

Geometric flows and Kähler reduction

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We investigate how to obtain various flows of Kähler metrics on a fixed manifold as variations of Kähler reductions of a metric satisfying a given static equation on a higher dimensional manifold. We identify static equations that induce the geodesic equation for the Mabuchi’s metric, the Calabi flow, the pseudo-Calabi flow of Chen-Zheng and the Kähler-Ricci flow. In the latter case we re-derive the V-soliton equation of La Nave-Tian.

1. Introduction

This note is concerned with the description, in various instances, of what geometry one should impose on the total space of a Kähler manifold (P, ω, J) endowed with a Hamiltonian holomorphic circle action with moment map $\mu : P \rightarrow \mathbf{R}$, so that the *variation* of symplectic quotients with induced metrics, sometimes called *Kähler reductions*, describes indeed a given flow on the initial quotient (naturally, up to diffeomorphisms). The mathematical literature is by now rich of examples of the interplay between geometry and Kähler (or merely symplectic) reductions, spanning from Guillemin and Sternberg fundamental papers [5–7] to the more recent work of Burns and Guillemin [1] passing through the work of Futaki [4].

Our motivation for this study is twofold: one is based on trying to create a theoretical set-up and machinery where one can naturally incorporate flows with *surgery* in a context where such surgeries occur naturally in the Morse-theory of variations of Kähler reductions; the other is to try and understand convergence (or lack thereof) at infinite time in terms of what happens to the Kähler reduction when one meets the first critical point of the moment map (which is not necessarily the first time the reduction becomes “singular”, cf. Example 2.2).

To the authors knowledge, the first instance in which Kähler reduction was used to analyze the nature of finite time singularities of a given flow of metrics is in the work of La Nave and Tian, in connection to the Kähler-Ricci flow [8]. There it is shown that the Kähler-Ricci flow on a Kähler manifold M is loosely speaking equivalent to a static equation, dubbed the

V -soliton equation since it is of soliton type, on a manifold P endowed with a Hamiltonian holomorphic circle action, which has M as one of its Kähler reductions.

We continue with this idea in this paper, where we investigate in fact how to obtain various flows of metrics as variations of Kähler reductions of a metric on the total space satisfying a given static equation. This approach is somewhat different from the usual one inspired by Kaluza-Klein theories, where one usually is interested in the rich nature of quotients of a given special metric, such as a Kähler-Einstein metric (cf. [1, 4]). Our take could be characterized as being somewhat opposite and more along the lines of a Geometric-flow version of variation of GIT or symplectic quotients (cf. [7] and references therein) whereby we analyze the *variation* of metrics under the variation of reductions and identify equations on the total space for which the variation of metrics follows a specified flow.

Specifically, we identify equations on the total space that induce:

- the Geodesic equation in the space of Kähler potentials (with respect to Mabuchi L^2 -metric) in Theorem 3.2;
- the Calabi flow in Theorem 3.4;
- the pseudo-Calabi flow of Chen and Zheng in Theorem 3.7;
- and finally we re-derive La Nave and Tian's V -soliton equation (cf. [8]) in Theorem 3.9 for the normalized Kähler-Ricci flow and its variant for the unnormalized one in Theorem 3.10.

Arguably, the major difference in approach between the current paper and [8] is the fact that for most of our applications, such as the Calabi flow, one needs only — and in fact must — take the space whose Kähler reductions represent the flow to simply be a product $P = M \times A$ with $A \subset \mathbf{C}^*$ some annulus (cf. Theorem 4.5 in Section 4). In the case of the normalized Kähler-Ricci flow this is also true, as long as the initial metric is canonical or anticanonical, as the case maybe. In particular, the case of interest in [8], namely the finite-time singularity of the Kähler-Ricci flow in non-(anti)-canonically polarized metrics, is never of this nature.

In the last section of this paper we explain how these new equations are indeed equivalent to the associated geometric flows; namely, given a solution to the flow on M , how to construct the relevant structures on the product P to get a solution to the new equations.

Along the way, we show how natural geometric quantities of the reduced metric g_τ on M comes from *reduction* (see Section 2.2 for precise definition)

of suitable quantities on P . For example in Proposition 2.8 and Corollary 2.9 we show that there exist a one form ρ and a smooth function R on P that descend respectively to the Ricci form and to the scalar curvature of the reduced metric g_τ on M . In the same circle of ideas, we show in Lemma 2.6 and Corollary 2.7 that the Monge-Ampère operator and, in particular, the Laplacian of the reduced metric g_τ are induced by suitable operators on P .

2. Hamiltonian circle actions on Kähler manifolds

We are mainly interested in Hamiltonian holomorphic circle actions on Kähler manifolds. The purpose of this section is to investigate the geometry of the Kähler reduction in connection with the one of the starting manifold. For simplicity we consider just the case of a semi-free action. This means that it is free away from the fixed point set, or equivalently that there are no finite non-trivial isotropy groups. In this case, any non-fixed point has a neighborhood equivariantly biholomorphic to the product of a fixed manifold with an annulus endowed with the standard circle action. The reduced manifold can be recovered by patching together the reductions of these invariant neighborhoods. Since we are just interested in local properties of the reduced manifolds, we can assume that our starting manifold is itself a product. In particular, this argument assures that all the statement of this sections are true for any Kähler manifold endowed with a semi-free holomorphic Hamiltonian circle action, with the exception of Lemma 2.1 and Proposition 2.5 (which will only hold locally).

2.1. The total space is a product

Given a (non-necessarily compact) connected complex n -fold M , consider the product $P = M \times A$, where $A = \{w \in \mathbf{C} \text{ s.t. } r < |w| < R\}$ is the annulus of radii $r, R > 0$. We allow $r = 0$ or $R = +\infty$. The standard circle action on P defined by $e^{i\theta} \cdot (x, w) = (x, e^{i\theta}w)$ is generated by the real vector field

$$V = i \left(w \frac{\partial}{\partial w} - \bar{w} \frac{\partial}{\partial \bar{w}} \right).$$

Moreover one has

$$(1) \quad JV = - \left(w \frac{\partial}{\partial w} + \bar{w} \frac{\partial}{\partial \bar{w}} \right) = -2s \frac{\partial}{\partial s},$$

where J denotes the complex structure of P and s is the smooth function on P defined by $s(x, w) = w\bar{w}$. We will consider Kähler metrics on P that makes Hamiltonian the standard circle action. By definition the Kähler form ω of such a metric satisfies $i_V\omega = d\mu$ for some moment map $\mu : P \rightarrow \mathbf{R}$. Let $\pi : P \rightarrow M$ be the projection on the first factor.

Lemma 2.1. *Let g be a Kähler metric on P with Kähler form ω . The standard circle action is Hamiltonian with moment map $\mu : P \rightarrow \mathbf{R}$ if and only if there exist a Kähler form σ on M , a smooth invariant function $\phi : P \rightarrow \mathbf{R}$, and a constant $c \in \mathbf{R}$ such that*

- i) $\omega = \pi^*\sigma + dd^c\phi$,
- ii) $\mu = JV(\phi) + c$.

Any other such triple $(\tilde{\sigma}, \tilde{\phi}, \tilde{c})$ satisfies

$$(2) \quad \begin{cases} \tilde{\sigma} = \sigma + dd^c u \\ \tilde{\phi} = \phi - \pi^*u + \frac{\tilde{c}-c}{2} \log s + b, \end{cases}$$

for some $b \in \mathbf{R}$, and some smooth function u on M such that $\sigma + dd^c u > 0$.

Proof. Assuming there exists (σ, ϕ, c) as in the statement such that (i) and (ii) hold, one has

$$i_V\omega = L_V d^c\phi - di_V d^c\phi = dJV(\phi) = d\mu.$$

Since ω is closed, the standard circle action turns out to be Hamiltonian with moment map μ .

