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Equivalence of the categories of modules over Lie algebroids

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We study the theory of geometric Morita equivalence in Poisson geometry. A new equivalence relation for integrable Lie algebroids is introduced and investigated. It is shown that two equivalent Lie algebroids have equivalent categories of infinitesimal actions of Lie algebroids. As an application, it is also shown that the Hamiltonian categories for gauge equivalent Dirac structures are equivalent as categories.

1. Introduction

Poisson geometry is considered to be intermediate between differential geometry and noncommutative geometry in the sense that it provides us with powerful techniques to study many geometric objects related to noncommutative algebras.

If (Q, Π_Q) and (P, Π_P) are Poisson manifolds, then a Poisson map $J : Q \to P$ induces a Lie algebra homomorphism by

$$C^\infty(P) \longrightarrow \mathfrak{X}(Q) \subset \mathrm{End}\big(C^\infty(Q)\big), \quad f \longmapsto -\Pi_Q(\,\cdot\,, J^* df\,). \tag{1.1}$$

From (1.1), $C^\infty(Q)$ can be regarded as a $C^\infty(P)$-module. This observation enables oneself to study geometric objects by connecting with a theory in algebra like Morita equivalence (refer to H. Bursztyn and A. Weinstein [5] for further discussion). Geometric Morita equivalence, which is introduced by P. Xu [16], plays a central role in Poisson geometry as Morita equivalence of $C^\star$-algebras does in noncommutative geometry. One of the remarkable properties is that Morita equivalence implies the equivalence of the categories of modules over Poisson manifolds: for an integrable Poisson manifold P, the category of modules over P is the category whose objects are complete symplectic realizations of P and whose morphisms are symplectic maps between complete symplectic realizations commuting with the realizations. This is just the analogy with Morita equivalence in algebra, first studied by K. Morita [14].

As is well-known, Poisson maps are always associated with Lie algebroid actions of cotangent bundles:

$$\Gamma^\infty(T^*P) \longrightarrow \mathfrak{X}(Q), \quad \alpha \longmapsto -\Pi_Q(\,\cdot,\, J^*\alpha\,). \tag{1.2}$$

The Lie algebra homomorphism (1.2) can be considered to be the representation of $\Gamma^\infty(T^*P)$ on $C^\infty(Q)$. More generally, if $A \to M$ is a Lie algebroid, the infinitesimal action of A on a smooth map $f : N \to M$ induces the representation of $\Gamma^\infty(A)$ on $C^\infty(N)$:

$$\Gamma^\infty(A) \longrightarrow \mathfrak{X}(N) \subset \mathrm{End}\big(C^\infty(N)\big). \tag{1.3}$$

Here, a natural question arises: what is an equivalence relation between Lie algebroids which implies an equivalence of the categories associating with Lie algebroid actions?

In this paper, we give a solution to the above question, that is, we introduce an equivalence relation for integrable Lie algebroids, called strong Morita equivalence, and show that the category consisting of the infinitesimal actions of Lie algebroids is invariant under strong Morita equivalence. Furthermore, applying the result to Dirac geometry, we partially recover the well-known proposition in H. Bursztyn and M. Crainic [2]. This study gives a general description of Morita equivalence for Poisson manifolds from the viewpoint of Lie algebroid, and is expected to have a connection with the study of quasi-Hamiltonian symmetry through the question presented by A. Weinstein [15].

The paper is organized as follows: in Section 2, we review the basics of Lie algebroids, including Lie algebroid morphisms and the construction of Lie algebroid from a given Lie groupoid. Section 3 is devoted to the study of the infinitesimal actions of Lie algebroids. The new equivalence relation for integrable Lie algebroids is introduced and discussed. In Section 4, we show that the category of the infinitesimal actions of Lie algebroid is invariant under strong Morita equivalence, and show also that two gauge equivalent Dirac structures are strongly Morita equivalent. Lastly, we find that the Hamiltonian categories for gauge equivalent Dirac structures are equivalent each other, by using the main theorem.

Throughout the paper, manifolds are assumed to be connected smooth manifolds. The set of smooth sections of a smooth vector bundle $E \to M$ is denoted by $\Gamma^\infty(E)$. Especially, we write $\mathfrak{X}(M)$ for $\Gamma^\infty(TM)$ when $E =$

TM. The space of smooth functions on a smooth manifold M is denoted by $C^\infty(M)$.

2. Basic terminologies of Lie algebroids

2.1. Lie algebroids

Let M be a smooth manifold. A Lie algebroid over M is a smooth vector bundle $A \to M$ with a bundle map $\rho : E \to TM$, called the anchor map, and a Lie bracket $[\![\cdot,\, \cdot]\!]$ on the space $\Gamma^\infty(A)$ of smooth sections of A such that

$$[\![\alpha,\, f\beta]\!] = \big(\rho(\alpha)f\big)\,\beta + f[\![\alpha,\, \beta]\!] \tag{2.1}$$

for any $f \in C^\infty(M)$ and $\alpha,\, \beta \in \Gamma^\infty(A)$. We denote a Lie algebroid by the triple $(A \to M,\, [\![\cdot,\, \cdot]\!],\, \rho)$ or, simply by A, and use the notation A^- for a Lie algebroid A with the opposite bracket.

The anchor map of a Lie algebroid A is a Lie algebra homomorphism. Indeed, from (2.1) and the Jacobi identity, it follows that

$$\begin{aligned}
0 &= [\![[\![\alpha,\, \beta]\!],\, f\gamma]\!] + [\![[\![\beta,\, f\gamma]\!],\, \alpha]\!] + [\![[\![f\gamma,\, \alpha]\!],\, \beta]\!] \\
&= f[\![[\![\alpha,\, \beta]\!],\, \gamma]\!] + \big(\rho\big([\![\alpha,\, \beta]\!]\big)f\big)\,\gamma \\
&\quad + f[\![[\![\beta,\, \gamma]\!],\, \alpha]\!] + \big(\rho(\beta)f\big)[\![\gamma,\, \alpha]\!] - \big(\rho(\alpha)f\big)[\![\beta,\, \gamma]\!] - \big(\rho(\alpha)\big(\rho(\beta)f\big)\big)\,\gamma \\
&\quad + f[\![[\![\gamma,\, \alpha]\!],\, \beta]\!] - \big(\rho(\beta)f\big)[\![\gamma,\, \alpha]\!] + \big(\rho(\alpha)f\big)[\![\beta,\, \gamma]\!] + \big(\rho(\beta)\big(\rho(\alpha)f\big)\big)\,\gamma \\
&= \Big(\big(\rho\big([\![\alpha,\, \beta]\!]\big) - [\![\rho(\alpha),\, \rho(\beta)]\!]\big)f\Big)\,\gamma
\end{aligned}$$

for any $f \in C^\infty(M)$ and $\alpha,\, \beta,\, \gamma \in \Gamma^\infty(A)$. Therefore, we have $\rho\big([\![\alpha,\, \beta]\!]\big) = [\![\rho(\alpha),\, \rho(\beta)]\!]$.

Example 2.1. A Lie algebra is a Lie algebroid over a point.

Example 2.2. (Tangent algebroids) A tangent bundle TM of a smooth manifold M is a Lie algebroid over M: the anchor map is the identity map id_{TM}, and the Lie bracket is the usual Lie bracket of vector fields. This Lie algebroid is called a tangent algebroid.

Example 2.3. (Cotangent algebroids) If $(P,\, \Pi)$ is a Poisson manifold, then a cotangent bundle T^*P is a Lie algebroid: the anchor map is the map $\Pi^\sharp$

induced from Π,

$$\Pi^\sharp : T^*P \longrightarrow TP, \quad \alpha \longmapsto \{\, \beta \mapsto \langle \beta,\, \Pi^\sharp(\alpha) \rangle = \Pi(\beta,\, \alpha) \,\}$$

and the Lie bracket is given by

$$[\![\alpha,\, \beta]\!] = \mathcal{L}_{\Pi^\sharp(\alpha)}\beta - \mathcal{L}_{\Pi^\sharp(\beta)}\alpha + d\big(\Pi(\alpha,\, \beta)\big),$$

where $\mathcal{L}_{\Pi^\sharp(\alpha)}\beta$ stands for the Lie derivative on β along $\Pi^\sharp(\alpha)$. The Lie algebroid $(T^*P \to P,\, [\![\cdot,\, \cdot]\!],\, \Pi^\sharp)$ is called a cotangent algebroid.

Example 2.4. (Transformation algebroids) Given an action $\varrho : \mathfrak{g} \to \mathfrak{X}(M)$ of a Lie algebra $(\mathfrak{g},\, [\cdot,\, \cdot])$ on a smooth manifold M, one can associate to it the Lie algebroid structure: the vector bundle is the trivial bundle $M \times \mathfrak{g} \to M$, the anchor map ρ is given by $\rho\,(p,\, V) \to (\varrho(V))_p \in T_pM$, $(\forall p \in M,\, V \in \mathfrak{g})$ and the Lie bracket on $\Gamma^\infty(M \times \mathfrak{g}) = C^\infty(M,\, \mathfrak{g})$ is defined as

$$[\![U,\, V]\!](p) := [U(p),\, V(p)] + \big(\varrho(U(p))\big)_p(V) - \big(\varrho(V(p))\big)_p(U).$$

This Lie algebroid is called a transformation algebroid, and denoted by $M \rtimes \mathfrak{g}$, for short.

Example 2.5. (Dirac structures) Let us consider a vector bundle $TM \oplus T^*M$ over a smooth manifold M. We endow the vector bundle with a bilinear operation

$$\langle \cdot,\, \cdot \rangle : \Gamma^\infty(TM \oplus T^*M) \times \Gamma^\infty(TM \oplus T^*M) \to C^\infty(M)$$

defined as

$$\langle\, (U, \alpha),\, (V, \beta)\, \rangle := \beta(U) + \alpha(V),$$

and a skew-symmetric bracket

$$[\![\cdot,\, \cdot]\!] : \Gamma^\infty(TM \oplus T^*M) \times \Gamma^\infty(TM \oplus T^*M) \to \Gamma^\infty(TM \oplus T^*M)$$

defined as

$$[\![(U, \alpha),\, (V, \beta)]\!] := (\, [U, V],\, \mathcal{L}_U\beta - \mathrm{i}_V d\alpha\,).$$

A subbundle $D_M \subset TM \oplus T^*M$ is called a Dirac structure if D_M satisfies the following three conditions:

(1) $\langle \cdot,\, \cdot \rangle|_{D_M} \equiv 0$;

(2) D_M has rank equal to $\dim(M)$;

(3) $[\![\Gamma^\infty(D_M), \Gamma^\infty(D_M)]\!] \subset \Gamma^\infty(D_M)$.

We call a pair (M, D_M) of a smooth manifold M and a Dirac structure $D_M \subset TM \oplus T^*M$ a Dirac manifold. A Dirac structure D_M, with the restriction of Courant bracket and the anchor map, is verified easily to be a Lie algebroid. We refer to H. Bursztyn and M. Crainic [1], and [5] for further discussions of Dirac structures and Courant algebroids.

2.2. Lie algebroid morphisms and the pull-back Lie algebroids

Let $(A_1 \to M_1, [\![\cdot, \cdot]\!]^1, \rho_1)$ and $(A_2 \to M_2, [\![\cdot, \cdot]\!]^2, \rho_2)$ be Lie algebroids. A Lie algebroid morphism from A_1 to A_2 is a vector bundle morphism $\Phi : A_1 \to A_2$ such that

$$\rho_2\big(\Phi(\alpha)\big) = \varphi_*\big(\rho_1(\alpha)\big), \quad \big(\forall \alpha \in \Gamma^\infty(A_1)\big), \tag{2.2}$$

and, for any smooth sections $\alpha, \beta \in \Gamma^\infty(A_1)$ written in the forms

$$\Phi \circ \alpha = \sum_i \xi_i(\gamma_i \circ \varphi), \quad \Phi \circ \beta = \sum_j \eta_j(\delta_j \circ \varphi), \tag{2.3}$$

where $\xi_i, \eta_j \in C^\infty(M_1)$ and $\gamma_i, \delta_j \in \Gamma^\infty(A_2)$,

$$\begin{aligned}\Phi \circ [\![\alpha, \beta]\!]^1 = \sum_{i,j} \xi_i\eta_j \left([\![\gamma_i, \delta_j]\!]^2 \circ \Phi\right) &+ \sum_j \big(\mathcal{L}_{\rho_1(\alpha)}\, \eta_j\big)(\delta_j \circ \Phi) \\ &- \sum_i \big(\mathcal{L}_{\rho_1(\beta)}\, \xi_i\big)(\gamma_i \circ \Phi)\end{aligned} \tag{2.4}$$

are satisfied (see K. Mackenzie [12]). Here, we denote the base map of Φ by φ.

