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A theorem on the removal of boundary singularities of pseudo-holomorphic curves

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We prove a theorem on the removal of singularities on the boundary of a pseudo-holomorphic curve. This theorem needs *no* assumption on the area of the curve.

1. Introduction

In this paper, we shall prove the following Theorem 1.1 due to Gromov [2]. This is a theorem on the removal of boundary singularities of pseudo-holomorphic curves. A precise statement of Theorem 1.1 is Theorem 2.4.

Let D^+ be the upper half disk on the complex plane and $\check{D}^+$ be the punctured upper half disk $D^+ - \{0\}$ (see (2.1) and (2.2)). Suppose M is a manifold with an almost complex structure J and W is an embedded totally real submanifold and a closed subset of M. Assume $f : \check{D}^+ \to M$ is a smooth and pseudo-holomorphic map such that $f(\partial \check{D}^+) \subseteq W$.

Theorem 1.1. *Suppose the image of f has a compact closure. Suppose α is a 1-from on M such that $d\alpha$ tames J. Furthermore, assume α is exact on W. Then f has a smooth extension over $0 \in D^+$.*

In Theorem 1.1, the assumption of the exactness of α on W was not stated by Gromov in [2, 1.3.C]. However, Theorem 1.1 will *no* longer be true if we drop this assumption. We construct the following simple counterexample to show that, without the assumption, there could be *no* continuous extension. (The proof of this example is given in Section 2.)

Example 1.2. Let $M = \mathbb{C}$ be the complex plane. Choose J to be the standard complex structure. Let $W = \{z = x + iy \in \mathbb{C} \mid |z| = 1\}$ be the unit circle. Define $\alpha = xdy$. Define a holomorphic function $f : \check{D}^+ \to \mathbb{C}$ as

$$f(z) = \exp\left(-\frac{i}{z}\right). \tag{1.1}$$

Then $f : \check{D}^+ \to (M, W, J, \alpha)$ satisfies every assumption in Theorem 1.1 with one exception: α is not exact on W.

The conclusion of Theorem 1.1 does not hold in this example.

To the best of our knowledge, up to now, there has been no work in the literature to give a correct statement of Theorem 1.1, let alone to prove it.

There has been much work in the literature to prove theorems on the removal of boundary singularities of pseudo-holomorphic curves, for example [8], [11], [10], [6] and [4]. We would like to point out that Theorem 1.1 is different from the previous work. All of those theorems require assumptions on the area of the image of the map f (see Definition 3.1). Most of them assume that the area is finite. However, Theorem 1.1 needs *no* assumption on the area at all. Theorem 1.1 does have certain advantages over the results obtained in previous work. This will be illustrated by a simple Example 2.6.

We describe now the main idea of our proof. Though Theorem 1.1 does not apriori assume that the image of the map f has a finite area, we shall prove that the image does have a finite area if f is restricted on a smaller punctured half disk $\check{D}^+(r) = \{z \in \check{D}^+ \mid |z| < r\}$, where $0 < r < 1$. Using this fact, we can apply the previous work on the removal of boundary singularities to the map $f|_{\check{D}^+(r)}$. As a result, we infer that the singularity of f can be removed.

In order to estimate the area of the image of $f|_{\check{D}^+(r)}$, we shall use a doubling argument. Actually, in [2, 1.3.C], Gromov suggests that a doubling argument would reduce a boundary singularity to an interior singularity. Our doubling argument follows this idea. More precisely, we double $\check{D}^+$ to the whole punctured disk $\check{D}$ (see (3.1)). After this, we pull the geometric data on M, such as metrics and forms, back to $\check{D}^+$ by using f, and then we extend the data symmetrically over $\check{D}$. Instead of constructing a map from $\check{D}$ to M which is an extension of f, we take $\check{D}$ as our ambient manifold. (Because of this, we call such a doubling an intrinsic doubling.) On $\check{D}$, we can adapt certain arguments which were used by Gromov on M in the case of an interior singularity. This leads to the desired estimate of the area of the image of $f|_{\check{D}^+(r)}$.

The outline of this paper is as follows. Section 2 precisely formulates the main result of this paper. Section 3 lists some technical results frequently used throughout this paper. In Section 4 we prove Theorem 1.1.

2. Main result

In this paper, all manifolds are without boundary if we don't say this explicitly. All manifolds, maps, functions, metrics, almost complex structures, forms and so on are smooth if we don't state this explicitly. Similarly, all submanifolds are assumed to be smoothly embedded submanifolds unless otherwise mentioned.