For any $(\tilde{\sigma}, \tilde{\phi}, \tilde{c})$ giving the same ω and μ , one has

$$(3) \quad \begin{cases} \pi^*(\tilde{\sigma} - \sigma) + dd^c(\tilde{\phi} - \phi) = 0 \\ JV(\tilde{\phi} - \phi) + \tilde{c} - c = 0 \end{cases}$$

Thanks to (1), from second equation one easily finds that

$$\tilde{\phi} - \phi = \frac{\tilde{c} - c}{2} \log s + b - \pi^*u,$$

for some constant $b \in \mathbf{R}$ and some smooth function u on M . Substituting in the first equation of (3) then proves (2).

It remains to show that any Kähler form ω on P making Hamiltonian the standard circle action with moment map μ satisfies (i) and (ii) for some

triple (σ, ϕ, c) . To this end consider the decomposition

$$(4) \quad \omega = \eta + \alpha \wedge d^c \log s + d \log s \wedge \beta + \gamma d \log s \wedge d^c \log s,$$

where η is a smooth section of the bundle $\pi^* \Lambda^{1,1} T^* M$, α, β are smooth sections of $\pi^* T^* M$, and γ is a smooth function on P . Imposing that μ is a moment map gives

$$\begin{aligned} d\mu &= i_V \omega = -2\alpha - 2\gamma d \log s, \\ d^c \mu &= i_{JV} \omega = -2\beta - 2\gamma d^c \log s, \\ JV(\mu) &= \omega(V, JV) = 4\gamma, \end{aligned}$$

whence, substituting in (4) after easy calculations it follows

$$(5) \quad \omega = \eta - \frac{1}{2} d\mu \wedge d^c \log s - \frac{1}{2} d \log s \wedge d^c \mu - \frac{1}{4} JV(\mu) d \log s \wedge d^c \log s.$$

The smooth circle invariant function ϕ on P defined by

$$\phi(x, s) = \int_{r^2}^s \frac{c - \mu(x, t)}{2t} dt,$$

clearly satisfies (ii). On the other hand, one can easily check that the following decomposition holds (cf. [8, Lemma 3.4]):

$$(6) \quad dd^c \phi = d_M d_M^c \phi - \frac{1}{2} d\mu \wedge d^c \log s - \frac{1}{2} d \log s \wedge d^c \mu - \frac{1}{4} JV(\mu) d \log s \wedge d^c \log s,$$

where $d_M d_M^c \phi$ denotes the projection of $dd^c \phi$ on the sub-bundle $\pi^* \Lambda^{1,1} T^* M \subset \Lambda^{1,1} T^* P$. Comparing $dd^c \phi$ with (5) gives

$$(7) \quad \omega - dd^c \phi = \eta - d_M d_M^c \phi.$$

The form $\omega - dd^c \phi$ is clearly circle invariant, and from the right hand side of equation above it follows that it vanishes on the distribution generated by V and JV . Moreover one has

$$\begin{aligned} L_{JV}(\omega - dd^c \phi) &= d(i_{JV} \omega - i_{JV} dd^c \phi) \\ &= dd^c \mu - di_{JV} \left(d_M d_M^c \phi - \frac{1}{2} d\mu \wedge d^c \log s - \frac{1}{2} d \log s \wedge d^c \mu \right. \\ &\quad \left. - \frac{1}{4} JV(\mu) d \log s \wedge d^c \log s \right) \\ &= 0. \end{aligned}$$

Thus there exists a Kähler form σ on M such that $\omega - dd^c\phi = \pi^*\sigma$, whence (i) follows. \square

2.2. Kähler reduction

Keeping notation of Section 2, fix a Kähler metric g on P that makes Hamiltonian the standard circle action with moment map μ . Thanks to Lemma 2.1, this amounts to choosing a Kähler form σ on M , a smooth invariant function ϕ on P , and a constant $c \in \mathbf{R}$ satisfying conditions (i) and (ii) of Lemma 2.1. Let τ be a regular value of μ , so that the circle action is free on the level set $S_\tau = \mu^{-1}(\tau)$. With no additional assumption on μ , the level set S_τ is not necessarily compact even if M is, and its quotient

$$M_\tau = \pi(S_\tau) \subset M$$

may not be closed. Such a situation is illustrated by the following example.

Example 2.2. Consider the family $\pi : X \rightarrow \mathbf{C}$, where

$$X = \{(z, w) \in \mathbf{CP}^2 \times \mathbf{C} \text{ s.t. } z_0 z_2 - z_1^2 w = 0\},$$

and π is just the coordinate w . It's easy to check that endowing \mathbf{CP}^2 with the circle action $e^{i\theta} \cdot z = (e^{i\theta} z_0, z_1, z_2)$, and \mathbf{C} with the standard one, makes X invariant for the diagonal action on $\mathbf{CP}^2 \times \mathbf{C}$. Thus X has a circle action and one readily verifies that π turns out to be an equivariant map. Moreover it is easy to see that removing the central fiber $\pi^{-1}(0)$ from X gives rise to an invariant submanifold X^* biholomorphic to $P = \mathbf{CP}^1 \times \mathbf{C}^*$. An explicit biholomorphism $\Phi : P \rightarrow X^*$ is given by

$$\Phi(x, w) = (wx_0^2, x_0 x_1, x_1^2, w).$$

Note that Φ becomes equivariant when P is endowed with the standard action on the \mathbf{C}^* factor, thus we are in the general situation considered above.

The product metric $\omega_{FS} + \omega_{Euclid}$ on $\mathbf{CP}^2 \times \mathbf{C}$ is Hamiltonian with moment map

$$(z, w) \mapsto -\frac{|z_0|^2}{|z|^2} - |w|^2,$$

whence it follows readily that a moment map for the circle action on P (endowed with the induced metric) is given by

$$\mu(x, w) = -|w|^2 \frac{(1 + |w|^2)|x_0|^4 + |x_0|^2|x_1|^2 + |x_1|^4}{|w|^2|x_0|^4 + |x_0|^2|x_1|^2 + |x_1|^4}.$$

By means of easy calculations one can check that the image of μ is the interval $(-\infty, 0)$, and one has biholomorphisms $M_\tau \simeq \mathbf{CP}^1$ for $\tau < -1$, and $M_\tau \simeq \mathbf{CP}^1 \setminus \{(1 : 0)\}$ for $-1 \leq \tau < 0$. Moreover the reduced metric ω_τ on M_τ is smooth for $\tau < -1$, conic of angle π for $\tau = -1$ and it is incomplete for $\tau \in (-1, 0)$. In the latter case, its completion recovers \mathbf{CP}^1 .

Finally note that μ has no critical points, since the circle action on P has no fixed points. Nonetheless μ is clearly not *proper*.

The example above shows that, due to the non-properness of the moment map μ , the topology of reductions can still change when τ varies. On the other hand, if the reduced manifold M_τ coincides with M , then the same holds for $M_{\tau+\varepsilon}$ for all ε sufficiently small.

This example suggests to define, when μ is not proper, the Kähler reduction (M_τ, ω_τ) as the metric completion of the standard reduction. On the other hand, for the purposes of this paper, we only need the following:

Assumption. $\tau \in \mu(P)$ is chosen so that M_τ coincides with M . When τ varies, it does so in an interval where all the reductions coincide with M .