Proposition 2.6. *If a vector bundle morphism $\Phi : A_1 \to A_2$ is the Lie algebroid morphism, then there exists a subbundle*

$$R \subset (A_1 \times A_2)|_{\mathrm{Gr}(\varphi)}$$

which satisfies the following conditions:

(1) *For any $z \in \mathrm{Gr}(\varphi)$, $(\rho_1 \times \rho_2)\left(R_z\right) \subset T_z\big(\mathrm{Gr}(\varphi)\big)$;*

(2) *For any* $\alpha,\,\beta \in \varGamma^{\infty}(A_1 \times A_2)$ *such that* $\alpha|_{\mathrm{Gr}(\varphi)}\,,\ \beta|_{\mathrm{Gr}(\varphi)} \in \varGamma^{\infty}(R)$, *it holds that*

$$[\![\,\alpha,\,\beta\,]\!]|_{\mathrm{Gr}(\varphi)} \in \varGamma^{\infty}(R),$$

where $[\![\cdot,\,\cdot]\!] = \big(\,[\![\cdot,\,\cdot]\!]^1,\,[\![\cdot,\,\cdot]\!]^2\,\big)$.

Proof. Suppose that $\Phi : A_1 \to A_2$ is a Lie algebroid morphism. Define the vector bundle $R \subset (A_1 \times A_2)|_{\mathrm{Gr}(\varphi)}$ as

$$R = \coprod_{p \in M_1} \Big\{ (a,\,\Phi(a)) \;\Big|\; a \in (A_1)_p \Big\}.$$

Using (2.2), we have

$$\begin{aligned}(\rho_1 \times \rho_2)\big(a,\,\Phi(a)\big) &= \Big(\,\rho_1(a),\,\rho_2\big(\Phi(a)\big)\,\Big)\\ &= \Big(\,\rho_1(a),\,\Phi_*\big(\rho_1(a)\big)\,\Big) \in T_p\big(\mathrm{Gr}(\varphi)\big).\end{aligned}$$

That is, the condition (1) holds.

For $\alpha,\,\beta \in \varGamma^{\infty}(A_1)$ which we assume to satisfy (2.3), we define the smooth sections $\widehat{\alpha},\,\widehat{\beta}$ of $A_1 \times A_2 \to M_1 \times M_2$ as

$$\widehat{\alpha}_{(p,\varphi(p))} := \big(\alpha_p,\,\Phi(\alpha_p)\big) \in (R)_{(p,\varphi(p))}, \quad \widehat{\beta}_{(p,\varphi(p))} := \big(\beta_p,\,\Phi(\beta_p)\big) \in (R)_{(p,\varphi(p))}.$$

From (2.1) and (2.4), it follows that

$$[\![\,\Phi(\alpha),\,\Phi(\beta)\,]\!]^2{}_{\Phi(p)} = \Phi\,\big([\![\,\alpha,\,\beta\,]\!]^1{}_p\big).$$

This leads us to the condition (2). □

The Lie algebroid morphism $\Phi : A_1 \to A_2$ is said to be a Lie algebroid isomorphism if Φ is an isomorphism of vector bundles. If there exists the Lie algebroid isomorphism from A_1 to A_2, we write $A_1 \cong A_2$.

Let $(A \to M,\,[\![\cdot,\cdot]\!],\,\rho)$ be a Lie algebroid and $f : M' \to M$ a smooth map from a smooth manifold M' to M. Assume that the differential of f is transversal to the anchor map $\rho : A \to TM$ in the sense that

$$\mathrm{Im}\,\rho_{f(x)} \,+\, \mathrm{Im}\,(df)_x \,=\, T_{f(x)}M, \quad (\forall x \in M').$$

Here, $\mathrm{Im}\,\rho_{f(x)}$ stands for the image of $\rho_{f(x)}$.

This assumption leads us to the following condition:

$$\text{(2.5)}\quad \mathrm{Im}\,(id_x \times \rho_{f(x)}) + T_{(x.f(x))}\big(\mathrm{Gr}(f)\big) = T_xM' \oplus T_{f(x)}M, \quad (\forall x \in M'),$$

where id_x means the identity map on T_xM'. The condition (2.5) ensures that the preimage

$$\text{(2.6)}\quad \begin{aligned} &(id \times \rho)^{-1} T\big(\mathrm{Gr}(f)\big) \\ &= \coprod_{x \in M'} \Big\{ (V, \alpha) \;\Big|\; V \in T_xM',\, \alpha \in A_{f(x)},\, (df)_x(V) = \rho(\alpha) \Big\} \end{aligned}$$

is a smooth subbundle of $(TM' \times A)|_{\mathrm{Gr}(f)}$. The vector bundle (2.6) over $\mathrm{Gr}(f) \cong M'$ has the structure of Lie algebroid whose anchor map is the natural projection proj_1. This vector bundle is called a pull-back of Lie algebroid and denoted by $f^!A$ (see P. Higgins and K. Mackenzie [10]).

Let $\Phi_1 : A_1 \to A$ and $\Phi_2 : A_2 \to A$ be Lie algebroid morphisms. We denote each base map by $\varphi_1 : M_1 \to M$ and $\varphi_2 : M_2 \to M$. Suppose that the following conditions:

(1) $\mathrm{Im}\,(\Phi_1)_p + \mathrm{Im}\,(\Phi_2)_q = A_r, \quad \big(r = \Phi_1(p) = \Phi_2(q)\big)$;

(2) The map $\varphi_1 \times \varphi_2$ is transversal to the submanifold $\Delta = \{\, (m, m) \,|\, m \in M \,\} \subset M \times M$:

$$\mathrm{Im}\,\big((d\varphi_1)_p \times (d\varphi_2)_q\big) + T_{(r,r)}\Delta = T_{(r,r)}(M \times M)$$

are satisfied. Then, one can obtain the Lie algebroid

$$A_1 \times_A A_2 := \coprod_{(p,q) \in M_1 \times_M M_2} \Big\{ (a, b) \;\Big|\; a \in (A_1)_p,\, b \in (A_2)_q,\, \Phi_1(a) = \Phi_2(b) \Big\}$$

over $M_1 \times_M M_2 = \big\{(p, q) \in M_1 \times M_2 \,|\, \varphi_1(p) = \varphi_2(q)\big\}$, whose Lie bracket $[\![\cdot, \cdot]\!]$ is given by $[\![\cdot, \cdot]\!] := \big([\![\cdot, \cdot]\!]^1, [\![\cdot, \cdot]\!]^2 \big)$, and whose anchor map $\widehat{\rho} : A_1 \times_A A_2 \to T(M_1 \times_M M_2)$ is defined as $\widehat{\rho}(a, b) := (\rho_1(a), \rho_2(b))$. We call this Lie algebroid the fibered product. The pull-back of a Lie algebroid $f^!A$ discussed can be the fibered product of two Lie algebroid morphisms $f_* : TM' \to TM$ and $\rho : A \to TM$. Hence, a fibered product Lie algebroid is a pull-back Lie algebroid in a general sense.

2.3. The Lie algebroid of a Lie groupoid

Let $\mathit{\Gamma} \rightrightarrows M$ be a Lie groupoid with an identity section $\boldsymbol{\varepsilon}$, a source map $\boldsymbol{s}$ and a target map $\boldsymbol{t}$. Denote by $\mathcal{A}(\mathit{\Gamma}) \to M$ the vector bundle consisting of tangent spaces to $\boldsymbol{s}$-fibers at X:

$$\mathcal{A}(\mathit{\Gamma})\big|_p = \ker(d\boldsymbol{s})_{\boldsymbol{\varepsilon}(p)} \quad (p \in M).$$

For any $\gamma \in \mathit{\Gamma}$, the differential of the right translation R_γ by γ induces a map

$$(dR_\gamma)_{\boldsymbol{\varepsilon}(\gamma')} : T_{\boldsymbol{\varepsilon}(\gamma')}\left(\boldsymbol{s}^{-1}\big(\boldsymbol{t}(\gamma)\big)\right) \longrightarrow T_{\boldsymbol{\varepsilon}(\gamma')}\left(\boldsymbol{s}^{-1}\big(\boldsymbol{s}(\gamma)\big)\right),$$

where $\gamma' = \boldsymbol{t}(\gamma)$. By the map, any smooth section $\alpha \in \mathit{\Gamma}^\infty(\mathcal{A}(\mathit{\Gamma}))$ gives rise to a right-invariant vector field

$$\widehat{\alpha}_\gamma := (dR_\gamma)_{\boldsymbol{\varepsilon}(\gamma')}(\alpha_{\boldsymbol{\varepsilon}(\gamma')}) \quad (\gamma \in \mathit{\Gamma}) \tag{2.7}$$

on $\mathit{\Gamma}$ (see [12]). Therefore, $\mathit{\Gamma}^\infty(\mathcal{A}(\mathit{\Gamma}))$ inherits the Lie bracket from $\mathfrak{X}(\mathit{\Gamma})$. One verifies that the vector bundle $\mathcal{A}(\mathit{\Gamma}) \to M$ with the above Lie bracket and the bundle map $d\boldsymbol{t} : \mathcal{A}(\mathit{\Gamma}) \to TM$ becomes a Lie algebroid. A Lie algebroid $A \to M$ is said to be integrable if there exists a Lie groupoid $\mathit{\Gamma} \rightrightarrows M$ whose Lie algebroid $\mathcal{A}(\mathit{\Gamma}) \to M$ is isomorphic to A as Lie algebroid. If A is integrable, there exists an unique source-simply-connected Lie groupoid integrating A (see I. Moerdijk and J. Mrčun [13]).

3. Infinitesimal actions of Lie algebroids and strongly Morita equivalence

We begin this section by recalling the actions of Lie algebroids. A Lie algebroid right (left) action of $(A \to M, [\![\cdot,\cdot]\!], \rho)$ on a smooth manifold N consists of a map $\mu : N \to M$ called the moment map and a Lie algebra (anti-) homomorphism $\xi : \mathit{\Gamma}^\infty(A) \to \mathfrak{X}(N)$ which satisfy

$$\rho\big(\alpha_{\mu(q)}\big) = (d\mu)_q\big(\xi(\alpha)\big) \quad (\forall q \in N) \tag{3.1}$$

for any $\alpha \in \mathit{\Gamma}^\infty(A)$, and

$$\xi(f\alpha) = (\mu^* f)\, \xi(\alpha) \quad (\forall f \in C^\infty(M)). \tag{3.2}$$

The right action of A is alternatively called the infinitesimal action of A. The action is said to be complete if $\xi(\alpha)$ is a complete vector field whenever $\alpha \in \mathit{\Gamma}^\infty(A)$ has compact support.

Example 3.1. Let $\mathfrak{g}$ be a Lie algebra. A Lie algebra action of $\mathfrak{g}$ on M is thought of a Lie algebroid action of $\mathfrak{g} \to \{*\}$ on $M \to \{*\}$.

Example 3.2. Any Poisson map $J : Q \to P$ and a cotangent algebroid T^*P over P is a Lie algebroid action by (1.2).

Example 3.3. Any smooth manifold X is thought of a Lie algebroid action of a trivial Lie algebroid $\{*\} \to \{*\}$ on a map $X \to \{*\}$. We call this action a trivial action.

Example 3.4. Given a Lie algebroid $A \to M$ with a surjective submersion $J : X \to M$ which satisfy

$$(J^!A)_{(x,J(x))} \cap (T_xX \oplus \{\mathbf{0}\}) = \{\mathbf{0}\} \quad (\forall x \in X), \tag{3.3}$$

we have the right action of Lie algebroid $\Gamma^\infty(A) \to \mathfrak{X}(X)$ by $\alpha \mapsto u$, where $u \in T_xX$ is the element such that $(u, \alpha) \in (J^!A)_{(x,J(x))}$. We remark that the element u is uniquely determined by (3.3). Indeed, if α and u, u' are the elements such that $(u, \alpha) \in (J^!A)_{(x,J(x))}$ and $(u', \alpha) \in (J^!A)_{(x,J(x))}$, then we have

$$(u - u', \mathbf{0}) \in (J^!A)_{(x,J(x))} \cap (T_xX \oplus \{\mathbf{0}\}).$$

It follows from (3.3) that $u = u'$.