Let's recall some basic definitions related to almost complex manifolds. Suppose M is a manifold with an almost complex structure J, that is a field of endomorphisms on T_pM for all $p \in M$ such that $J^2 = -\mathrm{Id}$. We call M an almost complex manifold and also denote it by (M, J). The dimension of M has to be even. We say a 2-form ω on M is a symplectic form if ω is closed and nondegenerate.

Definition 2.1. We say a 2-form ω tames J if $\omega(v, Jv) > 0$ for any nonzero tangent vector v on M.

Definition 2.2. A submanifold W of (M, J) is said to be totally real if $\dim(W) = \frac{1}{2}\dim(M)$ and $J(T_pW) \cap T_pW = \{0\}$ for all $p \in W$.

Definition 2.3. Suppose (S, J_1) and (N, J_2) are manifolds with almost complex structures J_1 and J_2. A C^1 map $h : S \to N$ is J-holomorphic or pseudo-holomorphic if the derivative of h is complex linear with respect to J_1 and J_2, i.e.

$$J_2 \cdot dh = dh \cdot J_1.$$

If S is a Riemann surface, we call such a map a J-holomorphic curve or a pseudo-holomorphic curve in N.

We fix now some notations for some subsets of the complex plane $\mathbb{C}$ which are frequently used throughout this paper. Denote by

$$D^+ = \{z \in \mathbb{C} \mid |z| < 1, \mathrm{Im} z \geq 0\}, \tag{2.1}$$

the half disk on the complex plane. Define

$$\check{D}^+ = D^+ - \{0\}. \tag{2.2}$$

as the punctured half disk. Clearly, $\check{D}^+$ is a Riemann surface with boundary. Its boundary is

$$\partial\check{D}^+ = \{z \in \mathbb{R} \mid 0 < |z| < 1\}. \tag{2.3}$$

The main goal of this paper is to present a proof of the following theorem due to Gromov [2, 1.3.C]. This is the precise version of Theorem 1.1 in the Introduction.

Theorem 2.4. *Suppose (M, J) is a smooth almost complex manifold. Suppose $f : \check{D}^+ \to (M, J)$ is a smooth J-holomorphic map such that $f(\partial \check{D}^+) \subseteq W$, where W is a smoothly embedded totally real submanifold and a closed subset of M. Denote by $\iota : W \hookrightarrow M$ the inclusion of W. Suppose J is tamed by $d\alpha$, where α is a smooth* 1*-from on M such that $\iota^*\alpha$ is exact on W. Suppose the image of f has a compact closure.*

Then f has a smooth extension over $0 \in D^+$.

Now we give a proof of our counterexample in the Introduction.

Proof of Example 1.2. We know that W is compact and $d\alpha = dx \wedge dy$ tames J. However, $\iota^*\alpha$ is not exact on the unit circle W. Actually, there exists no 1-form β such that $d\beta$ tames J and $\iota^*\beta$ is exact. Otherwise, by Stokes' formula, there would be no nonconstant compact J-holomorphic curve inside M whose boundary lies on W. However, the inclusion of the closed unit disk is such a curve, which results in a contradiction.

Clearly, f is J-holomorphic. By direct computation, we have $f(\partial \check{D}^+) \subseteq W$. Furthermore, $|f(z)| \leq 1$ for all $z \in \check{D}^+$, which implies that the image of f has a compact closure.

Nevertheless, the limit $\lim\limits_{z \to 0} f(z)$ does not exist. Actually, 0 is an essential singularity of $\exp(\frac{-i}{z})$. Therefore, f has no continuous extension over 0. $\square$

Remark 2.5. Readers are suggested to understand Example 1.2 together with Lemma 4.6, Example 4.7 and Lemma 4.10.

As mentioned in the Introduction, the results obtained in previous work on the removal of boundary singularities require assumptions on the area of the image of the map f (See Definition 3.1). Most of them assume that the area is finite. However, Theorem 2.4 does not need any assumption on the area at all. The following simple example shows that Theorem 2.4 has its advantage over those results in certain cases.

Example 2.6. Let f be a complex valued function on $\check{D}^+$ such that f is smooth and bounded. Suppose f is holomorphic in the interior of $\check{D}^+$. Suppose f takes real values on $\partial \check{D}^+$. By the Schwarz reflection principle, f extends to a bounded holomorphic function on $\check{D}$, which implies that 0 is a removable singularity.

This removal of singularities trivially follows from Theorem 2.4 by taking $M = \mathbb{C}$, $W = \mathbb{R}$ and $\alpha = xdy$.