Let $\iota_\tau : S_\tau \rightarrow P$ be the inclusion and let $\pi_\tau : S_\tau \rightarrow M_\tau$ be the projection on the quotient, namely the restriction of π to the level set S_τ , so that one has the following commutative diagram:

$$\begin{array}{ccc} S_\tau & \xrightarrow{\iota_\tau} & P \\ \downarrow \pi_\tau & & \downarrow \pi \\ M_\tau & \longrightarrow & M \end{array}$$

Any invariant smooth function f on P descends to a smooth function f_τ on M_τ defined by

$$\iota_\tau^* f = \pi_\tau^* f_\tau.$$

A simple but crucial example is constituted by the function s , which descends to a function s_τ . Given any f , one can express f_τ by means of s_τ . Indeed, thanks to invariance, f has the form $f = f(\pi, s)$, whence obviously it follows $f_\tau(x) = f(x, s_\tau(x))$.

More generally, any invariant differential r -form η on P satisfying $\iota_\tau^*(i_V\eta) = 0$, descends to an r -form η_τ on M_τ defined by

$$\iota_\tau^*\eta = \pi_\tau^*\eta_\tau.$$

This defines a reduction map, that is a linear map

$$\{\eta \in \Omega^*(P) \text{ s.t. } L_V\eta = 0, \iota_\tau^*(i_V\eta) = 0\} \rightarrow \Omega^*(M_\tau), \quad \eta \mapsto \eta_\tau,$$

satisfying

$$(8) \quad (\eta \wedge \xi)_\tau = \eta_\tau \wedge \xi_\tau, \quad (d\eta)_\tau = d\eta_\tau, \quad (\pi^*\zeta)_\tau = \zeta,$$

as can be readily checked.

In particular, the Kähler form ω of the metric g descends to a non-degenerate two form ω_τ on M_τ . Moreover ω_τ turns out to be compatible with the complex structure of M_τ induced by the one of M , so that it defines a Kähler metric g_τ on M_τ .

Lemma 2.3. *For every invariant functions $f \in C^\infty(P)$ one has:*

$$(9) \quad \frac{\partial f_\tau}{\partial \tau} = \left(\frac{JV(f)}{|V|^2} \right)_\tau.$$

Proof. Differentiating the identity $f_\tau(x) = f(x, s_\tau(x))$ gives

$$(10) \quad \frac{\partial f_\tau}{\partial \tau} = \left(\frac{\partial f}{\partial s} \right)_\tau \frac{\partial s_\tau}{\partial \tau} = -\frac{1}{2} (JV(f))_\tau \frac{\partial \log s_\tau}{\partial \tau}.$$

Taking $f = \mu$, since $\mu_\tau = \tau$ and $JV(\mu) = |V|^2$, one gets

$$(11) \quad \frac{\partial \log s_\tau}{\partial \tau} = \left(-\frac{2}{|V|^2} \right)_\tau.$$

Thus the thesis follows substituting (11) in (10). □

Lemma 2.4. *For every invariant function $f \in C^\infty(P)$, the following holds*

$$(12) \quad d^c f_\tau = \left(d^c f - \frac{JV(f)}{|V|^2} d^c \mu \right)_\tau.$$

Proof. Differentiating the identity $f_\tau(x) = f(x, s_\tau(x))$ gives

$$\begin{aligned} d^c f_\tau &= \left(d^c f - \frac{\partial f}{\partial s} d^c s \right)_\tau + \left(\frac{\partial f}{\partial s} \right)_\tau d^c s_\tau \\ &= \left(d^c f + \frac{1}{2} JV(f) d^c \log s \right)_\tau - \frac{1}{2} (JV(f))_\tau d^c \log s_\tau. \end{aligned}$$

Taking $f = \mu$, since $\mu_\tau = \tau$ and $JV(\mu) = |V|^2$, one gets

$$(13) \quad d^c \log s_\tau = \left(\frac{2}{|V|^2} d^c \mu + d^c \log s \right)_\tau,$$

whence the thesis follows substituting in equation above. □

Proposition 2.5. *In the situation of Lemma 2.1, the Kähler form of the reduced metric g_τ satisfies*

$$\omega_\tau = \sigma + dd^c \psi_\tau,$$

where

$$\psi = \phi + \frac{\mu - c}{2} \log s.$$

Proof. By Lemma 2.4, recalling that $JV(\phi) = \mu - c$ and $JV(\mu) = |V|^2$ it follows

$$d^c \psi_\tau = \left(d^c \phi + \frac{\mu - c}{2} d^c \log s \right)_\tau.$$

Thus, since $\omega = \pi^* \sigma + dd^c \phi$, by (8) one has

$$(14) \quad \sigma + dd^c \psi_\tau = \left(\pi^* \sigma + dd^c \phi + \frac{1}{2} d\mu \wedge d^c \log s \right)_\tau = \omega_\tau,$$

where we used $(d\mu \wedge d^c \log s)_\tau = 0$. □

The Monge-Ampère operator of the reduced manifold is given by the reduction of a non-linear operator on P . Indeed one has the following (which is a consequence of Lemma 3.3 and Lemma 3.7 in [8], but here we produce a different proof):

Lemma 2.6. *For all invariant functions $f \in C^\infty(P)$ one has:*

$$(15) \quad \frac{(\omega_\tau + dd^c f_\tau)^n}{\omega_\tau^n} = \left(\frac{\left(\omega + dd^c f - d \frac{JV(f)}{|V|^2} \wedge d^c \mu - \frac{JV(f)}{|V|^2} dd^c \mu \right)^{n+1}}{\omega^{n+1}} \right)_\tau.$$

Proof. Any top form θ on P can be written as

$$(16) \quad \theta = \eta \wedge d \log s \wedge d^c \log s,$$

where η is a section of $\pi^* K_M$. Clearly one can recover η from θ by contraction with V and JV :

$$(17) \quad i_{JV} i_V \theta = -2i_{JV} (\eta \wedge d \log s) = 4\eta.$$

Let

$$\xi = \omega + dd^c f - d \frac{JV(f)}{|V|^2} \wedge d^c \mu - \frac{JV(f)}{|V|^2} dd^c \mu.$$

Note that it can be also written as

$$\xi = \omega + d \left(d^c f - \frac{JV(f)}{|V|^2} d^c \mu \right).$$

We want to write ξ^{n+1} in the form (16). Since

$$(18) \quad \begin{aligned} i_{JV} i_V \xi^{n+1} &= (n+1) i_{JV} (i_V \xi \wedge \xi^n) \\ &= (n+1) i_{JV} i_V \xi \left(\xi - \frac{i_V \xi \wedge i_{JV} \xi}{i_{JV} i_V \xi} \right)^n, \end{aligned}$$

we need to calculate the contraction of ξ with V and JV . The one form $d^c f - \frac{JV(f)}{|V|^2} d^c \mu$ is circle invariant, thus

$$i_V \xi = d\mu + L_V \left(d^c f - \frac{JV(f)}{|V|^2} d^c \mu \right) + di_V \left(d^c f - \frac{JV(f)}{|V|^2} d^c \mu \right) = d\mu,$$

whence it follows readily that $i_{JV} i_V \xi = |V|^2$. As we will see, we do not need an explicit expression for $i_{JV} \xi$. Substituting in (18) gives

$$(19) \quad \begin{aligned} i_{JV} i_V \xi^{n+1} &= (n+1) |V|^2 \\ &\times \left(\omega + dd^c f - d \frac{JV(f)}{|V|^2} \wedge d^c \mu - \frac{JV(f)}{|V|^2} dd^c \mu - \frac{d\mu \wedge i_{JV} \xi}{|V|^2} \right)^n, \end{aligned}$$

and in particular, taking $f = 0$:

$$i_{JV} i_V \omega^{n+1} = (n+1) |V|^2 \left(\omega - \frac{d\mu \wedge d^c \mu}{|V|^2} \right)^n,$$

by (16) and (17) it follows

$$(20) \quad \frac{\xi^{n+1}}{\omega^{n+1}} = \frac{\left(\omega + dd^c f - d \frac{JV(f)}{|V|^2} \wedge d^c \mu - \frac{JV(f)}{|V|^2} dd^c \mu - \frac{d\mu \wedge i_{JV} \xi}{|V|^2}\right)^n}{\left(\omega - \frac{d\mu \wedge d^c \mu}{|V|^2}\right)^n},$$

whence the thesis follows by reducing at τ and using Lemma 2.4. □

An immediate consequence is the following fact (cf. formula (14) in [8]):

Corollary 2.7. *For all invariant functions $f \in C^\infty(P)$ the following holds:*

$$(21) \quad \Delta_\tau f_\tau = \left(\Delta f - (\Delta \mu - JV \log |V|) \frac{JV(f)}{|V|^2} - \frac{(JV)^2(f)}{2|V|^2} \right)_\tau,$$

where Δ and Δ_τ denote the Laplace operators of the metrics g and g_τ respectively.