Example 3.5. Let us assume that a Lie algebroid $A \to M$ is integrable and $\Gamma \rightrightarrows M$ be the Lie groupoid integrating A. As noted in Section 2, the fiber of A over $x \in M$ is the subspace $\ker (d\boldsymbol{s})_{\varepsilon(x)}$ of $T_{\varepsilon(x)}\Gamma$, and the anchor is given by $d\boldsymbol{t} : A \subset T\Gamma \to TM$. Given any section $\alpha \in \Gamma^\infty(A)$, the Formula (2.7) defines a right invariant vector field. The map ξ which assigns the right invariant vector field $\widehat{\alpha}$ on Γ to $\alpha \in \Gamma^\infty(A)$ is shown to be a Lie algebra homomorphism and satisfy (3.1) and (3.2). Therefore, the map $\Gamma^\infty(A) \to \mathfrak{X}(\Gamma)$ defines a right action of A on $\boldsymbol{t} : \Gamma \to M$. Similarly to this case, one can obtain a left action of A on $\boldsymbol{t} : \Gamma \to M$ by defining as $\Gamma^\infty(A) \ni \alpha \mapsto -\widehat{\alpha} \in \mathfrak{X}(\Gamma)$.

Proposition 3.6. *Let $(A \to M, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid and $J : X \to M$ a smooth map. Suppose that J is a surjective submersion. Then, we have a Lie algebroid action of A on $X/\mathcal{F}$, where $X/\mathcal{F}$ is the space of leaves induced from J.*

Proof. Since J is a surjective submersion, the space X has a foliation $\mathcal{F}$ whose leaves are J-fibers. We consider the space of leaves $X/\mathcal{F}$ and a map

$\overline{J} : X/\mathcal{F} \to M$ given by $\overline{J}(\overline{x}) = J(x)$ $(\forall x \in X)$. For any $\alpha_{J(x)} \in A_{J(x)}$ $(x \in X)$, there exists $u_x \in T_xX$ such that $(dJ)_x(u_x) = \rho(\alpha_{J(x)})$. A vector field $u = \{u_x\}_{x\in X} \in \mathfrak{X}(X)$ is J-related to $\rho(\alpha) \in \mathfrak{X}(M)$: $dJ \circ u = \rho(\alpha) \circ J$. We define a map $\xi : \Gamma^\infty(A) \to \mathfrak{X}(X/\mathcal{F})$ as

$$A_{J(x)} \longrightarrow T_{\overline{x}}(X/\mathcal{F}), \quad \alpha_{J(x)} \longmapsto \overline{u}_{\overline{x}} := (d\pi)_x(u_x),$$

where π stands for a natural projection $\pi : X \to X/\mathcal{F}$, $x \to \overline{x}$. Let $\xi(\alpha) = \overline{u}$ and $\xi(\beta) = \overline{v}$ for $\alpha, \beta \in \Gamma^\infty(A)$. The vector fields $\overline{u}$ and $\overline{v}$ on $X/\mathcal{F}$ are π-related to u and v, respectively. It follows from this that $[\xi(\alpha), \xi(\beta)] = \overline{[u, v]}$. On the other hand, we take a vector field w on X such that $\rho(\llbracket\alpha, \beta\rrbracket) \circ J = dJ \circ w$. Since the anchor map ρ is a Lie algebra homomorphism (see Section 2), we have

$$\begin{aligned} w(J^*g) &= (dJ \circ w)f = \big([\rho(\alpha),\, \rho(\beta)] \circ J\big)g = \big([dJ \circ u,\, dJ \circ v]\big)g \\ &= \big(dJ \circ [u,\, v]\big)\big)g = [u,\, v](J^*g) \end{aligned}$$

for any $g \in C^\infty(M)$. In other words, it holds that $w = [u, v]$ on each J-fiber. Hence, we have $\xi(\llbracket\alpha, \beta\rrbracket) = \overline{[u, v]}$. These result in that the map ξ is a Lie algebra homomorphism. It is shown easily that ξ also satisfies (3.1) and (3.2). □

Remark 3.1. If a Lie algebroid A acts on $\mu : N \to M$, then a pull-back vector bundle $\mu^*A \to N$ has a Lie algebroid structure whose anchor is the action map. We refer to [12] for further details.

From the definition of the Lie algebroid action, the space $C^\infty(N)$ can be regarded as a $\Gamma^\infty(A)$-module. In other words, one can think of actions of Lie algebroids as modules over Lie algebroids. We define a right (left) module over a Lie algebroid A to be the right (resp. left) action of A whose moment map is a surjective submersion. A right (left) module over A is said to be complete if the right (resp. left) action is complete.

Example 3.7. The action of $T^*P \to P$ given by $\Gamma^\infty(T^*P) \ni \alpha \mapsto \Pi_Q(\,\cdot, J^*\alpha\,) \in \mathfrak{X}(Q)$ is a left module over T^*P (see (1.2)).

Example 3.8. The Lie algebroid action of A in Proposition 3.6 is the right module over A.

Example 3.9. Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a Lie groupoid. Let us take points $x \in \Gamma_0$ and $h \in \Gamma_1$ such that $\boldsymbol{t}(h) = x$. For any smooth section $\alpha \in \Gamma^\infty(\mathcal{A}(\Gamma_1))$, we consider a smooth curve γ in $\boldsymbol{s}^{-1}(x)$ which satisfies

$$\left.\frac{d}{dt}\right|_{t=0}\gamma = \widehat{\alpha}_x \quad \text{and} \quad \gamma(0) = \varepsilon(x).$$

Since $\boldsymbol{s}(\gamma(t)) = x = \boldsymbol{t}(h)$ for each $t \in \mathbb{R}$, a smooth curve $t \mapsto \gamma(t)\cdot h$ can be defined. Then, the map

$$\Gamma^\infty(\mathcal{A}(\Gamma_1)) \longrightarrow \mathfrak{X}(\Gamma_1), \quad \alpha \longmapsto -\left\{\left.\frac{d}{dt}\right|_{t=0}\gamma(t)\cdot h\right\}_{h\in\Gamma_1} \tag{3.4}$$

defines a left module $\boldsymbol{t} : \Gamma_1 \to \Gamma_0$ over $\mathcal{A}(\Gamma_1) \to \Gamma_0$.

On the other hand, let us consider a smooth curve δ in $\boldsymbol{t}^{-1}(x)$ which satisfies

$$\left.\frac{d}{dt}\right|_{t=0}\delta = \widehat{\beta}_x \quad \text{and} \quad \delta(0) = \varepsilon(x).$$

for $x \in \Gamma_0$ and $g \in \Gamma_1$ such that $\boldsymbol{s}(g) = x$, and for any smooth section $\beta \in \Gamma^\infty(\mathcal{A}(\Gamma_1))$. Then, the map defined as

$$\Gamma^\infty(\mathcal{A}(\Gamma_1)) \longrightarrow \mathfrak{X}(\Gamma_1), \quad \beta \longmapsto \left\{\left.\frac{d}{dt}\right|_{t=0} g\cdot\delta(t)\right\}_{g\in\Gamma_1} \tag{3.5}$$

is a right module $\boldsymbol{s} : \Gamma_1 \to \Gamma_0$ over $\mathcal{A}(\Gamma_1) \to \Gamma_0$.

Suppose that we are given a right A-module $J : X \to M$ by

$$\xi : \Gamma^\infty(A) \longrightarrow \mathfrak{X}(X), \quad (A)_{J(x)} \ni \alpha_{J(x)} \longmapsto \xi\left(\alpha_{J(x)}\right) \in T_x X$$

and a left A-module $K : Y \to M$ by

$$\eta : \Gamma^\infty(A) \longrightarrow \mathfrak{X}(Y), \quad (A)_{K(y)} \ni \alpha_{K(y)} \longmapsto \eta\left(\alpha_{K(y)}\right) \in T_y Y.$$

Take the fiber product

$$X \times_M Y = \left\{\, (x,y) \in X \times Y \;\middle|\; J(x) = K(y) \,\right\},$$

then, a map

$$(A)_{J(x)} \ni \alpha_{J(x)} \longmapsto \big(\xi\,(\alpha)_x,\, \eta\,(\alpha)_y\big) \in T_{(x,y)}\big(X \times_M Y\big). \tag{3.6}$$

defines a singular distribution $D = \{D_{(x,y)}\}$ on $X \times_M Y$

$$\begin{aligned} X \times_M Y \ni (x,y) \longmapsto {} & D_{(x,y)} \\ & := \left\{\, \big(\xi(\alpha)_x,\, \eta(\alpha)_y\big) \;\middle|\; \alpha \in \Gamma^\infty(A_2) \,\right\} \subset T_{(x,y)}\big(X \times_M Y\big). \end{aligned}$$

The distribution D turns out to be integrable since the map (3.6) is thought of the anchor map of the fibered product $J^*A \times_A K^*A \to X \times_M Y$ (see Remark 3.1, and 8.1.4 in J.-P. Dufour and N. T. Zung [9]). We denote by $X \otimes_A Y$ the space of leaves $(X \times_M Y)/A$ obtained from D.

Definition 3.10. Two Lie algebroids $A_1 \to M_1$ and $A_2 \to M_2$ are said to be quasi-equivalent if there exists a smooth manifold X together with surjective submersions $J_k : X \to M_k$ $(k = 1, 2)$ such that

(Q1) A_1 has a left action ξ_1 on $J_1 : X \to M_1$ such that

$$\ker (dJ_2)_x = \left\{ \xi_1(\alpha)_x \,|\, \alpha \in \Gamma^\infty(A_1) \right\} \quad (\forall x \in X);$$

(Q2) A_2 has a right action ξ_2 on $J_2 : X \to M_2$ such that

$$\ker (dJ_1)_x = \left\{ \xi_2(\beta)_x \,|\, \beta \in \Gamma^\infty(A_2) \right\} \quad (\forall x \in X).$$

Example 3.11. Suppose that integrable Poisson manifolds P_1 and P_2 are Morita equivalent in the sense of Xu [16] each other, that is, there exists a symplectic manifold S together with two surjective submersions $P_1 \overset{\tau_1}{\leftarrow} S \overset{\tau_2}{\rightarrow} P_2$ such that

(1) τ_1 is a complete Poisson map and τ_2 is a complete anti-Poisson map;

(2) each τ_k has connected, simply-connected fibers $\quad (k = 1, 2)$;

(3) $\ker(d\tau_1)_z = \big(\ker(d\tau_2)_z\big)^\perp$ and $\ker(d\tau_2)_z = \big(\ker(d\tau_1)_z\big)^\perp$ $\quad (\forall z \in S)$.

Then, the cotangent algebroids $T^*P_1 \to P_1$ and $T^*P_2 \to P_2$ are quasi-equivalent: as is noted before, Poisson maps τ_1 and τ_2 induce the left and right actions of Lie algebroids by

$$\zeta_1 : \Gamma^\infty(T^*P_1) \longrightarrow \mathfrak{X}(S), \quad \alpha \longmapsto \Pi_S(\,\cdot,\, \tau_1^*\alpha\,).$$

and

$$\zeta_2 : \Gamma^\infty(T^*P_2) \longrightarrow \mathfrak{X}(S), \quad \beta \longmapsto -\Pi_S(\,\cdot,\, \tau_2^*\beta\,),$$

respectively. From the condition (3) it follows immediately that

$$\ker (d\tau_2)_z = \left\{ \zeta_1(\alpha)_z \,|\, \alpha \in \Gamma^\infty(T^*P_1) \right\} \quad (\forall z \in S)$$

and

$$\ker (d\tau_1)_z = \left\{ \zeta_2(\beta)_z \,|\, \beta \in \Gamma^\infty(T^*P_2) \right\} \quad (\forall z \in S).$$

In Example 3.11, the converse does not hold. Namely, P_1 and P_2 are not Morita equivalent even if their cotangent algebroids are quasi-equivalent. Indeed, we consider the 2-torus $\mathbb{T}^2 = S^1 \times S^1$ and the standard symplectic manifold $\mathbb{R}^2$. Then, both of the projections $\tau_1 : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{T}^2$ and $\tau_2 : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ are Poisson maps. As is mentioned before, Poisson maps τ_1 and τ_2 induce the actions of Lie algebroids

$$\zeta_1(\alpha) = \Pi^\sharp_{\mathbb{T}^2\times\mathbb{R}^2}(\tau_1^*\alpha) \quad (\forall\alpha \in \Gamma^\infty(T^*\mathbb{T}^2))$$

and

$$\zeta_2(\beta) = -\,\Pi^\sharp_{\mathbb{T}^2\times\mathbb{R}^2}(\tau_2^*\beta) \quad (\forall\beta \in \Gamma^\infty(T^*\mathbb{R}^2)),$$

respectively. Here, $\Pi_{\mathbb{T}^2\times\mathbb{R}^2}$ denote the natural Poisson structure on $\mathbb{T}^2 \times \mathbb{R}^2$ by the symplectic structure. It is easy to check that those actions satisfy conditions (Q1) and (Q2) in Definition 3.10. Consequently, the cotangent algebroid $T^*\mathbb{T}^2 \to \mathbb{T}^2$ and $T^*\mathbb{R}^2 \to \mathbb{R}^2$ are quasi-equivalent. However, those symplectic manifolds $\mathbb{T}^2$ and $\mathbb{R}^2$ are *not* Morita equivalent since their fundamental groups can not be isomorphic to each other (see Proposition 2.1 in [16]).