However, it's not easy to apply the results in previous work on the removal of boundary singularities. For instance, it's nontrivial to show that the area of the image of f is finite.

3. Set up

In this section, we shall give some notation, definitions and results which are frequently used throughout this paper.

First, together with (2.1) and (2.2), we define some subsets of the complex plane $\mathbb{C}$ as

$$D = \{z \in \mathbb{C} \mid |z| < 1\}, \qquad \check{D} = D - \{0\}, \tag{3.1}$$

$$D^- = \{z \in \mathbb{C} \mid |z| < 1, \mathrm{Im} z \leq 0\}, \qquad \check{D}^- = D^- - \{0\}, \tag{3.2}$$

$$D(r) = \{z \in \mathbb{C} \mid |z| < r\}, \qquad \check{D}^+(r) = \{z \in D(r) \mid z \neq 0, \mathrm{Im} z \geq 0\}, \tag{3.3}$$

and

$$\partial D(r) = \{z \in \mathbb{C} \mid |z| = r\}. \tag{3.4}$$

Definition 3.1. Suppose N is an n-dimensional C^1 Riemannian manifold, S is a k-dimensional C^1 manifold, and $h : S \to N$ is a C^1 map. Pulling back the Riemannian metric from N to S by h, we get a possibly singular C^0 metric on S. The volume of the image of h is defined as the volume of S with respect to this pull back metric. Denote this volume by $|h(S)|$. When $k = 1$ or 2, we also call it length or area of the image respectively.

As a subset of N, $h(S)$ has its k-dimensional Hausdorff measure with respect to the metric of N. By Federer's area formula, we know that $|h(S)|$ is no less than the Hausdorff measure of $h(S)$. It also could happen that $|h(S)|$ is strictly greater than the Hausdorff measure, for example, when h is a covering map.

If S is also a C^1 Riemannian manifold, then the volume of the image, $|h(S)|$, also equals the Riemannian integral of the (absolute value of the) Jacobian of h on S. In particular, suppose S is a Riemann surface and N is an almost complex manifold. Suppose both S and N are equipped with

Hermitian metrics with respect to their almost complex structures. Suppose h is J-holomorphic. Then h is conformal and therefore the Jacobian of h equals $\|dh\|^2$.

Suppose $\varphi : D \to N$ is a C^1 J-holomorphic map from the unit disk to a C^1 almost complex manifold N. Equip D with the standard Euclidean metric, and equip N with a C^1 Hermitian metric. For $0 < r \leq 1$, define

$$A(r) = |\varphi(D(r))|.$$

For $0 < r < 1$, define

$$L(r) = |\varphi(\partial D(r))|.$$

By the conformality of φ, we infer

$$A(r) = \int_{D(r)} \|d\varphi\|^2 \qquad \text{and} \qquad L(r) = \int_{\partial D(r)} \|d\varphi\|. \tag{3.5}$$

We see that $A(r)$ is C^1 on $[0,1)$ and $L(r)$ is C^0 in $[0,1)$. Furthermore, we can easily prove the following formula, for $r \in (0,1)$, (see the statement and the proof in the last line of [2, p. 315])

$$\frac{d}{dr} A(r) \geq (2\pi r)^{-1} L(r)^2. \tag{3.6}$$

This simple and powerful formula will play a key role in our argument.

Suppose (M, J) is an almost complex manifold. Let H be a Hermitian metric on M. Then, ReH, the real part of H is a Riemannian metric on M. In order to prove Theorem 2.4, Gromov suggests the following lemma ([2, 1.3.C]). This construction has been used in some other work (e.g. [6, Section 4.3]). It is important for us as well.

Lemma 3.2. *Suppose W is a totally real submanifold and a closed subset of (M, J). Then there exists a Hermitian metric H on M which satisfies the following properties.*

(1) *$TW \perp J(TW)$ with respect to ReH.*

(2) *W is totally geodesic with respect to ReH.*

Proof. A proof is presented in [6, Lemma 4.3.3]. One can construct such a H first in a neighborhood U of W and then extend it (using that W is closed) to a Hermitian metric H defined on M satisfying (1) and (2). $\square$

Now we point out that we may assume f is an embedding in the assumption of Theorem 2.4 when we prove it. The idea is to replace f by its graph. More precisely, we do the following graph construction.

Define

$$\hat{M} = \mathbb{C} \times M,$$

which is also an almost complex manifold. Define $\hat{f} : \check{D}^+ \to \hat{M} = \mathbb{C} \times M$ as

$$\hat{f}(z) = (z, f(z)).$$

Then $\hat{f}$ is certainly a J-holomorphic embedding. Let

$$\hat{W} = \mathbb{R} \times W.$$

Then $\hat{W}$ is a totally real submanifold and a closed subset of $\hat{M}$, and $\hat{f}(\partial D^+) \subseteq \hat{W}$.