Proof. As well known the linearization of the Monge-Ampère operator, i.e. the left hand side of (15), is twice the Laplacian. Thus by Lemma 2.6 one has

$$(22) \quad 2\Delta_\tau f_\tau = \left(\frac{(n+1) \left(dd^c f - d \frac{JV(f)}{|V|^2} \wedge d^c \mu - \frac{JV(f)}{|V|^2} dd^c \mu \right) \wedge \omega^n}{\omega^{n+1}} \right)_\tau.$$

Thanks to the identity $\nabla f \cdot \nabla g \omega^{n+1} = (n+1)df \wedge d^c g \wedge \omega^n$ and recalling that $\nabla \mu = JV$ one calculates

$$\begin{aligned} & (n+1) \left(dd^c f - d \frac{JV(f)}{|V|^2} \wedge d^c \mu - \frac{JV(f)}{|V|^2} dd^c \mu \right) \wedge \omega^n \\ &= \left(2\Delta f - JV \left(\frac{JV(f)}{|V|^2} \right) - 2 \frac{JV(f)}{|V|^2} \Delta \mu \right) \omega^{n+1} \\ &= \left(2\Delta f - \frac{JV^2(f)}{|V|^2} + \frac{JV(f)}{|V|^2} JV \log |V|^2 - 2 \frac{JV(f)}{|V|^2} \Delta \mu \right) \omega^{n+1}, \end{aligned}$$

whence the thesis follows by substitution in (22). □

The Ricci curvature of the reduced metric turns out to be the reduction of a two form on M . More precisely one has the following

Proposition 2.8. *The (1,1)-form on P*

$$\rho = \text{Ric}(\omega) + dd^c \log |V| + d \left(\frac{\Delta\mu - JV \log |V|}{|V|^2} d^c \mu \right)$$

satisfies $\rho_\tau = \text{Ric}(\omega_\tau)$.

Proof. In the situation of Lemma 2.1, on M one has

$$(23) \quad 2 \text{ Ric}(\omega_\tau) - 2 \text{ Ric}(\sigma) = -dd^c \log \frac{\omega_\tau^n}{\sigma^n}.$$

Letting $\psi = \phi - \frac{\mu-c}{2} \log s$, thanks to Proposition 2.5 $\omega_\tau = \sigma + dd^c \psi_\tau$ so that

$$(24) \quad 2 \text{ Ric}(\omega_\tau) - 2 \text{ Ric}(\sigma) = dd^c \log \frac{(\omega_\tau - dd^c \psi_\tau)^n}{\omega_\tau^n},$$

whence by Lemma 2.6 it follows

$$(25) \quad 2 \text{ Ric}(\omega_\tau) - 2 \text{ Ric}(\sigma) = dd^c \log \left(\frac{\left(\omega - dd^c \psi + d \frac{JV(\psi)}{|V|^2} \wedge d^c \mu + \frac{JV(\psi)}{|V|^2} dd^c \mu \right)^{n+1}}{\omega^{n+1}} \right)_\tau.$$

As one can easily verify $JV(\psi) = \frac{|V|^2}{2} \log s$, and

$$dd^c \psi = dd^c \phi + \frac{1}{2} \log s dd^c \mu + \frac{1}{2} d \log s \wedge d^c \mu + \frac{1}{2} d\mu \wedge d^c \log s.$$

Thus by substitution in (25), after some easy calculations and recalling that $\omega = \pi^* \sigma + dd^c \phi$, one gets

$$(26) \quad 2 \text{ Ric}(\omega_\tau) - 2 \text{ Ric}(\sigma) = dd^c \log \left(\frac{(\pi^* \sigma - \frac{1}{2} d\mu \wedge d^c \log s)^{n+1}}{\omega^{n+1}} \right)_\tau.$$

Let $F = (\pi^* \sigma - \frac{1}{2} d\mu \wedge d^c \log s)^{n+1} / \omega^{n+1}$, by Lemma 2.4 and recalling that the exterior derivative commute with the reduction map, one has

$$(27) \quad 2 \text{ Ric}(\omega_\tau) - 2 \text{ Ric}(\sigma) = \left(dd^c \log F - d \left(\frac{JV \log F}{|V|^2} d^c \mu \right) \right)_\tau.$$

Since $JV(\mu) = |V|^2$, it holds

$$\left(\pi^*\sigma - \frac{1}{2}d\mu \wedge d^c \log s\right)^{n+1} = \frac{n+1}{4}|V|^2\pi^*\sigma^n \wedge d \log s \wedge d^c \log s,$$

whence

$$(28) \quad \log F = \log \frac{n+1}{4} + \log |V|^2 - \log \frac{\omega^{n+1}}{\pi^*\sigma^n \wedge d \log s \wedge d^c \log s}.$$

The top form $\pi^*\sigma^n \wedge d \log s \wedge d^c \log s$ is, up to a positive constant factor, the volume form of the product metric defined by $\pi^*\sigma + d \log s \wedge d^c \log s$ on P . Since $\text{Ric}(\pi^*\sigma + d \log s \wedge d^c \log s) = \pi^* \text{Ric}(\sigma)$, it turns out

$$(29) \quad dd^c \log F = dd^c \log |V|^2 + 2 \text{Ric}(\omega) - 2\pi^* \text{Ric}(\sigma).$$

On the other hand, since $L_{JV}(\pi^*\sigma^n \wedge d \log s \wedge d^c \log s) = 0$, one calculates

$$(30) \quad JV \log F = JV \log |V|^2 - \frac{L_{JV}(\omega^{n+1})}{\omega^{n+1}} = JV \log |V|^2 - 2\Delta\mu.$$

The thesis follows substituting (29) and (30) in (27) and observing that $(\pi^* \text{Ric}(\sigma))_\tau = \text{Ric}(\sigma)$. □

The scalar curvature

$$\text{scal}(g_\tau) = \frac{n \text{Ric}(\omega_\tau) \wedge \omega_\tau^{n-1}}{\omega_\tau^n}$$

of the reduced metric g_τ can be computed by means of data on M . Indeed the following holds:

Corollary 2.9. *The function*

$$R = \text{scal}(g) + 2\Delta \log |V| + \frac{2}{|V|^2} (\Delta\mu - JV \log |V|)^2 + \frac{JV}{|V|^2} (\Delta\mu - JV \log |V|)$$

satisfies $R_\tau = \text{scal}(g_\tau)$.

Proof. Here the main point is that the trace of ρ with respect to ω descends to the trace of ρ_τ with respect to ω_τ . The thesis then follows directly from Proposition 2.8.