The quasi-equivalence in Definition 3.10 can be characterized in terms of the subbundles of the pull-backs of Lie algebroids.

Proposition 3.12. *Two Lie algebroids $A_1 \to M_1$ and $A_2 \to M_2$ are quasi-equivalent if and only if there exists a smooth manifold X together with surjective submersions $J_k : X \to M_k$ $(k = 1, 2)$ and a pair (L_1, L_2) of subbundles L_1 of ${J_1}^! A_1$ and L_2 of ${J_2}^! A_2^-$ which satisfy the following conditions*:

(A) $(L_k)_{(x,J_k(x))} \cap (T_xX \oplus \{\mathbf{0}\}) = \{\mathbf{0}\}$ for any $x \in X$ $(k = 1, 2)$;

(B) $\mathrm{pr}_1\big((L_1)_{(x,J_1(x))}\big) = T_x\big(J_2^{-1}\big(J_2(x)\big)\big)$ and
$\mathrm{pr}_1\big((L_2)_{(x,J_2(x))}\big) = T_x\left(J_1^{-1}\big(J_1(x)\big)\right)$ $(\forall x \in X)$;

(C) $\mathrm{pr}_2\big((L_1)_{(x,J_1(x))}\big) = (A_1)_{J_1(x)}$ and $\mathrm{pr}_2\big((L_2)_{(x,J_2(x))}\big) = (A_2)_{J_2(x)}$ $(\forall x \in X)$;

where pr_1 *and* pr_2 *are the natural projections from $TX \times A_i$ $(i = 1, 2)$ to the first component TX and the second component A_i, respectively.*

Proof. Suppose that two Lie algebroids $A_1 \to M_1$ and $A_2 \to M_2$ are quasi-equivalent by $M_1 \overset{J_1}{\leftarrow} X \overset{J_2}{\rightarrow} M_2$. We define subbundles L_1 of ${J_1}^! A_1$ and L_2 of

$(J_2{}^!A_2)^-$ as

$$L_1 = \coprod_{x\in X}\Big\{ \big(\xi_1(\alpha)_x,\, \alpha_{J_1(x)}\big) \;\big|\; \alpha \in \Gamma^\infty(A_1) \Big\}$$

and

$$L_2 = \coprod_{x\in X}\Big\{ \big(\xi_2(\beta)_x,\, \beta_{J_2(x)}\big) \;\big|\; \beta \in \Gamma^\infty(A_2) \Big\},$$

respectively. The condition (C) holds obviously. If we take a zero section $\alpha \equiv \mathbf{0} \in \Gamma^\infty(A_1)$, then $\xi_1(\alpha)_x = \mathbf{0}$. This shows that (A) holds. The condition (B) is verified by the assumptions that the images of the action ξ_1 (ξ_2) are tangent to J_2 (resp. J_1)-fibers.

Conversely, assume that there exists such a smooth manifold X and a pair (L_1, L_2) of subbundles $L_1 \subset J_1{}^!A_1$ and $L_2 \subset J_2{}^!A_2^-$. Let us choose any smooth section $\alpha \in \Gamma^\infty(A_1)$. From the conditions (A) and (C), there exists a unique element $u \in T_xX$ such that $(u, \alpha_{J_1(x)}) \in (L_1)_{(x,J_1(x))}$ (see Example 3.4). That is, we have a map

$$\xi_1 : \Gamma^\infty(A_1) \ni \alpha \longmapsto u \in \mathfrak{X}(X). \tag{3.7}$$

as assigning to $\alpha \in \Gamma^\infty(A_1)$ a unique element $u \in T_xX$ such as $(u,\, \alpha_{J_1(x)}) \in (L_1)_{(x,J_1(x))}$. The map (3.7) defines a left action of A_1. A right action ξ_2 of A_2 is defined in the obvious analogous way. It follows from (B) that

$$\big\{\, \xi_1(\alpha)_x \,|\, \alpha \in \Gamma^\infty(A_1) \,\big\} \;=\; T_x\Big(J_2^{-1}(J_2(x))\Big).$$

Similarly to this case, the right action ξ_2 of A_2 yields

$$\big\{\, \xi_2(\beta)_x \,|\, \beta \in \Gamma^\infty(A_2) \,\big\} \;=\; T_x\Big(J_1^{-1}(J_1(x))\Big).$$

This shows that A_1 and A_2 are quasi-equivalent to each other. □

In Example 3.11, let us take subbundles $L_1 \subset \tau_1{}^!(T^*P_1)$ and $L_2 \subset \tau_2{}^!(T^*P_2)^-$ as

$$L_1 = \big\{\, (\Pi_S^\sharp(\tau_1{}^*\alpha),\, \alpha) \,|\, \alpha \in T^*P_1 \,\big\}$$

and

$$L_2 = \big\{\, (\Pi_S^\sharp(\tau_2{}^*\beta),\, \beta) \,|\, \beta \in (T^*P_2)^- \,\big\},$$

respectively, where $\Pi_S^\sharp$ stands for the bundle map induced by the symplectic Poisson structure $\Pi_S \in \Gamma^\infty(\wedge^2 TS)$. The condition (A) and (C) in Proposition 3.12 are easily checked. The condition (B) follows from (3) in Example 3.11 that fibers of τ_1, τ_2 are symplectically orthogonal to one another.

The statement similar to this holds in a more general setting. Let D_{M_1} and D_{M_2} be Dirac structures over M_1 and M_2, respectively. A smooth map $F: M_1 \to M_2$ is called a forward Dirac map if it holds that

$$(D_{M_2})_{F(m)} = \Big\{ \big((dF)_m U,\, \beta\big) \in T_{F(m)}M_2 \oplus T^*_{F(m)}M_2 \\ \mid \big(U,\, (dF)^*_m \beta\big) \in (D_{M_1})_m \Big\}$$

for any point $m \in M_1$. In addition, a forward Dirac map $F: (M_1, D_{M_1}) \to (M_2, D_{M_2})$ is called a strong Dirac map if

$$\ker(dF)_m \cap \ker(D_{M_1})_m = \{\mathbf{0}\} \quad (\forall m \in M_1) \tag{3.8}$$

is satisfied, where $\ker(D_{M_1})_m = (D_{M_1})_m \cap T_m M_1$ (see H. Bursztyn and M. Crainic [2]).

Remark 3.2. A strong Dirac map is alternatively called a Dirac realization in [1].

A strong Dirac map $F: (M_1, D_{M_1}) \to (M_2, D_{M_2})$ induces a map

$$\zeta : \Gamma^\infty(D_{M_2}) \longrightarrow \mathfrak{X}(M_1), \quad (V,\, \beta) \longmapsto \hat{V}, \tag{3.9}$$

where $\hat{V}$ is a tangent vector such that $V = F_* \hat{V}$ which is determined uniquely by the condition (3.8). The map ζ defines an infinitesimal actions of the Lie algebroid D_{M_2} (see Proposition 2.3 in [2]). A strong Dirac map F is said to be complete if the infinitesimal action ζ is complete. As noted in Example 2.5, Dirac structures are regarded as Lie algebroids. The following proposition states the sufficient condition for two Dirac structures to be quasi-equivalent.

Proposition 3.13. *Two Dirac structures D_{M_1} and D_{M_2} are quasi-equivalent if there exists a Dirac manifold (N, D_N) together with surjective submersions $(M_1, D_{M_1}) \overset{F_1}{\leftarrow} N \overset{F_2}{\rightarrow} (M_2, {D_{M_2}}^-)$ satisfying*

(1) *each F_k is a strong Dirac map* $(k = 1, 2)$;

(2) $\mathrm{pr}_1\big((\Lambda_1)_n\big) = \ker(dF_2)_n$ and $\mathrm{pr}_2\big((\Lambda_2)_n\big) = \ker(dF_1)_n$ $(\forall n \in N)$,

where $(\Lambda_k)_n := (D_N)_n \cap \big(T_n N \oplus \mathrm{Im}\,(dF_k)^*_n\big)$ $(k = 1, 2)$.

Proof. We define subbundles $L_1 \subset F_1{}^! D_{M_1}$ and $L_2 \subset F_2{}^! D^-_{M_2}$ over N as

$$L_1 := \coprod_{n \in N} \Big\{ \big(u;\, (dF_1)_n(u),\, \beta\big) \;\Big|\; u \in T_nN,\, \beta \in T_{F_1(n)}M_1,\, \big(u,\, (dF_1)^*_n(\beta)\big) \in (D_N)_n \Big\}$$

and

$$L_2 := \coprod_{n \in N} \Big\{ \big(u;\, (dF_2)_n(u),\, \beta\big) \;\Big|\; u \in T_nN,\, \beta \in T_{F_2(n)}M_1,\, \big(u,\, (dF_2)^*_n(\beta)\big) \in (D_N)_n \Big\}.$$

From the assumption that each $F_k : (N, D_N) \to (M_k, D_{M_k})$ is a Dirac map, it follows that

$$\mathrm{pr}_2\Big((L_k)_{(n,F_k(n))}\Big) = (D_{M_1})_{F_k(n)} \quad (k = 1, 2).$$

This shows that condition (C) in Proposition 3.12 holds. If a point $\big(u;\, (dF_k)_n(u),\, \beta\big) \in (L_k)_{(n,F_k(n))}$ belongs to the space $T_nN \oplus \{\mathbf{0}\} \subset T_nN \oplus T^*_nN$, we find that $u \in \ker(df_k)_n$ and $\beta = \mathbf{0}$. Since the condition (3.8), we have $(u, \mathbf{0}) \in \ker(dF_k)_n \cap \ker(D_N)_n = \{\mathbf{0}\}$. This implies $u = \mathbf{0}$. Therefore, condition (A) in Proposition 3.12 holds. For any $n \in N$, each space $\mathrm{pr}_1\big((L_k)_{(n,F_k(n))}\big)$ coincides with $\mathrm{pr}_1\big((D_N)_n \cap \big(T_nN \oplus \mathrm{Im}\,(dF_k)^*_n\big)\big)$. Consequently, condition (B) in Proposition 3.12 holds. □

Basing on the above discussion, we introduce a new binary relation between integrable Lie algebroids.

Definition 3.14. Suppose that both Lie algebroids $A_1 \to M_1$ and $A_2 \to M_2$ are integrable. They are said to be strongly Morita equivalent if they are quasi-equivalent to each other, and satisfy the following conditions:

(S1) each of the moment maps has connected and simply-connected fibers;

(S2) both the left action ξ_1 and the right action ξ_2 are complete;

(S3) for any smooth section $\alpha \in \Gamma^\infty(A_1)$ and $\beta \in \Gamma^\infty(A_2)$,

$$[\xi_1(\alpha),\, \xi_2(\beta)] = \mathbf{0}.$$

It will be shown that strong Morita equivalence is indeed an equivalence relation between Lie algebroids integrated to the source-simply-connected Lie groupoids.

Remark 3.3. The second condition in Definition 3.14 indicates that if θ_t^1 and θ_t^2 are the flows of the vector fields $\xi_1(\alpha)$ and $\xi_2(\beta)$, respectively, then it holds that $\theta_t^1 \circ \theta_s^2 = \theta_s^2 \circ \theta_t^1$ for all t, s for which the flows are defined.

Example 3.15. If two integrable Poisson manifolds P_1 and P_2 are Morita equivalent, then $T^*P_1 \to P_1$ and $T^*P_2 \to P_2$ are also strongly Morita equivalent. Indeed, they are quasi-equivalent (see Example 3.11). A left action of T^*P_1 on $S \overset{\tau_1}{\to} P_1$ and a right action of $(T^*P_2)^-$ on $S \overset{\tau_2}{\to} P_2$ are given like as the action in Example 3.2. The completeness of Poisson maps τ_1 and τ_2 implies that both of the actions are complete (see [8]). Furthermore, it holds that

$$\left[\, \Pi_S^\sharp(\tau_1^* df),\, -\Pi_S^\sharp(\tau_2^* dg) \,\right] = \Pi_S\big(\,\cdot,\, \Pi_S(\tau_1^* df,\, \tau_2^* dg)\,\big) = \mathbf{0},$$

since fibers of τ_1 and τ_2 are symplectically orthogonal to one another.