It's easy to see that $\hat{f} : \check{D}^+ \to \hat{M}$ satisfies the assumption of Theorem 2.4 as long as f does. If $\hat{f}$ has a smooth extension over 0, then certainly so does f. Therefore, we get the following.

Observation 3.3. Without loss of generality, we may assume f is an embedding in the assumption of Theorem 2.4 when we prove it.

4. Proof of main theorem

The goal of this section is to prove the main Theorem 2.4. Based on the previous work [4] and [5], this theorem will be easily derived from the following lemma.

Lemma 4.1. *Under the assumption of Theorem 2.4, equip M with the Riemannian metric ReH in Lemma 3.2 and equip $\check{D}^+$ with the standard Euclidean metric. Then there exists a constant $C > 0$ such that*

$$\|df(z)\| \leq \frac{C}{|z| \log \frac{1}{|z|}}. \tag{4.1}$$

In particular $|f(\check{D}^+(r))| < +\infty$ for all $r \in (0, 1)$.

Our strategy to prove (4.1) starts from the following observation. An estimate on $\|df\|$ is equivalent to an estimate on the metric on $\check{D}^+$ pulled back by f. To estimate the pullback metric, we need the intrinsic doubling mentioned in the Introduction.

The pullback metric is extended to be a metric G on $\check{D}$. Following Gromov's approach in the interior case (see [2, 1.3.A']), we bound G in terms of the hyperbolic metric on $\check{D}$, which immediately implies (4.1). To bound G, we need to estimate the derivatives of the universal covering maps from the Euclidean disk D to $(\check{D}, G)$.

In this process, we use certain isoperimetric inequalities resulting from the Gaussian curvature and forms on $\check{D}$ (see Lemmas 4.9 and 4.10). We shall study the metrics, Gaussian curvature and forms on $\check{D}$ at first.

By Observation 3.3, we may assume that f is an embedding. Then the pull back metric $f^*\mathrm{Re}H$ makes sense and is conformal on $\check{D}^+$. Therefore, we have the following two lemmas given by [2].

Lemma 4.2. *The Gaussian (sectional) curvature of f^*ReH on $\check{D}^+$ has an upper bound.*

Lemma 4.2 is proved in [2, 1.1.B] under the assumption that M is compact. More details can be found in [7, p. 219]. This argument certainly works for us because $f(\check{D}^+)$ has a compact closure.

Lemma 4.3. *$\partial\check{D}^+$ is totally geodesic in $\check{D}^+$ with respect to f^*ReH.*

As mentioned in [2, 1.3.C], Lemma 4.3 follows from Lemma 3.2 and the fact that f is J-holomorphic. We omit a proof here since it's easy.

Define $\sigma : \mathbb{C} \to \mathbb{C}$ as the complex conjugation, i.e. $\sigma(z) = \bar{z}$. By (3.2), we have $\check{D}^- = \sigma(\check{D}^+)$. Define a Riemannian metric G on $\check{D}$ as

$$G = \begin{cases} f^*\mathrm{Re}H & \text{on } \check{D}^+, \\ \sigma^* f^*\mathrm{Re}H & \text{on } \check{D}^-. \end{cases} \tag{4.2}$$

The importance of (2) in Lemma 3.2, i.e. the fact that W is totally geodesic, lies in the following lemma.

Lemma 4.4. *The metric G in (4.2) is a well defined C^2 conformal metric on $\check{D}$. In particular, the Gaussian curvature of G makes sense. Furthermore, G is smooth on $\check{D}^+$ and $\check{D}^-$, and σ is an isometry.*

Proof. By (1) in Lemma 3.2 and the fact that f is J-holomorphic, we see that $f^*\mathrm{Re}H = \sigma^* f^*\mathrm{Re}H$ on $\partial\check{D}^+ = \partial\check{D}^-$. Therefore, G is well defined.

We also see that everything in this lemma is easily to be checked except maybe that G is C^2. Let's prove this.