First of all notice that by Cartan formula, circle invariance and by the identity $JV(\mu) = |V|^2$ it follows

$$(31) \quad \begin{aligned} i_V \rho &= i_V \operatorname{Ric}(\omega) - di_V d^c \log |V| - di_V \left(\frac{\Delta \mu - JV \log |V|}{|V|^2} d^c \mu \right) \\ &= i_V \operatorname{Ric}(\omega) + d\Delta \mu = 0, \end{aligned}$$

where the last equality is quite standard, but we include the proof for convenience of the reader.

$$(32) \quad \begin{aligned} i_V \operatorname{Ric}(\omega) &= i_V (\operatorname{Ric}(\omega) - \pi^* \operatorname{Ric}(\sigma)) \\ &= -\frac{1}{2} i_V dd^c \log \frac{\omega^{n+1}}{\pi^* \sigma^n \wedge d \log s \wedge d^c \log s} \\ &= -\frac{1}{2} dJV \log \frac{\omega^{n+1}}{\pi^* \sigma^n \wedge d \log s \wedge d^c \log s} \\ &= -\frac{1}{2} d \frac{L_{JV}(\omega^{n+1})}{\omega^{n+1}} \\ &= -\frac{1}{2} d \frac{(n+1) dd^c \mu \wedge \omega^n}{\omega^{n+1}} \\ &= -d\Delta \mu. \end{aligned}$$

Arguing as in the proof of Lemma 2.6, one has

$$\rho \wedge \omega^n = \eta \wedge d \log s \wedge d^c \log s,$$

where $4\eta = i_{JV} i_V (\rho \wedge \omega^n)$ is a section of $\pi^* K_M$. Thus by easy calculations one finds

$$(33) \quad \begin{aligned} \rho \wedge \omega^n &= \frac{n}{4} \left(i_{JV} \rho \wedge d\mu \wedge \omega^{n-1} \right. \\ &\quad \left. + |V|^2 \rho \wedge \left(\omega - \frac{d\mu \wedge d^c \mu}{|V|^2} \right)^{n-1} \right) \wedge d \log s \wedge d^c \log s. \end{aligned}$$

In the same way one gets

$$(34) \quad \omega^{n+1} = \frac{n+1}{4} |V|^2 \left(\omega - \frac{d\mu \wedge d^c \mu}{|V|^2} \right)^n \wedge d \log s \wedge d^c \log s.$$

Combining (33) and (34) then gives

$$(35) \quad \frac{(n+1)\rho \wedge \omega^n}{\omega^{n+1}} = \frac{n|V|^{-2}i_{JV}\rho \wedge d\mu \wedge \omega^{n-1} + n\rho \wedge \left(\omega - \frac{d\mu \wedge d^c \mu}{|V|^2}\right)^{n-1}}{\left(\omega - \frac{d\mu \wedge d^c \mu}{|V|^2}\right)^n}$$

and reducing at τ , recalling that $\rho_\tau = \text{Ric}(\omega_\tau)$, by Proposition 2.8 one then has

$$(36) \quad \left(\frac{(n+1)\rho \wedge \omega^n}{\omega^{n+1}}\right)_\tau = \frac{n\rho_\tau \wedge \omega_\tau^{n-1}}{\omega_\tau^n} = \text{scal}(g_\tau).$$

The thesis follows from equation above after observing that the trace of ρ with respect to ω coincides with the function R on the statement. Indeed one has

$$\begin{aligned} \frac{(n+1)\rho \wedge \omega^n}{\omega^{n+1}} &= \frac{(n+1)\left(\text{Ric}(\omega) + dd^c \log |V| + d\left(\frac{\Delta\mu - JV \log |V|}{|V|^2} d^c \mu\right)\right) \wedge \omega^n}{\omega^{n+1}} \\ &= \text{scal}(g) + 2\Delta \log |V| + \frac{2}{|V|^2}(\Delta\mu - JV \log |V|)\Delta\mu \\ &\quad + JV \frac{\Delta\mu - JV \log |V|}{|V|^2} \\ &= R. \end{aligned}$$

□

3. Geometric flows

In this section we consider the case when the reduced Kähler forms induce some geometrically meaningful path of cohomologous metrics on a compact manifold M . Thanks to Duistermaat-Heckman [3], this forces to assume P to be globally a product:

$$P = M \times \mathbf{C}^*.$$

By Lemma 2.1, choosing a Kähler metric ω on P making Hamiltonian the standard circle action with moment map μ is the same as choosing a triple (σ, ϕ, c) constituted by a Kähler form σ on M , a smooth invariant function ϕ on P , and a real constant c satisfying suitable compatibility conditions

with ω and μ . Moreover, by Lemma 2.5 the reduced metrics are given by

$$\omega_\tau = \sigma + dd^c\psi_\tau,$$

where ψ_τ is the reduction of the invariant function $\phi + \frac{\mu-c}{2} \log s$ on P . In particular ψ_τ defines a path in the space of Kähler potential

$$\mathcal{H}_\sigma = \{\varphi \in C^\infty(M) \mid \sigma_\varphi = \sigma + dd^c\varphi > 0\}.$$

Note that a fixed path ω_τ of Kähler metrics can induce many different paths in \mathcal{H}_σ . Indeed for any function $h = h(\tau)$, one has

$$\omega_\tau = \sigma + dd^c(\psi_\tau + h(\tau)).$$

On the other hand, given the path of reduced metrics $\{\omega_\tau\}$ one can ask if a path $\{\tilde{\omega}_t\}$ obtained just by reparametrization $\tau = \tau(t)$ of the time comes as path of reduced metrics on P . This is the case, and it holds the following

Proposition 3.1. *Let $f : \mu(P) \rightarrow \mathbf{R}$ be any smooth function invertible on its image. The path of metrics $\{\omega_{f^{-1}(t)}\}$ on M can be realized as path induced by reduction of the metric \tilde{g} associated to the Kähler form*

$$\tilde{\omega} = \omega + dd^c\Psi,$$

where Ψ is a smooth invariant function on P satisfying

$$JV(\Psi) = f(\mu) - \mu.$$

Proof. Since $i_V\tilde{\omega} = d(\mu + JV(\Psi))$, a moment map of the metrics \tilde{g} is given by

$$\tilde{\mu} = \mu + JV(\Psi) = f(\mu),$$

whence the thesis follows. □

3.1. Geodesics

As shown by Mabuchi [9], letting

$$\|v\|^2 = \int_M v^2 \frac{\sigma_\varphi^n}{n!}, \quad v \in T_\varphi\mathcal{H}_\sigma$$

defines a Riemannian metric on the space of Kähler potentials \mathcal{H}_σ . It is not difficult to see that a path $\{\varphi_t\} \subset \mathcal{H}_\sigma$ is a geodesic if it satisfies the equation

$$(37) \quad \varphi_t'' - \frac{1}{2} |\nabla \varphi_t'|^2 = 0,$$

where the prime denotes the derivative with respect to the time parameter t , and gradient and norm are taken with respect the variable metric σ_{φ_t} .

Theorem 3.2. *Suppose that for some smooth function $h = h(\mu)$ on P , the metric g satisfies*

$$(38) \quad \Delta s = hs.$$

Then the family of reduced metrics g_τ induces a geodesic in \mathcal{H}_σ .