Relating to Example 3.15, strong Morita equivalence does not necessarily induce Morita equivalence. For example, let us consider Poisson manifolds $\mathbb{R}^2$ with the standard Poisson structure $\Pi_{\mathbb{R}^2} = \partial/\partial x_1 \wedge \partial/\partial x_2$ and $\mathbb{R}$ with zero Poisson structure $\Pi_{\mathbb{R}} = 0$. It is easy to show that the natural projections pr_1 from $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ to $\mathbb{R}^2$ and pr_2 from $\mathbb{R}^2 \times \mathbb{R}$ to $\mathbb{R}$ are Poisson maps. Similarly to the case of the observation immediately after Example 3.11, we obtain the left module over $T^*\mathbb{R}^2$ and the right module over $T^*\mathbb{R}$ from pr_1 and pr_2, respectively. Consequently, we find that $T^*\mathbb{R}^2$ and $T^*\mathbb{R}$ are strongly Morita equivalent. However, $\mathbb{R}^2$ and $\mathbb{R}$ are *not* Morita equivalent by $\mathbb{R}^2 \overset{\mathrm{pr}_1}{\leftarrow} \mathbb{R}^3 \overset{\mathrm{pr}_2}{\to} \mathbb{R}$ since $\mathbb{R}^3$ can not be a symplectic manifold.

An (A_1, A_2)-bimodule, denoted by $A_1 \overset{J_1}{\leftarrow} X \overset{J_2}{\to} A_2$, is a pair of a complete left module $X \overset{J_1}{\to} M_1$ over A_1 and a complete right module $X \overset{J_2}{\to} M_2$ over A_2 which makes A_1 and A_2 be strongly Morita equivalent to each other as in Definition 3.14. Let us consider an (A_1, A_2)-bimodule $A_1 \overset{J_1}{\leftarrow} X \overset{J_2}{\to} A_2$ and an (A_2, A_3)-bimodule $A_2 \overset{K_2}{\leftarrow} Y \overset{K_3}{\to} A_3$. We use ξ_1 and ξ_2 for the left and right actions of A_1 and A_2 on X, and also η_2 and η_3 for the left and right actions of A_2 and A_3 on Y, respectively. Then, the map $\widehat{\xi}_1 : \Gamma^\infty(A_1) \to \mathfrak{X}(X \otimes_{A_2} Y)$,

$$\Gamma^\infty(A_1) \ni \alpha \longmapsto \overline{(\,\xi_1\,(\alpha)_x,\, \mathbf{0}\,)} \in T_{\overline{(x,y)}}(X \otimes_{A_2} Y)$$

and the map $\widehat{\eta}_3 : \Gamma^\infty(A_3) \to \mathfrak{X}(X \otimes_{A_2} Y)$,

$$\Gamma^\infty(A_3) \ni \beta \longmapsto \overline{(\,\mathbf{0},\, \eta_3\,(\beta)_y\,)} \in T_{\overline{(x,y)}}(X \otimes_{A_2} Y)$$

induce a complete left action of A_1 on $\widehat{J}_1 : X \otimes_{A_2} Y \to M_1$, $\overline{(x, y)} \mapsto J_1(x)$ and a complete right action of A_3 on $\widehat{K}_3 : X \otimes_{A_2} Y \to M_3$, $\overline{(x, y)} \mapsto K_3(y)$, respectively. Here, we notice that $X \otimes_{A_2} Y$ is the leaf space as mentioned in the line immediately before Definition 3.10. In addition, we use the bar notation for equivalence classes in a quotient space. Namely, $\overline{(x, y)}$ and $\overline{(\,\xi_1\,(\alpha)_x,\, \mathbf{0}\,)}$ stand for equivalence classes in the leaf space $X \otimes_{A_2} Y$ and the tangent space $T_{\overline{(x,y)}}(X \otimes_{A_2} Y)$, respectively. It is easily verified that those actions satisfy

$$\ker\,(d\widehat{K}_3)_{\overline{(x,y)}} \supset \left\{\, \widehat{\xi}_1(\alpha)_x \,|\, \alpha \in \Gamma^{\infty}(A_1)\,\right\}$$

and

$$\ker\,(d\widehat{J}_1)_{\overline{(x,y)}} \supset \left\{\, \widehat{\eta}_3(\beta)_x \,|\, \beta \in \Gamma^{\infty}(A_3)\,\right\}.$$

If $\overline{(u,\, v)}$ is any point in $\ker\,(d\widehat{K}_3)_{\overline{(x,y)}}$, then there exists a smooth section $\beta \in \Gamma^{\infty}(A_2)$ such that $v = \eta_2(\beta)_y$. Consequently, we have

$$\begin{aligned}(dJ_2)_x\big(u - \xi_2(\beta)_y\big) &= (dJ_2)_x(u) - \rho_2(\beta) = (dK_2)_y(v) - \rho_2(\beta)\\ &= (dK_2)_y(\eta_2(\beta)_y) - \rho_2(\beta) = \mathbf{0}.\end{aligned}$$

That is, $u - \xi_2(\beta) \in \ker\,(dJ_2)_x$. Therefore,

$$(u,\, v) = (u - \xi_2(\beta)_x,\, \mathbf{0}) + (\xi_2(\beta)_x,\, \eta_2(\beta)_y).$$

By the assumption, there exists a smooth section $\alpha \in \Gamma^{\infty}(A_1)$ such that $\xi_1(\alpha)_x = u - \xi_2(\beta)_x$. This implies that $\overline{(u,\, v)} = \overline{(\xi_1(\alpha),\, \mathbf{0})}$. As a result, we show that

$$\ker\,(d\widehat{K}_3)_x \;=\; \left\{\, \widehat{\xi}_1(\alpha)_x \,|\, \alpha \in \Gamma^{\infty}(A_1)\,\right\}.$$

Similarly,

$$\ker\,(d\widehat{J}_1)_x \;=\; \left\{\, \widehat{\eta}_3(\beta)_x \,|\, \beta \in \Gamma^{\infty}(A_3)\,\right\}.$$

The observation leads us to the conclusion that the leaf space $X \otimes_{A_2} Y$ is an (A_1, A_3)-bimodule.

Example 3.16. If $X \to M$ is the right module over A, then $\{*\} \leftarrow X \to M$ is the $(*, A)$-bimodule. Similarly, $M \leftarrow X \to \{*\}$ turns out to be the $(A, *)$-bimodule if $X \to M$ is the left module over A.

On the basis of those observations, we can show the following proposition.

Proposition 3.17. *Strong Morita equivalence for integrable Lie algebroids whose Lie groupoids are source-simply-connected is an equivalence relation.*

Proof. The transitivity holds obviously by the above observation. Let $A \to M$ be an integrable Lie algebroid and $\Gamma(A) \rightrightarrows M$ the source-simply-connected Lie groupoid integrating $A \to M$. From Example 3.9, we have the left action ξ by (3.4) and the right action η by (3.5). It is obvious that those actions are complete. As for the left action ξ, we have

$$(d\boldsymbol{s})_{\varepsilon(x)}\big(\xi(\alpha)\big) = \frac{d}{dt}\bigg|_{t=0} \boldsymbol{s}(\gamma(t)\cdot h) = \frac{d}{dt}\bigg|_{t=0} \boldsymbol{s}(h) = \mathbf{0} \quad (\forall x \in M).$$

Similarly,

$$(d\boldsymbol{t})_{\varepsilon(x)}\big(\eta(\beta)\big) = \frac{d}{dt}\bigg|_{t=0} \boldsymbol{t}(g\cdot \delta(t)) = \frac{d}{dt}\bigg|_{t=0} \boldsymbol{t}(h) = \mathbf{0} \quad (\forall x \in M).$$

From this, it follows that $\ker(d\boldsymbol{s})_{\varepsilon(x)} = \{\, \xi(\alpha)_x \mid \alpha \in \Gamma^\infty(A)\,\}$ and $\ker(d\boldsymbol{t})_{\varepsilon(x)} = \{\, \eta(\beta)_x \mid \beta \in \Gamma^\infty(A)\,\}$. Moreover,

$$\begin{aligned}(d\boldsymbol{s})_{\varepsilon(x)}\big(\eta(\beta)\big) &= \frac{d}{dt}\bigg|_{t=0} \boldsymbol{s}(g\cdot\delta(t)) = \frac{d}{dt}\bigg|_{t=0} \boldsymbol{s}(\delta(t)) \\ &= (d\boldsymbol{s})_{\varepsilon(x)}(\widehat{\beta}_{\varepsilon(x)}) = (d\boldsymbol{s})_{\varepsilon(x)}\big((dR_{\varepsilon(x)})(\beta_{\varepsilon(x)})\big) = \mathbf{0},\end{aligned}$$

since the right invariant vectors $\widehat{\beta}$ lie in the $\boldsymbol{s}$-fibers. Therefore, we have that $[\xi(\alpha), \eta(\beta)] = \mathbf{0}$ for any $\alpha, \beta \in \Gamma^\infty(A)$. This results in that any integrable Lie algebroid A is strongly Morita equivalent to itself. Lastly, suppose that A_1 and A_2 are strongly Morita equivalent by (A_1, A_2)-bimodule $A_1 \overset{J_1}{\leftarrow} X \overset{J_2}{\rightarrow} A_2$. Defining a left action ξ' of A_2 and a right action η' of A_1 as $\xi'(\beta) := -\xi_2(\beta)$ $(\forall \beta \in \Gamma^\infty({J_2}^*A_2))$ and $\eta'(\alpha) := -\xi_1(\alpha)$ $(\forall \alpha \in \Gamma^\infty({J_1}^*A_1))$, respectively, we obtain an (A_2, A_1)-bimodule $A_2 \overset{J_2}{\leftarrow} X \overset{J_1}{\rightarrow} A_1$. This shows that the symmetric property holds. □

Example 3.18. Suppose that a smooth manifold M is simply-connected. Then, the pair groupoid $M \times M \rightrightarrows M$ is isomorphic to the fundamental groupoid $\Pi(M) \rightrightarrows M$. The tangent algebroid TM of M is strongly Morita equivalent to itself by a (TM, TM)-bimodule $TM \overset{\boldsymbol{t}}{\leftarrow} \Pi(M) \overset{\boldsymbol{s}}{\rightarrow} TM$.

Before observing the next example, let us recall the fact that actions of Lie groupoids induce actions of Lie algebroids similarly to the case of Example 3.9: we let $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ be source-simply-connected Lie groupoids, and suppose that G_1 and H_1 act on $\mu : X \to G_0$ and $\nu : X \to H_0$ from the left and the right, respectively. For a point $x \in X$ and any smooth section $\alpha \in \Gamma^\infty(\mathcal{A}(G_1))$, we take a smooth curve $g(t)$ in $\boldsymbol{s}_G^{-1}(\mu(x))$ which satisfies

$$\left.\frac{d}{dt}\right|_{t=0} g = \widehat{\alpha}_{\mu(x)} \quad \text{and} \quad g(0) = \varepsilon_G(\mu(x)).$$

Then, it is verified that the map

$$\Gamma^\infty(\mathcal{A}(G_1)) \longrightarrow \mathfrak{X}(X), \quad \alpha \longmapsto -\left\{ \left.\frac{d}{dt}\right|_{t=0} g(t) \cdot x \right\}_{x \in X}$$

defines a complete left action of $\mathcal{A}(G_1) \to G_0$ on $\mu : X \to G_0$. Similarly, for $x \in X$ and any smooth section $\beta \in \Gamma^\infty(\mathcal{A}(H_1))$, a smooth curve $h(t)$ in $\boldsymbol{t}_H^{-1}(\nu(x))$ such that

$$\left.\frac{d}{dt}\right|_{t=0} h = \widehat{\beta}_{\nu(x)} \quad \text{and} \quad h(0) = \varepsilon_H(\nu(x))$$

turns out to induce the map

$$\Gamma^\infty(\mathcal{A}(H_1)) \longrightarrow \mathfrak{X}(X), \quad \beta \longmapsto \left\{ \left.\frac{d}{dt}\right|_{t=0} x \cdot h(t) \right\}_{x \in X}$$

and this defines a complete right action of $\mathcal{A}(H_1) \to H_0$ on $\nu : X \to H_0$.

Example 3.19. (Atiyah sequence) Let G be a simply-connected Lie group with Lie algebra $\mathfrak{g}$. For a principal G-bundle $P \xrightarrow{\pi} M$, we denote the gauge groupoid of P by $G(P)$. $G(P)$ integrates the Lie algebroid $TP/G \to M$ called the Atiyah sequence (see A. Canna da Silva and A. Weinstein [6] or [12]). Note that P carries a left $G(P)$-action with moment map $\pi : P \to M$. Actually, the action is defined as $[(p, q)] \cdot u := p$ by choosing representatives so that $q = u$. Such a choice of representatives is ensured since G acts transitively on π-fibers. According to the above observation, the action of $G(P)$ induces a left action ξ of $TP/G \to M$ on $\pi : P \to M$. Similarly, the right action of G on P induces a right action η of $\mathfrak{g} \to \{*\}$ on $c : P \to \{*\}$. It can be checked that those actions are complete and satisfy the conditions (1) and (2) in Definition 3.10. Moreover, from the direct computation, we have $c_*(\eta(V)) = \mathbf{0}$, and find that $[\xi(\alpha), \eta(V)] = \mathbf{0}$ ($\forall \alpha \in \Gamma^\infty(TP/G), \forall V \in \mathfrak{g}$). As a result, the Atiyah sequence $TP/G \to M$ is strongly Morita equivalent to Lie algebra $\mathfrak{g}$.

