

Localization and the link Floer homology of doubly-periodic knots

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A knot $\tilde{K} \subset S^3$ is q -periodic if there is a \mathbb{Z}_q -action preserving \tilde{K} whose fixed set is an unknot \tilde{U} . The quotient of \tilde{K} under the action is a second knot K . We construct equivariant Heegaard diagrams for q -periodic knots, and show that Murasugi’s classical condition on the Alexander polynomials of periodic knots is a quick consequence of these diagrams. For \tilde{K} a two-periodic knot, we show there is a spectral sequence whose E^1 page is $\widehat{HFL}(S^3, \tilde{K} \cup \tilde{U}) \otimes V^{\otimes(2n-1)} \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $(\widehat{HFL}(S^3, K \cup U) \otimes V^{\otimes(n-1)}) \otimes \mathbb{Z}_2((\theta))$, as $\mathbb{Z}_2((\theta))$ -modules, and a related spectral sequence whose E^1 page is $(\widehat{HFK}(S^3, \tilde{K}) \otimes V^{\otimes(2n-1)} \otimes W) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $(\widehat{HFK}(S^3, K) \otimes V^{\otimes(n-1)} \otimes W) \otimes \mathbb{Z}_2((\theta))$. As a consequence, we use these spectral sequences to recover a classical lower bound of Edmonds on the genus of \tilde{K} , along with a weak version of a classical fibredness result of Edmonds and Livingston. We give an example of a knot \tilde{K} which is not obstructed from being two-periodic with a particular quotient knot K by Edmonds’ and Murasugi’s conditions, but for which our spectral sequence cannot exist.

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1. Introduction

We say $\tilde{K} \subset S^3$ is a *periodic knot* if there is a \mathbb{Z}_q -action on (S^3, \tilde{K}) which preserves \tilde{K} and whose fixed set is an unknot \tilde{U} disjoint from \tilde{K} . Let τ be a generator for the action. An important special case is $\tau^2 = 1$, in which case \tilde{K} is said to be *doubly-periodic*.

The quotient of (S^3, \tilde{K}) under the group action is a second knot (S^3, K) such that the map $(S^3, \tilde{K}) \rightarrow (S^3, K)$ is a q -fold branched cover over U , the unknot which is the image of \tilde{U} in the quotient. The knot K is said to be the q -fold *quotient knot* of \tilde{K} .

Link Floer homology is an invariant of a link in S^3 introduced by Ozsváth and Szabó [27] as a generalization of the knot Floer homology developed in 2003 by Ozsváth and Szabó [23] and independently by Rasmussen [30]. The theory associates to (S^3, L) a bigraded \mathbb{Z}_2 -vector space $\widehat{HFL}(S^3, L)$ which

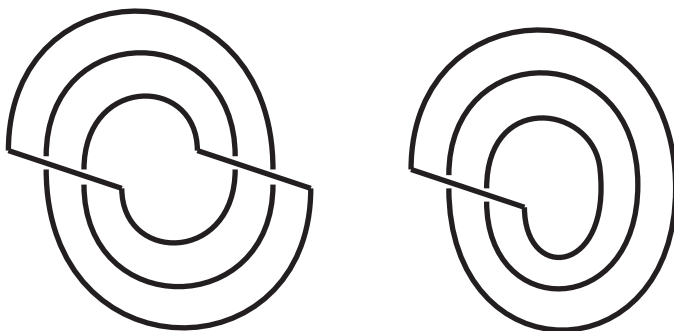


Figure 1. A doubly-periodic diagram for the trefoil, and its quotient knot (an unknot) under the \mathbb{Z}_2 -action.

arises as the homology of the Floer chain complex of two Lagrangian tori in the symmetric product of a punctured Heegaard surface for (S^3, L) . The graded Euler characteristic of link Floer cohomology is the multivariable Alexander polynomial of the knot multiplied by certain standard factors [27]; indeed, the theory categorifies the Thurston norm of the knot complement [28].

The purpose of this paper is to use localization theorems of Seidel and Smith to analyze the link Floer homology of a periodic (usually doubly-periodic) knot together with its axis compared to the link Floer homology of the quotient knot together with the axis. We will first construct Heegaard diagrams for $(S^3, \tilde{K} \cup U)$ which are preserved by the action of \mathbb{Z}_q and whose quotients under the action are Heegaard diagrams for (S^3, K) . These periodic Heegaard diagrams will allow us to give a simple Heegaard Floer reproof of one of Murasugi’s conditions for the Alexander polynomial of a periodic knot in the case that $q = p^r$ for some prime p . Let $\lambda = \text{lk}(\tilde{K}, \tilde{U}) = \text{lk}(K, U)$. Since \tilde{K} is a knot (not a link), it follows that λ is coprime to the periodicity q .

Theorem 1.1. [19, Corollary 1] $\Delta_{\tilde{K}}(t) \equiv t^{\pm i} (1+t+\dots+t^{\lambda-1})^{q-1} (\Delta_K(t))^q$ modulo p .

Restricting to the case that \tilde{K} is doubly periodic, we will proceed to prove the following localization theorem. Let $\mathbb{Z}_2((\theta)) = \mathbb{Z}_2[[\theta]](\theta^{-1})$. Let V and W be two-dimensional vector spaces over \mathbb{Z}_2 , which will later be distinguished by their gradings. Let A_1 be the Alexander grading relative to the knot and A_2 be the Alexander grading relative to the unknotted axis (these gradings will be defined in Section 2).

Theorem 1.2. *There is an integer n_1 less than half the number of crossings of a periodic diagram D for \tilde{K} such that there is a spectral sequence whose E^1 page is $(\widehat{HFL}(S^3, \tilde{K} \cup \tilde{U}) \otimes V^{\otimes(2n_1-1)}) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $(\widehat{HFL}(S^3, K \cup U) \otimes V^{\otimes(n_1-1)}) \otimes \mathbb{Z}_2((\theta))$ as $\mathbb{Z}_2((\theta))$ -modules. This sequence splits along Alexander multigradings, and carries the grading $(A_1, A_2) = (2a + \frac{1}{2}, b)$ to $(A_1, A_2) = (a + \frac{1}{2}, b)$ for all $a, b \in \mathbb{Z}$. In any grading (A_1, A_2) which cannot be written as $(2a + \frac{1}{2}, b)$, the last page of the spectral sequence is empty.*

We may also reduce the spectral sequence of Theorem 1.2 to contain only the information of knot Floer homology of \tilde{K} and K .

Theorem 1.3. *There is an integer n_1 less than half the number of crossings on a periodic diagram D for \tilde{K} such that there is a spectral sequence whose E^1 page is $(\widehat{HFK}(S^3, \tilde{K}) \otimes V^{\otimes(2n_1-1)} \otimes W) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $(\widehat{HFK}(S^3, K) \otimes V^{\otimes(n_1-1)} \otimes W) \otimes \mathbb{Z}_2((\theta))$. This sequence splits along Alexander gradings, and carries the Alexander grading $A_1 = 2a + \frac{1-\lambda}{2}$ to $A_1 = a + \frac{1-\lambda}{2}$ for all integers $s \in \mathbb{Z}$. In any grading A_1 which cannot be written as $2a + \frac{1-\lambda}{2}$, the last page of the spectral sequence is empty.*

Analyzing the behavior of the Alexander gradings in these spectral sequences yields some geometric consequences. In particular, it is worth noting the following rank inequalities implied by Theorems 1.2 and 1.3. In Section 2, we will construct a Heegaard diagram \mathcal{D} for $K \cup U$ with equivariant lift a Heegaard diagram $\tilde{\mathcal{D}}$ for $\tilde{K} \cup \tilde{U}$. Then Theorem 1.2 can be rephrased as asserting the existence of a spectral sequence from the theory $\widehat{HFL}(\tilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta))$ to $\widehat{HFL}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta))$, and Theorem 1.3 can be rephrased as a spectral sequence from $\widehat{HFK}(\tilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta))$ to $\widehat{HFK}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta))$. We have the following rank inequalities.

Corollary 1.4. *For all $a, b \in \mathbb{Z}$, there is a rank inequality*

$$rk \left(\widehat{HFL} \left(\tilde{\mathcal{D}}, \left(\frac{1}{2} + 2a, b \right) \right) \right) \geq rk \left(\widehat{HFL} \left(\mathcal{D}, \left(\frac{1}{2} + a, b \right) \right) \right).$$

Corollary 1.5. *For all $a \in \mathbb{Z}$, there is a rank inequality*

$$rk \left(\widehat{HFK} \left(\tilde{\mathcal{D}}, 2a + \frac{1-\lambda}{2} \right) \right) \geq rk \left(\widehat{HFK} \left(\mathcal{D}, a + \frac{1-\lambda}{2} \right) \right).$$

In particular, if Edmonds' condition is sharp, there is a rank inequality

$$\widehat{HFK}(S^3, \tilde{K}, g(\tilde{K})) \geq \widehat{HFK}(S^3, K, g(K)).$$

These relationships yield a reproof of a classical result, proved by Alan Edmonds using minimal surface theory.

Corollary 1.6. *[4, Theorem 4] Let \tilde{K} be a doubly-periodic knot in S^3 and K be its quotient knot. Then*

$$g(\tilde{K}) \geq 2g(K) + \frac{\lambda - 1}{2}.$$

We will also observe from Corollary 1.5 a proof of the following corollary.

Corollary 1.7. *Let \tilde{K} be a doubly-periodic knot in S^3 and K its quotient knot. If Edmonds' condition is sharp and \tilde{K} is fibered, K is fibered.*

This is a weaker version of the following theorem, proved by Edmonds and Livingston.

Theorem 1.8. *[5, Prop. 6.1] Let \tilde{K} be a doubly-periodic knot in S^3 and K its quotient knot. If \tilde{K} is fibered, K is fibered.*

In some cases, Corollary 1.5 may provide an obstruction to a knot being two-periodic with a particular quotient knot, even if Corollary 1.1 and Corollary 1.6 are satisfied. In Section 3.3 we will use it to show the following.

Example 1.9. Let \tilde{K} be the connect sum of the right-handed trefoil and the Kinoshita-Terasaka knot and K be the positive untwisted Whitehead double of the figure eight knot. The Alexander polynomials and genera of \tilde{K} and K satisfy the conditions of Corollaries 1.1 and 1.6, but the spectral sequence of Theorem 1.3 cannot exist.

The spectral sequences of Theorems 1.2 and 1.3 are analogs of the spectral sequence for the knot Floer homology of double branched covers of knots in S^3 constructed in [14]; their construction requires only slightly more technical complexity. As in that paper, the key technical tool in the proofs of Theorems 1.2 and 1.3 is a result of Seidel and Smith concerning equivariant Floer cohomology. Let M be an exact symplectic manifold, convex at infinity, containing exact Lagrangians L_0 and L_1 and equipped with an involution τ preserving (M, L_0, L_1) . Let $(M^{\text{inv}}, L_0^{\text{inv}}, L_1^{\text{inv}})$ be the submanifolds

of each space fixed by τ . Then under certain stringent conditions on the normal bundle $N(M^{\text{inv}})$ of M^{inv} in M , there is a rank inequality between the Floer cohomology $HF(L_0, L_1)$ of the two Lagrangians L_0 and L_1 in M and the Floer cohomology $HF(L_0^{\text{inv}}, L_1^{\text{inv}})$ of L_0^{inv} and L_1^{inv} in M^{inv} . More precisely, they consider the normal bundle $N(M^{\text{inv}})$ to M^{inv} in M and its pullback $\Upsilon(M^{\text{inv}})$ to $M^{\text{inv}} \times [0, 1]$. We ask that M satisfy a K -theoretic condition called *stable normal triviality* relative to two Lagrangian subbundles over $L_0^{\text{inv}} \times \{0\}$ and $L_1^{\text{inv}} \times \{1\}$. Seidel and Smith prove the following.

Theorem 1.10. *[32, Section 3f] If $\Upsilon(M^{\text{inv}})$ carries a stable normal trivialization, there is a spectral sequence whose E^1 page is $HF(L_0, L_1) \otimes \mathbb{Z}_2((\theta))$ and whose E^∞ page is isomorphic to $HF(L_0^{\text{inv}}, L_1^{\text{inv}}) \otimes \mathbb{Z}_2((\theta))$ as $\mathbb{Z}_2((\theta))$ modules.*

This paper is organized as follows: In Section 2 we recall the construction and important properties of link Floer homology. In Section 3 we construct equivariant Heegaard diagrams for periodic knots, prove Theorem 1.1 from these diagrams, and explain how Corollaries 1.6 and 1.7 follow from Theorems 1.2 and 1.3. We also explain Example 1.9. In Section 4 we briefly review Seidel and Smith's localization theory for Floer cohomology, and describe the symmetric products to which we will apply it. In Section 5 we provide a description of the homotopy type and cohomology of the most general of these symmetric products (which contains the symmetric products used to compute knot and link Floer homology as submanifolds). In Section 6 we give a proof that this general symmetric product carries a stable normal trivialization, which will imply that the symmetric products used in the computation of knot and link Floer homology do as well. In Section 7 we compute the spectral sequences of Theorems 1.2 and 1.3 for the unknot and the trefoil as doubly-periodic knots as examples.

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2. Heegaard Floer homology theories

We pause for some discussion of link Floer homology in the three sphere, first defined by Ozsváth and Szabó in [27]. Our purpose is primarily to give notation and to emphasize the points which are important to the proofs in this paper; for a complete treatment, we refer the reader to [27]. All work is done over \mathbb{Z}_2 .

Definition 2.1. A *multipointed Heegaard diagram* $\mathcal{D} = (S, \alpha, \beta, \mathbf{w}, \mathbf{z})$ consists of the following data.

- An oriented surface S of genus g .
- Two sets of basepoints $\mathbf{w} = (w_1, \dots, w_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$.
- Two sets of closed embedded curves $\alpha = \{\alpha_1, \dots, \alpha_{g+n-1}\}$ and $\beta = \{\beta_1, \dots, \beta_{g+n-1}\}$ such that each of α and β spans a g -dimensional subspace of $H_1(S)$, $\alpha_i \cap \alpha_j = \emptyset = \beta_i \cap \beta_j$ for $i \neq j$, each α_i and β_j intersect transversely, and each component of $S - \bigcup \alpha_i$ and of $S - \bigcup \beta_i$ contain exactly one point of \mathbf{w} and one point of \mathbf{z} .

From any Heegaard diagram \mathcal{D} , we may construct a unique 3-manifold Y containing a link L ; in this paper we are only interested in the case that $Y = S^3$. In particular, to recover the link, we connect the w basepoints to the z basepoints in the complement of the curves β_i , then connect the z basepoints to the w basepoints in the complement of the curves α_i , letting the first set of arcs overcross the second. We number our basepoints such that if $L = K_1 \cup \dots \cup K_\ell$ is the link produced, there are n_j pairs of basepoints on K_j , and there are integers $0 = k_0 < k_1 < \dots < k_\ell = n$ with $k_j - k_{j-1} = n_j$ such that $w_{k_{j-1}+1}, \dots, w_{k_j}, z_{k_{j-1}+1}, \dots, z_{k_j}$ are the basepoints on K_j . (In the examples we are actually interested in, we will have $L = K_1 \cup K_2$ with n_1 pairs of basepoints on K_1 and a single pair of basepoints on K_2 , so the notation will not be too bad.)

We impose one further technical condition on our Heegaard diagram. Call the components of $S - \bigcup \alpha_i - \bigcup \beta_i$ the *elementary regions* of the Heegaard diagram.

Definition 2.2. A *periodic domain* P on $S \setminus \mathbf{w}$ is a linear combination of elementary regions whose boundary may be expressed as a linear combination of the α and β curves.

We say that D is *weakly admissible* if every periodic domain on S has both positive and negative local multiplicities, and require that any Heegaard diagram we use to compute link Floer homology have this property.

The link Floer homology $\widehat{HFL}(Y, L)$ is the Lagrangian Floer cohomology of the two Lagrangian tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+n-1}$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+n-1}$ inside the symmetric product $\text{Sym}^{g+n-1}(S \setminus (\mathbf{w} \cup \mathbf{z}))$. Therefore the chain complex $\widehat{CFL}(\mathcal{D})$ for knot Floer homology is generated by the finite set of intersection points of \mathbb{T}_α and \mathbb{T}_β . (More concretely, a generator of $\widehat{CFL}(\mathcal{D})$ is a point $\mathbf{x} = (x_1 \cdots x_{g+n-1}) \in \text{Sym}^{g+n-1}(S)$ such that each α or β curve contains a single x_i .) The differential ∂ counts holomorphic representatives of homotopy classes of Whitney disks $[\phi] \in \pi_2(\mathbf{x}, \mathbf{y})$ whose image lies in the symmetric product of the punctured sphere. Here a Whitney disk is a map $\phi : B_1(0) \rightarrow \text{Sym}^{g+n-1}(S)$ which maps the left half of the boundary of the unit disk $B_1(0) \cup \mathbb{C}$ to \mathbb{T}_β and the right half to \mathbb{T}_α , and has $\phi(-i) = \mathbf{x}$, $\phi(i) = \mathbf{y}$.

2.1. The Maslov index and grading

Recall that if $\phi : B_1(0) \rightarrow \text{Sym}^{g+n-1}(S)$ is a Whitney disk, we can associate to ϕ its shadow $D = \sum a_i D_i$ on the Heegaard diagram, where D_i are the closures of the elementary regions of \mathcal{D} and a_i is the algebraic multiplicity of the intersection of the holomorphic submanifold $V_{x_i} = \{x_i\} \times \text{Sym}^{g+n-2}(S)$ with $\phi(B_1(0))$ for any interior point x_i of D_i . The boundary of D consists of α arcs from points of \mathbf{x} to points of \mathbf{y} and β arcs from points of \mathbf{y} to points of \mathbf{x} . If D_i contains a basepoint z_j , we let $a_i = n_{z_j}(\phi)$ be the algebraic intersection number of $z_j \times \text{Sym}^{g+n-2}(S)$ with the image of ϕ .

Given $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$, we define the *Maslov index* as follows. Recall that $\phi : D^2 \rightarrow \text{Sym}^{g+n-1}(S)$ maps the portion of the boundary of the unit disk D^2 in the right half of the complex plane $\mathbb{C} = \{u + iv : u, v \in \mathbb{R}\}$ to a loop in \mathbb{T}_α and the portion of the boundary in the left half to \mathbb{T}_β . Choose a constant trivialization of the orientable real vector bundle $\phi^*(T(\mathbb{T}_\alpha))$ over $\partial D|_{v \geq 0}$. We may tensor this real trivialization with \mathbb{C} and extend to a complex trivialization of $\phi^*(T(\text{Sym}^{g+n-1}(S)))$ by pushing across the disk linearly. Relative to this trivialization, the real bundle $\phi^*(T(\mathbb{T}_\beta))$ over $\partial D|_{v \geq 0}$ induces a loop of real subspaces of $\mathbb{C}^{g+n-1} = \phi^*(T(\text{Sym}^{g+n-1}(S)))$. The winding number of this loop is the Maslov index of the map ϕ . Notice that we could also have

used $\phi^*(J(T(\mathbb{T}_\alpha)))$ and $\phi^*(T(\mathbb{T}_\beta))$, where J is the complex structure on the vector bundle $T\text{Sym}^{g+n-1}(S)$, and obtained the same number.

The Maslov index $\mu(\phi)$ can equivalently be computed using the associated domain $\Sigma a_i D_i$ in a formula of Lipshitz’s [17, Proposition 4.2]. For each domain D_i , let $e(D_i)$ be the Euler measure of D_i . In particular, if D_i is a disk with $2k$ corners, $e(D_i) = 1 - \frac{k}{2}$. Let $p_{\mathbf{x}}(D)$ be the sum of the average of the multiplicities of D at the four corners of each point in \mathbf{x} and likewise for $p_{\mathbf{y}}(D)$. Then the Maslov index is

$$(2.1) \quad \mu(\phi) = \sum a_i e(D_i) + p_{\mathbf{x}}(D) + p_{\mathbf{y}}(D).$$

In the case that $[\phi] \in \pi_2(\mathbf{x}, \mathbf{x})$ is a domain from \mathbf{x} to itself, and therefore a periodic domain, we have the following alternate interpretation of the Maslov index, which will be very important to the proof of the main theorems of this paper. Because $\phi(i) = \phi(-i)$, we see that ϕ sends $\partial D|_{v \geq 0}$ to a loop in \mathbb{T}_α and $\partial D|_{v \leq 0}$ to a loop in \mathbb{T}_β . Therefore we may replace ϕ by a map $\widehat{\phi}: S^2 \times I \rightarrow \text{Sym}^{g+n-1}(S)$ which maps $S^1 \times \{1\}$ to \mathbb{T}_α and $S^1 \times \{0\}$ to \mathbb{T}_β . We then consider the complex pullback bundle $E = \widehat{\phi}^*(T(\text{Sym}^{g+n-1}(S)))$ to $S^2 \times I$ and the totally real subbundles $\widehat{\phi}|_{S^1 \times \{1\}}^*(T(\mathbb{T}_\alpha))$ of $E|_{S^1 \times \{1\}}$ and $\widehat{\phi}|_{S^1 \times \{0\}}^*(T(\mathbb{T}_\beta))$ of $E|_{S^1 \times \{0\}}$. The Maslov index is still calculated by trivializing $\widehat{\phi}|_{S^1 \times \{1\}}^*(T(\mathbb{T}_\alpha))$, complexifying, and computing the winding number of the loop of real-half dimensional subspaces in \mathbb{C}^{g+n-1} represented by $\widehat{\phi}|_{S^1 \times \{0\}}^*(T(\mathbb{T}_\beta))$ with respect to the trivialization. This number classifies the bundle in the following way: complex vector bundles over the annulus whose restriction to the boundary of the annulus carries a canonical real subbundle are in bijection with maps $[(S^1 \times I, \partial(S^1 \times I)), (BU, BO)] = \langle (S^1 \times I, \partial(S^1 \times I)), (BU, BO) \rangle \cong \mathbb{Z}$, where the map to \mathbb{Z} is the Maslov index $\mu(\phi) = \mu(\widehat{\phi})$ [18, Theorem C.3.7].

Now let us look at the bundle E over $S^1 \times I$ and its real subbundles over the boundary components of $S^1 \times I$ from a slightly different perspective. The real bundles $\widehat{\phi}|_{S^1 \times \{1\}}^*(T(\mathbb{T}_\alpha))$ and $\widehat{\phi}|_{S^1 \times \{0\}}^*(T(\mathbb{T}_\beta))$ are orientable, hence trivialisable over the circle, so we may choose real trivializations and tensor with \mathbb{C} to obtain a complex trivialization of $E|_{\partial(S^1 \times I)}$. We can now regard E as a relative vector bundle E_{rel} over $(S^1 \times I, \partial(S^1 \times I))$, and consider its relative first Chern class $c_1(E|_{\text{rel}})$. Equivalently, we may use this trivialization to construct a vector bundle \widetilde{E} over $(S^1 \times I)/\partial(S^1 \times I) \cong S^2$ such that the pullback $q^*(\widetilde{E})$ along the quotient map is E . Then $c_1(\widetilde{E})$ is the relative first Chern class $c_1(E|_{\text{rel}})$ under the identification $H^1(S^2) \simeq H^1(S^1 \times I, \partial(S^1 \times I))$. Moreover, isomorphism classes of vector bundles over

S^2 are in bijection with homotopy classes of maps $[S^2, BU] \cong \langle S^2, BU \rangle = \pi_2(BU) \cong \mathbb{Z}$, where the identification with \mathbb{Z} is via the first Chern class. Using the homotopy long exact sequence of the pair (BU, BO) , we observe the following relationship between μ and c_1 .

$$\begin{array}{ccccc} \pi_2(BU) & \longrightarrow & \pi_2(BU, BO) & \longrightarrow & \pi_1(BO) \\ \downarrow c_1 & & \downarrow \mu & & \downarrow \\ \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 \end{array}$$

Therefore the Maslov index $\mu(\phi)$ is twice the relative first Chern class $c_1(E|_{\text{rel}})$.