Proof. By Proposition 2.5 the reduced Kähler form is given by

$$\omega_\tau = \sigma + dd^c \psi_\tau,$$

where $\psi = \phi + \frac{\mu-c}{2} \log s$. Given a smooth function $\kappa(\tau)$, the path $\psi_\tau - \kappa(\tau)$ defines a geodesic in \mathcal{H}_σ if and only if

$$(39) \quad \psi_\tau'' - \frac{1}{2} |\nabla_\tau \psi_\tau'|_\tau^2 = \kappa''(\tau),$$

where the prime denotes the derivative with respect to τ . Using the identity $\Delta e^\varphi = e^\varphi (\Delta \varphi + \frac{1}{2} |\nabla \varphi|^2)$, one readily verifies that the equation above is equivalent to:

$$(40) \quad \psi_\tau'' - e^{-\psi_\tau'} \Delta_\tau e^{\psi_\tau'} + \Delta_\tau \psi_\tau' = \kappa''(\tau).$$

Thanks to Lemma 2.3 we have:

$$(41) \quad \psi_\tau' = (\log s)_\tau, \quad \psi_\tau'' = \left(-\frac{2}{|V|^2} \right)_\tau,$$

whence, by means of Corollary 2.7, one can infer that:

$$(42) \quad \Delta_\tau e^{\psi_\tau'} = \left(\Delta s + (\Delta \mu - JV \log |V|) \frac{2s}{|V|^2} - \frac{2s}{|V|^2} \right)_\tau$$

and

$$(43) \quad \Delta_\tau \psi_\tau' = \left(\Delta \log s + (\Delta \mu - JV \log |V|) \frac{2}{|V|^2} \right)_\tau.$$

Substituting in (40), recalling that $\Delta \log s = 0$, then gives

$$(44) \quad \left(-\frac{\Delta s}{s} \right)_\tau = \kappa''(\tau),$$

which is Equation (38) after choosing $h(\tau) = -\kappa''(\tau)$. □

Remark 3.3. It is clear from the proof of the result above that h appears in connection to the fact that reduced potentials ψ_τ may define a geodesic only up to normalization.

3.2. Calabi flow and its variants

Theorem 3.4. *Suppose that for some smooth function $h = h(\mu)$ on P , the metric g satisfies*

$$(45) \quad R = \log s + h,$$

where

$$(46) \quad R = \text{scal}(g) + 2\Delta \log |V| + \frac{2}{|V|^2} (\Delta\mu - JV \log |V|)^2 + \frac{JV}{|V|^2} (\Delta\mu - JV \log |V|).$$

Then the family of reduced metrics g_τ is a solution of the Calabi flow on M :

$$(47) \quad \frac{\partial \omega_\tau}{\partial \tau} = \frac{1}{2} dd^c \text{scal}(g_\tau).$$

Proof. By Proposition 2.5 the reduced Kähler form is given by

$$\omega_\tau = \sigma + dd^c \psi_\tau,$$

where $\psi = \phi + \frac{\mu - c}{2} \log s$. Equation (47) is then clearly equivalent to

$$(48) \quad dd^c \left(\frac{\partial \psi_\tau}{\partial \tau} - \frac{1}{2} \text{scal}(g_\tau) \right) = 0.$$

By Corollary 2.9 one has $\text{scal}(g_\tau) = R_\tau$. On the other hand, by the identity $2JV(\psi) = \log s$ and Lemma 2.3 it follows $\frac{\partial \psi_\tau}{\partial \tau} = \frac{1}{2} (\log s)_\tau$. Thus (48) turns

out to be equivalent to

$$(49) \quad dd^c(\log s - R)_\tau = 0,$$

and this equation is satisfied whenever (45) holds, since $h_\tau = h(\tau)$ is constant on M . □

Remark 3.5. The appearance of the function h in the result above is a consequence of the fact that the reduced potentials ψ_τ are defined up to a constant on M depending on τ . On the other hand, given a Kähler form ω on P , there is a unique function h candidate to solve Equation (45). Indeed, after reducing at τ that equation, integrating over M and dividing by the volume of ω_τ (which is independent of τ) gives

$$(50) \quad h(\tau) = \lambda - \frac{\int_M \log s_\tau \omega_\tau^n}{\int_M \omega_\tau^n},$$

where $\lambda = \frac{nc_1(M) \cup [\sigma]^{n-1}}{[\sigma]^n}$ is the mean scalar curvature of the Kähler class $[\sigma]$.

Remark 3.6. Equation (49) suggests there exist Kähler metrics on P for which condition (45) may fail, but still the reduced metrics induce the Calabi flow on the reduction. Clearly in this case the reduced manifold is non-compact.

The pseudo-Calabi flow has been introduced and studied by Chen and Zheng [2].

Theorem 3.7. *Suppose that g satisfies the following equation*

$$(51) \quad R + \frac{2}{|V|^2} (\Delta\mu - JV \log |V|) = \lambda,$$

where

$$(52) \quad R = \text{scal}(g) + 2\Delta \log |V| + \frac{2}{|V|^2} (\Delta\mu - JV \log |V|)^2 + \frac{JV}{|V|^2} (\Delta\mu - JV \log |V|),$$

and $\lambda = \frac{nc_1(M) \cup [\sigma]^{n-1}}{[\sigma]^n}$. Then the family of reduced metrics g_τ is a solution of the pseudo-Calabi flow on M :

$$(53) \quad \begin{cases} \omega_\tau = \sigma + dd^c \psi_\tau \\ \Delta_\tau \psi'_\tau + \text{scal}(g_\tau) = \lambda. \end{cases}$$

where the prime denotes the partial derivative with respect to τ .

Proof. By Proposition 2.5 the reduced Kähler form is given by

$$\omega_\tau = \sigma + dd^c \psi_\tau,$$

where $\psi = \phi + \frac{\mu - c}{2} \log s$. Thanks to (43), the path ψ_τ is a solution of (53) if and only if

$$(54) \quad \left(\Delta \log s + \frac{2}{|V|^2} (\Delta \mu - JV \log |V|) \right)_\tau + \text{scal}(g_\tau) = \lambda,$$

thus the thesis follows by Corollary 2.9 and observing that $\Delta \log s = 0$. \square

Remark 3.8. It is a notable fact that for the pseudo-Calabi flow, one need not introduce the function h in order to renormalize the reduced potentials ψ_τ .

3.3. Kähler-Ricci flow

The approach to Kähler-Ricci flow by means of symplectic reduction is due to La Nave and Tian with the introduction of the V-soliton equation. In particular, the following result is due to them [8, Theorem 3.7]. Here we give a proof resting on results of Section 2. The reader is referred to their work for more details.

Theorem 3.9. *Suppose that for some smooth function $f = f(\mu)$ on P , the metric g satisfies*

$$(55) \quad \text{Ric}(\omega) + dd^c (\log |V| + f) = \lambda \omega,$$

being $\lambda = \frac{nc_1(M) \cup [\sigma]^{n-1}}{[\sigma]^n}$. Then $a = \Delta \mu - JV \log |V| - JV(f) + \lambda \mu$ is constant, and the family of reduced metrics g_τ satisfies:

$$(56) \quad (\lambda \tau - a) \frac{\partial \omega_\tau}{\partial \tau} = -\text{Ric}(\omega_\tau) + \lambda \omega_\tau,$$

which is the Kähler-Ricci flow up to reparametrizing τ .