Suppose that $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ are Morita equivalent. That is, there is a biprincipal (G_1, H_1)-bibundle X with moment maps $G_0 \xleftarrow{\mu} X \xrightarrow{\nu} H_0$ (see [5]).

Proposition 3.20. *If each of the moment maps has connected and simply-connected fibers, then $\mathcal{A}(G_1) \to G_0$ and $\mathcal{A}(H_1) \to H_0$ are strongly Morita equivalent.*

Proof. G_1 and H_1 are source-simply-connected Lie algebroids acting on X from the left and right. From the above observation, we obtain the complete left module ξ over $\mathcal{A}(G_1) \to G_0$ and the complete right module η over $\mathcal{A}(H_1) \to H_0$ to verify that they are quasi-equivalent to each other and satisfy the conditions (S1) and (S2) in Definition 3.14. For $\alpha \in \Gamma^\infty(G_1)$, $\beta \in \Gamma^\infty(H_1)$, denoting by Φ_t and Ψ_s the flows of $\xi(\alpha) \in \mathfrak{X}(X)$ and $\eta(\beta) \in \mathfrak{X}(X)$, respectively, we have that

$$\Phi_t(\Psi_s(x)) = \Phi_t(x \cdot h(s)) = g(t)(x \cdot h(s)) = (g(t) \cdot x) \cdot h(s) = \Psi_t(\Phi_s(x))$$

for $x \in X$ since the actions commute. Consequently, the condition (S3) in Definition 3.14 holds. This completes the proof. □

Example 3.21. We let $\Gamma_1 \rightrightarrows \Gamma_0$ be a source-simply-connected Lie groupoid and assume that it is transitive. Denoting by Γ_x the isotropy group at given $x \in \Gamma_0$, we consider the source-fiber $E_x = \boldsymbol{s}^{-1}(x)$ over x. Then, $\Gamma_1 \rightrightarrows \Gamma_0$ and $\Gamma_x \rightrightarrows \{x\}$ are Morita equivalent by $\Gamma_0 \xleftarrow{\boldsymbol{t}} E_x \xrightarrow{\boldsymbol{s}} \{x\}$ (see Example 4.15 in [5]). Therefore, it follows from Proposition 3.20 that $\mathcal{A}(\Gamma_1) \to \Gamma_0$ and $T_{\varepsilon(x)}(\Gamma_x) \to \{x\}$ are strongly Morita equivalent.

4. Equivalence of the categories of infinitesimal actions

4.1. The Weinstein groupoid

Before proceeding to the main theorem, let us review briefly the basics of A-path and the Weinstein groupoid. For further details, we refer to M. Crainic and R. J. Fernandes [7], [9].

Definition 4.1. Let $A \xrightarrow{\pi} M$ be a Lie algebroid with an anchor map $\rho : A \to TM$. An A-path is a smooth path $a : I \to A$ which projects to a base path $\pi \circ a : I \to M$ such that

$$\rho\big(a(t)\big) \;=\; \frac{d}{dt}\pi\big(a(t)\big) \quad (\forall t \in I).$$

Here, $I = [0, 1]$ stands for the unit interval. We denote by $P(A)$ the space of A-paths of class C^1 for A.

The space $P(A)$ has the subspace topology of the larger space $\tilde{P}(A)$ of all C^1-curves with the base paths of class C^2. The space $\tilde{P}(A)$ is endowed with the C^1-topology. For a compact set $K \subset I$ and an open set $W \subset J^1(I, A)$, let $C(K, W)$ denote the set of all $a \in \tilde{P}(A)$ such that $(j^1 a)(K) \subset W$:

$$C(K, W) \;=\; \{\, a \in \tilde{P}(A) \mid (j^1 a)(K) \subset W \,\} \subset \tilde{P}(A).$$

Here, $J^1(I, A)$ and $j^1 a$ stand for the 1-jet space and the 1-jet extension, respectively. The C^1-topology of $\tilde{P}(A)$ is defined as the topology generated by the collection of such $C(K, W)$.

A map $a_\epsilon(t) := a(\epsilon, t) : I \times I \to A$ is called a variation of A-paths if a_ϵ is a family of A-paths of class C^2 on ϵ with the property that the base paths $c_\epsilon(t) = \pi \circ a_\epsilon(t) : I \times I \to M$ have fixed end points. Denoting by ∇ and T_∇ a connection of $A \to M$ and the torsion of ∇ respectively, we consider the differential equation

$$(\partial_t b)(\epsilon, t) - (\partial_\epsilon a)(\epsilon, t) = T_\nabla(a,\, b), \qquad b(\epsilon, 0) = 0.$$

It is shown that the solution $b(\epsilon, t)$ does not depend on ∇. In addition, it turns out that, for a time-dependent section σ_ϵ of A such that $\sigma_\epsilon(t,\, c_\epsilon(t)) = a_\epsilon(t)$, the solution $b(\epsilon, t)$ is given by

$$b(\epsilon, t) = \int_0^t \phi^{t,s}_{\sigma_\epsilon}(c_\epsilon(s)) \left(\frac{d\sigma_\epsilon}{d\epsilon}\big(s,\, c_\epsilon(s)\big) \right) ds \,\in A_{c_\epsilon(t)}, \tag{4.1}$$

where $\phi^{t,s}_{\sigma_\epsilon}$ denotes the infinitesimal flow of the time-depending section σ_ϵ (see Proposition 1.3 in [7]).

Definition 4.2. Two A-path a_0 and a_1 are A-homotopic if there exists a variation a_ϵ whose base paths $\pi \circ a_0$ and $\pi \circ a_1$ have the same end points and such that $b(\epsilon, 1) = 0$ for $\epsilon \in I$. We write $a_0 \sim a_1$ when a_0 and a_1 are A-homotopic.

Let $\mathit{\Gamma} \rightrightarrows M$ be a Lie groupoid. A $\mathit{\Gamma}$-path is a path $\gamma : I \to \mathit{\Gamma}$ which satisfies $\boldsymbol{s}(\gamma(t)) = p$ $(\forall t)$ and $\gamma(0) = \boldsymbol{\varepsilon}(p)$ for some $p \in M$. We denote by $P(\mathit{\Gamma})$ the space of $\mathit{\Gamma}$-paths equipped with C^2-topology. Suppose that $\mathit{\Gamma} \rightrightarrows M$ integrates the Lie algebroid $A \to M$. Then, there exists a homeomorphism $D^R : P(\mathit{\Gamma}) \to P(A)$ by

$$(D^R\gamma)(t) := (dR_{\gamma(t)^{-1}})_{\gamma(t)} \left(\frac{d}{dt}\gamma(t) \right)$$

(see Proposition 1.1 in [7]). The homeomorphism D^R is called the differentiation of Γ-paths, and its inverse map is called the integration of A-path. The inverse D^{-R} of D^R is given as follows: for $a \in P(A)$, we choose a time-dependent section σ such that $a(t) = \sigma\big(t, \pi \circ a(t)\big)$ and denote by $\phi_\sigma^{t,0}$ the flow of the right-invariant vector field which corresponds to σ. Then, it is verified that

$$(D^{-R}a)(t) = \phi_\sigma^{t,1}\big(\pi(a(1))\big). \tag{4.2}$$

Indeed, putting $\gamma(t) = (D^{-R}a)(t)$, we have

$$\begin{aligned}(D^R\gamma)(t) &= (dR_{\gamma(t)^{-1}})_{\gamma(t)}\left(\frac{d}{dt}\phi_\sigma^{t,1}\big(\pi(a(1))\big)\right) = (dR_{\gamma(t)^{-1}})_{\gamma(t)}\Big(\sigma\big(t, \gamma(t)\big)\Big)\\ &= \sigma\big(t, \pi(a(t))\big) = a(t).\end{aligned}$$

If a_ϵ is the variation of A-paths and a family γ_ϵ of Γ-paths satisfies $D^R\gamma_\epsilon = a_\epsilon$, one finds that $b(\cdot, t) = D^R(\gamma^t)$, where $\gamma^t(\epsilon) := \gamma_\epsilon(t)$. (see Proposition 1.3 in [7]).

The quotient $\mathcal{G}(A) := P(A)/\sim$ is a smooth manifold (see [7]). As a matter of fact, $\mathcal{G}(A)$ turns out to be the unique source-simply-connected Lie groupoid over M with the structure maps $\boldsymbol{s}([a]) := \pi(a(0))$ and $\boldsymbol{t}([a]) := \pi(a(1))$ (see [7]). This Lie groupoid is called the Weinstein groupoid.

4.2. Main result

In algebra, Morita equivalence implies an equivalence of the categories of modules. To be concrete, if two algebras R, S are Morita equivalent, then $\mathcal{M}_R$ and $\mathcal{M}_S$ are equivalent. Here, $\mathcal{M}_R$ denotes the category of right R-modules, whose objects are right R-modules and whose morphisms between M_1 and M_2 are R-homomorphisms. Basing on this well-known fact, we introduce the category of modules over Lie algebroids as follows:

Definition 4.3. Let $(A \to M, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid. The category of modules over a Lie algebroid is the category $\mathcal{M}(A)$ whose objects are right modules over A and whose morphisms between $\mu : N \to M$ and $\mu' : N' \to M$ are smooth map $f : N \to N'$ such that $\mu' \circ f = \mu$ and, for each $\alpha \in \Gamma^\infty(A)$, their respective vector fields $\xi(\alpha) \in \mathfrak{X}(N)$ and $\xi'(\alpha) \in \mathfrak{X}(N')$ are f-related:

$$\xi'(\alpha)_{f(n)} = (df)_n(\xi(\alpha)_n) \quad (\forall n \in N)$$

Let $(A_1 \to M_1, [\![\cdot,\cdot]\!]_1, \rho_1)$ and $(A_2 \to M_2, [\![\cdot,\cdot]\!]_2, \rho_2)$ be integrable Lie algebroids and assume that A_1 and A_2 are strongly Morita equivalent. Let $N \overset{\mu}{\to} M_1$ be a right module over A_1. From the assumption, there exists an (A_1, A_2)-bimodule $A_1 \overset{J_1}{\leftarrow} X \overset{J_2}{\to} A_2$. We remark that A_1 acts on $\mu : N \to M_1$ and $J_1 : X \to M_1$ from the right and the left, respectively. The right module $N \to M_1$ over A_1 is thought of a $(*, A_1)$-bimodule (see Example 3.16). Hence, as discussed earlier, the tensor product

$$\nu : N \otimes_{A_1} X \longrightarrow M_2, \quad \overline{(n, x)} \longmapsto J_2(x)$$

turns out to be a $(*, A_2)$-module, that is, a right module over A_2. In addition, given a morphism $f : N \to N'$ in $\mathcal{M}(A_1)$, we define the map $\widehat{f} : N \otimes_{A_1} X \to N' \otimes_{A_1} X$ as $\widehat{f}(\overline{n,x}) = (\overline{f(n),x})$ for any $(\overline{n,x}) \in N \otimes_{A_1} X$. The map $\widehat{f}$ turns out easily to satisfy $\nu' \circ \widehat{f} = \nu$. As a result, we obtain a functor $\mathcal{S}$ from $\mathcal{M}(A_1)$ to $\mathcal{M}(A_2)$ which assigns to each object $N \to M_1$ an object $N \otimes_{A_1} X \to M_2$, and to each morphism $f : N \to N'$ in $\mathcal{M}(A_1)$ a morphism $N \otimes_{A_1} X \to N' \otimes_{A_1} X$.

This observation leads us to the following proposition.

Proposition 4.4. *If A_1 and A_2 are strongly Morita equivalent, then there exists a covariant functor from $\mathcal{M}(A_1)$ to $\mathcal{M}(A_2)$.*

In a similar way, we can obtain a covariant functor $\mathcal{T}$ from $\mathcal{M}(A_2)$ to $\mathcal{M}(A_1)$.