The complex $\widehat{CFL}(\mathcal{D})$ carries a (relative, for our purposes) homological grading called the Maslov grading $M(\mathbf{x})$ which uses the Maslov index and takes values in \mathbb{Z} . Suppose \mathbf{x} and \mathbf{y} are connected by a Whitney disk ϕ . Then the relative Maslov grading is determined by

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2 \sum_i n_{w_i}(\phi).$$

A formula for the absolute Maslov grading may be found in [26, Theorem 3.3], but will not be needed in this paper.

2.2. The Alexander grading and relative spin^c structures

The complex $\widehat{CFL}(\mathcal{D})$ also carries an Alexander multigrading $\mathbf{A} = (A_1, \dots, A_\ell)$. This multigrading takes values in an affine lattice \mathbb{H} over $H_1(S^3 - L; \mathbb{Z}) \cong H_1(L)$. Recall that $H_1(S^3 - L; \mathbb{Z}) \cong \mathbb{Z}^\ell$ generated by the homology classes of meridians μ_{K_j} of the component knots K_j of L . Define the lattice \mathbb{H} to consist of elements

$$\sum_{i=1}^{\ell} a_i [\mu_{K_i}]$$

where $a_i \in \mathbb{Q}$ satisfies the property that $2a_i + \text{lk}(K_i, L - K_i)$ is an even integer. To determine the relative Alexander multigrading, recall that the basepoints $w_{k_{j-1}+1}, \dots, w_{k_j}, z_{k_{j-1}+1}, \dots, z_{k_j}$ lie on K_j (with the convention that $k_0 = 0$). Then once again if ϕ is a Whitney disk connecting \mathbf{x} and \mathbf{y} ,

$$A_j(\mathbf{x}) - A_j(\mathbf{y}) = \sum_{i=k_{j-1}+1}^{k_j} n_{z_i}(\phi) - \sum_{i=k_{j-1}+1}^{k_j} n_{w_i}(\phi).$$

We can also see the relative Alexander multigrading as a linking number. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we find paths

$$a: [0, 1] \rightarrow \mathbb{T}_\alpha \quad \text{and} \quad b: [0, 1] \rightarrow \mathbb{T}_\beta$$

such that $\partial a = \partial b = \mathbf{x} - \mathbf{y}$. (For example, $a \cup b$ may be the boundary of a Whitney disk ϕ from \mathbf{y} to \mathbf{x} .) View these paths as as one-chains on $S \setminus (\mathbf{w} \cup \mathbf{z})$. Since attaching one- and two-handles to the α and β curves on S and filling in three-balls at the basepoints yields Y , we obtain a trivial one-cycle in Y . Indeed, a domain D on \mathcal{D} is the shadow of a Whitney disk if and only if its boundary, viewed as a cycle on S , descends to a trivial cycle on Y . However, if we attach α and β circles to $S \setminus (\mathbf{w} \cup \mathbf{z})$ (and no three balls) we obtain the manifold $Y - L$ and a one cycle $\epsilon(\mathbf{x}, \mathbf{y})$ in $Y - L$.

$$\epsilon_{\mathbf{w}, \mathbf{z}}: (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \rightarrow H_1(Y - L; \mathbb{Z})$$

We obtain the following lemma (which has only been very slightly adjusted from the original to account for the possibility of multiple pairs of basepoints on a link component).

Lemma 2.3. [27, Lemma 3.10] *An oriented ℓ -component link L in Y induces a map*

$$\Pi: H_1(Y - L) \rightarrow \mathbb{Z}^\ell$$

where $\Pi_i(\gamma)$ is the linking number of γ with the i th component K_i of L . In particular, for $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we have

$$\Pi_i(\epsilon_{\mathbf{w}, \mathbf{z}}(\mathbf{x}, \mathbf{y})) = \sum_{n_{i-1}+1}^{n_i} n_{z_j}(\phi) - \sum_{n_{i-1}+1}^{n_i} n_{z_j}(\phi).$$

Proof. The proof is nearly identical to the original: ϕ induces a nulhomology of $\epsilon(\mathbf{x}, \mathbf{y})$, which meets the i th component K_i of L with intersection number $\sum_{n_{i-1}}^{n_i} n_{z_i}(\phi) - \sum_{n_{i-1}}^{n_i} n_{w_i}(\phi)$. □

There is another less practical, but more intrinsic, method of thinking about Alexander multigradings via *relative spin^c structures*. Let $(N, \partial N)$ be a manifold with toroidal boundary $T_1 \cup \dots \cup T_\ell$. The boundary of a torus contains, up to isotopy, a canonical nowhere-vanishing vector field preserved by translation. Therefore we consider nowhere-vanishing vector fields v on N which restrict to the canonical vector field on each boundary component

of N . In this case vector fields v and v' are homologous if they are homotopic on $N - B$ for B a ball in N° . The set of such homotopy classes is the set of relative spin^c structures, and is an affine space for the relative homology $H^2(N, \partial N)$. This space is denoted $\underline{\text{Spin}}^c(N, \partial N)$. If v is a nowhere-vanishing vector field on N with canonical restriction to the toroidal boundary ∂N , the restriction of the field of two-planes v^\perp has a canonical trivialization along ∂N . Therefore there is a well-defined notion of the relative first Chern class $c_1(\underline{s})$ of a relative spin^c structure.

There is a canonical way of associating to every generator \mathbf{x} in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ a relative spin^c structure $\underline{s}_{\mathbf{w}, \mathbf{z}}(\mathbf{x})$ on $(S^3 - \nu(L))$ by modifying the gradient vector field of the Morse function on S^3 corresponding to the Heegaard diagram near flowlines corresponding to \mathbf{x} . This is described in detail in [27, Section 3.3]. Then the relationship between this relative spin^c structure and the Alexander grading as defined previously is

$$c_1(\underline{s}_{\mathbf{w}, \mathbf{z}}(\mathbf{x})) + \sum_{i=1}^{\ell} PD[\mu_{K_i}] = 2PD(A_1[\mu_1] + \cdots + A_\ell[\mu_\ell]).$$

Finally, before moving on, observe that if \mathcal{D} is a Heegaard diagram for (Y, L) , then there is an identification

$$\begin{aligned} H_2(Y - \nu(L), \partial(Y - \nu(L))) &= H_1(Y - L) \\ &\cong \frac{H_1(S \setminus \{\mathbf{w} \cup \mathbf{z}\})}{\langle [\alpha_1], \dots, [\alpha_{g_n-1}], [\beta_1], \dots, [\beta_{g+n-1}] \rangle} \cong \frac{H_1(\text{Sym}^{g+n-1}(S \setminus \{\mathbf{w} \cup \mathbf{z}\}))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)}. \end{aligned}$$

Under this identification the set $\widehat{\underline{\text{Spin}}^c}(Y - \nu(L), \partial(Y - \nu(L)))$ is canonically identified with the set of homotopy classes of paths $\mathcal{P}(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. We will discuss the homology and cohomology of punctured symmetric products at greater length in Section 5.

2.3. Link Floer homology and an alternate differential

The differential ∂ lowers the Maslov grading by one and preserves the Alexander multigrading. Therefore $\widehat{CFL}(\mathcal{D})$ also splits along Alexander grading.

The homology of $\widehat{CFL}(\mathcal{D})$ with respect to the differential ∂ is very nearly the link Floer homology of (Y, K) . There is, however, a slight subtlety having to do with the number of pairs of basepoints z_i and w_i on \mathcal{D} . Let V_i be a vector space over \mathbb{Z}_2 with generators in gradings $(M, (A_1, \dots, A_\ell)) =$

$(0, (0, \dots, 0))$ and $(M, (A_1, \dots, A_j, \dots, A_\ell)) = (-1, (0, \dots, -1, \dots, 0))$, with the -1 in the j th component. As before, let K_j carry n_j pairs of basepoints.

Definition 2.4. The homology of the complex $\widehat{CFL}(\mathcal{D})$ with respect to the differential ∂ is

$$\widehat{HFL}(\mathcal{D}) = \widehat{HFL}(S^3, L) \otimes V_1^{\otimes(n_1-1)} \otimes \dots \otimes V_\ell^{\otimes(n_\ell-1)}.$$

The theory $\widehat{HFL}(S^3, L)$ is symmetric with respect to the Alexander multigrading as follows. Let $\widehat{HFL}_d(S^3, L, (A_1, \dots, A_\ell))$ be the summand of the link Floer homology of L in Alexander multigrading (A_1, \dots, A_ℓ) and Maslov grading d .

Proposition 2.5. [27, Proposition 8.2] *There is an isomorphism*

$$\widehat{HFL}_d(S^3, L, (A_1, \dots, A_\ell)) \cong \widehat{HFL}_{d-\sum A_i}(S^3, L, (-A_1, \dots, -A_\ell)).$$

In particular, ignoring Maslov gradings, we see that the link Floer homology is symmetric in each of its Alexander gradings.

Before moving on, let us consider one addition differential on the complex $\widehat{CFL}(\mathcal{D})$. Suppose that in addition to the disks we counted previously, we also include disks passing over z basepoints on the component K_j of L . In other words, let the differential ∂_{K_j} be the differential of Lagrangian Floer cohomology in $\text{Sym}^{g+n-1}(S \setminus (\mathbf{w} \cup \{z_i : z_i \notin K_j\}))$. This has the effect of discounting the contribution of the component K_j to the link Floer homology, but of maintaining the effect of an extra n_j pairs of basepoints on the Heegaard surface. Ergo we have the following proposition. Let W be a two-dimensional vector space over \mathbb{Z}_2 with summands in gradings $(M, (A_1, \dots, \widehat{A}_j, \dots, A_\ell)) = (0, (0, \dots, 0))$ and $(M, (A_1, \dots, \widehat{A}_j, \dots, A_\ell)) = (-1, (0, \dots, 0))$.

Proposition 2.6. [27, Proposition 7.2] *The homology of the complex $\widehat{CFL}(S^3, L)$ with respect to the differential ∂_{K_j} is isomorphic to $\widehat{HFL}(S^3, L - K_j) \otimes V_1^{\otimes(n_1-1)} \otimes \dots \otimes W^{\otimes n_j} \otimes \dots \otimes V_\ell^{\otimes(n_\ell-1)}$.*

We may think of Proposition 2.6 as the assertion that there is a spectral sequence from the \mathbb{Z}^ℓ graded theory $\widehat{HFL}(S^3, L)$ to the $\mathbb{Z}^{\ell-1}$ graded theory $\widehat{HFL}(S^3, L - K_j)$ by computing all differentials that change the k_j th entry of the \mathbb{Z}^ℓ multigrading. This spectral sequence comes with an overall shift in relative Alexander gradings, which is computed by considering fillings of relative spin^c structures on $S^3 - L$ to relative spin^c structures on $S^3 - (L -$

K_j). The generally slightly complicated formula admits a simple expression in the case of two-component links, which is the only case of interest to this paper.

Lemma 2.7. [27, Lemma 3.13] *Let $L = K_1 \cup K_2$, and $\lambda = \text{lk}(K_1, K_2)$. Then suppose \mathcal{D} is a Heegaard diagram for (S^3, L) , and $\mathbf{x} \in \widehat{CFL}(\mathcal{D})$. If $(A_1(\mathbf{x}), A_2(\mathbf{x})) = (i, j)$ in the complex $\widehat{CFL}(\mathcal{D})$ with differential ∂ , then in the complex $\widehat{CFK}(\mathcal{D})$ with differential ∂_{K_2} , the Alexander grading of \mathbf{x} is $A_1(\mathbf{x}) = i - \frac{\lambda}{2}$.*

That is, forgetting one component of a two-component link has the effect of shifting Alexander gradings of the other component downward by $\frac{\lambda}{2}$. The proof comes from an analysis of filling relative spin^c structures; the effect of extending a relative spin^c structure \mathfrak{s} on $S^3 - \nu L$ to $S^3 - \nu(L - K_j)$ is to shift the Chern class $c_1(\mathfrak{s})$ by the Poincaré dual of the homology class of K_j in $S^3 - \nu(L - K_j)$. For a two-component link this is a shift by the linking number.

2.4. Link Floer homology, the multivariable Alexander polynomial, and the Thurston norm

Recall that the *multivariable Alexander polynomial* of an oriented link $L = K_1 \cup \dots \cup K_\ell$ is a polynomial invariant $\Delta_L(t_1, \dots, t_\ell)$ with one variable for each component of the link. While its relationship to the Alexander polynomials of the component knots is in general slightly complicated, in the case of a two-component $L = K_1 \cup K_2$ Murasugi proved the following using Fox calculus.

Lemma 2.8. [19, Proposition 4.1] *Let $L = K_1 \cup K_2$ be an oriented two-component link with $\text{lk}(K_1, K_2) = \lambda$. If $\Delta_L(t_1, t_2)$ is the multivariable Alexander polynomial of L and $\Delta_{K_1}(t_1)$ is the ordinary Alexander polynomial of K_1 , then*

$$\Delta_L(t_1, 1) = (1 + t + t^2 + \dots + t^{\lambda-1})\Delta_{K_1}(t)$$

The Euler characteristic of link Floer homology encodes the multivariable Alexander polynomial of the link as follows. Let

$$\chi(\widehat{HFL}(S^3, L)) = \sum_{(A_1, \dots, A_\ell) \in \mathbb{H}} t_1^{A_1(\mathbf{x})} \dots t_\ell^{A_\ell(\mathbf{x})} \chi(\widehat{HFL}(S^3, L, (A_1, \dots, A_\ell))).$$

Proposition 2.9. [27, Theorem 1.3] *If L is an oriented link, the Euler characteristic of its link Floer homology is equal to*

$$\chi(\widehat{HFL}(S^3, L)) = \begin{cases} \left(\prod_{i=1}^{\ell} (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})\right) \Delta_L(t_1, \dots, t_{\ell}) & \ell > 1 \\ \Delta_L(t_1) & \ell = 1. \end{cases}$$

Link Floer homology also categorifies the Thurston seminorm of the link complement. Let us recall the definition of the Thurston seminorm in this context. Recall that the *complexity* of a compact, oriented, surface with boundary F with components F_i is

$$\chi_-(F) = \sum_{F_i: \chi(F_i) \leq 0} \chi(F_i).$$

For any homology class in $H_2(S^3, L)$, there is some compact, oriented surface with boundary in $S^3 - \bigcup \nu(K_i)$ which represents h . We can consider a function

$$x_L: H_2(S^3, L) \rightarrow \mathbb{Z} \\ h \mapsto \min_{F \hookrightarrow S^3 \cup \nu(K_i), [F]=h} \chi_-(F).$$

Thurston [33] shows this extends to a seminorm on x_L on $H_2(S^3, L; \mathbb{R})$, called the Thurston seminorm.

Let us pause to be a little more concrete about what surfaces we will consider in computing the Thurston seminorm of a link. Notice that $H_2(S^3, L)$ is generated by the duals of the meridians μ_{K_i} of each component K_i of the link. Through an abuse of notation, and to avoid confusion with cohomology classes later, we will also refer to these duals as $[\mu_{K_i}]$. Computing the Thurston seminorm of the element of $H_2(S^3, L)$ which is dual to $\sum a_i [\mu_{K_i}] \in H_1(L)$ is a matter of computing the minimal Euler characteristic of an embedded surface F with no sphere components whose intersection with a meridian μ_{K_i} of K_i is a_i for each i . In particular, $x([\mu_{K_i}])$ is the minimal Euler characteristic of surface F with boundary one longitude of K_i and an arbitrary number of meridians of the components of L . (For practical purposes, one may consider taking a Seifert surface F for K_i and puncturing F wherever it intersects some other component of L . However, take note that puncturing a minimal Seifert surface for K_i does not necessarily result in a Thurston-norm minimizing surface.)

Link Floer homology yields a related function. Recall that $\mathbb{H} \subset H^2(S^3, L; \mathbb{R}) \cong H_1(L; \mathbb{R})$ is the affine lattice of real second cohomology classes $h =$

$\sum A_i[\mu_{K_i}]$ for which $\widehat{HF\!L}(S^3, L, \mathbf{A})$ is defined. We have

$$y: H^1(S^3 - L; \mathbb{R}) \rightarrow \mathbb{R}$$

which is defined by

$$y(\gamma) = \max_{\{\sum A_i[\mu_{K_i}] \in \mathbb{H}: \widehat{HF\!L}(L, \mathbf{A}) \neq 0\}} \left| \left\langle \sum A_i \widehat{[\mu_{K_i}]}, \gamma \right\rangle \right|.$$

The categorification considers the case of links with no *trivial components*, that is, unknotted components unlinked with the rest of the link.

Proposition 2.10. [28, Thm 1.1] *Let L be an oriented link with no trivial components. Given $\gamma \in H^1(S^3 - L; \mathbb{R})$, the link Floer homology groups determine the Thurston norm of L via the relationship*

$$x_L(\text{PD}[\gamma]) + \sum_{i=1}^{\ell} |\langle \gamma, \mu_{K_i} \rangle| = 2y(\gamma).$$

Here μ_{K_i} is the homology class of the meridian for the i th component of L in $H_1(S^3 - L; \mathbb{R})$, and therefore $|\langle \gamma, \mu_{K_i} \rangle|$ is the absolute value of the Kronecker pairing of h with μ_{K_i} .

We will primarily evaluate this equality on the dual classes to the meridians μ_{K_i} themselves. In the case that K is a knot, so that $x_K([\mu_K]) = 2g(K) - 1$, Proposition 2.10 reduces to the familiar theorem of [22, Theorem 1.2] that the top Alexander grading i for which $\widehat{HF\!K}(S^3, K, i)$ is nontrivial is the genus of the knot. In general, observe that if we evaluate on μ_{K_i} , we obtain $x_L([\mu_{K_i}]) + 1 = 2y(\mu_{K_i})$. In other words, the total breadth of the A_i Alexander grading in the link Floer homology is the Thurston norm of the dual to K_i plus one.

Before leaving the realm of link Floer homology background, we will require one further result concerning the knot Floer homology of fibred knots.

Proposition 2.11. [20, Thm 1.1], [10, Thm 1.4] *Let K be a knot, and $g(K)$ its genus. Then K is fibred if and only if $\widehat{HF\!K}(S^3, K, g(K)) = \mathbb{Z}_2$.*

The forward direction (that if K is fibred, then the knot Floer homology in the top nontrivial Alexander grading is \mathbb{Z}_2) is due to Ozsváth and Szabó [25, Theorem 1.1], whereas the other direction was proved by Ghiggini [10] in the case $g = 1$ and Ni [20] in the general case.

We are now ready to consider the specific case of periodic knots.

3. Proofs of Murasugi's and Edmonds' conditions

Let $\tilde{K} \subset S^3$ be an oriented q -periodic knot and K its quotient knot. We will begin by constructing a Heegaard diagram for $(S^3, \tilde{K} \cup \tilde{U})$ which is preserved by the action of \mathbb{Z}_q on $(S^3, \tilde{K} \cup \tilde{U})$ and whose quotient under this action is a Heegaard diagram for $(S^3, K \cup U)$.

3.1. Heegaard diagrams for periodic knots

As in [14], it will be necessary to work with Heegaard diagrams for $(S^3, K \cup U)$ on the sphere S^2 . Regard S^3 as $\mathbb{R}^2 \cup \{\infty\}$ and arrange \tilde{K} such that the unknotted axis of periodicity \tilde{U} is the z -axis together with the point at infinity, and the action is given by rotation by $2\pi/q$. Then the projection of \tilde{K} to the xy -plane together with the point at infinity is a periodic diagram \tilde{E} for \tilde{K} . Taking the quotient of (S^3, \tilde{K}) by the action of \mathbb{Z}_q and similarly projecting to the xy -plane together with the point at infinity produces a quotient diagram E for K .

Construct a Heegaard diagram for $K \cup U$ as follows: Begin with the diagram E on $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Place a basepoint w_0 at ∞ and z_0 at 0; these will be the sole basepoints on U . (This is a slight departure from the notation of Section 2; it will be more convenient to have the indexing start at w_0 rather than w_1 for the diagrams we construct.) Arrange basepoints $z_1, w_1, \dots, z_{n_1}, w_{n_1}$ on K such that traversing K in the chosen orientation, one passes through the basepoints in that order. Moreover, we insist that while travelling from z_i to w_i one passes only through undercrossings and travelling from w_i to z_{i+1} or from w_{n_1} to z_1 one passes only through overcrossings. In other words, we choose basepoints so as to make E into a bridge diagram for K . Notice that n_1 is at most the number of crossings on the diagram E , or half the number of crossings on \tilde{E} . Encircle the portion of the knot running from z_i to w_i with a curve α_i , oriented counterclockwise in the complement of w_0 . Similarly, encircle the portion of the knot running from w_i to z_{i+1} (or from w_{n_1} to z_1) with a curve β_i , oriented counterclockwise in the complement of w_0 . Notice that both α_i and β_i run counterclockwise around w_i , and moreover for each i , $S^2 \setminus \{\alpha_i, \beta_i\}$ has four components: one each containing z_i, w_i , and z_{i+1} , and one containing all other basepoints. This yields a Heegaard diagram $\mathcal{D} = (S^2, \alpha, \beta, \mathbf{w}, \mathbf{z})$ for $(S^3, K \cup U)$.

We may now take the branched double cover of \mathcal{D} over z_0 and w_0 to produce a Heegaard diagram $\tilde{\mathcal{D}}$ for $(S^3, \tilde{K} \cup \tilde{U})$ compatible with \tilde{E} . This diagram has basepoints w_0 and z_0 for \tilde{U} and basepoints $z_1^1, w_1^1, \dots, z_{n_1}^1, w_{n_1}^1, z_1^2, \dots, w_{n_1}^2, \dots, z_1^q, \dots, w_{n_1}^q$ arranged in that order along the oriented knot

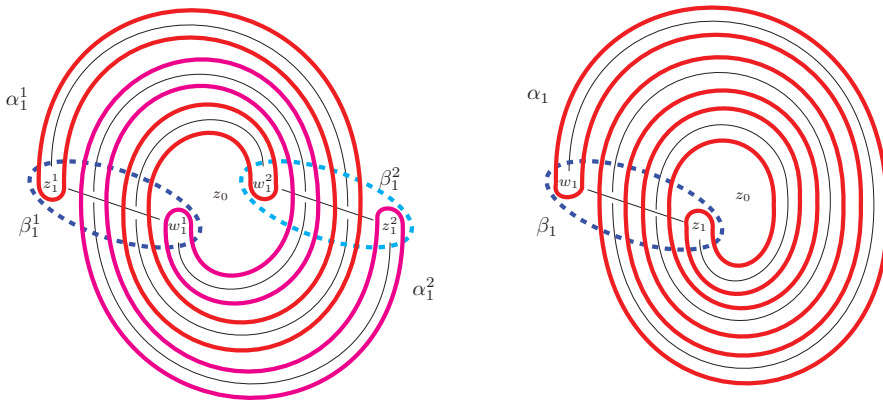


Figure 2. An equivariant Heegaard diagram $\tilde{\mathcal{D}}$ for the trefoil together with the unknotted axis, and its quotient Heegaard diagram \mathcal{D} for the Hopf link.

\tilde{K} . Each adjacent pair z_i^a and w_i^a is encircled by α_i^a a lift of α_i , and each adjacent pair w_i^a and z_{i+1}^a is encircled by β_i^a a lift of β_i . (Pairs $w_{n_1}^a$ and z_1^{a+1} , as well as $w_{n_1}^a$ and z_1^1 , are encircled by lifts $\beta_{n_1}^a$ of β_{n_1} .) This yields a diagram $\tilde{\mathcal{D}} = (S^2, \tilde{\alpha}, \tilde{\beta}, \tilde{w}, \tilde{z})$ with qn_1 each of α and β curves and $qn_1 + 1$ pairs of basepoints.

Remark 3.1. The notation above is not quite the notation of [14], in which the two lifts of a curve α_i in a Heegaard surface \mathcal{D} to its double branched cover $\tilde{\mathcal{D}}$ were $\tilde{\alpha}_i$ and $\tau(\tilde{\alpha}_i)$. In the new slightly more streamlined notation, adopted in view of the need to work with q -fold branched covers and to consider multiple lifts of some of the basepoints, these two curves would be α_i^1 and α_i^2 .