Since G is conformal, using the coordinate $z = x + iy$, we have

$$G = g(x,y)dx \otimes dx + g(x,y)dy \otimes dy.$$

We know that g is continuous on $\check{D}$, and smooth on $\check{D}^{\pm}$. Since σ is an isometry, we also have

$$g(x,y) = g(x,-y). \tag{4.3}$$

By Lemma 4.3, we infer that

$$\frac{\partial g}{\partial y^+}(x,0) = 0, \tag{4.4}$$

where $\frac{\partial g}{\partial y^+}$ is the upper half partial derivative with respect to y. By (4.3) and (4.4), we infer that $\frac{\partial g}{\partial y}(x,0)$ exists. Then $\frac{\partial^2 g}{\partial y^2}(x,0)$ automatically exists because (4.3) tells us g is an even function of y.

Now it's easy to check that g is C^2 on $\check{D}$. □

Lemmas 4.2 and 4.4 immediately imply the following.

Lemma 4.5. *The Gaussian curvature of G on $\check{D}$ has an upper bound.*

Now we consider differential forms. We shall construct a bounded 1-form α' on $\check{D}$ which is a piecewise primitive of a symplectic form η on $\check{D}$. Both forms α' and η are obtained by a symmetric construction involving the pullbacks of α and $d\alpha$ to $\check{D}$. The exactness of $\iota^*\alpha$ enables us to establish a good property for (α', η) which ensures a Stokes' formula (see Example 4.7). Such a property will be useful in Lemma 4.9 to do certain arguments on $\check{D}$ which are similar to arguments of Gromov used on M.

Lemma 4.6. *There exists a 1-form α_1 on M such that $\iota^*\alpha_1 = 0$ on W and $d\alpha_1 = d\alpha$.*

Proof. Since $\iota^*\alpha$ is exact, there exists a function μ on W such that $\iota^*\alpha = d\mu$. Since W is closed, we can extend μ to be a function μ_1 on M such that $\iota^*\mu_1 = \mu$. We finish the proof by defining $\alpha_1 = \alpha - d\mu_1$. □

Lemma 4.6 tells us that we may replace α in Theorem 2.4 by α_1. Therefore, from now on, we assume that $\iota^*\alpha = 0$.

Define a 2-form η on $\check{D}$ as

$$\eta = \begin{cases} f^*d\alpha & \text{on } \check{D}^+, \\ -\sigma^* f^* d\alpha & \text{on } \check{D}^-. \end{cases} \tag{4.5}$$

It's easy to check that $f^*d\alpha|_{\partial\check{D}^+} = -\sigma^* f^* d\alpha|_{\partial\check{D}^+}$. Therefore, η is well defined. We infer that η is smooth on $\check{D}^+$ and $\check{D}^-$, and continuous on $\check{D}$.

Similarly, define a 1-form α' on $\check{D}$ as

$$\alpha' = \begin{cases} f^*\alpha & \text{on } \check{D}^+, \\ -\sigma^* f^* \alpha & \text{on } \check{D}^-. \end{cases} \tag{4.6}$$

Let $\iota_0 : \partial\check{D}^+ \hookrightarrow \check{D}$ be the inclusion. By the assumptions that $\iota^*\alpha = 0$ and $f(\partial\check{D}^+) \subseteq W$, we get

$$\iota_0^* f^* \alpha = 0.$$

Therefore, it's easy to check that α' is well defined and continuous on $\check{D}$.

It's important to observe that, if the assumption $\iota^*\alpha = 0$ is dropped, one cannot expect that α' is continuous on $\check{D}$.

Clearly, on $\check{D}^+$ and $\check{D}^-$, α' is smooth and

$$d\alpha' = \eta. \tag{4.7}$$

By Example 1.2, Theorem 2.4 will no longer be true if we drop the assumption that $\iota^*\alpha = 0$ (or more generally $\iota^*\alpha$ is exact). How does this assumption help our proof? The following is a quintessential example.

Example 4.7. Suppose Ω is a closed disk in $\check{D}$. Then $\partial\check{D}^+$ divides Ω into two parts Ω^+ and Ω^-. (See Figure 1. The shadowed part is Ω.)

By (4.7), we have

$$\int_{\Omega^\pm} \eta = \int_{\Omega^\pm} d\alpha' = \int_{\partial\Omega^\pm} \alpha'$$

and

$$\int_{\Omega} \eta = \int_{\Omega^+} \eta + \int_{\Omega^-} \eta = \int_{\partial\Omega^+} \alpha' + \int_{\partial\Omega^-} \alpha'.$$

Since $\iota_0^*\alpha' = 0$, we get the integrals of α' along the real line is 0. Therefore, we get the important formula

$$\int_{\Omega} \eta = \int_{\partial\Omega} \alpha'. \tag{4.8}$$

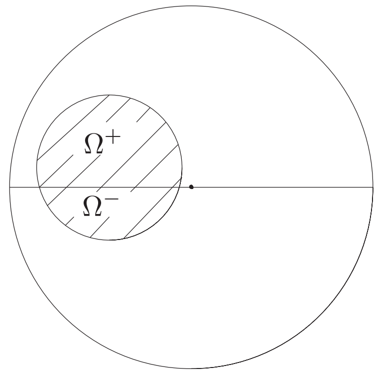

Figure 1

Actually, as far as α' is continuous on $\check{D}$, we shall also get (4.8) because the integrals on $\partial\check{D}^+$ and $\partial\check{D}^-$ cancel each other. However, as mentioned before, α' as defined above would not be continuous on $\check{D}$ if $\iota^*\alpha \neq 0$.