Proof. Recall that $i_V \text{Ric}(\omega) + d\Delta\mu = 0$ as proved in (32). Thus contracting (55) with V gives

$$(57) \quad -d\Delta\mu + d(JV \log |V| + JV(f)) = \lambda d\mu,$$

whence, by connectedness of P , it follows that a is constant. On the other hand, thanks to the expression of ρ in Proposition 2.8, Equation (55) turns out to be equivalent to

$$\rho + d \left(d^c f - \frac{\Delta\mu - JV \log |V|}{|V|^2} d^c \mu \right) = \lambda \omega,$$

and by expression of a even to

$$(58) \quad \rho + d \left(d^c f - \frac{JV(f) - \lambda\mu + a}{|V|^2} d^c \mu \right) = \lambda \omega.$$

By Proposition 2.5 the reduced Kähler form is given by

$$\omega_\tau = \sigma + dd^c \psi_\tau,$$

where $\psi = \phi + \frac{\mu-c}{2} \log s$. By the identity $2JV(\psi) = |V|^2 \log s$ and Lemma 2.3 it follows $\frac{\partial \psi_\tau}{\partial \tau} = \frac{1}{2} \log s_\tau$. Therefore Lemma 2.4 and the equality $JV \log s = -2$ imply

$$\frac{\partial \omega_\tau}{\partial \tau} = d \left(\frac{1}{2} d^c \log s + \frac{d^c \mu}{|V|^2} \right)_\tau,$$

whence it follows that $d \left(\frac{\lambda\mu-a}{|V|^2} d^c \mu \right)$ reduces to $(\lambda\tau - a) \frac{\partial \omega_\tau}{\partial \tau}$. On the other hand, by Lemma 2.3 the form $d \left(d^c f - \frac{JV(f)}{|V|^2} d^c \mu \right)$ reduces to $dd^c f(\tau) = 0$. Thus reducing (58) gives

$$\text{Ric}(\omega_\tau) + (\lambda\tau - a) \frac{\partial \omega_\tau}{\partial \tau} = \lambda \omega_\tau.$$

□

Finally we consider the un-normalized Kähler-Ricci flow. Since the Kähler class of the reduced metric may change along the flow, according to Duistermaat-Heckman [3, Theorem 1.1] here we assume $\pi : P \rightarrow M$ to be the holomorphic \mathbf{C}^* -principal bundle on M associated to the canonical bundle K_M of M . Note that $c_1(TP) = 0$. Moreover V will denote the generator of the induced circle action on P , and g will be a Kähler metric on P making Hamiltonian that action. We still assume that the moment map μ is proper

so that, even in this case, any Kähler quotient is canonically biholomorphic to M via the projection π .

Theorem 3.10. *In the situation above, assume the metric g on P satisfies*

$$(59) \quad \text{Ric}(\omega) + dd^c \log |V| + d \left(\frac{\Delta\mu - JV \log |V| + 1}{|V|^2} d^c \mu \right) = 0.$$

Then the family of reduced metrics g_τ is a solution of the Kähler-Ricci flow on M :

$$(60) \quad \frac{\partial \omega_\tau}{\partial \tau} = -\text{Ric}(\omega_\tau).$$

Proof. Let $U \subset M$ be a trivializing open set, so that $\pi^{-1}(U) \simeq U \times \mathbf{C}^*$. Over U , by Proposition 2.5 the reduced Kähler form is given by

$$\omega_\tau = \sigma + dd^c \psi_\tau,$$

where $\psi = \phi + \frac{\mu-c}{2} \log s$. Arguing as in the proof of Theorem 3.9 one has

$$\frac{\partial \omega_\tau}{\partial \tau} = d \left(d^c \log s + \frac{d^c \mu}{|V|^2} \right)_\tau.$$

On the other hand, by Proposition 2.8 one has $\text{Ric}(\omega_\tau) = \rho_\tau$, where

$$\rho = \text{Ric}(\omega) + dd^c \log |V| + d \left(\frac{\Delta\mu - JV \log |V|}{|V|^2} d^c \mu \right).$$

Thus the thesis follows by substituting in (60), after noting that $dd^c \log s = 0$, so that the resulting equation is global. □

4. The structure of the total space and the converse theorems

The following result says that any path of cohomologous Kähler metrics can be realized as a family of reduced metrics from a bigger Kähler manifold.

Theorem 4.1. *Let g_t be a smooth path of cohomologous Kähler metrics on a compact complex manifold M with $t \in [0, T)$, for some $T \in (0, +\infty]$. Then there is $r \in [0, 1)$ and a circle invariant Kähler metric on $P = M \times A$, where $A = \{r < |w| < 1\} \subset \mathbf{C}$, inducing the path g_t on M via Kähler reduction.*

Proof. Let σ_t be the Kähler form of g_t . By dd^c -lemma there exists a smooth path of σ -plurisubharmonic functions ψ_t such that $\sigma_t = \sigma + dd^c\psi_t$, being $\sigma = \sigma_0$. Note that ψ_t is just defined up to an additive constant possibly depending on t . We will make use of this arbitrariness later.

Recalling that $\pi : P \rightarrow M$ is the projection on the first factor, and s is the smooth function on P defined by $s(p) = |w|^2$ if $p = (x, w)$, let F be the function on $[0, T) \times P$ defined by

$$(61) \quad F(t, p) = \psi_t(\pi(p)) - \frac{t}{2} \log s(p).$$

We look for a circle invariant function ϕ on P such that

$$(62) \quad F(JV(\phi), p) = \phi.$$

for all $p \in P$, and $JV(\phi)(p) = 0$ if $s(p) = 1$. This equation is clearly equivalent to

$$(63) \quad \phi + \frac{JV(\phi)}{2} \log s = \psi_{JV(\phi)} \circ \pi,$$

thus by Proposition 2.5 and Lemma 2.1 the function ψ_τ is the Kähler potential of reduced metrics whenever one can find a circle invariant solution ϕ to (62) satisfying $\pi^*\sigma + dd^c\phi > 0$. If a solution to (62) exists, letting $\mu = JV(\phi)$, by (6) and (63) it holds

$$(64) \quad \begin{aligned} dd^c\phi &= d \left(\pi^* d^c\psi_t + \frac{\partial\psi_t \circ \pi}{\partial t} d^c\mu \Big|_{t=\mu} - \frac{1}{2} \log s d^c\mu - \frac{\mu}{2} d^c \log s \right) \\ &= d \left(\pi^* d^c\psi_t \Big|_{t=\mu} - \frac{\mu}{2} d^c \log s \right) \\ &= \pi^* dd^c\psi_t \Big|_{t=\mu} + d\mu \wedge \left(\pi^* d^c \frac{\partial\psi_t}{\partial t} \Big|_{t=\mu} - \frac{1}{2} d^c \log s \right) \\ &= \pi^* dd^c\psi_t - \frac{\partial^2\psi_t \circ \pi}{\partial t^2} d\mu \wedge d^c\mu \Big|_{t=\mu}, \end{aligned}$$

where the second and last equalities follow from the identity $\frac{1}{2} \log s = \frac{\partial\psi_t \circ \pi}{\partial t} \Big|_{t=\mu}$, which in turn descends readily from applying JV to (63). Thus

for any vector field X on P one has

$$\begin{aligned}
 & (\pi^*\sigma + dd^c\phi)(X, JX) \\
 &= (\sigma + dd^c\psi_t)|_{t=\mu}(\pi_*X, \pi_*JX) - \frac{\partial^2\psi_t \circ \pi}{\partial t^2} \Big|_{t=\mu} (X(\mu)^2 + JX(\mu)^2),
 \end{aligned}$$

whence it is clear that $\pi^*\sigma + dd^c\phi > 0$ if and only if $\frac{\partial^2\psi_t \circ \pi}{\partial t^2} < 0$. The latter inequality can be made to hold with no additional assumption simply replacing ψ_t by $\psi_t + a_t$ where

$$a_t = -t^2 - \int_0^t \int_0^v \sup_M \frac{\partial^2\psi_u}{\partial u^2} dudv.$$

It remains then to show that (62) can be solved. This follows by a standard application of method of characteristics. It is not difficult to see that projected characteristics are integral curves of JV in P . □

Remark 4.2. The Theorem above clearly implies that any of geometric flows considered in the previous section — with the notable exception of the Kähler-Ricci flow for which $\frac{\partial\omega_\tau}{\partial\tau}$ is a nontrivial class (see Formula (67) below), i.e. when the initial metric is not canonical (e.g., the flow on projective manifolds with non ample canonical bundles), which is treated in [8] — can be realized as a path of reduced Kähler metric coming from a suitable Kähler metric on P .