Let us pause to introduce some notation on the diagram \mathcal{D} that will be useful in Section 6. Let x_i be the single positive intersection point between α_i and β_i and y_i the negative intersection point. Moreover, let F_i be the closure of the component of $S - \alpha_i - \beta_i$ containing z_i and E_i be the closure of the component of $S - \alpha_i - \beta_i$ containing z_{i+1} (or z_1 if $i = n_1$). Then $P_i = E_i - F_i$ is a periodic domain of index zero on \mathcal{D} . Finally, let γ_i be the union of the arc of α_i running from x_i to y_i and the arc of β_i running from x_i to y_i . In particular, this specifies that γ_i has no intersection with any α or β curves other than α_i and β_i , and moreover the component of $S - \gamma_i$ which does not contain w_0 contains only a single basepoint w_i .

Our next goal will be to investigate the behavior of the relative Maslov and (particularly) Alexander gradings of generators of $\widehat{CFK}(\mathcal{D})$ and $\widehat{CFK}(\widetilde{\mathcal{D}})$. We begin with two relatively simple lemmas. As before, let τ be the involution on (S^3, \widetilde{K}) (and on $\widetilde{\mathcal{D}}$). Let $\tau_{\#}$ be the induced involution on $\widehat{CFK}(\widetilde{\mathcal{D}})$.

Lemma 3.2. *The induced map $\tau_{\#}$ preserves Alexander and Maslov gradings.*

Proof. Let $\mathbf{s} \in \mathbb{T}_{\widetilde{\alpha}} \cap \mathbb{T}_{\widetilde{\beta}}$. Choose a generator $\mathbf{x} \in \widehat{CFK}(\mathcal{D})$, and let $\widetilde{\mathbf{x}} = \pi^{-1}(\mathbf{x})$, such that $\widetilde{\mathbf{x}}$ is a generator in $\widehat{CFK}(\widetilde{\mathcal{D}})$ which is invariant under $\tau_{\#}$. Choose a Whitney disk ϕ in $\pi_2(\mathbf{s}, \widetilde{\mathbf{x}})$ and let D be its shadow on \mathcal{D} . Then $\tau \circ \phi$ is a Whitney disk in $\pi_2(\tau(\mathbf{s}), \widetilde{\mathbf{x}})$ with shadow $\tau(D)$. Furthermore, since w_0 and z_0 are fixed by the involution, $n_{z_0}(D) = n_{z_0}(\tau(D))$ and $n_{w_0}(D) = n_{w_0}(\tau(D))$, whereas since the remaining basepoints are interchanged by the involution, we have $\sum_{i=0}^{n_1} n_{z_i}(D) = \sum_{i=0}^{n_1} n_{z_i}(\tau(D))$ and $\sum_{i=0}^{n_1} n_{w_i}(D) = \sum_{i=0}^{n_1} n_{w_i}(\tau(D))$. These equalities imply that $M(\widetilde{\mathbf{x}}) - M(\mathbf{s}) = M(\widetilde{\mathbf{x}}) - M(\tau_{\#}(\mathbf{s}))$, and similarly for A_1 and A_2 . Therefore \mathbf{s} and $\tau_{\#}(\mathbf{s})$ are in identical gradings. \square

Lemma 3.3. *Let $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ be a Whitney disk between generators \mathbf{x} and \mathbf{y} in $\widehat{CFK}(\mathcal{D})$ with Maslov index $\mu(\phi)$, with shadow the domain D on \mathcal{D} . There is a Whitney disk $\widetilde{\phi} \in \pi_2(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ with shadow the domain $\pi^{-1}(D)$ on $\widetilde{\mathcal{D}}$, and $\mu(\widetilde{\phi}) = q\mu(\phi) - (q - 1)(n_{z_0}(\phi) + n_{w_0}(\phi))$.*

Proof. The boundary of the lift $\pi^{-1}(D)$ is trivial as a one cycle in S^3 , implying that $\pi^{-1}(D)$ is the shadow of a Whitney disk $\widetilde{\phi} \in \pi_2(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$. We will compare the Maslov index of ϕ with the Maslov index of $\widetilde{\phi}$ using the formula 2.1. As in that formula, we will write D as a sum of the closures of the components of $S^2 - \alpha - \beta$. Say there are m such components, and label them as follows. There are two domains in $S^2 - \alpha - \beta$ which contain a branch point. Let these be D_1 containing z_0 and D_2 containing w_0 . Let the shadow of ϕ be $a_1D_1 + a_2D_2 + \sum_{i=3}^m a_iD_i$. Then the Maslov index of ϕ is

$$\mu(\phi) = \sum_i a_i e(D_i) + p_{\mathbf{x}}(D) + p_{\mathbf{y}}(D)$$

Let us now consider applying the same formula to $\pi^{-1}(D) = \sum_i a_i \pi^{-1}(D_i)$. For $i \geq 3$, the lift $\pi^{-1}(D_j)$ of D_j consists of q copies of D_i , and by additivity of the Euler measure we see that $e(\widetilde{D}_i) = qe(D_i)$. For $i = 1, 2$, let $2k_i$ be the number of corners of D_i . Then $\pi^{-1}(D_i) = \widetilde{D}_i$ is a single component of $S^2 - \widetilde{\alpha} - \widetilde{\beta}$ with $2qk_i$ corners and Euler measure $e(\widetilde{D}_i) = 1 - \frac{qk_i}{2} =$

$q(e(D_i)) - (q - 1)$. Notice, furthermore, that $p_{\tilde{\mathbf{x}}}(\tilde{\mathcal{D}}) = qp_{\mathbf{x}}(D)$ and $p_{\tilde{\mathbf{y}}}(\tilde{\mathcal{D}}) = qp_{\mathbf{y}}(D)$. Therefore we compute

$$\begin{aligned} \mu(\tilde{\phi}) &= \sum_i a_i e(\pi^{-1}(D_i)) + p_{\tilde{\mathbf{x}}}(\tilde{\mathcal{D}}) + p_{\tilde{\mathbf{y}}}(\tilde{\mathcal{D}}) \\ &= a_1 e(\tilde{D}_1) + a_2 e(\tilde{D}_2) + q \sum_{i=3}^m e(D_i) + qp_{\mathbf{x}}(D) + qp_{\mathbf{y}}(D) \\ &= a_1(qe(D_1) - (q - 1)) + a_2(qe(D_2) - (q - 1)) \\ &\quad + q \sum_{i=3}^m e(D_i) + qp_{\mathbf{x}}(D) + qp_{\mathbf{y}}(D) \\ &= q\mu(\phi) - (q - 1)(a_1 + a_2). \end{aligned}$$

Since $a_1 + a_2$ is exactly $n_{z_0}(D) + n_{w_0}(D)$ the total algebraic intersection of ϕ with the branch points, this proves the result. \square

We can now construct the relationship between the Alexander gradings of the generators of $\widehat{CFL}(\tilde{\mathcal{D}})$ and $\widehat{CFL}(\mathcal{D})$. For the case of a q -periodic knot, we will look only at the relative gradings; later in the particular case of a doubly-periodic knot we will fix the absolute gradings using symmetries of link Floer homology. Let $\pi: \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ be the restriction of the branched covering map $(S^3, \tilde{K} \cup \tilde{U}) \rightarrow (S^3, K \cup U)$. We have the following, which is analogous to [16, Lemma 3.1].

Lemma 3.4. *Let $\mathbf{s} \in \widehat{CFL}(\tilde{\mathcal{D}})$, thought of as a qn_1 -tuple of points on S^2 with one point on each α_i^k and one on each β_i^k . Consider its projection $\pi(\mathbf{s})$ to qn_1 points on \mathcal{D} . There is a (not at all canonical) way to write $\pi(\mathbf{s})$ as $\mathbf{s}_1 \cup \mathbf{s}_2 \cup \dots \cup \mathbf{s}_q$ a union of generators in $\widehat{CFL}(\mathcal{D})$.*

The proof of this lemma (pointed out by Adam Levine) is an application of the following combinatorial result of Hall [11]. Let A be a set, and $\{A_i\}_{i=1}^m$ be a collection of finite subsets (a version also exists for infinitely many A_i). A *system of distinct representatives* is a choice of elements $j_i \in A_i$ for each i such that $j_{i_1} \neq j_{i_2}$ if $i_1 = i_2$. Hall’s theorem gives conditions under which a system of distinct representatives exists.

Theorem 3.5. [11, Theorem 1] *Let $\{A_i\}_{i=1}^m$ be finitely many subsets of a set A . Then a system of distinct representatives exists if and only if, for any $1 \leq s \leq m$ and $1 < i_1 < \dots < i_s < m$, $A_{i_1} \cup \dots \cup A_{i_s}$ contains at least s elements.*

Using this, we may prove Lemma 3.4

Proof of Lemma 3.4. Let $A = \{1, \dots, q\}$. For $1 \leq i \leq n_1$, let A_i be a set of integers j , with $1 \leq j \leq n_1$, such that each j appears once in A_i for every point of $\pi(\mathbf{s}) \cap (\alpha_i \cap \beta_j)$. That is, the sets A_i record how many intersection points on α_i also lie on β_j . Notice that there are q elements in each A_i , and each j appears exactly q times in $\coprod_i A_i$. We claim the sets A_i satisfy the condition of Hall's theorem. For given $1 \leq i_1 \leq \dots \leq i_m \leq n_1$, the disjoint union $\coprod_{k=1}^m A_{i_k}$ contains qm elements, and therefore must contain at least m different integers j . Therefore $\bigcup_{k=1}^m A_{i_k}$ contains at least m elements. Hence we can choose a set of distinct representatives j_i in A_i . There is a generator \mathbf{s}_1 consisting of points in $\pi(\mathbf{s})$ on $\alpha_i \cap \beta_{j_i}$. Remove these points from $\pi(\mathbf{s})$ (and the individual j_i from the sets A_i , producing new sets A'_i) and repeat the argument, now with $q - 1$ elements in each A'_i and $q - 1$ appearances of each symbol j in $\coprod_i A'_i$. After q repetitions, we have broken \mathbf{s} into $\mathbf{s}_1 \cup \dots \cup \mathbf{s}_q$, where each \mathbf{s}_i is a generator for $\widehat{CFL}(\mathcal{D})$. This choice of partition is not at all unique. \square

We can start by determining the relative Alexander gradings of generators of $\widehat{CFL}(\tilde{\mathcal{D}})$ which are invariant under the action of \mathbb{Z}_q on $\tilde{\mathcal{D}}$; that is, exactly those generators which are total lifts of generators in $\widehat{CFL}(\mathcal{D})$ under the projection map π .

Lemma 3.6. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ be their total lifts in $\mathbb{T}_{\tilde{\alpha}} \cap \mathbb{T}_{\tilde{\beta}}$. Then*

$$\begin{aligned} A_1(\tilde{\mathbf{y}}) - A_1(\tilde{\mathbf{x}}) &= q(A_1(\mathbf{y}) - A_1(\mathbf{x})) \\ A_2(\tilde{\mathbf{y}}) - A_2(\tilde{\mathbf{x}}) &= A_2(\mathbf{y}) - A_2(\mathbf{x}). \end{aligned}$$

Proof. Let D be a domain from \mathbf{x} to \mathbf{y} on \mathcal{D} . Then $\pi^{-1}(D) = \tilde{D}$ is a domain from $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{y}}$ on $\tilde{\mathcal{D}}$. Since z_0 and w_0 are branch points of π , we see that $n_{z_0}(\tilde{D}) = n_{z_0}(D)$ and $n_{w_0}(\tilde{D}) = n_{w_0}(D)$. Therefore

$$\begin{aligned} A_2(\tilde{\mathbf{x}}) - A_2(\tilde{\mathbf{y}}) &= n_{z_0}(\tilde{D}) - n_{w_0}(\tilde{D}) \\ &= n_{z_0}(D) - n_{w_0}(D) \\ &= A_2(\mathbf{x}) - A_2(\mathbf{y}) \end{aligned}$$

However, for $i \neq 0$, each of z_i and w_i basepoints on \mathcal{D} has q preimages in $\tilde{\mathcal{D}}$. Moreover $n_{z_i}(\tilde{D}) = n_{z_i}(D)$ for all $1 \leq i \leq n_1$ and $1 \leq j \leq q$, and similarly

for w_i^j , so we compute as follows.

$$\begin{aligned} A_1(\tilde{\mathbf{x}}) - A_1(\tilde{\mathbf{y}}) &= \sum_{i=1}^{n_1} \sum_{j=1}^q n_{z_i^j}(\tilde{D}) - \sum_{i=1}^{n_1} \sum_{j=1}^q n_{w_i^j}(\tilde{D}) \\ &= q \left(\sum_{i=1}^{n_1} n_{z_i}(D) - \sum_{i=1}^{n_1} n_{w_i}(D) \right) \\ &= q(A_1(\mathbf{x}) - A_1(\mathbf{y})) \quad \square \end{aligned}$$

Finally, we can use this fact to compute the relative Alexander gradings in $\widehat{CFL}(\tilde{D})$.

Lemma 3.7. *Let \mathbf{s}, \mathbf{r} be generators of $\widehat{CFL}(\tilde{D})$, whose projection to \mathcal{D} can be written as $\pi_*(\mathbf{s}) = \mathbf{s}_1 \cup \dots \cup \mathbf{s}_q$ and $\mathbf{r} = \mathbf{r}_1 \cup \dots \cup \mathbf{r}_q$. Then the relative Alexander gradings between \mathbf{s} and \mathbf{r} is described by*

$$\begin{aligned} A_1(\mathbf{s}) - A_1(\mathbf{r}) &= \sum_{j=1}^q (A_1(\mathbf{s}_j) - A_1(\mathbf{r}_j)) \\ A_2(\mathbf{s}) - A_2(\mathbf{r}) &= \frac{1}{q} \left(\sum_{j=1}^q (A_2(\mathbf{s}_j) - A_2(\mathbf{r}_j)) \right). \end{aligned}$$

Proof. The proof is quite similar to the first half of the argument of [16, Proposition 3.4]. As in Section 2, choose paths $a: [0, 1] \rightarrow \mathbb{T}_{\tilde{\alpha}}$, $b: [0, 1] \rightarrow \mathbb{T}_{\tilde{\beta}}$, with $\partial a = \partial b = \mathbf{s} - \mathbf{r}$. Then let $\epsilon(\mathbf{s}, \mathbf{r}) = a - b$ be a one-cycle in $H_1(S^3 - (K \cup \tilde{U}); \mathbb{Z})$. Consider the projection $\pi_*(\epsilon(\mathbf{s}, \mathbf{r}))$ to \mathcal{D} . The restriction of this one-cycle to any α or β curve consists of q possibly overlapping arcs. By adding copies of the α or β circle if necessary, we may arrange that these arcs connect a point in \mathbf{s}_j to a point in \mathbf{r}_j . That is, modulo α and β curves, which have linking number zero with the knot, $\pi_*\epsilon(\mathbf{s}, \mathbf{r}) = \epsilon(\mathbf{r}_1, \mathbf{s}_1) + \dots + \epsilon(\mathbf{r}_q, \mathbf{s}_q)$. Notice also that $\text{lk}(\pi_*\epsilon(\mathbf{s}, \mathbf{r}), K) = \text{lk}(\epsilon(\mathbf{s}, \mathbf{r}), \tilde{K})$, whereas $\text{lk}(\pi_*\epsilon(\mathbf{s}, \mathbf{r}), U) = q \text{lk}(\epsilon(\mathbf{s}, \mathbf{r}), \tilde{U})$. Therefore we compute

$$\begin{aligned} A_1(\mathbf{r}) - A_1(\mathbf{s}) &= \text{lk}(\epsilon(\mathbf{s}, \mathbf{r}), \tilde{K}) \\ &= \text{lk}(\pi_*\epsilon(\mathbf{s}, \mathbf{r}), K) \\ &= \sum_{j=1}^q \text{lk}(\epsilon(\mathbf{s}_j, \mathbf{r}_j), K) \\ &= \sum A_1(\mathbf{r}) - A_1(\mathbf{s}) \end{aligned}$$

and moreover

$$\begin{aligned}
 A_2(\mathbf{r}) - A_2(\mathbf{s}) &= \text{lk}(\epsilon(\mathbf{s}, \mathbf{r}), \tilde{U}) \\
 &= \frac{1}{q} \text{lk}(\pi_*(\epsilon(\mathbf{s}, \mathbf{r})), U) \\
 &= \frac{1}{q} \left(\sum_{j=1}^q (\text{lk}(\epsilon(\mathbf{s}_j, \mathbf{r}_j), U)) \right) \\
 &= \frac{1}{q} \left(\sum_{j=1}^q (A_2(\mathbf{r}_j) - A_2(\mathbf{s}_j)) \right)
 \end{aligned}
 \tag*{\square}$$

We may now give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let \tilde{K} be a periodic knot with period $q = p^r$ for some prime p , and K be its quotient knot, and let $\lambda = \text{lk}(\tilde{K}, \tilde{U}) = \text{lk}(K, U)$. Choose a periodic diagram \tilde{D} for $(S^3, \tilde{K} \cup \tilde{U})$ and its quotient diagram D for $(S^3, K \cup U)$ as outlined above.

Consider the Euler characteristic of $\widehat{CFL}(\tilde{\mathcal{D}})$ computed modulo p . Let \mathbf{s} be a generator in $\widehat{CFL}(\tilde{\mathcal{D}})$. Either $\mathbf{s} = \tilde{\mathbf{y}} = \pi^{-1}(\mathbf{y})$ for some $\mathbf{y} \in \widehat{CFL}(\mathcal{D})$, and thus \mathbf{s} is invariant under the action of τ , or the order of the orbit of \mathbf{s} under the action of τ is a multiple of p . Since the action preserves the Alexander and Maslov gradings, modulo p the terms of the Euler characteristic of $\widehat{CFL}(\tilde{\mathcal{D}})$ corresponding to noninvariant generators sum to zero. Moreover, there is a one-to-one correspondence between generators \mathbf{y} of $\widehat{CFL}(\mathcal{D})$ and their total lifts $\pi^{-1}(\mathbf{y}) = \mathbf{s}$ in $\widehat{CFL}(\tilde{\mathcal{D}})$.

This correspondence preserves relative Alexander A_2 -gradings, and multiplies Alexander A_1 -gradings by a factor of q . We also claim that it preserves the parity of relative Maslov gradings if p is odd. In particular, any two generators \mathbf{x} and \mathbf{y} of $\widehat{CFL}(\mathcal{D})$ are joined by a domain D which does not pass over w_0 . Let \tilde{D} be the lift of this domain. Then the Maslov index $\mu(\tilde{D})$ is equal to $q\mu(D) - (q - 1)n_{z_0}(D)$ by Lemma 3.3, and we have

$$\begin{aligned}
 M(\tilde{\mathbf{x}}) - M(\tilde{\mathbf{y}}) &= \mu(\tilde{D}) - 2 \sum_{i=1}^n \left(n_{w_i^1}(\tilde{D}) + \dots + n_{w_i^q}(\tilde{D}) \right) \\
 &= q\mu(D) - (q - 1)n_{z_0}(D) - 2q \sum_{i=1}^n n_{w_i}(D) \\
 &\equiv \mu(D) - 2 \sum_{i=1}^n n_{w_i}(D) \pmod{2} \\
 &\equiv M(\mathbf{x}) - M(\mathbf{y}) \pmod{2}
 \end{aligned}$$

Here we have used the assumption that p , and therefore q , is odd, and $q - 1$ is even. Ergo if p is odd, $(-1)^{M(\tilde{\mathbf{x}})} = \pm(-1)^{M(\mathbf{x})}$ where the choice of sign is the same for all generators \mathbf{x} of $\widehat{CFL}(\mathcal{D})$. (If $p = 2$ the sign is of course immaterial.)

Therefore, we capture the following equality.

$$\chi(\widehat{CFL}(\tilde{\mathcal{D}}))(t_1, t_2) \doteq \chi(\widehat{CFL}(\tilde{\mathcal{D}}))(t_1^q, t_2) \pmod p$$

Here \doteq denotes equivalence up to an overall factor of $\pm t_1^{i_1} t_2^{i_2}$.

Notice that $\chi(\widehat{CFL}(\tilde{\mathcal{D}})) = \chi(\widehat{HFL}(\tilde{\mathcal{D}}))$ and similarly $\chi(\widehat{CFL}(\mathcal{D})) = \chi(\widehat{HFL}(\mathcal{D}))$. Hence we may express the Euler characteristics of the chain complexes as the multivariable Alexander polynomials of $\tilde{L} = \tilde{K} \cup \tilde{U}$ and $L = K \cup U$ multiplied by appropriate powers of $(1 - t_1^{-1})$ and $(1 - t_2^{-1})$ according to the number of basepoints on each component of each link. Therefore the equality above reduces to

$$\begin{aligned} \Delta_{\tilde{L}}(t_1, t_2)(1 - t_1^{-1})^{qn_1}(1 - t_2^{-1}) &\doteq \Delta_L(t_1^q, t_2)(1 - (t_1^{-1})^q)^{n_1}(1 - t_2^{-1}) \pmod p \\ \Delta_{\tilde{L}}(t_1, t_2)(1 - t_1^{-1})^{qn_1} &\doteq \Delta_L(t_1^q, t_2)(1 - (t_1^{-1})^q)^{n_1} \pmod p. \end{aligned}$$

Recalling that q is a power of p , and that therefore $(a + b)^q \equiv a^q + b^q \pmod p$, we may reduce farther.

$$\begin{aligned} \Delta_{\tilde{L}}(t_1, t_2)(1 - t_1^{-1})^{qn_1} &\doteq \Delta_L(t_1^q, t_2)(1 - t_1^{-1})^{qn_1} \pmod p \\ \Delta_{\tilde{L}}(t_1, t_2) &\doteq \Delta_L(t_1^q, t_2) \pmod p \end{aligned}$$

We now set $t_2 = 1$. By Lemma 2.8, this reduces the equality above to

$$\Delta_{\tilde{K}}(t_1)(1 + t_1 + \dots + t_1^{\lambda-1}) \doteq \Delta_K(t_1^q)(1 + t_1^q + (t_1^q)^2 + \dots + ((t_1^q)^\lambda)^{\lambda-1}) \pmod p.$$

Again using the fact that q is a power of p , we produce

$$\begin{aligned} \Delta_{\tilde{K}}(t_1)(1 + t_1 + \dots + t_1^{\lambda-1}) &\doteq (\Delta_K(t_1))^q(1 + t_1 + t_1^2 + \dots + t_1^{\lambda-1})^q \pmod p \\ \Delta_{\tilde{K}}(t_1) &\doteq (\Delta_K(t_1))^q(1 + t + \dots + t^{\lambda-1})^{q-1} \pmod p. \end{aligned}$$

This last is Murasugi’s condition. □

3.2. Spectral sequences for doubly-periodic knots

From now on we restrict ourselves entirely to the case of \tilde{K} a doubly-periodic knot. Moreover, we insist that \tilde{K} be oriented such that $\text{lk}(\tilde{K}, \tilde{U}) = \lambda$ is positive. Recall that λ is necessarily odd; otherwise \tilde{K} would be disconnected.