Lemma 4.8. *There exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$G(v, v) \leq C_1 \eta(v, iv)$$

and

$$\|\alpha'\|_G \leq C_2,$$

where v is any tangent vector on $\check{D}$, i is the complex structure on $\check{D}$ and $\|\alpha'\|_G$ is the norm of α' with respect to G.

Proof. Since $f(\check{D}^+)$ has a compact closure and $d\alpha$ tames J, we infer there exist $C_1 > 0$ and $C_2 > 0$ such that, on $f(\check{D}^+)$,

$$H(w, w) \leq C_1 d\alpha(w, Jw)$$

and

$$\|\alpha\|_{\mathrm{Re}H} \leq C_2,$$

where w is any tangent vector tangent to $f(\check{D}^+)$.

By the definitions of G, η and α', the lemma follows. $\square$

Let $\varphi : D \to N$ be a C^1 J-holomorphic map, where N is a C^1 almost complex manifold. Let $A(r) = |\varphi(D(r))|$ and $L(r) = |\partial\varphi(D(r))|$. Suppose

$A(r) \to 0$ when $r \to 0$. As we can see in [2, 1.2 and 1.3] and in this paper, studying the decaying rate of $A(r)$ gives very important geometric information.

How to obtain such a decaying rate? Gromov tells us it's sufficient to use two tools. The first one is the formula (3.6). The second one is an isoperimetric inequality.

Following [2, 1.2 and 1.3], we shall build two types of isoperimetric inequalities in this paper. The first one is $L^2 \geq 4\pi A - CA^2$. The second one is $L^2 \geq CA^2$. Comparing them, the first one is better when A is small, while the second one becomes better when A is large. The second one is important when we cannot bound $A(1)$, which lies at the heart of how the form α plays a role in Theorem 2.4 and Lemma 4.1 (see also Remark 4.12).

For $z_0 \in \check{D}$, let $\varphi_{z_0} : D \to \check{D}$ be a holomorphic universal covering such that $\varphi_{z_0}(0) = z_0$. It's well-known that D carries a hyperbolic metric $2(1 - |z|^2)^{-1}|dz|$ and $\check{D}$ carries a hyperbolic metric $\left(|z| \log \frac{1}{|z|}\right)^{-1} |dz|$, and

$$\varphi_{z_0} : \left(D, \frac{2|dz|}{1-|z|^2}\right) \to \left(\check{D}, \frac{|dz|}{|z| \log \frac{1}{|z|}}\right) \tag{4.9}$$

is a local isometry. Even though we use above the hyperbolic metrics to describe the map φ_{z_0}, whenever not otherwise indicated, the metric on D is the Euclidean metric, and the metric on $\check{D}$ is G. In particular the subsequent isoperimetric inequalities are for the metric G on $\check{D}$. (Compare also Definition 3.1).

By Lemma 4.5 and the classical isoperimetric inequality in terms of Gaussian curvature [1, Theorem (1.2)] (see also [9, p. 1206, (4.25)]), we obtain the following isoperimetric inequality.

Lemma 4.9. *There exists a constant $C > 0$ such that, for all $r \in (0,1)$ and $z_0 \in \check{D}$,*

$$|\varphi_{z_0}(\partial D(r))|^2 \geq 4\pi \left(|\varphi_{z_0}(D(r))| - C|\varphi_{z_0}(D(r))|^2\right).$$

In the proof of the following isoperimetric inequality, we shall see the importance of the assumption on α in Theorem 2.4. (Compare [2, p. 317, (8)].)

Lemma 4.10. *There exists a constant $C > 0$ such that, for all $r \in (0,1)$ and $z_0 \in \check{D}$,*

$$|\varphi_{z_0}(\partial D(r))|^2 \geq 2\pi C|\varphi_{z_0}(D(r))|^2.$$

Proof. We know that $\varphi_{z_0}^{-1}(\partial\check{D}^+)$ is a countable union of circular arcs whose ends are on the boundary of D. (Figure 2 shows some of these circular arcs. They are in fact geodesics of the hyperbolic metric on D.)

