vertices (i.e. vertices with no edges attached to them) we may assume that the number of $+$ and $-$ vertices are equal. Similar to part (d) of Example 3.9, associated with any embedding of $\imath:G\to X$ of G inside a closed 3-manifold X and any coloring as above, one may construct a weakly balanced sutured manifold as follows. Let $X(G)=X-\mathrm{nd}(G)$ and suppose that $\tau(G)$ is a union of simple closed curves on the boundary of $X(G)$ which correspond to meridians of the edges of the G. If the edges $e\in E(G)$ are oriented so that they start from $V^-(G)$ and end at $V^+(G)$, the curves in $\tau(G)$ may be oriented accordingly. The resulting weakly balanced sutured manifold will be denoted by $(X(G),\tau(G))$. The choice of the coloring of the vertices of G and the embedding $\imath:G\to X$ are suppressed from the notation. The corresponding algebra will be denoted by $\mathbb{A}_G$. The suture associated with an edge $e\in E(G)$ will be denoted by τ_e and the variable associated with τ_e will be denoted by $\mathfrak{u}_e$.

Let us assume that G is a tree. Thus $\partial X(G)$ may be identified as the 2-sphere. Fix a degree one vertex v of G, which is connected to $G-v$ by an edge e which connects v to another vertex w. Alternatively, the weakly balanced sutured manifold $(X(G),\tau(G))$ may be constructed as the connected sum of $(S^3(G),\tau(G))$ and $(X(1),\tau(1))$, where 1 denotes the graph with two vertices and a single edge connecting them. The connected sum is done along $e\in E(G)$ and the single edge of the graph 1.

We would like to use Proposition 9.3 to compute $\mathrm{CF}(X(G),\tau(G),\mathfrak{s})$. Suppose that $v\in V^+(G)$. Let $(X^1,\tau^1)=(S^3(G),\tau(G))$ and $(X^2,\tau^2)=(X(1),\tau(1))$. We then have $\mathbb{A}\simeq\mathbb{A}_G$. Note that $\mathrm{Spin}^c(S^3(G),\tau(G))$ is an affine space over

$$\mathrm{H}^2(S^3(G),\partial S^3(G);\mathbb{Z})=\mathrm{H}^2(D^3,S^2;\mathbb{Z})=0.$$

Thus, there is a unique relative Spin^c class in $\mathrm{Spin}^c(S^3(G),\tau(G))$, and

$$\begin{aligned}&\mathrm{Spin}^c\left(X(G),\tau(G)\right)=\mathrm{Spin}^c\left(X(1),\tau(1)\right)\quad\text{and}\\&\mathrm{Spin}^c\left(\overline{X(G)}\right)=\mathrm{Spin}^c\left(\overline{X(1)}\right).\end{aligned}$$

By Proposition 9.3, for $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X(1)})$ the chain complex

$$\mathrm{CF}(X(G),\tau(G),\mathfrak{s})$$

is filtered chain homotopic to

$$\mathrm{CF}(X(1),\tau(1),\mathfrak{s};\mathbb{A}_G)\otimes_{\mathbb{A}_G}\mathrm{CF}(S^3(G),\tau(G),\mathfrak{s}_0),$$

where $\mathfrak{s}_0$ denotes the unique Spin^c class in $\mathrm{Spin}^c(\overline{S^3(G)})$, and $\mathbb{A}_G$ is a test ring for $\mathbb{A}_1=\mathbb{Z}[\mathfrak{u}]$ via the algebra homomorphism sending $\mathfrak{u}$ to $\mathfrak{u}_e$. Consequently, $\mathrm{CF}(X(G),\tau(G),\mathfrak{s})$ only depends on $\mathrm{CF}(X(1),\tau(1),\mathfrak{s})=\mathrm{CF}^-(X,\mathfrak{s})$ and the tree G.

Example 9.7. Let G be a connected path with four vertices inside S^3. We may color its vertices with $+$ and $-$ and the number of $+$ vertices is equal to the number of $-$ vertices. The algebra associated to the boundary of $(S^3(G),\tau(G))$ is

$$\mathbb{A}=\frac{\mathbb{Z}[\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3]}{\langle\mathfrak{u}_1\mathfrak{u}_2+\mathfrak{u}_3=\mathfrak{u}_1+\mathfrak{u}_2\mathfrak{u}_3\rangle}$$

A Heegaard diagram $H=(S^2,\alpha,\beta,\mathbf{z}=\{z_1,z_2,z_3\})$ corresponding to the weakly balanced sutured manifold $(S^3(G),\tau(G))$ may be constructed as follows. The curves α and β are simple closed curves bounding disks A and B on S^2 and intersecting each other in two points $\{x,y\}$. Furthermore, we assume

$$z_2 \in A \cap B, \quad z_1 \in A - B, \quad z_3 \in B - A.$$

We then have

$$\begin{aligned}&\mathrm{CF}(S^2, \alpha, \beta, \mathbf{z}) = \mathbb{A}\langle x, y\rangle\\ &\partial x = (\mathfrak{u}_1 - \mathfrak{u}_3)y, \quad \partial y = (\mathfrak{u}_2 - 1)x.\end{aligned}$$

Thus the chain homotopy type of $\mathrm{CF}(S^3(G), \tau(G), \mathfrak{s}_0)$ is non-trivial, while $\mathrm{SFH}(S^3(G), \tau(G), \mathfrak{s}_0) = 0$.

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Department of Mathematics, Sharif University of Technology
P.O. Box 11155-9415, Tehran, Iran
E-mail address: akram.alishahi@gmail.com

School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran
E-mail address: eaman@ipm.ir

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