We proceed to explain how Corollary 1.6 follows from Theorem 1.3. Consider the map $\tau_{\#}: \widehat{CFL}(\widetilde{\mathcal{D}}) \rightarrow \widehat{CFL}(\widetilde{\mathcal{D}})$ induced by the involution τ on $\widetilde{\mathcal{D}}$. As a consequence of our application of Seidel and Smith’s localization theory to the symmetric products $\text{Sym}^{2n_1}(S^2 \setminus (\widetilde{\mathbf{w}} \cup \widetilde{\mathbf{z}}))$ and $\text{Sym}^{n_1}(S^2 \setminus (\mathbf{w} \cup \mathbf{z}))$, we will replace $\widehat{CFL}(\widetilde{\mathcal{D}})$ with a chain homotopy equivalent complex $\widehat{CFL}(\widetilde{\mathcal{D}})'$ with the same invariant generators and $\tau_{\#}$ with an involution $\tau'_{\#}$, not necessarily chain homotopy equivalent, which has the same fixed set as $\tau_{\#}$ and is a chain map. (For more on this replacement, see Section 4). This map $\tau'_{\#}$ also preserves Alexander gradings. Consider the double complex

$$\begin{array}{ccccccc}
 & & \downarrow \partial' & & \downarrow \partial' & & \downarrow \partial' \\
 0 & \longrightarrow & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} & \widehat{CFL}_{i+1}(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} \\
 & & \downarrow \partial' & & \downarrow \partial' & & \downarrow \partial' \\
 0 & \longrightarrow & \widehat{CFL}_i(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} & \widehat{CFL}_i(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} & \widehat{CFL}_i(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} \\
 & & \downarrow \partial' & & \downarrow \partial' & & \downarrow \partial' \\
 0 & \longrightarrow & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} & \widehat{CFL}_{i-1}(\widetilde{\mathcal{D}})' & \xrightarrow{1+\tau'_{\#}} \\
 & & \downarrow \partial' & & \downarrow \partial' & & \downarrow \partial'
 \end{array}$$

Definition 3.8. The homology of the complex $(\widehat{CFL}(\widetilde{\mathcal{D}})' \otimes \mathbb{Z}_2[[\theta]], d + \theta(1 + \tau'_{\#}))$ is $\widehat{HFL}_{\text{Borel}}(\widetilde{\mathcal{D}})$.

Computing vertical differentials first, we obtain a spectral sequence from $\widehat{HFL}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2[[\theta]]$ to $\widehat{HFL}_{\text{Borel}}(\widetilde{\mathcal{D}})$. Moreover, our application of Seidel and Smith’s localization theory will lead us to the following theorem.

Theorem 3.9. *There is a localization map*

$$\widehat{HFL}_{\text{Borel}}(\widetilde{\mathcal{D}}) \rightarrow \widehat{HFL}(\mathcal{D}) \otimes \mathbb{Z}_2[[\theta]]$$

which becomes an isomorphism after tensoring with $\mathbb{Z}_2((\theta))$.

Therefore after tensoring the spectral sequence from $\widehat{HFL}(\widetilde{\mathcal{D}}) \otimes \mathbb{Z}_2[[\theta]]$ to $\widehat{HFL}_{\text{Borel}}(\widetilde{\mathcal{D}})$ with $\mathbb{Z}_2((\theta))$, we obtain the spectral sequence of Theorem 1.2. The proof reduces to finding a *stable normal trivialization* for the triple $(\text{Sym}^{2n_1}(\Sigma(S) \setminus \widetilde{\mathbf{w}}), \mathbb{T}_{\widetilde{\alpha}}, \mathbb{T}_{\widetilde{\beta}})$; see Sections 4–6 for details. Since ∂ and $\tau'_{\#}$ also

preserve both Alexander gradings on $\widehat{CFL}(\tilde{\mathcal{D}})$, the spectral sequence splits along the grading (A_1, A_2) .

The knot Floer homology spectral sequence of Theorem 1.3 arises from a similar double complex. Recall that for a link $L = K_1 \cup \dots \cup K_\ell$, the differential ∂_{K_j} corresponds to forgetting the component K_j of the link. Therefore for the link $\tilde{L} = \tilde{K} \cup \tilde{U}$ we begin with the chain complex $(\widehat{CFK}(\tilde{\mathcal{D}}), \partial_{\tilde{U}})$ and again replace with a chain homotopy equivalent complex $(\widehat{CFK}(\tilde{\mathcal{D}})'', \partial'_{\tilde{U}})$ and a possibly not chain homotopy equivalent involution $\tau''_{\#}$ which is a chain map, preserves the Alexander grading A_1 , and has the same fixed set as $\tau_{\#}$. Consider the following double complex.

$$\begin{array}{ccccccc}
 & & \downarrow \partial'_{\tilde{U}} & & \downarrow \partial'_{\tilde{U}} & & \downarrow \partial'_{\tilde{U}} \\
 0 & \longrightarrow & \widehat{CFK}_{i+1}(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \widehat{CFK}_{i+1}(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \widehat{CFK}_{i+1}(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \\
 & & \downarrow \partial'_{\tilde{U}} & & \downarrow \partial'_{\tilde{U}} & & \downarrow \partial'_{\tilde{U}} & & \\
 0 & \longrightarrow & \widehat{CFK}_i(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \widehat{CFK}_i(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \widehat{CFK}_i(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \\
 & & \downarrow \partial'_{\tilde{U}} & & \downarrow \partial'_{\tilde{U}} & & \downarrow \partial'_{\tilde{U}} & & \\
 0 & \longrightarrow & \widehat{CFK}_{i-1}(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \widehat{CFK}_{i-1}(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \widehat{CFK}_{i-1}(\tilde{\mathcal{D}})'' & \xrightarrow{1+\tau''_{\#}} & \\
 & & \downarrow \partial'_{\tilde{U}} & & \downarrow \partial'_{\tilde{U}} & & \downarrow \partial'_{\tilde{U}} & &
 \end{array}$$

Definition 3.10. The homology of the complex $(\widehat{CFK}(\mathcal{D})'' \otimes \mathbb{Z}_2[[\theta]], \partial'_{\tilde{U}} + \theta(1 + \tau''_{\#}))$ is $\widehat{HFK}_{\text{Borel}}(\tilde{\mathcal{D}})$.

Computing vertical differentials first gives a spectral sequence from $E^1 = \widehat{HFK}(\tilde{\mathcal{D}}) \otimes \mathbb{Z}_2[[\theta]]$ to $\widehat{HFK}_{\text{Borel}}(\tilde{\mathcal{D}})$. As before, our application of Seidel and Smith’s localization theory will lead us to the following theorem.

Theorem 3.11. *There is a localization map*

$$\widehat{HFK}_{\text{Borel}}(\tilde{\mathcal{D}}) \rightarrow \widehat{HFK}(\mathcal{D}) \otimes \mathbb{Z}_2[[\theta]]$$

which becomes an isomorphism after tensoring with $\mathbb{Z}_2((\theta))$.

Therefore after tensoring the spectral sequence from $\widehat{HFK}(\tilde{\mathcal{D}}) \otimes \mathbb{Z}_2[[\theta]]$ to $\widehat{HFK}_{\text{Borel}}(\tilde{\mathcal{D}})$ with $\mathbb{Z}_2((\theta))$, we obtain the spectral sequence of 1.3. As $\partial'_{\tilde{U}}$

does not preserve Alexander A_2 gradings relative to the axis U , neither does the spectral sequence; however, it still splits along Alexander A_1 gradings, the grading relative to the knot itself.

Let us now complete the proofs of Theorems 1.2 and 1.3, and Corollaries 1.5 and 1.4, by fixing the relationship between the absolute Alexander gradings of $\widehat{HFL}(\mathcal{D})$ and $\widetilde{HFL}(\mathcal{D})$.

For \mathbf{x} any intersection point in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})$ be the relative spin^c structure on $(S^3 - \nu(L), \partial(S^3 - \nu(L)))$ associated to \mathbf{x} as in Section 2. Then since the map $\pi: S^3 - \nu(\tilde{L}) \rightarrow S^3 - \nu(L)$ is a local diffeomorphism, we can pull back relative spin^c structures along this map, and examining the construction in [27, Section 3.6] we see that the pullback of $\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})$ is the relative spin^c structure on $(S^3 - \nu(\tilde{L}), \partial(S^3 - \nu(\tilde{L})))$ associated to $\tilde{\mathbf{x}}$ the lift of \mathbf{x} . In particular, taking the first Chern class of both relative spin^c structures, we conclude that $\pi^*(c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x}))) = c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\tilde{\mathbf{x}}))$.

Now let γ be an arbitrary element of $H_2(S^3 - \nu(\tilde{L}), \partial(S^3 - \nu(\tilde{L}))) \simeq H_2(S^3, \tilde{L})$. As before, let this group be generated by the classes dual to $[\mu_{\tilde{K}}]$ and $[\mu_{\tilde{U}}]$ in $H_1(L)$, and called by the same names via our previous abuse of notation, and let $\gamma = c[\mu_{\tilde{K}}] + d[\mu_{\tilde{U}}]$ for integers c, d . Moreover, let $H_2(S^3 - \nu(L), \partial(S^3 - \nu(L))) \simeq H_2(S^3, L)$ be similarly generated by classes $[\mu_K]$ and $[\mu_U]$, and let $H^2(S^3 - \nu(L), \partial(S^3 - \nu(L))) \simeq H^2(S^3, L)$ be generated by their duals $[\mu_K]^*$ and $[\mu_U]^*$. Similarly, we have $H^2(S^3 - \nu(\tilde{L}), \partial(S^3 - \nu(\tilde{L}))) \simeq H^2(S^3, \tilde{L})$ generated by the cohomology classes $[\mu_{\tilde{K}}]^*$ and $[\mu_{\tilde{U}}]^*$. Finally, suppose that $c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})) = e[\mu_K]^* + f[\mu_U]^*$. Then we compute:

$$\begin{aligned} \langle c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\tilde{\mathbf{x}})), \gamma \rangle &= \langle \pi^*(c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x}))), \gamma \rangle = \langle c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})), \pi_*(\gamma) \rangle \\ &= \langle c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})), 2c[\mu_K] + d[\mu_U] \rangle \\ &= 2ec + fd \end{aligned}$$

Since the equation above holds for all $c, d \in \mathbb{Z}$, we conclude that if $c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})) = e[\mu_K]^* + f[\mu_U]^*$, we must then have $c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\tilde{\mathbf{x}})) = 2e[\mu_{\tilde{K}}]^* + f[\mu_{\tilde{U}}]^*$. This means that upstairs we have

$$\begin{aligned} c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\tilde{\mathbf{x}})) + PD([\mu_{\tilde{K}}] + [\mu_{\tilde{U}}]) &= 2PD(A_1[\mu_{\tilde{K}}] + A_2[\mu_{\tilde{U}}]) \\ (2e + 1)[\mu_{\tilde{K}}]^* + (f + 1)[\mu_{\tilde{U}}]^* &= 2A_1[\mu_{\tilde{K}}]^* + 2A_2[\mu_{\tilde{U}}]^* \end{aligned}$$

whereas downstairs we have

$$\begin{aligned} c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})) + PD([\mu_{\tilde{K}}] + [\mu_{\tilde{U}}]) &= 2PD(A_1[\mu_{\tilde{K}}] + A_2[\mu_{\tilde{U}}]) \\ (e + 1)[\mu_{\tilde{K}}]^* + (f + 1)[\mu_{\tilde{U}}]^* &= 2A_1[\mu_{\tilde{K}}]^* + 2A_2[\mu_{\tilde{U}}]^* \end{aligned}$$

This justifies the assertion that the spectral sequence carries the gradings $(A_1, A_2) = (2a + \frac{1}{2}, b)$ upstairs to $(A_1, A_2) = (a + \frac{1}{2}, b)$ downstairs, where $a = e$ and $b = \frac{f+1}{2}$.

There is an alternate proof, more complicated but also more helpful to our geometric intuition, using the link Floer homology categorification of the Thurston norm. Let $x_L([\mu_K])$ be the Thurston seminorm of the class dual to $[\mu_K]$ in $H_2(S^3, L)$ and $x_{\tilde{L}}([\mu_{\tilde{K}}])$ be the Thurston seminorm of the class dual to $[\mu_{\tilde{K}}]$ in $H_2(S^3, \tilde{L})$. Notice that if F is a Thurston-norm minimizing surface for the class dual to $[\mu_K]$ of Euler characteristic $\chi(F) = -x_L([\mu_K])$, then the preimage $\tilde{F} = \pi^{-1}(F)$ under the ordinary double cover $\pi: S^3 - \nu(\tilde{L}) \rightarrow S^3 - \nu(L)$ is an embedded surface representing the class dual to $[\mu_{\tilde{K}}]$ in $H_2(S^3, \tilde{L})$, and $\chi(\tilde{F}) = 2\chi(F)$. Hence $x_{\tilde{L}}([\mu_{\tilde{K}}]) \leq 2x_L([\mu_K])$.

However, recall that $x_L([\mu_K]) + 1$ is exactly the breadth of the Alexander A_1 grading in $\widehat{HFL}(S^3, L)$, and therefore that the breadth of the A_1 grading in $\widehat{HFL}(\mathcal{D})$ is $x_L([\mu_K]) + 1 + (n_1 - 1) = x_L([\mu_K]) + n_1$. Similarly, the breadth of the A_1 grading in $\widehat{HFL}(S^3, \tilde{L})$ is $x_{\tilde{L}}([\mu_{\tilde{K}}]) + 1$, and therefore the total breadth of the A_1 grading in $\widehat{HFL}(\tilde{\mathcal{D}})$ is $x_{\tilde{L}}([\mu_{\tilde{K}}]) + 1 + 2n_1 - 1 = x_{\tilde{L}}([\mu_{\tilde{K}}]) + 2n_1$. Moreover, we have seen that in the spectral sequence from $E^1 = \widehat{HFL}(\tilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta))$ to $E^\infty \cong \widehat{HFL}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta))$, the relative A_1 grading of two elements on the E^1 page is twice the relative A_1 grading of their residues on the E^∞ page. Therefore the total breadth of the A_1 grading on the E^1 page of the spectral sequence is at least twice the breadth of the A_1 grading on the last page of the spectral sequence. We thus have the inequality

$$\begin{aligned} x_{\tilde{L}}([\mu_{\tilde{K}}]) + 2n_1 &\geq 2x_L([\mu_K]) + 2n_1 \\ &\geq 2(x_L([\mu_K]) + n_1) \\ \text{i.e. } x_{\tilde{L}}([\mu_{\tilde{K}}]) &\geq 2x_L([\mu_K]). \end{aligned}$$

Consequently we see directly from the spectral sequence that

$$x_{\tilde{L}}([\mu_{\tilde{K}}]) = 2x_L([\mu_K]).$$

This implies that the breadth of the A_1 grading on the E^1 page is exactly twice the breadth of the A_1 grading on the E^∞ page. Therefore the breadth of the A_1 grading cannot decrease over the course of the spectral sequence. A similar argument, sans the factors of two, shows that $x_{\tilde{L}}([\mu_{\tilde{U}}]) = x_L([\mu_U])$ and that the breadths of the Alexander A_2 grading of the E^1 and E^∞ pages of the spectral sequence are the same. Therefore the breadth of the A_2 grading does not change either over the course of the spectral sequence.

In particular, the top A_1 grading in $\widehat{HFL}(\widetilde{\mathcal{D}})$ is sent to the top A_1 grading in $\widehat{HFL}(\mathcal{D})$. However, by the symmetry of \widehat{HFL} and the determination of the breadth of the A_1 grading by the Thurston norm, the top A_1 grading of $\widehat{HFL}(\widetilde{\mathcal{D}})$ is the same as the top A_1 grading of $\widehat{HFL}(S^3, \widetilde{L})$, which is $A_1 = \frac{x_{\widetilde{L}}([\mu_{\widetilde{K}}]) + 1}{2} = \frac{2x_L([\mu_K]) + 1}{2}$. Similarly, the top A_1 grading of $\widehat{HFL}(\mathcal{D})$ is the top A_1 grading of $\widehat{HFL}(S^3, L)$, to wit, $A_1 = \frac{x_L([\mu_K]) + 1}{2}$. Therefore the spectral sequence carries the A_1 Alexander grading $\frac{2x_L([\mu_K]) + 1}{2}$ on the E^1 page to the A_1 grading $\frac{x_L([\mu_K]) + 1}{2}$ on the E^∞ page. Therefore in general the A_1 grading $\frac{2x_L([\mu_K]) + 1}{2} + 2a$ on the E^1 page is sent to the A_1 grading $\frac{x_L([\mu_K]) + 1}{2} + a$ for any integer a . Notice that $x_L([\mu_K])$ is an even number: suppose F is a Thurston-seminorm minimizing surface for K in $S^3 - L$ of genus g' with geometric intersection number $\#(F \cap U) = \Lambda$. Since the algebraic intersection number λ is odd, so is Λ . Then $x_L([\mu_K]) = 1 - 2g' - \Lambda$ is even. Therefore $\frac{x_L([\mu_K])}{2}$ is an integer, and we can take $a = \frac{x_L([\mu_K])}{2}$ to see that the A_1 grading $\frac{1}{2}$ on the E^1 page is sent to the A_1 grading $\frac{1}{2}$ on the E^∞ page.

A parallel but simpler argument for the A_2 gradings shows that the A_2 grading b on the E^1 page is sent precisely to the A_2 grading b on the E^∞ page, using the fact that relative A_2 gradings of elements on the E^1 page that survive in the spectral sequence are preserved rather than doubled on the E^∞ page.

By Lemma 2.7, computing the knot Floer homology complex using $\partial_{\widetilde{U}}$ yields a downward shift in Alexander gradings by $\frac{\text{lk}(\widetilde{K}, \widetilde{U})}{2}$ on the E^1 page and $\frac{\text{lk}(K, U)}{2}$ on the E^∞ page. Since both of these numbers are $\frac{\lambda}{2}$, we obtain an overall downward shift of $\frac{\lambda}{2}$ between that link and knot Floer homology spectral sequences.

This nearly completes the proofs of Theorems 1.2 and 1.3. Corollaries 1.4 and 1.5 follow almost immediately, but the last statement in Corollary 1.5 deserves a few more words. If Edmonds' condition is sharp, then $g(\widetilde{K}) = 2g(K) + \frac{\lambda - 1}{2}$. Since the knot Floer spectral sequence sends the Alexander grading $2a + \frac{1 - \lambda}{2}$ to the Alexander grading $a + \frac{1 - \lambda}{2}$, it sends the Alexander grading $g(\widetilde{K})$ to $g(K)$. Therefore there is a rank inequality between $\widehat{HFK}(\widetilde{\mathcal{D}}, g(\widetilde{K}))$ to $\widehat{HFK}(\mathcal{D}, g(K))$. However, recall that the summands of the vector space V carry Alexander gradings 0 and 1, so $\widehat{HFK}(\widetilde{\mathcal{D}}, g(\widetilde{K})) \simeq \widehat{HFK}(S^3, \widetilde{K}, g(\widetilde{K})) \otimes W$ and $\widehat{HFK}(\mathcal{D}, g(K)) \simeq \widehat{HFK}(S^3, K, g(K)) \otimes W$. So ignoring a consistent factor of two from W , we obtain the last inequality in Corollary 1.5.

Remark 3.12. The observations that $x_{\tilde{L}}([\mu_{\tilde{K}}]) = 2x_L([\mu_K])$ and $x_{\tilde{L}}([\mu_{\tilde{U}}]) = x_L([\mu_U])$ are a special case of Gabai’s theorem [9, Corollary 6.13] that the Thurston norm is multiplicative for ordinary finite covers. Indeed, by appealing to Gabai’s theorem (or by constructing a \mathbb{Z}_p analog of Seidel and Smith’s localization spectral sequence) we could similarly fix the relationship between the absolute gradings of $\widehat{HFL}(\tilde{\mathcal{D}})$ and $\widehat{HFL}(\mathcal{D})$ for p -periodic knots.

Remark 3.13. This alternate proof illustrates the important role of the vector space V_1 in the existence of the spectral sequence; the breadth of the A_1 grading in $\widehat{HFL}(S^3, \tilde{L})$ is one less than twice the breadth of the A_1 grading in $\widehat{HFL}(S^3, L)$, a trouble which is corrected for by increasing the breadth upstairs by $2n_1 - 1$ and the breadth downstairs by $n_1 - 1$. While one might hope to produce a spectral sequence in which $n_1 = 1$, it seems impossible to produce a link Floer homology spectral sequence for doubly periodic knots which does not involve at least one copy of V_1 on the E^1 page.

Having fixed the Alexander gradings in the spectral sequence, we may provide a proof of Corollary 1.6 (Edmonds’ Condition) from 1.3.

Proof of Corollary 1.6. Once again, let V_1 denote a two-dimensional vector space over \mathbb{Z}_2 whose two sets of gradings $(M, (A_1, A_2))$ are $(0, (0, 0))$ and $(-1, (-1, 0))$, and likewise let W be a two-dimensional vector space over \mathbb{Z}_2 whose two sets of gradings $(M, (A_1, A_2))$ are $(0, (0, 0))$ and $(-1, (0, 0))$. By Theorem 1.3, there is a spectral sequence whose E^1 page is $\widehat{HFK}(S^3, \tilde{K}) \otimes V_1^{\otimes n_1 - 1} \otimes W \otimes \mathbb{Z}_2((\theta))$ to a theory whose E^∞ page is $\mathbb{Z}((\theta))$ -isomorphic to $\widehat{HFK}(S^3, K) \otimes V_1^{\otimes 2n_1 - 1} \otimes W \otimes \mathbb{Z}((\theta))$. Moreover, this spectral sequence splits along the A_1 grading, and by Theorem 1.5 the subgroup of the E^1 page in A_1 grading $\frac{1-\lambda}{2} + 2a$ is carried to the subgroup of the E^∞ page in grading $\frac{1-\lambda}{2} + a$. Moreover, the top A_1 grading on the E^1 page is $g(\tilde{K})$ and the top A_1 grading on the E^∞ page is $g(K)$. Since there must be something on the E^1 page in the A_1 grading which converges to the A_1 grading $g(K)$ on the E^∞ page, we have the following inequality.

$$g(\tilde{K}) - \frac{1-\lambda}{2} \geq 2 \left(g(K) - \frac{1-\lambda}{2} \right)$$

i.e. $g(\tilde{K}) \geq 2g(K) + \frac{\lambda-1}{2}$ □

Finally, let us prove Corollary 1.7.

Proof of Corollary 1.7. Suppose Edmonds’ condition is sharp, that is, that $g(\tilde{K}) = 2g(K) + \frac{\lambda-1}{2}$. As in the proof of Corollary 1.5, this implies that the

knot Floer spectral sequence sends the Alexander grading $g(\tilde{K}) = 2g(K) + \frac{\lambda-1}{2}$ to the Alexander grading $g(K)$. That is, sharpness of Edmonds' condition exactly says that the top Alexander grading on the E^1 page is not killed in the spectral sequence.

Suppose now that \tilde{K} is fibered. Then the top Alexander grading of $\widehat{HFK}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta)) = (\widehat{HFK}(S^3, K) \otimes V_1^{\otimes(2n_1-1)} \otimes W) \otimes \mathbb{Z}_2((\theta))$ has rank two as a $\mathbb{Z}_2((\theta))$ module. (The knot Floer homology of fibered knots is monic in the top Alexander grading by the forward direction of Lemma 2.11, and the factor of W doubles the number of entries in each Alexander grading.) Since this Alexander grading is not killed in the spectral sequence, the top Alexander grading of $\widehat{HFK}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta)) = \widehat{HFK}(S^3, K) \otimes V_1^{\otimes(n_1-1)} \otimes W) \otimes \mathbb{Z}_2((\theta))$ also has rank two as a $\mathbb{Z}_2((\theta))$ -module. Therefore K is also fibered. □

Remark 3.14. The converse of Corollary 1.7, that when Edmonds' condition is sharp, the quotient knot K being fibered implies \tilde{K} is fibered, is false. Consider the following counterexample: the knot $\tilde{K} = 10_{144}$ is doubly periodic with quotient knot $K = 3_1$, the trefoil. The linking number $\lambda = \text{lk}(\tilde{K}, \tilde{U}) = 1$. Since $g(10_{144}) = 2$ and $g(3_1) = 1$, we see that $g(\tilde{K}) = 2 = 2g(K) + \frac{\lambda-1}{2}$. Therefore Edmonds' condition is sharp. However, the trefoil is fibered, whereas 10_{144} is not [3].

3.3. An example of an obstruction not given by Alexander polynomials and genera

We now give an example of a knot \tilde{K} which is not obstructed from being two-periodic with a specific quotient knot K by Edmonds' and Murasugi's conditions, taken together, but for which the spectral sequence of 1.3 cannot exist. Indeed, we can do slightly better. There is a second condition of Murasugi (*not* recovered by the spectral sequences of this paper), as follows.

Theorem 3.15. [19, Corollary 1] *If \tilde{K} is q -periodic with quotient knot K , then $\Delta_K(t) | \Delta_{\tilde{K}}(t)$.*

Our example will have $\Delta_K(t) = 1$, and therefore will pass both of Murasugi's conditions.