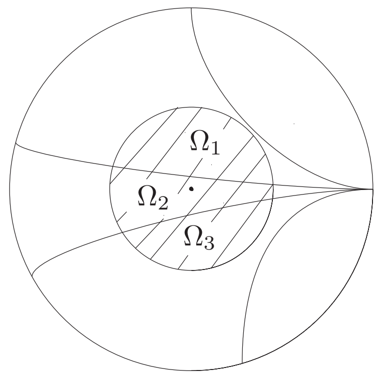

Figure 2

We consider the integral $\int_{D(r)} \varphi_{z_0}^* \eta$. Clearly, $\varphi_{z_0}^{-1}(\partial\check{D}^+)$ divides $D(r)$ into several domains $\Omega_1, \ldots, \Omega_k$, which are compact submanifolds with corners insider D. (This is illustrated by Figure 2. The shadowed part is $D(r)$.) In each Ω_i, by (4.7), we have

$$\varphi_{z_0}^* \eta = d\varphi_{z_0}^* \alpha'.$$

Applying Stokes' formula, we get

$$\int_{\Omega_i} \varphi_{z_0}^* \eta = \int_{\partial\Omega_i} \varphi_{z_0}^* \alpha'.$$

Similar to Example 4.7, the line integral of $\varphi_{z_0}^* \alpha'$ vanishes on $\varphi_{z_0}^{-1}(\partial\check{D}^+)$. Thus, summing up these integrals, we get

$$\int_{D(r)} \varphi_{z_0}^* \eta = \int_{\partial D(r)} \varphi_{z_0}^* \alpha'.$$

By Lemma 4.8, we get

$$|\varphi_{z_0}(D(r))| \leq C_1 \int_{D(r)} \varphi_{z_0}^* \eta = C_1 \int_{\partial D(r)} \varphi_{z_0}^* \alpha' \leq C_1 C_2 |\varphi_{z_0}(\partial D(r))|,$$

where C_1 and C_2 are the constants in Lemma 4.8. This finishes the proof. $\square$

Lemma 4.11. *There exists a constant $C > 0$ such that $\|d\varphi_{z_0}(0)\| \le C$ for all maps φ_{z_0}.*

Proof. Let $A(t) = |\varphi_{z_0}(D(t))|$ and $L(t) = |\varphi_{z_0}(\partial D(t))|$. By (3.6) and Lemmas 4.10 and 4.9, there exist constants $C_3 > 0$ and $C_4 > 0$ independent of φ_{z_0} such that

$$\dot{A} \ge t^{-1} C_3 A^2, \tag{4.10}$$

and

$$\dot{A} \ge 2t^{-1}(A - C_4 A^2). \tag{4.11}$$

Since $A > 0$ for $t \in (0,1)$, by (4.10), we infer

$$A^{-2}\dot{A} \ge C_3 t^{-1}.$$

So, for $0 < r < \frac{1}{2}$, we have

$$\int_r^{\frac{1}{2}} A^{-2}\dot{A}dt \ge \int_r^{\frac{1}{2}} C_3 t^{-1} dt$$

or

$$A(r) \le \frac{1}{A(\frac{1}{2})^{-1} - C_3 \log 2r}, \tag{4.12}$$

where $A(\frac{1}{2}) < +\infty$.

Choose r_0 such that

$$-C_3 \log 2r_0 = 2C_4,$$

we get

$$r_0 = \frac{1}{2} e^{-\frac{2C_4}{C_3}}. \tag{4.13}$$

By (4.12) and (4.13), we get

$$A(r_0) \le \frac{1}{A(\frac{1}{2})^{-1} - C_3 \log 2r_0} \le \frac{1}{-C_3 \log 2r_0} = \frac{1}{2C_4}. \tag{4.14}$$

For $0 < r \le r_0$, since $A(r) \le A(r_0)$, we have

$$A(r) - C_4 A(r)^2 \ge A(r)(1 - C_4 A(r_0)) \ge 2^{-1} A(r) > 0. \tag{4.15}$$

By (4.11) and (4.15), we get, for $0 < r \leq r_0$,

$$(A - C_4 A^2)^{-1} \dot{A} \geq 2t^{-1}. \tag{4.16}$$

Integrating both sides of (4.16) on $[r, r_0]$, we get

$$\log \frac{A(r_0)}{1 - C_4 A(r_0)} - \log \frac{A(r)}{1 - C_4 A(r)} \geq \log \left(\frac{r_0}{r}\right)^2$$

or

$$\frac{A(r)}{1 - C_4 A(r)} \leq \frac{A(r_0)}{1 - C_4 A(r_0)} \left(\frac{r}{r_0}\right)^2. \tag{4.17}$$