Suppose that A_1 and A_2 be strongly Morita equivalent by an (A_1, A_2)-bimodule $A_1 \overset{J_1}{\leftarrow} X \overset{J_2}{\to} A_2$. According to the observation in the previous section, one can obtain a (A_1, A_1)-bimodule $A_1 \overset{\widehat{J_1}}{\leftarrow} X \otimes_{A_2} X \overset{\widehat{J_1}}{\to} A_1$. Denote by ζ the right action of A_1 on $J_1 : X \to M_1$ and let $t \mapsto a(t)$ be an A_1-path with a base path $t \mapsto c(t) := \pi(a(t))$ starting at $m \in M$. For a point $x \in J_1^{-1}(m) \subset X$, we consider the following differential equation with the initial value problem:

$$\frac{d}{dt}u(t) = \zeta_{u(t)}\big(a(t)\big), \quad u(0) = x. \tag{4.3}$$

To verify that the Equation (4.3) has a unique solution defined on the entire unit interval, we choose a time-dependent smooth section σ of A_1 which has compact support, and satisfies $\sigma\big(t, c(t)\big) = a(t)$. A solution of (4.3) is an integral curve of a time-dependent vector field

$$V_{(t,x)} := \zeta_x\big(\sigma(t, J_1(x))\big) \quad \big((t,x) \in I \times X\big)$$

induced from σ. Conversely, suppose that u is an integral curve of V. Then, it is verified that $J_1 \circ u$ is an integral curve of $\rho_1(\sigma)$ with the initial point m. Indeed, we have

$$\frac{d}{dt}(J_1 \circ u)(t) = (dJ_1)_{u(t)}\Big(\zeta_{u(t)}\big(\sigma\big(t,\ J_1 \circ u(t)\big)\big)\Big) = \rho_1\big(\sigma(t,\ J_1 \circ u(t))\big).$$

by (3.1) and $J_1(u(0)) = J_1(x) = m$. Consequently, the curve $J_1 \circ u$ coincides with the base path c. From this, it follows that

$$\frac{d}{dt}u(t) = \zeta_{u(t)}\big(\sigma\big(t,\ c(t)\big)\big) = \zeta_{u(t)}\big(a(t)\big). \tag{4.4}$$

Therefore, the Equation (4.3) has a unique solution. Moreover, u is defined on the entire I since the completeness of the action ζ implies that $\zeta(\sigma(t,\ \cdot))$ is complete whenever $\sigma(t,\ \cdot)$ has compact supported. Now, let us consider the homotopy class $[a] \in \mathcal{G}(A_1)$ for any A_1-path a. Again, we remark that $\mathcal{G}(A_1) \rightrightarrows M_1$ is the Weinstein groupoid with the property that each source fiber is simply-connected. For a, we take a point x in X which satisfies $J_1(x) = \pi(a(0))$ and the integral curve u which is determined uniquely by the above observation. Then, the map

$$h : \mathcal{G}(A_1) \ni [a] \longmapsto \overline{(x, u(1))} \in X \otimes_{A_2} X, \tag{4.5}$$

which assigns to $[a]$ an equivalent class of a point $(x, u(1)) \in X \times_{M_2} X$ is well-defined. To verify that, suppose that $x, y \in X$ are distinct points such that $J_1(x) = \pi(a(0)) = J_1(y)$. They are on the same J_1-fiber. Denoting by η the right action on $J_2 : X \to M_2$, we note that

$$\ker(dJ_1)_z \ = \ \{\, \eta\,(\beta)_z \mid \beta \in \Gamma^\infty(A_2)\,\} \quad (\forall z \in X).$$

Accordingly, it turns out that x, y are on the same orbit of the flows by the action η. It follows from this that $\overline{(x, u(1))} = \overline{(y, u(1))}$. As a result, the element in $X \otimes_{A_2} X$ is uniquely determined for the homotopy class $[a]$ without contradiction. In addition, the map (4.5) will be shown to be independent of the choice of the homotopy class $[a]$ of a by the next lemma.

Lemma 4.5. *Let a_0 and a_1 be A_1-paths. If they are A_1-homotopic to each other, their solutions u_0 and u_1 of (4.3) satisfy $u_0(1) = u_1(1)$.*

Proof. Let a_ϵ be the variation which joins a_0 to a_1 with the property that $\pi \circ a_0(0) = \pi \circ a_1(0) = x$ and $\pi \circ a_0(1) = \pi \circ a_1(1)$. For the base paths $c_\epsilon = \pi \circ a_\epsilon$ ($\epsilon \in I$), there is a family u_ϵ of the integral curves (4.4) from the above discussion. Remark that $J_1 \circ u_\epsilon(t) = c_\epsilon(t)$ ($\forall t \in \mathbb{R}$) for each $\epsilon \in I$. By (4.1), we have a vector field

$$\zeta\big(b(\epsilon, t)\big) = \int_0^t \phi^{t,s}_{U_\epsilon}(u_\epsilon(s)) \left(\frac{dU_\epsilon}{d\epsilon}\big(s,\, u_\epsilon(s)\big) \right) ds \ \in T_{u_\epsilon(t)}X. \tag{4.6}$$

on the curve $t \to u_\epsilon(t)$. Here, $U_\epsilon = \zeta\,(\sigma_\epsilon)$ is the time-dependent vector field by

$$U_{\epsilon,(t,p)} = \zeta_p\,\big(\sigma_\epsilon(t, J_1(p))\big) \quad (\,(t,p) \in I \times X\,).$$

Putting $t = 1$ in (4.6), we get a tangent vector $\zeta\big(b(\epsilon, 1)\big)$ in $T_{u_\epsilon(1)}X$ to find that

$$\frac{d}{d\epsilon} u_\epsilon(1) \ = \ \zeta\big(b(\epsilon, 1)\big).$$

Since the variation a_ϵ satisfies $b(\epsilon, 1) = 0$, it turns out that $u_\varepsilon(1) \ = \ \text{const.}$. This shows that $u_0(1) = u_1(1)$. □

Remark 4.1. We remark that a point $(x, u(1))$ belongs to the fiber-product $X \times_{M_2} X$. Indeed, it follows from (4.3) and the assumption that

$$\left.\frac{d}{dt}\right|_t J_2(u(t)) = (dJ_2)_{u(t)}\big(\zeta(a(t))\big) = \mathbf{0}.$$

This implies that $J_2(u(t)) = \text{const.}$ for each $t \in I$. Therefore, $J_2(u(1)) = J_2(u(0)) = J_2(x)$.

This results in that $h([a]) = h([b])$ holds if $a \sim b$ for two A_1-paths, that is, the map (4.5) is well-defined. Conversely, we show that $a \sim b$ holds if $h([a]) = h([b])$ in what follows. For two A_1-paths $a(t), b(t) \in P(A_1)$, assume that points $(x, u(1))$ and $(y, v(t))$ are in the same leaf by the action η, i.e., $\overline{(x, u(1))} = \overline{(y, v(1))} \in X \otimes_{A_2} X$. Here, $u(t)$ and $v(t)$ are the integral curves obtained from paths $a(t)$ and $b(t)$ with the initial values $u(0) = x$ and $v(0) = y$, respectively. Note that $J_1 \circ u = \pi \circ a$ and $J_2 \circ v = \pi \circ b$. By the assumption, $c(t) := \pi \circ a(t)$ and $d(t) := \pi \circ b(t)$ have the same end points: $\pi(a(0)) = \pi(b(0))$, $\pi(a(1)) = \pi(b(1))$. Let us consider the integration $\gamma = D^{-R}a$ and $\delta = D^{-R}b \in P\big(\mathcal{G}(A_1)\big)$ of the A-paths. From (4.2), γ and δ

are concretely given by

$$\gamma(t) = \phi_\sigma^{t,1}\big(\pi(a(1))\big), \qquad \delta(t) = \psi_\varsigma^{t,1}\big(\pi(b(1))\big),$$

where σ and ς stand for the time-dependent sections such that $a(t) = \sigma(t, c(t))$ and $b(t) = \varsigma(t, d(t))$, and moreover, $\phi_\sigma^{t,1}$ and $\psi_\varsigma^{t,1}$ are the flow of the right-invariant vector fields corresponding to σ and ς, respectively. Therefore, we find that

$$\gamma(1) = \phi_\sigma^{1,1}\big(\pi(a(1))\big) = \pi(a(1)) = \pi(b(1)) = \psi_\varsigma^{1,1}\big(\pi(b(1))\big) = \delta(1).$$

On the other hand, $a = D^R\gamma$ and $b = D^R\delta$ are calculated to be

$$a(t) = \frac{d}{dt}\boldsymbol{\varepsilon}\big(\boldsymbol{t}(\gamma(t))\big), \qquad b(t) = \frac{d}{dt}\boldsymbol{\varepsilon}\big(\boldsymbol{t}(\delta(t))\big).$$

This implies that $\pi(a(t)) = \boldsymbol{t}(\gamma(t))$ and $\pi(b(t)) = \boldsymbol{t}(\delta(t))$ hold for any $t \in I$. It follows from this that

$$\boldsymbol{t}(\gamma(0)) = \pi(a(0)) = \pi(b(0)) = \boldsymbol{t}(\delta(0)).$$

Consequently, we find that $\gamma(0) = \delta(0)$ since $\boldsymbol{s}(\gamma(0)) = \boldsymbol{s}(\delta(0))$ holds as well. This shows that $\gamma(t)$ and $\delta(t)$ have the same end points $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$.

Those $P(\mathcal{G}(A))$-paths γ and δ are homotopic to each other in the corresponding source-fiber since the source-fiber is simply-connected. Let g_ϵ be the homotopy between them ($g_0 = \gamma, g_1 = \delta$). Then, $D^R g_\epsilon$ is a variation joining a and b whose base paths are the same end points. As mentioned in 4.1, it holds that $b(\epsilon, t) = (D^R g^t)(\epsilon)$ for each t. It follows from this that

$$b(\epsilon, 1) = (dR_{g^1(\epsilon)^{-1}})_{g^1(\epsilon)}\left(\frac{d}{d\epsilon}g^1(\epsilon)\right) = (dR_{g^1(\epsilon)^{-1}})_{g^1(\epsilon)}\left(\frac{d}{d\epsilon}\gamma(1)\right) = 0.$$

Therefore, a and b are A_1-homotopic by $D^R g_\epsilon$ if $h([a]) = h([b])$.

To sum up, two A_1-paths a and b are A_1-homotopic if and only if $h([a]) = h([b])$. Consequently, we obtain the following proposition.

Proposition 4.6. *The map $h : \mathcal{G}(A_1) \to X \otimes_{A_2} X$ is a homeomorphism.*

Proof. From the above observation, it follows that h is injective. In addition, we take any point $\overline{(x, y)}$ in $X \otimes_{A_2} X$. There exists a piecewise smooth curve $c : t \mapsto c(t) \in M_1$ which connect $J_1(x)$ to $J_1(y)$ since M_1 is connected.

We denote by $c_j := c|_{[t_{j-1},t_j]}$ the piecewise smooth curve restricted to the segment $[t_{j-1}, t_j]$ of the partition $t_0 < t_1 < \cdots < t_{n-1} < t_n$ and let a_j ($j = 1, 2, \ldots, n$) be A_1-paths over c_j. Those A_1-paths are obviously composable, i.e., $\pi(a_j(t_j)) = \pi(a_{j+1}(t_j))$ ($j = 1, 2, \ldots, n-1$), and then one obtains a new A_1-path a by the multiplication $a(t) = a_n \cdot a_{n-1} \cdots a_1(t)$. Here, for the multiplication of composable A-paths, we refer to [7]. Following the discussion immediately after Proposition 4.4, we get the unique solution u of differential equation:

$$\frac{d}{dt}u(t) = \zeta_{u(t)}\big(a(t)\big), \quad u(0) = x.$$

Remark that $J_1 \circ u = \pi \circ a = c$. By re-parameterizing, we have that $u(1) = y$. As a result, the A_1-homotopy class $[a]$ of a satisfies $h([a]) = \overline{(x, y)}$. This shows that h is surjective. Consequently, we find that h is bijective.