Let \tilde{K} be the connect sum of the Kinoshita-Terasaka knot $11n42$ and the right-handed trefoil 3_1 . Then $g(\tilde{K}) = g(11n42) + g(3_1) = 3$. Moreover since the Kinoshita-Terasaka knot has trivial Alexander polynomial, $\Delta_{\tilde{K}}(t) =$

$\Delta_{3_1}(t) = 1 - t + t^2 \equiv (1 + t + t^2)(1)^2$ modulo two. Suppose that \tilde{K} is two-periodic. Then by Murasugi’s condition, we must have $\lambda = 3$. If K is any genus one knot with trivial Alexander polynomial, then we see that $\Delta_{\tilde{K}}(t) \equiv (1 + t + t^2)(\Delta(K))^2$ modulo two, $\Delta_K(t)|\Delta_{\tilde{K}}(t)$, and finally $g(\tilde{K}) = 2g(K) + \frac{\lambda-1}{2}$. Therefore neither Edmonds’ nor either of Murasugi’s conditions obstructs \tilde{K} from being two-periodic with quotient knot K .

Now, consider candidate quotient knot $K = D(4_1)$ the untwisted positive Whitehead double of the figure-eight knot, which is a genus one knot with trivial Alexander polynomial. Suppose \tilde{K} is two-periodic with quotient knot K . Since Edmonds’ condition is sharp, by Corollary 1.4, there is a rank inequality between $\widehat{HFK}(S^3, \tilde{K}, 3)$ and $\widehat{HFK}(S^3, K, 1)$. Let us consider the ranks of these groups.

First, let us compute the rank of the group $\widetilde{HFK}(\tilde{\mathcal{D}}, 3)$. Recall that the knot Floer homology of a connect sum of knots in the three-sphere obeys a Kunneth formula [23, Theorem 7.1]. Therefore the top-dimensional knot Floer homology of \tilde{K} is the tensor product of the top-dimensional knot Floer homologies of the trefoil and the Kinoshita-Terasaka knot. The knot Floer homology of the latter was first computed in the extremal gradings in [24, Theorem 1.1], and later in full by Baldwin and Gillam [2, Section 4]. The fact from these computations that we will need is that $\text{rk}(\widehat{HFK}(S^3, 11n42, 2)) = 2$. Therefore since $\text{rk}(\widehat{HFK}(S^3, 3_1, 1)) = 1$, we conclude that $\text{rk}(\widehat{HFK}(S^3, \tilde{K}, 3)) = 2$.

However, Hedden [13, Proposition 7.1] has computed the knot Floer homology of Whitehead doubles. Relevantly, from his work we know that $\text{rk}(\widehat{HFK}(S^3, D(4_1), 1)) = 4$. Therefore there cannot be a spectral sequence sending $\widetilde{HFK}(\tilde{\mathcal{D}}, 3)$ to $\widetilde{HFK}(\mathcal{D}, 1)$, implying that \tilde{K} cannot be two-periodic with quotient knot K .

4. Spectral sequences for Lagrangian Floer cohomology

Floer cohomology is an invariant for Lagrangian submanifolds in a symplectic manifold introduced by Floer [6–8]. In this section we briefly recall the setting and statement of Seidel and Smith’s localization theorem for Floer cohomology. For further detail, we refer the reader to the longer exposition in [14, Section 2].

Let M be a manifold equipped with an exact symplectic form $\omega = d\theta$ and a compatible almost complex structure J which is convex at infinity. Let L_0 and L_1 be two exact Lagrangian submanifolds of M . For our purposes we can restrict to the case that L_0 and L_1 are compact and intersect transversely.

Let $CF(L_0, L_1) = \mathbb{Z}_2\langle L_0 \cap L_1 \rangle$ be the Floer cohomology chain group with differential ∂ , such that $HF(L_0, L_1) = H_*(CF(L_0, L_1), \partial)$ is the Lagrangian Floer cohomology of L_0 and L_1 in M .

Now, suppose that M carries a symplectic involution τ preserving (M, L_0, L_1) and the forms ω and θ . Let the submanifold of M fixed by τ be M^{inv} , and similarly for L_i^{inv} for $i = 0, 1$. The Floer chain complex $CF(L_0, L_1)$ carries an induced involution $\tau_{\#}$ which takes $x \in L_0 \cap L_1$ to the intersection point $\tau(x) \in L_0 \cap L_1$. This map $\tau_{\#}$ is not a chain map with respect to a generic family of complex structures on M . However, suppose that we are in the nice case that we can find a family of complex structures \mathbf{J} on M such that $\tau_{\#}$ commutes with the differential on $CF(L_0, L_1)$. Then $1 + \tau_{\#}$ is a second differential on $CF(L_0, L_1)$, and we can use the double complex below to define the Borel (or equivariant) cohomology of (M, L_0, L_1) with respect to this involution.

$$0 \rightarrow CF(L_0, L_1) \xrightarrow{1+\tau_{\#}} CF(L_0, L_1) \xrightarrow{1+\tau_{\#}} CF(L_0, L_1) \cdots$$

Definition 4.1. If $CF(L_0, L_1)$ is the Floer chain complex and $\tau_{\#}$ is a chain map with respect to the complex structure on M , $HF_{\text{Borel}}(L_0, L_1)$ is the homology of the complex $CF(L_0, L_1) \otimes \mathbb{Z}_2[[\theta]]$ with respect to the differential $\partial + (1 + \tau_{\#})\theta$.

As in the original, the choice of $\mathbb{Z}_2[[\theta]]$ instead of $\mathbb{Z}_2[\theta]$ is largely irrelevant since only finitely many powers of θ appear in each degree, but was chosen by Seidel and Smith to agree with more general contexts [32, Section 2].

Let us now set up notation for Seidel and Smith’s main definition and theorem. Consider the normal bundle $N(M^{\text{inv}})$ to M^{inv} in M and its Lagrangian subbundles $N(L_i^{\text{inv}})$ the normal bundles to each L_i^{inv} in L_i . We pull back the bundle $N(M^{\text{inv}})$ along the projection map $M^{\text{inv}} \times [0, 1] \rightarrow M^{\text{inv}}$. Call this pullback $\Upsilon(M^{\text{inv}})$. This bundle is constant with respect to the interval $[0, 1]$. Its restriction to each $M^{\text{inv}} \times \{t\}$ is a copy of $N(M^{\text{inv}})$ which will occasionally, by a slight abuse of notation, be called $N(M^{\text{inv}}) \times \{t\}$; similarly, for $i = 0, 1$ the copy of $N(L_i^{\text{inv}})$ above $L_i^{\text{inv}} \times \{t\}$ will be referred to as $N(L_i^{\text{inv}}) \times \{t\}$.

We make a note here of the correspondence between our notation and Seidel and Smith’s original usage. Our bundle $\Upsilon(M^{\text{inv}})$ is their TM^{anti} ; while our $N(L_0^{\text{inv}}) \times \{0\}$ is their TL_0^{inv} and our $N(L_1^{\text{inv}}) \times \{1\}$ is their TL_1^{anti} . (The name TL_1^{anti} is also used for the bundle that we denote $N(L_1^{\text{inv}}) \times \{0\}$, using the obvious isomorphism between the bundles.)

Definition 4.2. [32, Defn 18] A stable normal trivialization of the vector bundle $\Upsilon(M^{\text{inv}})$ over $M^{\text{inv}} \times [0, 1]$ consists of the following data.

- A stable trivialization of unitary vector bundles $\phi: \Upsilon(M^{\text{inv}}) \oplus \mathbb{C}^K \rightarrow \mathbb{C}^{k_{\text{anti}}+K}$ for some K .
- A Lagrangian subbundle $\Lambda_0 \subset (\Upsilon(M^{\text{inv}}))|_{[0,1] \times L_0^{\text{inv}}}$ such that $\Lambda_0|_{\{0\} \times L_0^{\text{inv}}} = (N(L_0^{\text{inv}}) \times \{0\}) \oplus \mathbb{R}^K$ and $\phi(\Lambda_0|_{\{1\} \times L_0^{\text{inv}}}) = \mathbb{R}^{k_{\text{anti}}+K}$.
- A Lagrangian subbundle $\Lambda_1 \subset (\Upsilon(M^{\text{inv}}))|_{[0,1] \times L_1^{\text{inv}}}$ such that $\Lambda_1|_{\{0\} \times L_1^{\text{inv}}} = (N(L_1^{\text{inv}}) \times \{0\}) \oplus \mathbb{R}^K$ and $\phi(\Lambda_1|_{\{1\} \times L_1^{\text{inv}}}) = i\mathbb{R}^{k_{\text{anti}}+K}$.

The crucial theorem of [32], proved through extensive geometric analysis and comparison with the Morse theoretic case, is as follows.

Theorem 4.3. [32, Thm 20] *If $\Upsilon(M^{\text{inv}})$ carries a stable normal trivialization, then after an equivariant exact Lagrangian isotopy which replaces L_0 and L_1 with \tilde{L}_0 and \tilde{L}_1 and fixes the invariant sets, $HF_{\text{borel}}(\tilde{L}_0, \tilde{L}_1)$ is well-defined and there are localization maps*

$$\Delta^{(m)}: HF_{\text{borel}}(\tilde{L}_0, \tilde{L}_1) \rightarrow HF(L_0^{\text{inv}}, L_1^{\text{inv}})[[\theta]]$$

defined for $m \gg 0$ and satisfying $\Delta^{(m+1)} = \theta \Delta^{(m)}$. Moreover, after tensoring over $\mathbb{Z}_2[[\theta]]$ with $\mathbb{Z}_2((\theta))$ these maps are isomorphisms.

Let us say a few words about the appearance of the isotopy carrying L_i to \tilde{L}_i in Theorem 4.3, and explain how Theorem 4.3 implies Theorem 1.10. One of the uses of the stable normal trivialization condition in the proof of Theorem 4.3 is to construct both this isotopy and a family of complex structures \mathbf{J} on M with respect to which $HF_{\text{Borel}}(\tilde{L}_0, \tilde{L}_1)$ is well-defined. After applying the isotopy, we replace $(CF(L_0, L_1), \partial)$ with a chain homotopy equivalent complex $(CF(\tilde{L}_0, \tilde{L}_1), \partial')$ and $\tau_{\#}$ with some possibly *not* chain homotopy equivalent chain map $\tau'_{\#}$ which induced by the action of τ on the generators of $CF(\tilde{L}_0, \tilde{L}_1)$. The first page of the Seidel–Smith spectral sequence is the chain complex $(CF(\tilde{L}_0, \tilde{L}_1) \otimes \mathbb{Z}_2[[\theta]], \partial' + (1 + \tau'_{\#})\theta)$; computing vertical differentials first gives a spectral sequence from $HF(L_0, L_1) \otimes \mathbb{Z}_2[[\theta]] = HF(\tilde{L}_0, \tilde{L}_1) \otimes \mathbb{Z}_2[[\theta]]$ to $HF_{\text{Borel}}(\tilde{L}_0, \tilde{L}_1)$, which after tensoring with θ^{-1} becomes a spectral sequence from $HF(L_0, L_1) \otimes \mathbb{Z}_2((\theta))$ to $HF(L_0^{\text{inv}}, L_1^{\text{inv}}) \otimes \mathbb{Z}_2((\theta))$.

We can in fact dispense with the symplectic structure on M and work on the level of the complex normal bundle $\Upsilon(M^{\text{inv}})$ with its totally real subbundles $NL_0^{\text{inv}} \times \{1\}$ and $J(NL_1^{\text{inv}} \times \{1\})$. The following lemma is mentioned in [32, Section 3d]; a detailed proof is laid out in [14, Proposition 7.1].

Lemma 4.4. *The existence of a stable normal trivialization of (M, L_0, L_1) is implied by the existence of a nullhomotopy of the map*

$$(M, L_0, L_1) \rightarrow (BU, BO)$$

which classifies the complex normal bundle $\Upsilon(M^{\text{inv}}) = NM^{\text{inv}} \times [0, 1]$ and its totally real subbundles $NL_0^{\text{inv}} \times \{0\}$ over $L_0^{\text{inv}} \times \{0\}$ and $J(NL_1^{\text{inv}} \times \{1\})$ over $L_1^{\text{inv}} \times \{1\}$.

Let $\tilde{\mathcal{D}} = (S^2, \tilde{\alpha}, \tilde{\beta}, \tilde{\mathbf{w}}, \tilde{\mathbf{z}})$ be a multipointed Heegaard diagram for \tilde{K} defined using the method of Section 3, and $\mathcal{D} = (S^2, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be its quotient under the involution τ . Given x a point on D , let x^1, x^2 be its two lifts to \tilde{D} in some order. There is a natural map

$$\begin{aligned} \iota: \text{Sym}^{n_1}(S^2) &\rightarrow \text{Sym}^{2n_1}(S^2) \\ (x_1 \cdots x_{n_1}) &\mapsto (x_1^1 x_1^2 \cdots x_{n_1}^1 x_{n_1}^2). \end{aligned}$$

This map is a holomorphic embedding; for a proof, see [14, Appendix 1]. Moreover, consider the induced involution on $\text{Sym}^{2n_1}(S^2)$, which through a slight abuse of notation we will also call τ . The fixed set of τ is exactly our embedded copy of $\text{Sym}^{n_1}(S^2)$; moreover, τ preserves the two tori $\mathbb{T}_{\tilde{\alpha}}$ and $\mathbb{T}_{\tilde{\beta}}$, with fixed sets $T_{\tilde{\alpha}}^{\text{inv}} = \mathbb{T}_{\alpha}$ and $T_{\tilde{\beta}}^{\text{inv}} = \mathbb{T}_{\beta}$.

Perutz has shown that for an arbitrary Heegaard diagram $D = (S, \alpha, \beta, \mathbf{w}, \mathbf{z})$, there is a symplectic form ω on $\text{Sym}^{g+n-1}(S)$ which is compatible with the complex structure induced by a complex structure on S , and with respect to which the submanifolds \mathbb{T}_{α} and \mathbb{T}_{β} are in fact Lagrangian and the various Heegaard Floer homology theories are their Lagrangian Floer cohomologies [29, Thm 1.2]. In particular, the knot Floer homology is the Floer cohomology of these two tori in the ambient space $\text{Sym}^{g+n-1}(S \setminus (\mathbf{z} \cup \mathbf{w}))$, where the removal of the basepoints accounts for the restriction that holomorphic curves not be permitted to intersect the submanifolds V_{w_i} and V_{z_j} of the symmetric product.

In order to apply Theorem 4.3 to the case of doubly periodic knots, we will work with three different subspaces of $\text{Sym}^{2n_1}(S^2)$ and their fixed sets

under the involution τ , as follows.

$$\begin{aligned} M_0 &= \text{Sym}^{2n_1}(S^2 \setminus \tilde{\mathbf{w}}) & M_0^{\text{inv}} &= \text{Sym}^{n_1}(S^2 \setminus \mathbf{w}) \\ M_1 &= \text{Sym}^{2n_1}(S^2 \setminus (\tilde{\mathbf{w}} \cup (\tilde{\mathbf{z}} - z_0))) & M_1^{\text{inv}} &= \text{Sym}^{n_1}(S^2 \setminus (\mathbf{w} \cup (\mathbf{z} - z_0))) \\ M_2 &= \text{Sym}^{2n_1}(S^2 \setminus (\tilde{\mathbf{w}} \cup \tilde{\mathbf{z}})) & M_2^{\text{inv}} &= \text{Sym}^{n_1}(S^2 \setminus (\mathbf{w} \cup \mathbf{z})) \end{aligned}$$

In all cases the Lagrangians and their invariant sets under the involution will be as follows.

$$\begin{aligned} L_0 &= \mathbb{T}_{\tilde{\beta}} & L_0^{\text{inv}} &= \mathbb{T}_{\beta} \\ L_1 &= \mathbb{T}_{\tilde{\alpha}} & L_1^{\text{inv}} &= \mathbb{T}_{\alpha} \end{aligned}$$

The following is immediate from the definitions.

Lemma 4.5. *With respect to our choice of symplectic manifolds M_i for $i = 1, 2, 3$ and Lagrangians L_0 and L_1 , we have the following Floer cohomology groups.*

In M_1 ,

$$\begin{aligned} HF(L_0, L_1) &= HF(\mathbb{T}_{\tilde{\beta}}, \mathbb{T}_{\tilde{\alpha}}) = \widehat{HF}L(S^3, \tilde{K} \cup \tilde{U}) \otimes V^{\otimes(2n_1-1)}. \\ HF(L_0^{\text{inv}}, L_1^{\text{inv}}) &= HF(\mathbb{T}_{\beta}, \mathbb{T}_{\alpha}) = \widehat{HF}L(S^3, K \cup U) \otimes V^{\otimes(n_1-1)}. \end{aligned}$$

In M_2 ,

$$\begin{aligned} HF(L_0, L_1) &= HF(\mathbb{T}_{\tilde{\beta}}, \mathbb{T}_{\tilde{\alpha}}) = \widehat{HF}K(S^3, \tilde{K}) \otimes V^{\otimes(2n_1-1)} \otimes W. \\ HF(L_0^{\text{inv}}, L_1^{\text{inv}}) &= HF(\mathbb{T}_{\beta}, \mathbb{T}_{\alpha}) = \widehat{HF}K(S^3, K) \otimes V^{\otimes(n_1-1)} \otimes W. \end{aligned}$$

Each of the triples (M_i, L_0, L_1) satisfies the basic symplectic conditions of Seidel and Smith’s theory, by arguments similar to [14, Section 4]. Therefore to demonstrate that that (M_i, L_0, L_i) carries a stable normal trivialization, it suffices to check the existence of a nulhomotopy of the maps in Lemma 4.4. We will prove in Section 6 that the map $(M_0, L_0, L_1) \rightarrow (BU, BO)$ admits a nulhomotopy; this then naturally restricts to a nulhomotopy of $(M_i, L_0, L_1) \rightarrow (M_0, L_0, L_1) \rightarrow (BU, BO)$ for $i = 1, 2$. In some moral sense, this is the correct level of generality: the most important feature of our punctured Heegaard diagram is that no periodic domain has nonzero index, and \mathbf{w} (or \mathbf{z}) is the smallest set of points at which we may puncture S^2 and produce a diagram for which this property holds. Moreover, M_0^{inv} conveniently deformation retracts onto each of \mathbb{T}_{α} and \mathbb{T}_{β} , making certain

cohomology computations in Sections 5 and 6 cleaner than they might otherwise be.

Let us pause to consider gradings in these theories. Recall that in general the involution $\tau_{\#}$ on $\widehat{CFK}(\tilde{\mathcal{D}})$ arising from the action of τ on $\tilde{\mathcal{D}}$ is replaced by an involution $\tau'_{\#}$ on a chain homotopy equivalent complex $\widehat{CFK}(\tilde{\mathcal{D}})'$. To see that $\tau'_{\#}$ preserves Alexander (multi)gradings, consider that Alexander gradings are determined by relative spin^c structures on $S^3 - \mu(\tilde{L})$, which correspond to homotopy classes of paths between \mathbb{T}_{α} and \mathbb{T}_{β} in $\text{Sym}^{2n_1}(\Sigma(S) \setminus (\tilde{\mathbf{w}} \cup \tilde{\mathbf{z}}))$ [27]. If the isotopy of Theorem 4.3 replaces \mathbb{T}_{α} and \mathbb{T}_{β} with \mathbb{T}'_{α} and \mathbb{T}'_{β} , then homotopy classes of paths between the original Lagrangians are in canonical bijection with homotopy classes of paths between the new Lagrangians, and therefore $\tau'_{\#}$ also preserves relative spin^c structures, and therefore Alexander gradings. Moreover, within any fixed Alexander grading, $\tau'_{\#}$ must preserve the Maslov grading, because the composition of any holomorphic disk with τ is another holomorphic disk with the same Maslov index, and within any Alexander grading the Maslov grading is entirely determined by the Maslov indices of Whitney disks. In some simple cases, such as those computed in Section 7, knowing these properties of $\tau'_{\#}$ constrains the behavior of the spectral sequence very strongly.

Finally, the localization map of Theorem 4.3 is entirely constructed by counting pseudoholomorphic disks of varying index and by multiplication and division by powers of θ . In particular, on M_1^{inv} , the localization isomorphism preserves the Alexander multigrading (A_1, A_2) on the E^{∞} page, because no flowlines can pass over the missing basepoint divisors V_{z_i} and V_{w_i} for $0 \leq i \leq n_1$, and on M_2^{inv} , the localization isomorphism preserves the Alexander grading A_1 since flowlines can pass over V_{z_0} and V_{w_0} but no other basepoint divisors. In general, however, we should expect that the localization isomorphisms will not preserve the data of the Maslov grading.

5. The geometry of the symmetric product

Over the next two sections, we will prove that the triple $(M_0^{\text{inv}}, L_0^{\text{inv}}, L_1^{\text{inv}})$ satisfies the the complex conditions of Lemma 4.4, and therefore has a stable normal trivialization. The first thing to do is describe the homotopy type and cohomology of M_0^{inv} .

We claim M_0^{inv} deformation retracts onto each of \mathbb{T}_{α} and \mathbb{T}_{β} . To check this, we refer to a lemma whose proof is outlined in [14, Lemma 5.1] following an argument of Ong [21].

Lemma 5.1. *The r th symmetric product of a wedge of k circles $\bigvee_{i=1}^k S_i^1$ deformation retracts onto the r -skeleton of the k torus $\prod_{i=1}^k S_i^1$, where each circle is given a CW structure consisting of the wedge point and a single one-cell, and the torus has the natural product CW structure.*

We will apply this observation to M_0^{inv} . In S^2 , let $\nu_i: [0, 1] \rightarrow S^2$ be a small closed curve around z_i for $0 \leq i \leq n_1$, such that ν_i is oriented counterclockwise in the complement of w_0 . Then

$$H_1(S^2 \setminus \mathbf{z}) = \mathbb{Z}\langle \nu_0, \nu_1, \dots, \nu_{n_1} \rangle.$$

Now $S^2 \setminus \{\mathbf{w}\}$ deformation retracts onto a wedge of n_1 circles $\bigvee_{i=1}^{n_1} \nu'_i$, where ν'_i is a closed curve homotopic to ν_i which passes once through the origin. Therefore $M_0^{\text{inv}} = \text{Sym}^{n_1}(S^2 \setminus \mathbf{w})$ deformation retracts onto the symmetric product of $\bigvee_{i=1}^{n_1} \nu'_i$, which in turn deformation retracts onto the product $\prod_{i=1}^{n_1} \nu'_i$. However, this product is homotopy equivalent to $\prod_{i=1}^{n_1} \nu_i$, and indeed to $\prod_{i=1}^{n_1} \alpha_i$ and to $\prod_{i=1}^{n_1} \beta_i$. We conclude that M_0^{inv} has the homotopy type of an n_1 -torus and in particular admits a deformation retraction onto each of \mathbb{T}_α and \mathbb{T}_β .