By (4.13), (4.14) and (4.17), we get, for $r \leq r_0$,

$$\begin{aligned} A(r) &\leq \frac{A(r)}{1 - C_4 A(r)} \leq \frac{A(r_0)}{1 - C_4 A(r_0)} \left(\frac{r}{r_0}\right)^2 \\ &\leq \frac{(2C_4)^{-1}}{1 - C_4 (2C_4)^{-1}} \left(\frac{1}{2} e^{-\frac{2C_4}{C_3}}\right)^{-2} r^2 = 4C_4^{-1} e^{\frac{4C_4}{C_3}} r^2. \end{aligned} \tag{4.18}$$

Therefore,

$$\|d\varphi_{z_0}(0)\|^2 = \lim_{r \to 0} (\pi r^2)^{-1} A(r) \leq 4\pi^{-1} C_4^{-1} e^{\frac{4C_4}{C_3}},$$

which finishes the proof. $\square$

Remark 4.12. In the proof of Lemma 4.11, by using Lemma 4.10, we get the uniform estimate (4.14) for $A(r_0)$, where both (4.14) and r_0 are independent of the maps φ_{z_0}. Based on this estimate, Lemma 4.9 yields the decaying rate (4.18).

Now the following lemma implies Lemma 4.1 immediately.

Lemma 4.13. *Using the standard coordinate $z = x + iy$ on $\check{D}$, the metric G has the form $g dx \otimes dx + g dy \otimes dy$. There exists a constant $C > 0$ such that*

$$\sqrt{g(z)} \leq \frac{C}{|z| \log \frac{1}{|z|}}.$$

Proof. Following the proof of [2, 1.3.A'], the idea of this proof is to compare G with the hyperbolic metric on $\check{D}$. (See also [7, p. 227] and [3, p. 42].)

Consider the above $\varphi_{z_0} : D \to \check{D}$ with $\varphi_{z_0}(0) = z_0$. When φ_{z_0} is viewed as a holomorphic function, we denote its derivative by $\varphi'_{z_0}(z)$. Clearly,

$$\|d\varphi_{z_0}(0)\| = \sqrt{g(z_0)}|\varphi'_{z_0}(0)|.$$

Since (4.9) is a local isometry, we infer

$$|\varphi'_{z_0}(0)| = 2|z_0| \log \frac{1}{|z_0|}.$$

By Lemma 4.11, there exists a constant $C_0 > 0$ such that

$$\|d\varphi_{z_0}(0)\| \leq C_0.$$

Then we get

$$\sqrt{g(z_0)} \leq \frac{C_0}{2} \frac{1}{|z_0| \log \frac{1}{|z_0|}},$$

which finishes the proof. □

Finally, we are ready to prove the main Theorem 2.4.

Proof of Theorem 2.4. Equip M with the Riemannian metric ReH from Lemma 3.2. By Lemma 4.1, we have $|f(\check{D}^+(r))| < +\infty$ for all $r \in (0,1)$.

For $r \in (0,1)$, define $f_r : \check{D}^+ \to M$ as

$$f_r(z) = f(rz).$$

Then f_r is a J-holomorphic map such that $f_r(\partial\check{D}^+) \subseteq W$ and $f_r(\check{D}^+)$ has a compact closure. Furthermore, $|f_r(\check{D}^+)| < +\infty$.

By Corollary 3.2 in [4], f_r extends to $0 \in D^+$ as a $W^{1,p}$-map $f_r : D^+ \to M$ for some $p > 2$, where $W^{1,p}$ is a Sobolev space. (The assumptions of Corollary 3.2 in [4] are quite general. Besides the assumption on the area of the image of f_r, in order to apply that corollary, one has to check additionally whether the conditions "ReH-uniformly continuous", "ReH-complete", "ReH-uniformly totally real" and "ReH-uniformly transversal" in the sense of [4] are satisfied in our situation. This can be done by using the fact that $f_r(\check{D}^+)$ has a compact closure, (1) in Lemma 3.2 and the closedness of W.)

By Sobolev's embedding theorem, $f_r : D^+ \to M$ is a $C^{0,\delta}$ Hölder continuous map for some $\delta \in (0,1)$. By (i) of Theorem 1.3 in [5], we infer that $f_r : D^+ \to M$ is a smooth map. This implies the theorem. □

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