Let $\mathcal{S}(X)$ denote the set of all solutions of differential Equations (4.3) given for $a \in P(A_1)$:

$$\mathcal{S}\,(X) = \Big\{\, u \in C^1(\mathbb{R}, X) \;\Big|\; \frac{d}{dt}u(t) = \zeta_{u(t)}(a(t)),\; u(0) = x,\, x \in X,\, a \in P(A_1) \,\Big\}.$$

The space $\mathcal{S}\,(X)$ is endowed with the subspace topology induced from the space of all C^1-curves $u : I \to X$ with the C^1-topology. That is, denoting by $C'(K, W')$ the set of all C^1-curves $u : I \to X$ for a compact subset $K \subset I$ and an open set $W' \subset J^1(\mathbb{R}, X)$ such that $(j^1u)(K) \subset W'$, any open set V in $\mathcal{S}\,(X)$ is given by the intersection of a finite collection of those subsets $C'(K_j, W'_j)$:

$$V = \bigcap_{j=1}^{r} G'(K_j, W'_j) \cap \mathcal{S}\,(X) \quad (r \in \mathbb{N}). \tag{4.7}$$

Let $\tilde{h} : P(A_1) \to \mathcal{S}\,(X)$ denote the map which assigns to a the solution of (4.3) determined uniquely by a. Given a in $P(A_1)$, we let V be any open set in $\mathcal{S}\,(X)$ containing $u = \tilde{h}(a)$. Again, we remark that V is written in the form (4.7). For the sake of simplicity, replacing $J^1(\mathbb{R}, X)$ by coordinate domains $W'_j \subset \mathbb{R} \times \mathbb{R}^{\dim X} \times \mathbb{R}^{\dim X}$ for each j and replacing $\pi : A_1 \to M_1$ and $J_1 : X \to M_1$ by their coordinate representations, we consider two maps

$$\varpi := id \times \pi \times \pi_* : \mathbb{R} \times \mathbb{R}^{\dim A_1} \times \mathbb{R}^{\dim A_1} \to \mathbb{R} \times \mathbb{R}^{\dim M_1} \times \mathbb{R}^{\dim M_1}$$

and

$$\mathcal{J}_1 := id \times J_1 \times J_{1*} : \mathbb{R} \times \mathbb{R}^{\dim X} \times \mathbb{R}^{\dim X} \to \mathbb{R} \times \mathbb{R}^{\dim M_1} \times \mathbb{R}^{\dim M_1}.$$

We remark that

$$\varpi\left(t, a(t), \frac{d}{dt}a(t)\right) = \mathcal{J}_1\left(t, u(t), \frac{d}{dt}u(t)\right) =: q_0(t) \in \mathcal{J}(W_j') \quad (\forall t \in K_j)$$

for each j. Since $\mathcal{J}_1$ is a submersion, there exist open neighborhoods $N(q_0(t))$ of $q_0(t)$ such that $(t, u(t), \frac{d}{dt}u(t))$ are in the images of a smooth sections on $N(q_0(t))$. Furthermore, by the continuity of ϖ, there exist open neighborhoods $W_j(t)$ of $a(t)$ in $\mathbb{R} \times \mathbb{R}^{\dim A_1} \times \mathbb{R}^{\dim A_1}$ such that $\varpi(W_j(t)) \subset N(q_0(t))$. Consequently, we obtain an open neighborhood $W_j = \bigcup_{t \in K_j} W_j(t)$ of $j^1 a$ in $\mathbb{R} \times \mathbb{R}^{\dim A_1} \times \mathbb{R}^{\dim A_1}$, and then verify that $(j^1 a)(K_j) \subset W_j$. Therefore,

$$\bigcap_{j=1}^{r} C(K_j, W_j) \cap P(A_1)$$

is a open set in $P(A_1)$ containing a. This shows that $\tilde{h} : P(A_1) \to \mathcal{S}(X)$ is continuous. It follows from this that $h : \mathcal{G}(A_1) \to X \otimes_{A_2} X$ is continuous.

On the other hand, we let G be any open subset in $\mathcal{G}(A_1)$ and consider the map $\widehat{J}_1 : X \otimes_{A_2} X \to M_1$, $\overline{(x, y)} \mapsto J_1(x)$. Remark that $\boldsymbol{s} \circ h = \widehat{J}_1$. Since the source map $\boldsymbol{s}$ of $\mathcal{G}(A_1)$ is an open map (see [7]), $\widehat{J}_1^{-1}\big(\boldsymbol{s}(G)\big) \subset X \otimes_{A_2} X$ is open. Therefore, h^{-1} is continuous. This completes the proof. □

The next proposition can be shown in a similar way to the proof of Proposition 4.6.

Proposition 4.7. *The map $h : \mathcal{G}(A_1) \to X \otimes_{A_2} X$ is a diffeomorphism.*

Theorem 4.8. *Let A_1 and A_2 be integrable Lie algebroids. If A_1 and A_2 are strongly Morita equivalent, then their categories of modules $\mathcal{M}(A_1)$ and $\mathcal{M}(A_2)$ are equivalent.*

Proof. Let $N \xrightarrow{\mu} M_1$ and $N' \xrightarrow{\mu'} M_1$ be objects in $\mathcal{M}(A_1)$, and $f : N \to N'$ a morphism in $\mathcal{M}(A_1)$. From Proposition 4.4, there exist the covariant functors $\mathcal{S} : \mathcal{M}(A_1) \to \mathcal{M}(A_2)$ and $\mathcal{T} : \mathcal{M}(A_2) \to \mathcal{M}(A_1)$, and then, one can obtain two right modules

$$\mathcal{T}\circ\mathcal{S}\,(N) = (N\otimes_{A_1} X)\otimes_{A_2} X \longrightarrow M_1, \quad \overline{(\overline{(n,x')}_1,\,x)}_2 \longmapsto J_1(x),$$

and

$$\mathcal{T}\circ\mathcal{S}\,(N') = (N'\otimes_{A_1} X)\otimes_{A_2} X \longrightarrow M_1, \quad \overline{(\overline{(n',x')}_1,\,x)}_2 \longmapsto J_1(x),$$

over A_1, and a morphism

$$\mathcal{T}\circ\mathcal{S}\,(f) : \mathcal{T}\circ\mathcal{S}\,(N) \to \mathcal{T}\circ\mathcal{S}\,(N'), \quad \overline{(\overline{(n',x')}_1,\,x)}_2 \mapsto \overline{(\overline{(f(n),x')}_1,\,x)}_2$$

from $\mathcal{T}\circ\mathcal{S}\,(N)$ to $\mathcal{T}\circ\mathcal{S}\,(N')$. Here, $A_1 \overset{J_1}{\leftarrow} X \overset{J_2}{\rightarrow} A_2$ is an (A_1, A_2)-bimodule.

On the other hand, for any right module $N \overset{\mu}{\to} M_1$ over A_1, we define a map $\Psi_N : \mathcal{T}\circ\mathcal{S}\,(N) \to N$ as

$$\Psi_N : (N\otimes_{A_1} X)\otimes_{A_2} X \overset{\simeq}{\longrightarrow} N, \quad \overline{(\overline{(n,x')}_1,\,x)}_2 \longmapsto \overline{(x',x)}_2\cdot n,$$

where the element $\overline{(x',x)}_2$ in $X\otimes_{A_2} X$ is thought of an element in $\mathcal{G}(A_1)$ by Proposition 4.7, and where $\overline{(x,x')}_2\cdot n$ means the point $\phi_1(n)$ on the integral curve $\phi_t(n)$ starting at $n\in N$ which is determined by $[b] := \overline{(x,x')}_2 \in \mathcal{G}(A_1)$. Remark that the map Ψ_N is well-defined by Remark 3.3. Since their respective vector fields induced those actions ϱ and ϱ' of A_1 are f-related, we have

$$\frac{d}{dt}(f\circ\phi_t)(n) = (df)_{\phi_t(n)}\left(\frac{d}{dt}\phi_t(n)\right) = (df)_{\phi_t(n)}\big(\varrho(b(t))_{b(t)}\big) = \varrho'_{f(b(t))}(b(t)).$$

That is, $t\mapsto f\circ\phi_t(n)$ is an integral curve of $\varrho'(b(t))$. From the uniqueness, it follows that $f(\overline{(x',x)}_2\cdot n) = \overline{(x',x)}_2\cdot f(n)$. In other words, the diagram

$$\begin{array}{ccc} \mathcal{T}\circ\mathcal{S}\,(N) & \xrightarrow{\Psi_N} & N \\ {\scriptstyle \mathcal{T}\circ\mathcal{S}\,(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{T}\circ\mathcal{S}\,(N') & \xrightarrow[\Psi_{N'}]{} & N'. \end{array}$$

commutes. Consequently, the functor $\mathcal{T}\circ\mathcal{S}$ is natural isomorphic to the identity functor $\mathrm{Id}_{\mathcal{M}(A_1)}$. Similarly, it can be shown that there exists also a natural isomorphism between $\mathcal{S}\circ\mathcal{T}$ and $\mathrm{Id}_{\mathcal{M}(A_2)}$. This completes the proof. □

4.3. Application to the Hamiltonian category

Let (M, D_M) be a Dirac manifold. The Hamiltonian category [2] of (M, D_M), denoted by $\overline{\mathcal{M}}\,(M, D_M)$, is a category whose objects are strong Dirac maps $F : N \to M$ and whose morphisms are forward Dirac maps $\varphi : N \to N'$ satisfying $F = F' \circ \varphi$.

Let us focus on the case where (M, D_M) is integrable. We define the subcategory $\mathcal{H}\,(M, D_M)$ of $\overline{\mathcal{M}}\,(M, D_M)$ as the category whose objects are complete strong Dirac maps $F : N \to M$ which are surjective submersions and whose morphisms are forward Dirac maps $\varphi : N \to N'$ such that $F = F' \circ \varphi$. Suppose that we are given a morphism $\varphi : N \to N'$ in $\mathcal{H}\,(M, D_M)$. As mentioned in Section 3, each strong Dirac map $N \xrightarrow{F} M$ and $N' \xrightarrow{F'} M$ induces an infinitesimal action $\zeta : \Gamma^\infty(D_M) \to \mathfrak{X}(N)$ and $\zeta' : \Gamma^\infty(D_M) \to \mathfrak{X}(N')$ by (3.9). That is, for $(Y, \beta) \in \Gamma^\infty(D_M)$, we can obtain $X_n \in T_nN$ and $X'_{n'} \in T_{n'}N'$ such that $Y_{F(n)} = (dF)_n(X_n)$ and $Y_{F'(n')} = (dF')_{n'}(X_{n'})$. Note that $Y_{F(n)} = d(F' \circ \varphi)_n(X_n) = (dF')_{\varphi(n)}\big((d\varphi)_n(X_n)\big)$ by $F = F' \circ \varphi$ and we have $(dF')_{\varphi(n)}\big(X'_{\varphi(n)} - (d\varphi)_n(X_n)\big) = \mathbf{0}$. That is, $X'_{\varphi(n)} - (d\varphi)_n(X_n) \in \ker{(dF')_{\varphi(n)}}$. In addition, $X'_{\varphi(n)} - (d\varphi)_n(X_n)$ is also an element in $\ker{(D_{N'})}$ since φ is a forward-Dirac map. Consequently,

$$X'_{\varphi(n)} - (d\varphi)_n(X_n) \in \ker{(dF')_{\varphi(n)}} \cap \ker{(D_{N'})_{\varphi(n)}} = \{\mathbf{0}\}.$$

This shows that the vector fields $\zeta(Y, \beta)$ and $\zeta'(Y, \beta)$ are φ-related. Therefore, the subcategory $\mathcal{H}\,(M, D_M)$ of the Hamiltonian category $\overline{\mathcal{M}}\,(M, D_M)$ can be regarded as the category of modules $\mathcal{M}\,(D_M)$ over D_M.

For a closed 2-form B on M, a subbundle

$$\tau_B(D_M) := \big\{\, (Y,\, \beta + \mathrm{i}_Y B) \,|\, (Y,\, \beta) \in \Gamma^\infty(D_M) \,\big\} \subset TM \oplus T^*M.$$

satisfies the conditions in Example 2.5. In other words, $\tau_B(D_M)$ is a Dirac structure on M. The Dirac structure $\tau_B(D_M)$ associated to a closed 2-form B on M is called a gauge transformation by B (see H. Bursztyn and O. Radko [4]). In Example 6.6 [3], it is proven that $\tau_B(D_M)$ is integrated to source-simply-connected Lie groupoid $\mathcal{G}(\tau_B(D_M)) \rightrightarrows M$ associating with $\tau_B(D_M)$, which is isomorphic to $\mathcal{G}(D_M) \rightrightarrows M$ associating with D_M. Accordingly, we obtain the complete left action by (3.4) of $D_M \to M$ with moment map $\boldsymbol{t} : \mathcal{G}(D_M) \to M$ and the complete right action by (3.5) of $\tau_B(D_M) \to M$ with moment map $\boldsymbol{s} : \mathcal{G}(D_M) \to M$. In a similar way to the proof of the reflexive property in Proposition 3.17, we can obtain the following proposition:

Proposition 4.9. *If* (M, D_M) *is an integrable Dirac manifold, then two Lie algebroids* D_M *and* $\tau_B(D_M)$ *are strongly Morita equivalent to each other.*

Using this proposition and Theorem 4.8, we can recover partially Proposition 2.8 in [2].

Corollary 4.10. (cf. H. Bursztyn and M. Crainic [2]) *Let* D_M *be an integrable Dirac structure and* B *a closed 2-form on* M*. Then, the subcategories* $\mathcal{H}(M.D_M)$ *and* $\mathcal{H}(M.\tau_B(D_M))$ *are equivalent.*

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