We will require a concrete description of the cohomology rings of M_0^{inv} , \mathbb{T}_α , and \mathbb{T}_β for the computations of Section 6, so we supply one now. Consider the one cycles

$$\begin{aligned} \bar{\nu}_i: [0, 1] &\rightarrow M_0^{\text{inv}} \\ t &\mapsto (\nu_i(t)x_0 \cdots x_0) \end{aligned}$$

where x_0 is any choice of basepoint. The $[\bar{\nu}_i]$ form a basis for $H_1(M_0^{\text{inv}})$. Ergo, letting $[\bar{\nu}_i]^*$ denote the dual of $\bar{\nu}_i$, we have

$$\begin{aligned} H^1(M_0^{\text{inv}}) &= \mathbb{Z}\langle [\bar{\nu}_0]^*, \dots, [\bar{\nu}_{n_1}]^* \rangle \\ H^k(M_0^{\text{inv}}) &= \bigwedge^k H^1(M_0^{\text{inv}}) \end{aligned}$$

Similarly, we can write down the homology of the tori \mathbb{T}_α and \mathbb{T}_β . Through a slight abuse of notation, let us insist that we have parametrizations $\alpha_i: [0, 1] \rightarrow S^2$ running counterclockwise in $S^2 \setminus \{w_0\}$ and β_i running clockwise. The first homology of \mathbb{T}_α is generated by one-cycles

$$\begin{aligned} \bar{\alpha}_i: [0, 1] &\rightarrow \alpha_1 \times \cdots \times \alpha_{n_1} \\ t &\mapsto (y_1, \dots, y_{i-1}, \alpha_i(t), y_{i+1}, \dots, y_{n_1}) \end{aligned}$$

where $y_j = \alpha_j(0)$, and thus the cohomology of this torus is

$$\begin{aligned} H^1(\mathbb{T}_\alpha) &= \mathbb{Z}\langle [\overline{\alpha_1}]^*, \dots, [\overline{\alpha_{n_1}}]^* \rangle \\ H^k(\mathbb{T}_\alpha) &= \bigwedge^k H^1(\mathbb{T}_\alpha) \end{aligned}$$

We apply analagous naming conventions to \mathbb{T}_β , obtaining

$$\begin{aligned} H^1(\mathbb{T}_\beta) &= \mathbb{Z}\langle [\overline{\beta_1}]^*, \dots, [\overline{\beta_{n_1}}]^* \rangle \\ H^k(\mathbb{T}_\beta) &= \bigwedge^k H^1(\mathbb{T}_\beta) \end{aligned}$$

Let $X \subset M_0^{\text{inv}} \times [0, 1]$ be $(L_0^{\text{inv}} \times \{0\}) \cup (L_1^{\text{inv}} \times \{1\})$. Observe that under the map on homology induced by inclusion $\iota: X \hookrightarrow M_0^{\text{inv}} \times [0, 1]$, both $[\overline{\alpha_i}]$ and of $[\overline{\beta_i}]$ are sent to $[\overline{\nu_i}]$. Therefore the map ι^* on cohomology $H^k(M_0^{\text{inv}} \times [0, 1]) \rightarrow H^k(X)$ is precisely the diagonal map, with

$$\bigwedge_{j=1}^m [\overline{\nu_{i_j}}]^* \mapsto \bigwedge_{j=1}^m [\overline{\alpha_{i_j}}]^* + \bigwedge_{j=1}^m [\overline{\beta_{i_j}}]^*$$

Corollary 5.2. *The relative cohomology $H^*(M_0^{\text{inv}} \times [0, 1], X)$ is the cohomology of the torus $(S^1)^{n_1}$, and in particular is torsion-free.*

Proof. Consider the long exact sequence

$$\dots H^{m-1}(X) \longrightarrow H^m(M_0^{\text{inv}} \times [0, 1], X) \xrightarrow{q^*} H^m(M_0^{\text{inv}}) \xrightarrow{\iota^*} H^m(X) \dots$$

Taking into account that the obvious isomorphism between $H^*(\mathbb{T}_\alpha)$ and $H^*(\mathbb{T}_\beta)$ respects our labelling of the cohomology classes, ι^* is the diagonal map, and in particular an injection. When $m \geq 2$ we have the following short exact sequences.

$$\begin{aligned} 0 \longrightarrow H^{m-1}(M_0^{\text{inv}} \times [0, 1]) &\xrightarrow{\iota^*} H^{m-1}(X) \\ &\longrightarrow H^m(M_0^{\text{inv}} \times [0, 1], X) \longrightarrow 0 \end{aligned}$$

Here the map i^* sends the wedge $\bigwedge_{j=1}^{m-1} [\overline{\nu_{i_j}}]^*$ to the sum $\bigwedge_{j=1}^{m-1} [\overline{\alpha_{i_j}}]^* + \bigwedge_{j=1}^{m-1} [\overline{\beta_{i_j}}]^*$. We therefore see that for $m \geq 2$, $H^m(M_0^{\text{inv}} \times [0, 1], X) \cong H^{m-1}(\mathbb{T}_\alpha) \cong H^{m-1}(\mathbb{T}_\beta)$. (We will have occasion to be careful about the generators in Section 6.) The same result follows for $H^1(M_0^{\text{inv}} \times [0, 1], X)$ trivially. \square

Let us consider the implications of this result for the relative K -theory of $(M_0^{\text{inv}} \times [0, 1], X)$, which is isomorphic to the reduced K -theory $\widetilde{K}((M_0^{\text{inv}} \times$

$[0, 1]/X$). (For a review of K -theory, see [12, Chapter 2]; all of the facts used in this paper are also summarized in [14, Section 6].) Notice that $(M_0^{\text{inv}} \times [0, 1], X)$ deformation retracts onto the compact CW pair $(\text{Sym}^{n_1}(S^2 \setminus \bigcup_i \nu(w_i)) \times [0, 1], X)$, where $\nu(w_i)$ is a small open neighborhood. This deformation retraction makes it legitimate to consider the reduced K -theory of the pair by identifying it with $\tilde{K}((\text{Sym}^{n_1}(S^2 \setminus \bigcup_i \nu(w_i)) \times [0, 1])/X)$. From now on we will apply this trick without mention.

Recall that there is a rational ring isomorphism

$$\tilde{\text{ch}}: (\tilde{K}^0(X) \oplus \tilde{K}^1(X)) \otimes \mathbb{Q} \rightarrow \tilde{H}^*(X; \mathbb{Q})$$

between the rational reduced K -theory and rational reduced cohomology of any space X chosen such that if V is a line bundle over X and $c_1(V)$ is the first Chern class of X , $\tilde{\text{ch}}(V) = \sum_{i=1}^{\infty} \frac{c_1(V)^i}{i!}$ and $\tilde{\text{ch}}$ is a ring homomorphism. Using this isomorphism and the Atiyah-Hirzebruch spectral sequence, Atiyah and Hirzebruch have shown that if the reduced cohomology $\tilde{H}^*(X)$ is torsion-free, the reduced K -theory $\tilde{K}^*(X)$ is as well, and the stable isomorphism class of a vector bundle is determined entirely by its Chern classes [1, Section 2.5]. In particular, since $H^*(M_0^{\text{inv}} \times [0, 1], X) = \tilde{H}^*((M_0^{\text{inv}} \times [0, 1])/X)$ is torsion-free, the stable isomorphism class of a complex vector bundle over (M_0^{inv}, X) — that is, a bundle whose restriction to X is stably trivial — is entirely determined by its Chern classes. To show such a bundle is stably trivial, it suffices to show that all its Chern classes are zero.

6. Stable normal triviality of the normal bundle

Having remarked that the symplectic geometry conditions of Seidel and Smith’s theory are satisfied for (M_i, L_0, L_1) , when $i = 0, 1, 2$, we now proceed to check that (M_0, L_0, L_1) fulfills the complex conditions that, by Lemma 4.4, imply the existence of a stable normal trivialization.

Proposition 6.1. *Consider the complex manifold $M_0 = \text{Sym}^{2n_1}(S^2 \setminus \tilde{\mathbf{w}})$ together with its totally real submanifolds $L_0 = \mathbb{T}_{\beta}$ and $L_1 = \mathbb{T}_{\alpha}$, and the holomorphic involution τ which preverves L_0 and L_1 . The map*

$$(M_0^{\text{inv}} \times [0, 1], (L_0 \times \{0\}) \cup (L_1 \times \{1\})) \rightarrow (BU, BO)$$

which classifies the pullback $\Upsilon(M_0^{\text{inv}}) = N(M_0^{\text{inv}}) \times [0, 1]$ of the complex normal bundle of M_0^{inv} together with the totally real subbundles $NL_0^{\text{inv}} \times \{0\}$ over $L_0^{\text{inv}} \times \{0\}$ and $J(NL_1^{\text{inv}}) \times \{1\}$ over $L_1^{\text{inv}} \times \{1\}$ is nulhomotopic.

As a first step, we must establish the complex triviality of NM_0^{inv} (and thus of $\Upsilon(M_0^{\text{inv}})$) and the real triviality of NL_i^{inv} for $i = 0, 1$.

Lemma 6.2. *The complex bundle NM_0^{inv} is stably trivial.*

Proof. The inclusion map $\iota_1: (S^2 \setminus \mathbf{w}) \hookrightarrow S^2$ is nulhomotopic. Therefore the induced inclusion $\text{Sym}^{n_1}(\iota_1): \text{Sym}^{n_1}(S^2 \setminus \mathbf{w}) \hookrightarrow \text{Sym}^{n_1}(S^2)$ is also nulhomotopic. The normal bundle of $M_0^{\text{inv}} = \text{Sym}^{n_1}(S^2 \setminus \mathbf{w})$ in $M_0 = \text{Sym}^{2n_1}(S^2 \setminus \tilde{\mathbf{w}})$ is exactly the restriction of the normal bundle to $\text{Sym}^{n_1}(S^2)$ in $\text{Sym}^{2n_1}(S^2)$ along the inclusion map $\text{Sym}^{n_1}(\iota_1)$. As the map $\text{Sym}^{n_1}(\iota_1)$ is nulhomotopic, NM_0^{inv} is stably trivial. \square

The proof of the next lemma proceeds exactly as in [14, Lemma 7.3].

Lemma 6.3. *The normal bundles of $\mathbb{T}_\alpha \subset \mathbb{T}_{\tilde{\alpha}}$ and $\mathbb{T}_\beta \subset \mathbb{T}_{\tilde{\beta}}$ are trivial.*

We now turn to the question of relative triviality. Let $X = (L_0 \times \{0\}) \cup (L_1 \times \{1\})$ as in Section 5. Choose preferred trivializations of the totally real bundles $NL_0^{\text{inv}} \times \{0\}$ and $J(NL_1^{\text{inv}}) \times \{1\}$ and tensor with \mathbb{C} to extend to a preferred trivialization of the complex bundle $\Upsilon(M_0^{\text{inv}})|_X$. We use this trivialization to pull back $[\Upsilon(M_0^{\text{inv}})] \in \tilde{K}^0(M_0^{\text{inv}} \times [0, 1])$ to a relative bundle $[\Upsilon(M_0^{\text{inv}})]_{\text{rel}} \in \tilde{K}^0((M_0^{\text{inv}} \times [0, 1])/X)$. Because the reduced cohomology, and therefore the reduced K -theory, of $(M_0^{\text{inv}} \times [0, 1])/X$ have no torsion, to verify triviality of the relative bundle, it suffices to show that the Chern classes of $[\Upsilon(M_0^{\text{inv}})]_{\text{rel}}$ are trivial.

Remark 6.4. It may be helpful to draw attention to a minor problem of notation: in earlier sections, \tilde{K} is a doubly periodic knot, but also in the present section $\tilde{K}^0(B)$ is the reduced K -theory of the topological space B . We hope this will not occasion confusion.

Fundamentally, the argument for relative triviality rests on the fact that each of the n_1 linearly independent periodic domains in $S^2 \setminus \mathbf{w}$ has Maslov index zero. Therefore, we pause here to recall the notation of Section 3. Let x_i be the single positive intersection point in $\alpha_i \cap \beta_i$ and y_i the negative intersection point. Let F_i be the closure of the component of $S - \alpha_i - \beta_i$ containing z_i and E_i be the closure of the component of $S - \alpha_i - \beta_i$ containing z_{i+1} (or z_1 if $i = n_1$). Then $P_i = E_i - F_i$ is a periodic domain of index zero on \mathcal{D} with boundary $\beta_i - \alpha_i$. Finally, let γ_i be the union of the arc of α_i running from x_i to y_i and the arc of β_i running from x_i to y_i . In particular, this specifies that γ_i has no intersection with any α or β curves other than

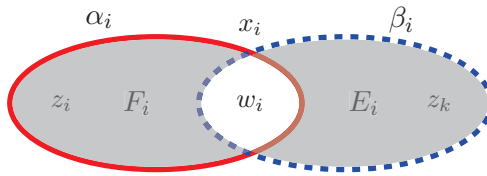


Figure 3. The periodic domain $P_i = E_i - F_i$ has Maslov index zero.

α_i and β_i , and moreover the component of $S - \gamma_i$ which does not contain w_0 contains only a single basepoint w_i . See Figure 3 for an illustration of the domain P_i .

The structure of the argument is as follows: for each even k such that $1 \leq k \leq 2n_1$, we will use the periodic domains P_i to produce a set of k -chains $\{W_{\mathbf{I}}\}$ in $(M_0^{\text{inv}} \times [0, 1], X)$ whose relative homology classes generate the k th homology of $H_k(M_0^{\text{inv}} \times [0, 1], X)$. We will then show that the restriction of $\Upsilon(M_0^{\text{inv}})|_{\text{rel}}$ to each $W_{\mathbf{I}}$ is trivial as a relative vector bundle, and that therefore $\langle c_k(\Upsilon(M_0^{\text{inv}})|_{\text{rel}}), [W_{\mathbf{I}}] \rangle = 0$. Since the $[W_{\mathbf{I}}]$ generate $H_k(M_0^{\text{inv}} \times [0, 1], X)$, we will have proved that $c_k(\Upsilon(M_0^{\text{inv}})|_{\text{rel}})$ is identically zero.

More specifically, we will describe the chains $W_{\mathbf{I}}$ as a subset of a product of two-chains Y_i in $(S^2 \setminus \mathbf{w}) \times [0, 1]$ such that the projection to $S^2 \setminus \mathbf{w}$ is the periodic domain P_i and the Y_i are pairwise disjoint. We will then show that the restriction of the bundle $\Upsilon(M_0^{\text{inv}})|_{\text{rel}}$ to $W_{\mathbf{I}}$ also breaks up as the restriction of a product of relative bundles $\Upsilon(Y_i)$ over the Y_i which are known to be trivial through Maslov index arguments. Let us begin by constructing these manifolds Y_i .

For $1 \leq i \leq n_1$, let Y_i be a subspace of $S^2 \times [0, 1]$ with the following properties:

- Y_i is topologically $S^1 \times [0, 1]$, and $Y_i \cap (S^2 \times \{t\})$ is S^1 for all t .
- The boundary of Y_i is $\beta_i \times \{0\} - \alpha_i \times \{1\}$.
- The projection of Y_i to the punctured sphere is a copy of the periodic domain P_i .
- $Y_i \cap Y_j = \emptyset$ if $i \neq j$.
- If x_i is the point of positive intersection of α_i and β_i , $\{x_i\} \times [0, 1] \subset Y_i$.

The 2-chains Y_i are constructed as follows: for $t \in [0, \frac{1}{2}]$, choose a linear homotopy H_t from β_i to γ_i inside the domain E_i which fixes $\beta_i \cap \gamma_i$. Let

the intersection of Y_i with $S^2 \times \{t\}$ be the embedded circle $H_t(\beta_i) \times \{t\}$. Similarly, for $t \in [\frac{1}{2}, 1]$, choose a linear homotopy J_t from γ_i to α_i inside the domain F_i which fixes $\alpha_i \cap \gamma_i$, and let the intersection of Y_i with $S^2 \times \{t\}$ be $J_t(\gamma_i)$.

Observe that this description of Y_i has the following properties: first, Y_i is contained in $(S^2 \setminus \mathbf{w}) \times [0, 1]$ as promised. Second, Y_i contains the line segment $\{x_i\} \times [0, 1]$. Third, the intersection of Y_i with $(\alpha_i \cup \beta_i) \times (0, 1)$ is entirely contained in the cylinder $\gamma_i \times (0, 1)$. Finally, the projection of $Y_i \cap (S^2 \times [0, \frac{1}{2}])$ to S^2 lies entirely “inside” β_i - that is, on the component of $S^2 \setminus \beta_i$ not containing w_0 - implying that the sets $Y_i \cap (S^2 \times [0, \frac{1}{2}])$ are pairwise disjoint. Similarly, the projection of $Y_i \cap (S^2 \times [\frac{1}{2}, 1])$ to S^2 lies entirely “inside” α_i , and therefore the sets $Y_i \cap (S^2 \times [\frac{1}{2}, 1])$ are pairwise disjoint. Ergo the Y_i are pairwise disjoint.

We are now ready to define the complex line bundles $\Upsilon(Y_i)$. Recall from Section 4 that there is a holomorphic embedding

$$\begin{aligned} \iota: M_0^{\text{inv}} = \text{Sym}^{n_1}(S^2 \setminus \mathbf{w}) &\rightarrow \text{Sym}^{n_1}(S^2 \setminus \tilde{\mathbf{w}}) = M_0 \\ (x_1 \cdots x_{n_1}) &\mapsto (x_1^1 x_1^2 \cdots x_{n_1}^1 x_{n_1}^2). \end{aligned}$$

Similarly, we have a holomorphic embedding

$$\begin{aligned} \iota': (S^2 \setminus \mathbf{w}) &\rightarrow \text{Sym}^2(S^2 \setminus \tilde{\mathbf{w}}) \\ x &\mapsto (x^1 x^2) \end{aligned}$$

which takes a point x on $S^2 \setminus \mathbf{w}$ to the unordered pair consisting of its two (not necessarily distinct) lifts on $S^2 \setminus \tilde{\mathbf{w}}$ under the projection map $\pi: (S^2 \setminus \tilde{\mathbf{w}}) \rightarrow (S^2 \setminus \mathbf{w})$. Notice that for each $1 \leq i \leq n_1$, $\iota'(\alpha_i) \subset \alpha_i^1 \times \alpha_i^2$ and $\iota'(\beta_i) \subset \beta_i^1 \times \beta_i^2$. Let $N(S^2 \setminus \mathbf{w})$ be the normal bundle to $S^2 \setminus \mathbf{w}$ in the second symmetric product $\text{Sym}^2(S^2 \setminus \tilde{\mathbf{w}})$. This is a complex line bundle over a punctured sphere, hence trivial. For each $1 \leq i \leq n_1$, let $N\alpha_i$ be the real normal bundle to α_i in $\alpha_i^1 \times \alpha_i^2$ and $N\beta_i$ be the real normal bundle to β_i in $\beta_i^1 \times \beta_i^2$. An argument similar to that of Lemma 6.3 shows that $N\alpha_i$ and $N\beta_i$ are trivial real line bundles for all i .

Consider the pullback of the normal bundle $N(S^2 \setminus \mathbf{w})$ to $(S^2 \setminus \mathbf{w}) \times [0, 1]$. The complex line bundle over Y_i that interests us is the restriction of this pullback to Y_i , which we shall by analogy denote $\Upsilon(Y_i)$. That is, $\Upsilon(Y_i) = (N(S^2 \setminus \mathbf{w}) \times [0, 1])|_{Y_i}$. This complex line bundle has totally real subbundles $J(N\alpha_i \times \{1\})$ over $\alpha_i \times \{1\}$ and $N\beta_i \times \{0\}$ over $\beta_i \times \{0\}$. Choose preferred

real trivializations of these real line bundles, and extend to a complex trivialization of $\Upsilon(Y_i)|_{(\beta_i \times \{0\}) \cup (\alpha_i \times \{1\})}$ by tensoring with \mathbb{C} . For convenience, let this subspace $(\beta_i \times \{0\}) \cup (\alpha_i \times \{1\})$ be X_i .

Lemma 6.5. *For $1 \leq i \leq n_1$, given any preferred trivialization of $\Upsilon(Y_i)|_{X_i}$ as a complex vector bundle, the relative vector bundle $\Upsilon(Y_i)|_{\text{rel}}$ over (Y_i, X_i) is stably trivial.*

Proof. Let ι_i be the inclusion of Y_i into $(S^2 \setminus \mathbf{w}) \times [0, 1]$, and p be the projection of $(S^2 \setminus \mathbf{w}) \times [0, 1]$ to $S^2 \setminus \mathbf{w}$. Then the image of $p \circ \iota_i(Y_i)$ is the periodic domain P_i .

Consider the commutative diagram below. The top horizontal inclusion of $S^2 \setminus \{w\}$ into $\text{Sym}^{n_1}(S^2 \setminus \mathbf{w})$ is defined by mapping a point x to $(xx_1 \cdots \widehat{x}_i \cdots x_{n_1})$, where again each x_j is the positively oriented point in $\alpha_j \cup \beta_j$. The bottom inclusion of $\text{Sym}^2(S^2 \setminus \widetilde{\mathbf{w}})$ into $\text{Sym}^{2n_1}(S^2 \setminus \widetilde{\mathbf{w}})$ sends an unordered pair (xy) to $(xyx_1^1 x_1^2 \cdots \widehat{x}_i^1 x_i^2 \cdots x_{n_1}^1 x_{n_1}^2)$.

$$\begin{array}{ccccc}
 Y_i & \xrightarrow{p \circ \iota_i} & S^2 \setminus \mathbf{w} & \hookrightarrow & \text{Sym}^{n_1}(S^2 \setminus \mathbf{w}) \\
 & \searrow & \downarrow \iota' & & \downarrow \iota \\
 & & \text{Sym}^2(S^2 \setminus \widetilde{\mathbf{w}}) & \hookrightarrow & \text{Sym}^{2n_1}(S^2 \setminus \widetilde{\mathbf{w}})
 \end{array}$$

Consider the map $\phi: Y_i \rightarrow \text{Sym}^{n_1}(S^2 \setminus \mathbf{w})$ given by composition along the top row of the diagram. This is a topological annulus representing the periodic domain P_i . Since P_i has Maslov index zero, by the discussion in Section 2, the first Chern class of the pullback of the complex tangent bundle $T \text{Sym}^{n_1}(S^2 \setminus \mathbf{w})$ to Y_i relative to the complexification of the pullbacks of the real tangent bundles $J(T(\mathbb{T}_\alpha))$ to one component of the boundary of Y_i and $T(\mathbb{T}_\beta)$ to the other is zero. However, the pullback $\phi^*(T \text{Sym}^{n_1}(S^2 \setminus \mathbf{w}))$ of the tangent bundle of the total symmetric product to Y_i is exactly $(p \circ \iota_1)_*(T(S^2 \setminus \mathbf{w}) \oplus \mathbb{C}^{n_1-1})$, where the factors of \mathbb{C} are the restriction of the tangent bundle of the punctured sphere to the points x_j such that $j \neq i$. Moreover, $(p \circ \iota_1)^*(T(S^2 \setminus \mathbf{w}))$ is precisely the restriction of the pullback bundle $p^*(T(S^2 \setminus \mathbf{w})) = T((S^2 \setminus \mathbf{w}) \times [0, 1])$ over $(S^2 \setminus \mathbf{w}) \times [0, 1]$ to the subspace Y_i . Therefore $\phi^*(T \text{Sym}^{n_1}(S^2 \setminus \mathbf{w})) = T((S \setminus \mathbf{w}) \times [0, 1])|_{Y_i} \oplus \mathbb{C}^{n_1-1}$.

Similarly, the pullback $\phi^*(J(T(\mathbb{T}_\alpha)))$ to the boundary component $\alpha_i \times \{1\} \subset Y_i$ is $(p \circ \iota_i)^*(J(T(\alpha_i) \oplus \mathbb{R}^{n_1-1}))$, where the factors of \mathbb{R} are the canonical real subspace of the tangent bundle to the punctured sphere at the points x_j for $j \neq i$. The pullback of this bundle to $\alpha_i \times \{1\} \subset Y_i$ is $J((T(\alpha_i) \times \{1\}) \oplus \mathbb{R}^{n_1-1})$. A similar argument shows that $\phi^*(J(T(\mathbb{T}_\beta)))$ is $(T(\beta_i) \times$

$\{0\} \oplus \mathbb{R}^{n_1-1}$. Therefore we have seen that the complex vector bundle $T((S^2 \setminus \mathbf{w}) \times [0, 1])|_{Y_i} \oplus \mathbb{C}^{n_1-1}$ relative to the complexification of its totally real subbundles $(T(\beta_i) \times \{0\}) \oplus \mathbb{R}^{n_1+1}$ over $\beta_i \times \{0\}$ and $J((T(\alpha_i) \times \{1\}) \oplus \mathbb{R}^{n_1+1})$ over $\alpha_i \times \{1\}$ has relative first Chern class zero. Hence the same is true of the complex line bundle $T((S^2 \setminus \mathbf{w}) \times [0, 1])|_{Y_i}$ relative to the complexification of its totally real subbundles $J(T(\alpha_i) \times \{1\})$ and $T(\beta_i) \times \{0\}$. As this is a line bundle, triviality of the first relative Chern class suffices to show stable triviality of the relative vector bundle.