This can be proved in a more geometrical way, as illustrated for the Calabi flow in the following subsection. This proof is less elegant and less general (in that one needs to reformulate the proof in every case) but it has the advantage of working even when one is forced to take a P consisting of a family of non-trivial principal S^1 -bundles.

4.1. Converse to Theorem 3.4

Let be given a Hamiltonian holomorphic circle action on a Kähler manifold (P, ω, J) with associated metric g and with moment map μ . Fix a regular value a of the moment map. Let z_1, \dots, z_m , with $m = n - 1$, be holomorphic coordinates on the quotient manifold $M = \mu^{-1}(a)/S^1$, and let τ be the “moment map coordinate”, i.e., $\mu = \tau$. By definition, we have $d\tau = i_V\omega$.

Let Q be the *horizontal distribution*, i.e., the horthogonal complement (with respect to the metric g) of the span of V and JV (clearly the quotient

map $\pi_a : \mu^{-1}(a) \rightarrow M$ induces an isomorphism $d\pi_a : Q \rightarrow TM$. Then we can define a 1-form θ by:

$$\theta(V) = 1, \theta(JV) = 0, \theta|_Q = 0,$$

One can show (cf. Lemma 3.1 in [8]) that:

Lemma 4.3. *For the above local coordinates, we have $g(dz_i, d\tau) = 0$, $g(dz_i, \theta) = 0$ and $g(\theta, d\tau) = 0$, where g also denotes the induced metric on the cotangent bundle of M . In particular, in these coordinates g and ω take the form (reap.):*

$$(65) \quad g = h_{i\bar{j}} dz_i d\bar{z}_j + w d\tau^2 + \frac{1}{w} \theta^2$$

and

$$(66) \quad \omega = \sqrt{-1} h_{i\bar{j}} dz_i \wedge d\bar{z}_j - d\tau \wedge \theta$$

where $\frac{1}{w} = |V|^2$.

One can see that J preserves $Q^{1,0} := Q \cap T(1,0)P$ and that $\alpha := w d\tau - \sqrt{-1}\theta$ is of type $(1,0)$. We can then rewrite g as

$$\omega = \sqrt{-1} h_{i\bar{j}} dz_i d\bar{z}_j + |V|^2 \alpha \wedge \bar{\alpha}.$$

Also, we have the decomposition: $T^{(1,0)}M = Q^{(1,0)} \oplus \langle \alpha \rangle$. Following this decomposition one can write, for any S^1 -invariant function f :

$$\partial f = \partial^h f + JV(f) \alpha \quad \bar{\partial} f = \bar{\partial}^h f + JV(f) \bar{\alpha}$$

and these identities define $\partial^h f \in Q^{(1,0)}$ and $\bar{\partial}^h f \in Q^{(0,1)}$. We have the following important formula about the connection 1-form of the principle S^1 -bundle:

Lemma 4.4. *One has:*

$$(67) \quad d\theta = \sqrt{-1} \left\{ -\frac{\partial h_{i\bar{j}}}{\partial \tau} dz_i \wedge d\bar{z}_j + \partial^h \log \left(\frac{1}{|V|^2} \right) \wedge \bar{\alpha} + \alpha \wedge \bar{\partial}^h \log \left(\frac{1}{|V|^2} \right) - d \log \left(\frac{1}{|V|^2} \right) \wedge \theta \right\}.$$

Proof. This is Lemma 3.2 in [8], after using that:

$$\partial^h \log \left(\frac{1}{|V|^2} \right) \wedge \bar{\alpha} = \partial^h w \wedge d\tau + \partial^h \log w \wedge \theta$$

and the analogous:

$$\alpha \wedge \bar{\partial}^h \log \left(\frac{1}{|V|^2} \right) = d\tau \wedge \bar{\partial}^h w + \bar{\partial}^h \log w \wedge \theta.$$

□

We can prove the sought after converse to Theorem 3.4:

Theorem 4.5. *Maintaining notations as in Theorem 3.4, let $(M, \omega(t))$ be a solution of the Calabi-flow — i.e., Eq. (47) — and assume that along the flow:*

$$\frac{\partial \text{scal}(g_\tau)}{\partial \tau} > -c$$

for some uniform $c > 0$. Then there exists a metric g and a function $h = h(\mu)$ such that:

$$R = \log s + h$$

Proof. Let $\omega_\tau = \sqrt{-1} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$. Naturally we set $\tau = -Ct + C_0$ where C is to be determined (C_0 is arbitrary). Imposing Equation (67), simply define (for now just formally as we don't know as of yet that thus defined, $d\theta$ is closed):

$$d\theta = \sqrt{-1} \left\{ -\frac{\partial(g_\tau)_{i\bar{j}}}{\partial \tau} dz_i \wedge d\bar{z}_j + \partial^h \log \left(\frac{1}{|V|^2} \right) \wedge \bar{\alpha} + \alpha \wedge \bar{\partial}^h \log \left(\frac{1}{|V|^2} \right) - d \log \left(\frac{1}{|V|^2} \right) \wedge \theta \right\}$$

where $\frac{1}{|V|_g^2}$ is yet to be defined.

Then clearly:

$$d\theta|_{\tau=const} = \frac{\partial \omega_\tau}{\partial \tau}$$

which is in the trivial class, due to the fact that $\omega(t)$ satisfies the Calabi flow. We can therefore take $P = M \times A$ where $A \subset \mathbb{C}^*$ is an annulus and the rest is clear, modulo defining the value of $|V|_g^2$. Equations (48) and (49) imply that the equation $R = \log s + h$ is satisfied in the *horizontal directions*, meaning that it holds up to the addition of a function C which is constant

in the horizontal variables (i.e. $dC = 0$). It is then readily seen that, in order for equation $R = \log s + h$ to be satisfied, one must have:

$$\nabla (JV (\text{scal}(g_\tau) - h)) = 0$$

and also that in fact one must take the constant C for which $\tau = -Ct + C_0$ to be:

$$-C := JV (\text{scal}(g_\tau) - h).$$

This very equation defines $|V|_g^2$ though, as one has $JV = |V|_g^2 \frac{\partial}{\partial \tau}$ which yields:

$$(68) \quad |V|_g^2 = \frac{-C}{\frac{\partial}{\partial \tau} (\text{scal}(g_\tau) - h)} = \frac{1}{\frac{\partial}{\partial t} (\text{scal}(g_\tau) - h)}$$

One can therefore choose h such that $|V|_g^2 > 0$, provided:

$$\frac{\partial \text{scal}(g_\tau)}{\partial \tau} > -c$$

for some uniform $c > 0$. It is now also easy to see from the Calabi flow equation that the form $d\theta$ is closed — since $\frac{1}{|V|_g^2} = \frac{\partial}{\partial t} (\text{scal}(g_\tau) - h)$. In fact, the closedness of $d\theta$ is equivalent to requiring:

$$\frac{\partial^2 (g_\tau)_{i\bar{j}}}{\partial \tau^2} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \left(\frac{1}{|V|_g^2} \right)$$

Using Equation (68) and the equation of the Calabi flow, this is readily seen to be equivalent to the condition:

$$\frac{\partial}{\partial \tau} \left(\frac{\partial^2 (\text{scal}(g_\tau) - h)}{\partial z_i \partial \bar{z}_j} \right) = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \left(\frac{\frac{\partial}{\partial \tau} (\text{scal}(g_\tau) - h)}{-C} \right)$$

which clearly holds. □

The converse to Theorem 3.9 has been treated in [8], while the converse to Theorem 3.10 is immediately adapted from there.

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