Now consider the map $\tilde{\phi} = \iota \circ \phi$ from Y_i to $\text{Sym}^{2n_1}(S^2 \setminus \tilde{\mathbf{w}})$. This is a topological annulus representing the periodic domain $\pi^{-1}(P_i)$ in \tilde{D} , which also has Maslov index zero. Therefore the pullback along $\tilde{\phi}$ of the complex tangent bundle $T(\text{Sym}^{2n_1}(S^2 \setminus \tilde{\mathbf{w}}))$ to Y_i relative to complexifications of pullbacks of the totally real subbundles $J(T(\mathbb{T}_{\tilde{\alpha}}))$ and $T(\mathbb{T}_{\tilde{\beta}})$ to $\alpha_i \times \{1\}$ and $\beta_i \times \{0\}$ has trivial relative first Chern class. However, once again the pullback of this relative bundle along the inclusion map $\text{Sym}^2(S^2 \setminus \tilde{\mathbf{w}}) \hookrightarrow \text{Sym}^{2n_1}(S^2 \setminus \tilde{\mathbf{w}})$ decomposes into a much smaller tangent bundle together with a trivial summand. Observe that $\tilde{\phi}^*(T(\text{Sym}^{2n_1}(S^2 \setminus \tilde{\mathbf{w}})))$ is

$$(\iota' \circ (p \circ \iota_i))^*(T(\text{Sym}^2(S^2 \setminus \tilde{\mathbf{w}})) \oplus \mathbb{C}^{2n_1-2}).$$

Pulling back along the inclusion map ι' , we see that this bundle breaks up as $T(S^2 \setminus \mathbf{w}) \oplus N(S^2 \setminus \mathbf{w}) \oplus \mathbb{C}^{2n_1-2}$ over $S^2 \setminus \mathbf{w}$, and that therefore its ultimate pullback to Y_i is $T((S^2 \setminus \mathbf{w}) \times [0, 1]) \oplus \Upsilon(Y_i) \oplus \mathbb{C}^{2n_1-2}$.

Similarly, the pullback along $\tilde{\phi}$ of $J(T(\mathbb{T}_{\tilde{\alpha}}))$ to $\alpha_i \times \{1\}$ is precisely $J(T\alpha_i \times \{1\}) \oplus J(N\alpha_i \times \{1\}) \oplus J(\mathbb{R}^{2n_1-2})$. Finally, the pullback along $\tilde{\phi}$ of $T(\mathbb{T}_{\tilde{\beta}})$ to $\beta_i \times \{0\}$ is $(T\beta_i \times \{0\}) \oplus (N_{\beta_i} \times \{0\}) \oplus \mathbb{R}^{2n_1-2}$. Dropping the trivial summands, we conclude that the first relative Chern class of $T((S^2 \setminus \mathbf{w}) \times [0, 1]) \oplus \Upsilon(Y_i)$ relative to complexifications of the totally real subbundle $(T\beta_i \times \{0\}) \oplus (N_{\beta_i} \times \{0\})$ over $\beta_i \times \{0\}$ and the totally real subbundle $J(T\alpha_i \times \{1\}) \oplus J(N\alpha_i \times \{1\})$ over $\alpha_i \times \{1\}$ has relative first Chern class zero. This, combined with our previous conclusions concerning relative triviality of $T((S^2 \setminus \mathbf{w}) \times [0, 1])$ with respect to its subbundles $T\beta_i \times \{1\}$ over $\beta_i \times \{1\}$ and $J(T\alpha_i \times \{0\})$ over $\alpha_i \times \{0\}$, implies that $\Upsilon(Y_i)$ has relative first Chern class zero with respect to complexifications of its subbundles $N\beta_i \times \{0\}$ and $J(N\alpha_1 \times \{1\})$, as promised. \square

We are now ready to construct generators for $H_k(M_0^{\text{inv}} \times [0, 1], X)$, and use them to prove that the relative Chern classes of $[\Upsilon(M_0^{\text{inv}})]$ are identically

zero. Recall that for $k > 1$ there are short exact sequences on homology

$$\begin{aligned} 0 \longrightarrow H_{k+1}(M_0^{\text{inv}} \times [0, 1], X) &\longrightarrow H_k(\mathbb{T}_\beta) \oplus H_k(\mathbb{T}_\alpha) \\ &\xrightarrow{\iota_*} H_k(M_0^{\text{inv}}) \longrightarrow 0. \end{aligned}$$

As in Section 5, for $k > 1$, the $(k + 1)$ st homology group $H_{k+1}(M_0^{\text{inv}} \times [0, 1], X)$ is the kernel of the map ι_* . Let us take a closer look at generators for this group. We have seen that the kernel of ι_* is $\mathbb{Z}\langle \bigwedge_{j=1}^k [\beta_{i_j}] \oplus -\bigwedge_{j=1}^k [\alpha_{i_j}] : 1 \leq i_1 < \dots < i_k \leq n_1 \rangle$. Therefore for each $\mathbf{I} = (i_1, \dots, i_k)$ such that $1 \leq i_1 < \dots < i_k \leq n_1$, there is a $(k + 1)$ -chain $W_{\mathbf{I}}$ in $M_0^{\text{inv}} \times [0, 1]$ whose boundary is $\prod_{j=1}^k \beta_{i_j} - \prod_{j=1}^k \alpha_{i_j}$. We will describe this chain in terms of the two-chains Y_i .

Construct the k chain $W_{\mathbf{I}}$ as follows. Let $W_{\mathbf{I}}$ be topologically $(S^1)^k \times [0, 1]$, and insist that the intersection of $W_{\mathbf{I}}$ with $M_0^{\text{inv}} \times \{t\}$ is the product $\prod_{j=1}^k Y_{i_j} \cap (S^2 \setminus \mathbf{w}) \times \prod_{i \notin \mathbf{I}} \{x_i\}$. Since the intersection of each Y_{i_j} with $(S^2 \setminus \mathbf{w}) \times \{t\}$ is a circle for each t , this says that the intersection of $W_{\mathbf{I}}$ with $M_0^{\text{inv}} \times \{t\}$ is a k -torus for all t .

Notice that for all t , $W_{\mathbf{I}} \cap (M_0^{\text{inv}} \times \{t\})$ is a subset of the product $\prod_{i=1}^n (Y_i \cap ((S^2 \setminus \mathbf{w}) \times [0, 1]))$. (Recall here that the line segments $\{x_i\} \times [0, 1]$ are subsets of Y_i for each i .) Since the Y_i are pairwise disjoint, each $W_{\mathbf{I}} \cap (M_0^{\text{inv}} \times \{t\})$ is a submanifold of $M_0^{\text{inv}} = \text{Sym}^{n_1}(S^2 \setminus \mathbf{w})$ and is disjoint from the fat diagonal. Furthermore, since $W_{\mathbf{I}}$ is homeomorphic to $(S^1)^k \times [0, 1]$ and $\partial W_{\mathbf{I}}$ is $\prod_{j=1}^k \beta_{i_j} - \prod_{j=1}^k \alpha_{i_j}$, we see that the collection $\{W_{\mathbf{I}} : 1 \leq i_1 < \dots < i_k \leq n_1\}$ generates $H^k(M_0^{\text{inv}} \times [0, 1], X)$.

Let us consider the restriction of $\Upsilon(M_0^{\text{inv}})$ to $W_{\mathbf{I}}$ for some particular \mathbf{I} . We pick preferred trivializations of $N(L_0) \times \{0\} = N(\mathbb{T}_\beta) \times \{0\}$ and $J(N(L_1) \times \{1\}) = J(N(\mathbb{T}_\alpha) \times \{1\})$ which are products of preferred trivializations of the real subbundles $N(\beta_i) \times \{0\}$ and $J(N(\alpha_i) \times \{1\})$ in $\Upsilon(Y_i)$.

Because $W_i \cap (M_0^{\text{inv}} \times \{t\})$ lies entirely off the fat diagonal for each t , the restriction of $\Upsilon(M_0^{\text{inv}})$ to each W_i also decomposes as a product bundle. In other words, regard W_i as a subset of the abstract product $\prod_{i=1}^{n_1} Y_i$ (which is *not* a submanifold of $M_0^{\text{inv}} \times [0, 1]$). Then $\Upsilon(M_0^{\text{inv}})|_{W_{\mathbf{I}}}$ is the restriction of the product bundle $(\prod \Upsilon(Y_i))|_{W_{\mathbf{I}}}$. Therefore, since each $\Upsilon(Y_i)$ admits a trivialization with respect to a choice of trivialization of the bundle restricted to α_i and β_i , we conclude that $\Upsilon(M_0^{\text{inv}})|_{W_{\mathbf{I}}}$ is trivialisable with respect to preferred trivializations of the bundle over $\mathbb{T}_\beta \times \{0\}$ and $J(\mathbb{T}_\alpha \times \{1\})$.

Remark 6.6. We could as easily have shown that $(\text{Sym}^{2n_1}(S^2 \setminus \mathbf{z}), \mathbb{T}_{\tilde{\beta}}, T_{\tilde{\alpha}})$ has a stable normal trivialization. In the case studied here this would produce

a spectral sequence from $\widehat{HF}(S^3) \otimes W^{\otimes 2n_1}$ to $\widehat{HF}(S^3) \otimes W^{\otimes n_1}$, which is not especially interesting.

7. Examples

In this section we produce some examples of the behavior of the spectral sequences of Theorems 1.2 and 1.3. We present the cases of the unknot and the trefoil as doubly-periodic knots. In general, it is difficult to show that the Seidel-Smith action on $\widehat{HFL}(\mathcal{D})$ agrees with the action induced by the map $\tau_{\#}$ on $\widehat{CFL}(\mathcal{D})$ derived from the Heegaard diagram, and therefore difficult to compute the higher differentials in the spectral sequence. However, in these simple examples the higher differentials are determined by the homological grading, so the spectral sequences we compute here are in fact the same as the Seidel-Smith spectral sequence.

To simplify matters, we will compute all spectral sequences using appropriate chain complexes tensored with $\mathbb{Z}_2((\theta))$, instead of tensoring with $\mathbb{Z}_2[[\theta]]$ and later further tensoring the E^∞ page with θ^{-1} . Then the E^1 page of each spectral sequence is a free module with generators the link Floer or knot Floer homology of our doubly periodic knot, and the E^∞ page is a free module over $\mathbb{Z}((\theta))$ with generators the link or knot Floer homology of the quotient. (The only information we lose by this change is that the E^∞ page is not the Borel Heegaard Floer link or knot homology, but rather the Borel homology tensored with θ^{-1} .)

7.1. The case of the unknot

In the simplest possible case, let \tilde{K} be an unknot, and K its quotient knot, a second unknot. Consider the diagram \mathcal{D} for $K \cup U$ on the sphere $S^2 = \mathbb{C} \cup \{\infty\}$ in which K is the unit circle with basepoints w_0, z_0 on U and basepoints w_1, z_1 on K such that w_0 lies at ∞ , z_0 lies at 0 , w_1 lies at i , and z_1 lies at $-i$. Supply a single α_1 and β_1 as in Figure 4 coherently with a clockwise orientation of K .

Label the intersection points a, b, c, e vertically down the diagram as in the figure. (Since we plan to discuss the differentials d_i in the spectral sequence, we will not also use d to label any intersection points.) There are no differentials that count for $\widehat{HFL}(\mathcal{D})$ — which is exactly the link Floer homology of the Hopf link with positive linking number — and three differentials that count for $\widehat{HFK}(\mathcal{D}) = \widehat{HFK}(S^3, K) \otimes W$. See the table of Figure 5 for the Alexander gradings of these entries.

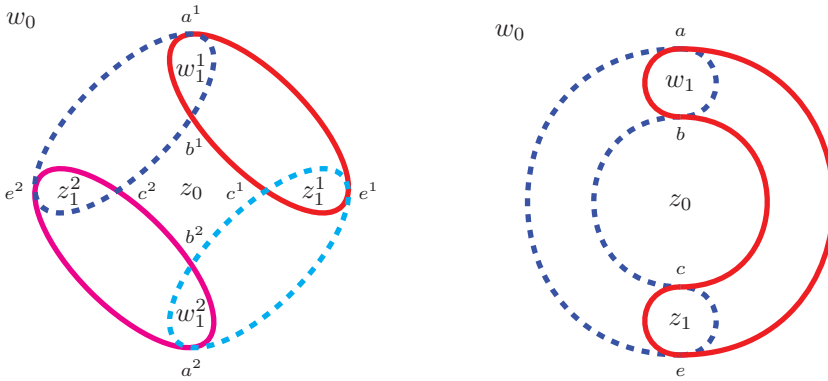


Figure 4. An equivariant Heegaard diagram for the unknot together with an axis (i.e. a Hopf link), and its quotient Heegaard diagram (another Hopf link).

A_1	$-\frac{1}{2}$	$\frac{1}{2}$
A_2	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	b	a
$-\frac{1}{2}$	c	e

A_1	-1	0
	b	a
	\vdots	\vdots
	\downarrow	\downarrow
	c	e

Figure 5. Alexander gradings and differentials for $\widehat{CFL}(\mathcal{D})$ (on the left) and $\widehat{CFK}(\mathcal{D})$ (on the right).

Now lift to a diagram $\widetilde{\mathcal{D}}$ for the same Hopf link, which has basepoints w_0, z_0 on the axis \widetilde{U} and $w_1^1, w_1^2, z_1^1, z_1^2$ on the lifted unknot \widetilde{K} . These basepoints lie on $S^2 = \mathbb{C} \cup \{\infty\}$ as follows: w_0 and z_0 lie at 0 and ∞ as previously, and $w_1^1, w_1^2, z_1^1, z_1^2$ lie at $i, -i, 1, -1$ respectively. There are two curves α_1^1 and α_1^2 encircling the pairs w_1^1, z_1^1 and w_1^2, z_1^2 , and two curves β_1^1 and β_1^2 encircling pairs z_1^2, w_1^1 and z_1^1, w_1^2 . The intersection points a, b, c, e lift to eight points $a^1, a^2, b^1, b^2, c^1, c^2, e^1, e^2$ on the diagram in $\cup(\alpha_1^i \cap \beta_1^j)$. (The numbering of each pair is arbitrarily determined by insisting that a^i lie on α_1^i , and so on.) The complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ has eight generators, whose Alexander gradings are laid out in the left table of Figure 6; there are no differentials that count

$A_1 \backslash A_2$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	b^1b^2	$b^1a^2 \quad a^1b^2$	a^1a^2
$-\frac{1}{2}$	c^1c^2	$c^1e^2 \quad e^1c^2$	e^1e^2

A_1	-2	-1	0
	b^1b^2	$b^1a^2 \quad a^1b^2$	a^1a^2
	\downarrow	$\downarrow \quad \swarrow \quad \searrow$	$\swarrow \quad \searrow$
	c^1c^2	$c^1e^2 \quad e^1c^2$	e^1e^2

Figure 6. Alexander gradings and differentials of $\widehat{CFL}(\widetilde{\mathcal{D}})$ (left) and $\widehat{CFK}(\widetilde{\mathcal{D}})$ (right).

for the theory $\widehat{HFL}(\widetilde{\mathcal{D}}) = \widehat{HFL}(S^3, \widetilde{K} \cup \widetilde{U}) \otimes V_1$. Allowing differentials that pass over the basepoint z_0 , we obtain the complex of the right table Figure 6 which computes $\widehat{HFK}(\mathcal{D}) = \widehat{HFK}(S^3, \widetilde{K}) \otimes V_1 \otimes W$.

First, consider the spectral sequence for link Floer homology of the double complex $(\widehat{CFL}(\widetilde{\mathcal{D}} \otimes \mathbb{Z}_2, \partial + (1 + \tau_{\#})\theta)$. Since $\partial = 0$ on this complex, the only nontrivial differential occurs on the E^1 page and is exactly $d_1 = 1 + \tau_{\#}$. Therefore the E^2 page of the spectral sequence is $\mathbb{Z}_2((\theta))\langle a^1a^2, b^1b^2, c^1c^2, e^1e^2 \rangle$, which is isomorphic to $\widehat{HFL}(S^3, K \cup U) \otimes \mathbb{Z}_2((\theta))$, as expected.

Next, consider the spectral sequence of the double complex $(\widehat{CFK}(\widetilde{\mathcal{D}} \otimes \mathbb{Z}_2, \partial_{\widetilde{U}} + (1 + \tau_{\#})\theta)$. We have the complex of Figure 6; the E^1 page of the spectral sequence is equal to $\mathbb{Z}_2\langle [a^1a^2], [e^1e^2], [a^1b^2 + b^1a^2], [c^1e^2] \rangle \otimes \mathbb{Z}_2((\theta))$. Bearing in mind that $[c^1e^2] = [c^2e^1]$, we see that τ_* is the identity on each of these four elements, and therefore the E^2 page of the spectral sequence is the same as the E^1 page. Since the differential d_2 must raise the Maslov grading by two, the only possible nontrivial differential on the E^2 page is $d_2([c^1e^2]\theta^n)$. Let us compute this differential. On the chain level, we have $(1 + \tau_{\#})(c^1e^2) = (c^1e^2 + c^2e^1)$. We observe that $c^1e^2 + c^2e^1 = \partial(a^1b^2)$, so

$$\begin{aligned} d_2([c^1e^2]\theta^n) &= [(1 + \tau_{\#})(a^1b^2)]\theta^{n+1} \\ &= [a^1b^2 + a^2b^1]\theta^{n+2} \end{aligned}$$

Therefore the E^3 page of this spectral sequence is exactly $\mathbb{Z}_2((\theta))\langle [a^1a^2], [e^1e^2] \rangle$, which is isomorphic to $(\widehat{HFK}(S^3, K) \otimes W) \otimes \mathbb{Z}_2((\theta))$ as promised, and unchanged thereafter.

7.2. The case of the trefoil

Let us now compute some of the spectral sequence for the trefoil as a doubly-periodic knot with quotient the unknot, using the Heegaard diagrams \mathcal{D} and $\tilde{\mathcal{D}}$ of Figure 2. Recall that \mathcal{D} is a Heegaard diagram for a link L in S^3 consisting of two unknots with linking number $\lambda = 3$ and $\tilde{\mathcal{D}}$ is a Heegaard diagram for a link \tilde{L} in S^3 consisting of the left-handed trefoil and the unknotted axis also with linking number $\lambda = 3$. We label the twelve intersection points of α_1 and β_1 in \mathcal{D} and lift to twenty-four intersection points in $\tilde{\mathcal{D}}$ in Figure 7. The Alexander gradings and differentials of $\widehat{CFL}(\mathcal{D})$ and $\widehat{CFK}(\mathcal{D})$ are laid out in the tables of Figure 8.

There are no differentials on $\widehat{CFL}(\mathcal{D})$, so $\widehat{HFL}(\mathcal{D}) = \widehat{HFL}(S^3, L)$ has the twelve generators and gradings of Figure 8. From the remainder of that diagram, we observe that the group $\widehat{HFK}(\mathcal{D}) = \widehat{HFK}(S^3, U) \otimes W$ is $\mathbb{Z}_2\langle [a], [m] \rangle$.

Now consider the seventy-two generators of $\widehat{CFL}(\tilde{\mathcal{D}})$, whose Alexander A_1 and A_2 gradings are laid out in Figure 9. It so happens that $\tilde{\mathcal{D}}$ is a nice diagram in the sense of Sarkar and Wang [31], although the equivariant diagrams for periodic knots introduced in Section 3 are not in general, so we may compute $\widehat{HFL}(\tilde{\mathcal{D}})$ with relative ease. The chain complexes in each Alexander A_1 grading are shown in Figures 16, 17, 18, 19, 20, 21 and 22 at the close of this section. Each of these figures shows the generators in a particular A_1 grading of $\widehat{CFL}(\tilde{\mathcal{D}})$, which are also the generators of the $A_1 - \frac{3}{2}$ grading of $\widehat{CFK}(\tilde{\mathcal{D}})$. Solid arrows denote differentials that count for the differential ∂ and thus exist in both complexes, whereas dashed arrows denote differentials corresponding to disks with nontrivial intersection with the divisor $V_{z_0} = \{z_0\} \times \text{Sym}^{2n_1}(S^2)$. Therefore dashed differentials only count for the knot Floer complex $(\widehat{CFK}(\tilde{\mathcal{D}}), \partial_{\tilde{\mathcal{D}}})$.

The link Floer homology spectral sequence associated to $\tilde{\mathcal{D}}$ arises from the double complex $(\widehat{CFL}(\mathcal{D}), \partial + (1 + \tau_{\#})\theta)$. Computing homology of $\widehat{CFL}(\tilde{\mathcal{D}})$ with respect to the differential ∂ , we obtain a set of generators for the E^1 page of the spectral sequence, which is $\widehat{HFL}(\tilde{\mathcal{D}}) \otimes \mathbb{Z}_2((\theta)) = (\widehat{HFL}(S^3, 3_1 \cup \tilde{U}) \otimes V_1) \otimes \mathbb{Z}_2((\theta))$. These generators and their gradings may be found in Figure 10. Whenever an element of $\widehat{HFK}(\tilde{\mathcal{D}})$ is invariant under the involution τ_* but has no representative which is invariant under the chain map $\tau_{\#}$, we have included two representatives of that element to make the τ_* invariance clear. (For example, observe that $[h^1c^2] = [c^1h^2]$ is invariant under τ_* .)

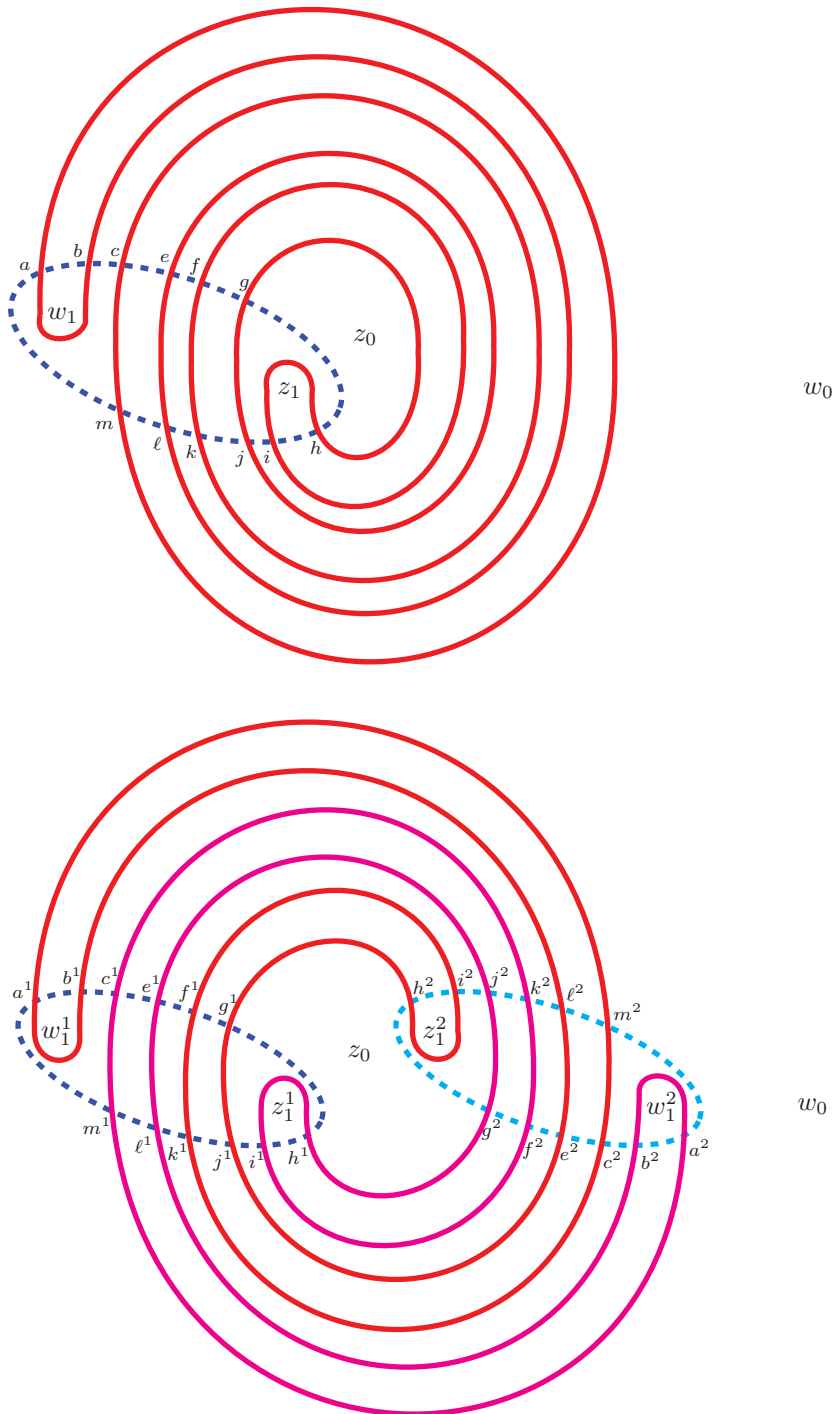


Figure 7. Intersection points of α and β curves in \mathcal{D} and $\tilde{\mathcal{D}}$.

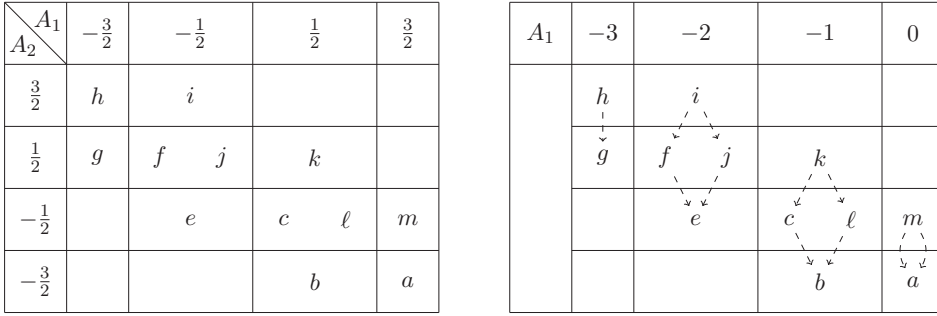


Figure 8. Alexander gradings and differentials of $\widehat{CFL}(\mathcal{D})$ (left) and $\widehat{CFK}(\mathcal{D})$ (right).

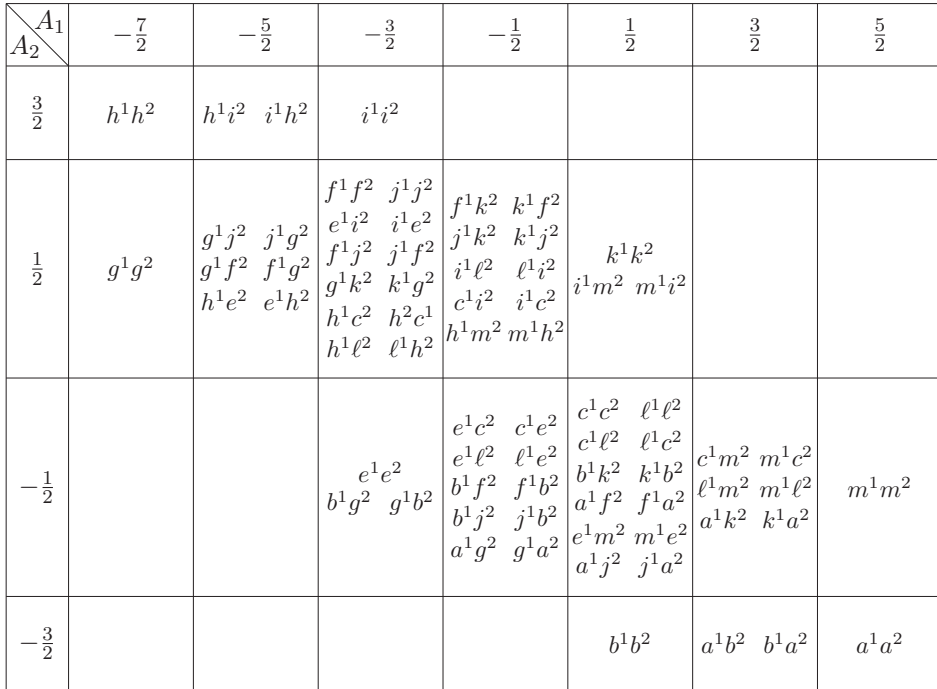


Figure 9. Alexander A_1 and A_2 gradings of the generators of $\widehat{CFL}(\tilde{\mathcal{D}})$. These generate the E^0 page of the link Floer homology spectral sequence.

$A_1 \backslash A_2$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\frac{3}{2}$	$[h^1 h^2]$	$[h^1 i^2] [i^1 h^2]$	$[i^1 i^2]$				
$\frac{1}{2}$	$[g^1 g^2]$	$[g^1 j^2 + f^1 g^2]$ $[j^1 g^2 + g^1 f^2]$	$[f^1 f^2 + k^1 g^2 +$ $g^1 k^2 + j^1 j^2]$ $[h^1 c^2] = [c^1 h^2]$	$[i^1 c^2] [c^1 i^2]$	$[i^1 m^2] = [m^1 i^2]$		
$-\frac{1}{2}$			$[b^1 g^2 + g^1 b^2]$	$[a^1 g^2] [g^1 a^2]$	$[c^1 c^2 + e^1 m^2 +$ $m^1 e^2 + \ell^1 \ell^2]$ $[a^1 f^2] = [f^1 a^2]$	$[c^1 m^2 + m^1 \ell^2]$ $[m^1 c^2 + \ell^1 m^2]$	$[m^1 m^2]$
$-\frac{3}{2}$					$[b^1 b^2]$	$[a^1 b^2] [b^1 a^2]$	$[a^1 a^2]$

Figure 10. The homology $\widetilde{HFL}(\widetilde{\mathcal{D}})$. These elements generate the E^1 page of the link Floer spectral sequence as a $\mathbb{Z}_2((\theta))$ module.

The differential d_1 on the E_1 page of the link Floer spectral sequence for $\widetilde{\mathcal{D}}$ is $(1 + \tau_*)\theta$; in particular, computing the homology of d_1 has the effect of killing all elements of $\widetilde{HFL}(\mathcal{D})$ not invariant under τ_* . Ergo we see that the E^2 page of this spectral sequence is generated as a $\mathbb{Z}_2((\theta))$ module by the elements of Figure 11. Notice that the ranks of the E^2 page in each Alexander grading $A_1 = 2k + \frac{1}{2}$ correspond precisely to the ranks of each Alexander grading $A_1 = k + \frac{1}{2}$ of Figure 8, which is the link Floer homology $\widetilde{HFL}(\mathcal{D})$. Therefore the link Floer homology spectral sequence converges on the E^2 page.

We now turn our attention to the knot Floer homology spectral sequence for $\widetilde{\mathcal{D}}$, which arises from the double complex $(\widehat{CFK}(\mathcal{D}) \otimes \mathbb{Z}_2((\theta)), \partial_{\widetilde{U}} + (1 + \tau_{\#})\theta)$. The Alexander A_1 gradings of the seventy-two generators in $\widehat{CFK}(\widetilde{\mathcal{D}})$ are laid out in Figure 12. Notice that these gradings are exactly the A_1 gradings of $\widehat{CFL}(\widetilde{\mathcal{D}})$ shifted downward by $\frac{\text{lk}(3_1, U)}{2} = \frac{3}{2}$. These elements generate the E^0 page of the link Floer spectral sequence as a $\mathbb{Z}_2((\theta))$ module.

As before, the chain complexes in each Alexander grading may be found in 16, 17, 18, 19, 20, 21 and 22. Computing the homology of these complexes, we obtain $\widehat{HFK}(\widetilde{\mathcal{D}}) = \widehat{HFK}(3_1) \otimes V_1 \otimes W$, whose generators are described

$A_1 \backslash A_2$	$-\frac{7}{2}$	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\frac{3}{2}$	$[h^1 h^2]$		$[i^1 i^2]$				
$\frac{1}{2}$	$[g^1 g^2]$		$[f^1 f^2 + k^1 g^2 + g^1 k^2 + j^1 j^2]$ $[h^1 c^2] = [c^1 h^2]$		$[i^1 m^2] = [m^1 i^2]$		
$-\frac{1}{2}$			$[b^1 g^2 + g^1 b^2]$		$[c^1 c^2 + e^1 m^2 + m^1 e^2 + \ell^1 \ell^2]$ $[a^1 f^2] = [f^1 a^2]$		$[m^1 m^2]$
$-\frac{3}{2}$					$[b^1 b^2]$		$[a^1 a^2]$

Figure 11. Generators for the $E^2 = E^\infty$ page of the link Floer spectral sequence for $\tilde{\mathcal{D}}$ as a $\mathbb{Z}_2((\theta))$ -module.

in Figure 13. These elements generate the E^1 page of the knot Floer homology spectral sequence of $\tilde{\mathcal{D}}$ as a $\mathbb{Z}_2((q))$ -module. Once again, homology classes which are equivalent under the induced involution τ_* but have no representative which is invariant under the chain map $\tau_\#$ have been included with two equivalent descriptions to emphasize their invariance.

The differential $d_1 = (1 + \tau_*)\theta$ on the E^1 page of the spectral sequence has the effect of eliminating all elements of $\widehat{HFK}(\tilde{\mathcal{D}})$ which are not invariant under the action of τ_* . Computing homology with respect to d_1 yields the set of generators of Figure 14.

The knot Floer spectral sequence stabilizes on the E^2 page in every Alexander grading except $A_1 = -2$. There is a single nontrivial d_2 differential which behaves similarly to the nontrivial d_2 differential we earlier saw in the case of the unknot as a doubly-periodic knot. In particular: consider $d_2([a^1 g^2])$. We compute this differential as follows: first, apply $1 + \tau_\#$ to $a^1 g^2$, obtaining $(a^1 g^2 + g^1 a^2)\theta$. Next, we choose an element whose boundary under $\partial_{\tilde{\mathcal{C}}}$ is $(a^1 g^2 + g^1 a^2)\theta$; one such is $(h^1 m^2)\theta$. Then $d_2([a^1 g^2]) = [1 + \tau_\#(h^1 m^2)]\theta = [h^1 m^2 + m^1 h^2]\theta^2$. Moreover, we then have $d_2([a^1 g^2]\theta^n) =$

A_1	-5	-4	-3	-2	-1	0	1
	$h^1 h^2$	$h^1 i^2 \quad i^1 h^2$	$i^1 i^2$				
	$g^1 g^2$	$g^1 j^2 \quad j^1 g^2$ $g^1 f^2 \quad f^1 g^2$ $h^1 e^2 \quad e^1 h^2$	$f^1 f^2 \quad j^1 j^2$ $e^1 i^2 \quad i^1 e^2$ $f^1 j^2 \quad j^1 f^2$ $g^1 k^2 \quad k^1 g^2$ $h^1 c^2 \quad h^2 c^1$ $h^1 \ell^2 \quad \ell^1 h^2$	$f^1 k^2 \quad k^1 f^2$ $j^1 k^2 \quad k^1 j^2$ $i^1 \ell^2 \quad \ell^1 i^2$ $c^1 i^2 \quad i^1 c^2$ $h^1 m^2 \quad m^1 h^2$		$k^1 k^2$ $i^1 m^2 \quad m^1 i^2$	
			$e^1 e^2$ $b^1 g^2 \quad g^1 b^2$	$e^1 c^2 \quad c^1 e^2$ $e^1 \ell^2 \quad \ell^1 e^2$ $b^1 f^2 \quad f^1 b^2$ $b^1 j^2 \quad j^1 b^2$ $a^1 g^2 \quad g^1 a^2$	$c^1 c^2 \quad \ell^1 \ell^2$ $c^1 \ell^2 \quad \ell^1 c^2$ $b^1 k^2 \quad k^1 b^2$ $a^1 f^2 \quad f^1 a^2$ $e^1 m^2 \quad m^1 e^2$ $a^1 j^2 \quad j^1 a^2$	$c^1 m^2 \quad m^1 c^2$ $\ell^1 m^2 \quad m^1 \ell^2$ $a^1 k^2 \quad k^1 a^2$	$m^1 m^2$
					$b^1 b^2$	$a^1 b^2 \quad b^1 a^2$	$a^1 a^2$

Figure 12. Alexander A_1 gradings of elements of $\widehat{CFK}(\widetilde{\mathcal{D}})$. These elements generate the E^0 page of the link Floer spectral sequence as a $\mathbb{Z}_2((\theta))$ module.

$[h^1 m^2 + m^1 h^2] \theta^{n+2}$ in general. Therefore the Alexander grading -2 vanishes on the E^3 page of the knot Floer spectral sequence, and the $E^3 = E^\infty$ page is isomorphic to $\widehat{HFK}(\mathcal{D})$ after an appropriate shift and rescaling in Alexander A_1 gradings. Generators for this page of the knot Floer spectral sequence appear in Figure 15.

Notice that for the left-handed trefoil considered as a two-periodic knot, Edmonds' condition is sharp: $g(3_1) = 1 = 2(0) + \frac{3-1}{2} = 2g(U) + \frac{\lambda-1}{2}$. We also see here the realization of Corollary 1.7: the left-handed trefoil is fibred, and the highest nontrivial Alexander grading in $\widehat{HFK}(\widetilde{\mathcal{D}})$, namely $A_1 = 1$ has rank two. The two generators in this Alexander grading, $[a_1 a_2]$ and $[m_1 m_2]$, are preserved over the course of the spectral sequence and become the two generators of $\widehat{HFK}(\mathcal{D})$ in grading $A_1 = 0$ under the isomorphism between the E^∞ page of the knot Floer spectral sequence and $\widehat{HFK}(\mathcal{D})$. Then the highest nontrivial Alexander grading of $\widehat{HFK}(\mathcal{D})$, $A_1 = 0$, also has rank two, corresponding to the unknot being a fibred knot.

A_1	-5	-4	-3	-2	-1	0	1
				$[h^1m^2 + m^1h^2]$	$\begin{matrix} [i^1m^2 + m^1e^2] \\ [m^1i^2 + e^1m^2] \end{matrix}$		
				$[a^1g^2] = [g^1a^2]$	$[a^1f^2] \quad [f^1a^2]$	$\begin{matrix} [\ell^1m^2 + m^1c^2] \\ [m^1\ell^2 + c^1m^2] \end{matrix}$	$[m^1m^2]$
						$[a^1b^2] \quad [b^1a^2]$	$[a^1a^2]$

Figure 13. The homology $\widehat{HFK}(\widetilde{\mathcal{D}})$, which is $\widehat{HFK}(3_1) \otimes V_1 \otimes W$. These elements generate the E^1 page of the knot Floer homology spectral sequence of $\widetilde{\mathcal{D}}$ as a $\mathbb{Z}_2((q))$ -module.

A_1	-5	-4	-3	-2	-1	0	1
				$[h^1m^2 + m^1h^2]$			
				$[a^1g^2] = [g^1a^2]$			$[m^1m^2]$
							$[a^1a^2]$

Figure 14. Generators for the E^2 page of the knot Floer spectral sequence associated to $\widetilde{\mathcal{D}}$ as a $\mathbb{Z}_2((q))$ module.

A_1	-5	-4	-3	-2	-1	0	1
							$[m^1 m^2]$
							$[a^1 a^2]$

Figure 15. Generators for the $E^3 = E^\infty$ page of the knot Floer spectral sequence associated to $\tilde{\mathcal{D}}$.



Figure 16. The chain complex $\widehat{CFL}(\tilde{\mathcal{D}})$ in Alexander grading $A_1 = -\frac{7}{2}$, and the chain complex $\widehat{CFK}(\tilde{\mathcal{D}})$ in Alexander grading $A_1 = -5$. Dashed arrows denote differentials appearing only in the latter.

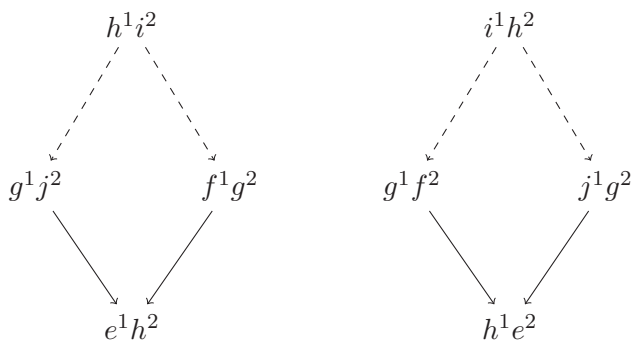


Figure 17. The chain complex $\widehat{CFL}(\tilde{\mathcal{D}})$ in Alexander grading $A_1 = -\frac{5}{2}$, and the chain complex $\widehat{CFK}(\tilde{\mathcal{D}})$ in Alexander grading $A_1 = -4$. Dashed arrows denote differentials appearing only in the latter.

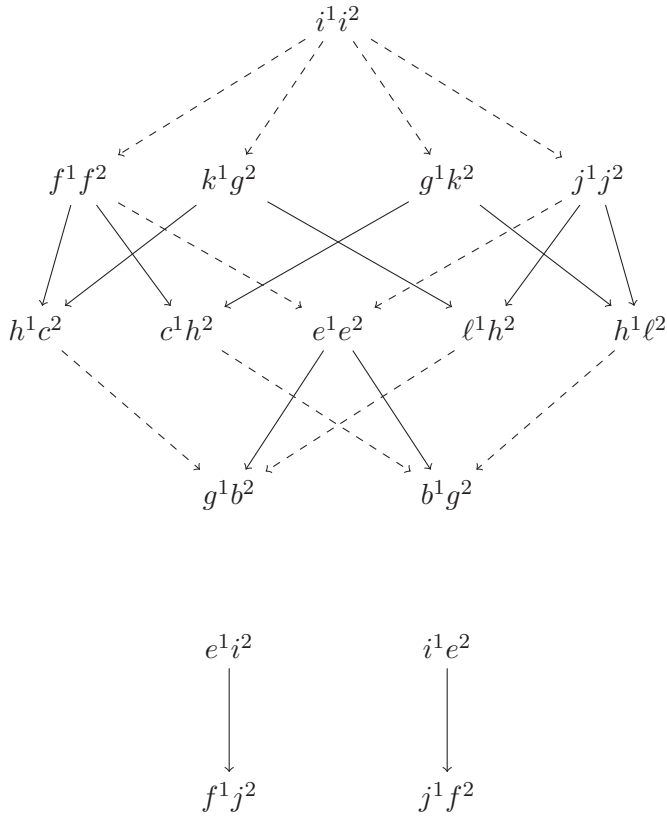


Figure 18. The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -\frac{3}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -3$. Dashed arrows denote differentials appearing only in the latter.

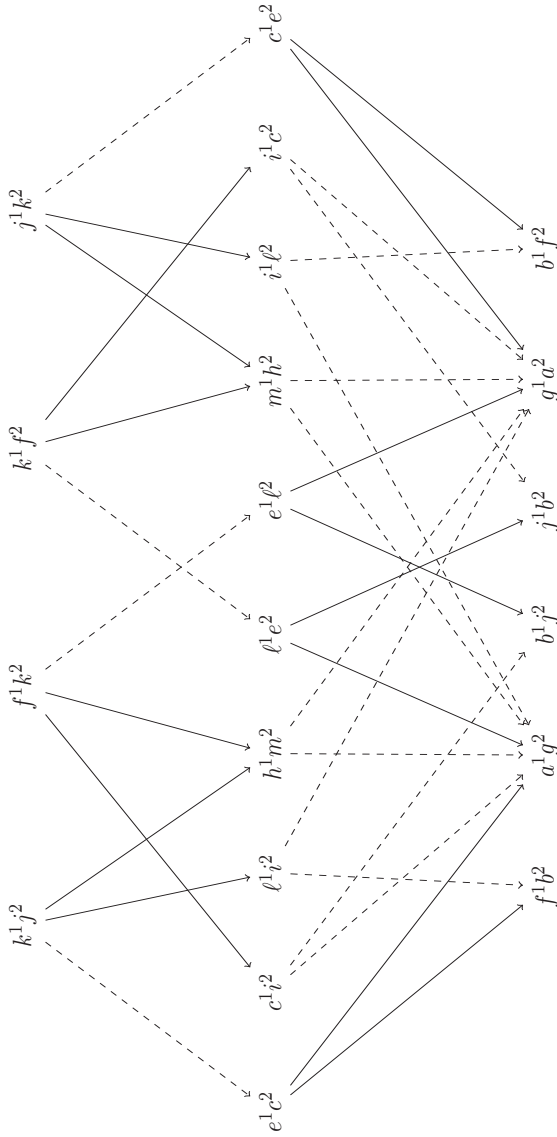


Figure 19. The chain complex $\widehat{CFL}(\tilde{\mathcal{D}})$ in Alexander grading $A_1 = -\frac{1}{2}$, and the chain complex $\widehat{CFK}(\tilde{\mathcal{D}})$ in Alexander grading $A_1 = -2$. Dashed arrows denote differentials appearing only in the latter.

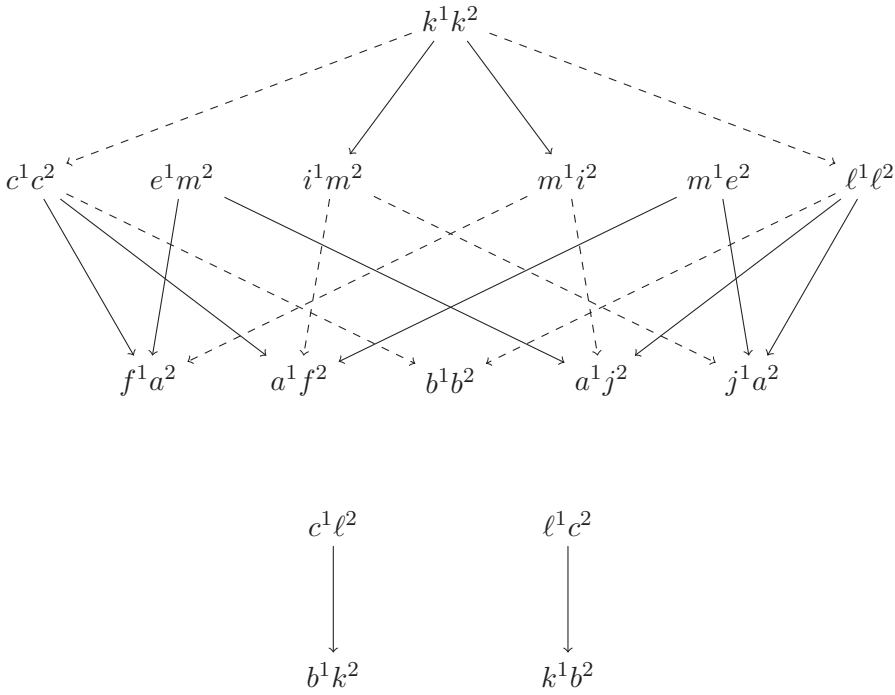


Figure 20. The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = \frac{1}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = -1$. Dashed arrows denote differentials appearing only in the latter.

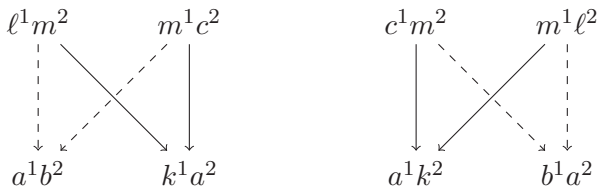


Figure 21. The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = \frac{3}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = 0$. Dashed arrows denote differentials appearing only in the latter.

$$\begin{array}{c}
 m^1 m^2 \\
 \text{---} \\
 \text{---} \\
 \text{---} \\
 \text{---} \\
 \text{---} \\
 \text{---} \\
 \text{---} \\
 \text{---} \\
 \text{---} \\
 a^1 a^2
 \end{array}$$

Figure 22. The chain complex $\widehat{CFL}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = \frac{5}{2}$, and the chain complex $\widehat{CFK}(\widetilde{\mathcal{D}})$ in Alexander grading $A_1 = 1$. Dashed arrows denote differentials appearing only in the latter.

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