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Fiber connectivity and bifurcation diagrams of almost toric integrable systems

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We describe the bifurcation diagrams of almost toric integrable Hamiltonian systems on a four dimensional symplectic manifold M, not necessarily compact. We prove that, under a weak assumption, the connectivity of the fibers of the induced singular Lagrangian fibration $M \to \mathbb{R}^2$ can be detected from the bifurcation diagram alone. In this case, it is possible to give a detailed description of the image of the fibration.

1. Introduction

A broad question in symplectic geometry and Hamiltonian mechanics is the classical inverse problem: what properties of the system can be detected on the image of the classical system (often called the *classical spectrum* by physicists)?

A rather extreme case, where the most complete answer can be given, is the case of toric manifolds, using the results of Atiyah [2], Guillemin-Sternberg [13], and Delzant [8]. They showed that the image of a Hamiltonian torus action is a convex polytope and that this polytope completely characterizes the system up to a symplectic isomorphism.

As it is now clearly understood, the convexity property is tightly linked to the connectivity of the fibers of the momentum map. Of course, it is clear that being able to detect connectivity is the crucial starting point for any type of classical inverse problem. However, in non-toric integrable cases, there is no general result to check this connectivity. This issue will be the main theme of our article.

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On the other hand, in the 1980s and 1990s, the Fomenko school developed a Morse theory for regular energy surfaces of integrable systems. They have set up a technique to encode the topological properties of the systems in a combinatorial graph, in relation with the *bifurcation diagram*, which is, in some sense, the “skeleton” of the classical spectrum.

Our paper aims at bringing together these viewpoints to treat integrable systems that are not necessarily toric, while keeping in mind the power of the symplectic techniques based on the momentum map. Under some weak transversality hypothesis, we shall prove that the fibers of a two degrees of freedom integrable system with non-degenerate non-hyperbolic singularities are connected. This will enable us to describe the structure of the image of the system which, in general, is very far from a polytope, but has some local convexity properties in terms of its induced affine structure.

The systems in which we are interested in this paper are two-degrees of freedom integrable systems with non-degenerate and non-hyperbolic singularities. They are called *almost toric systems* [26]. They generalize the momentum maps of toric manifolds and retain some of their rigidity while, at the same time, incorporate a very interesting type of singularity, namely of focus-focus type. Almost toric systems include the semitoric integrable systems (semitoric systems come endowed with a Hamiltonian S^1-action with proper momentum map and have been recently completely classified in [18, 19]). The set of symplectic manifolds on which almost toric systems can exist form a subset of *almost toric manifolds* which were introduced by Symington [23]. Compact almost toric manifolds were classified, up to diffeomorphisms, in [15, Table 1] by Leung and Symington.

In [26], the connectivity of the level sets of the system was proven for semitoric systems. In this article, we investigate connectivity without any S^1-action. Our main assumption is that the map from the four dimensional symplectic manifold to the plane is proper. However, there are results that do not require properness of this map (Theorems 3.6, 5.2 and Proposition 5.8), and which fit in the larger program of building a classification theory for integrable systems under minimal assumptions (see [21]).

In a future article, we intend to explore the consequences of these results to construct new symplectic invariants from the image of integrable systems that are not necessarily semitoric.

2. Statement of the results

In this paper, manifolds are assumed to be C^∞ and second countable. Let us recall here some standard definitions. A map $f\colon X \to Y$ between topological spaces is *proper* if the preimage of every compact set is compact. Let X, Y be smooth manifolds and $A \subset X$. A map $f : A \to Y$ is said to be *smooth* if every point in A admits an open neighborhood on which f can be smoothly extended. The map f is called a *diffeomorphism onto its image* if f is injective, smooth, and its inverse $f^{-1}\colon f(A) \to A$ is a smooth map, in the sense that that for every point $y \in f(A)$ there exist an open subset $U_y \subseteq Y$ containing y, and a smooth map $F\colon U_y \to X$ such that $F(y') = f^{-1}(y')$ for every $y' \in f(A) \cap U_y$. If X and Y are smooth manifolds, the *bifurcation set* Σ_f of a smooth map $f : X \to Y$ consists of the points of X where f is not locally trivial (see Definition 4.1). It is known that the set of critical values of f is included in the bifurcation set and that if f is proper this inclusion is an equality (see [1, Proposition 4.5.1] and the comments following it).

Recall that an integrable system $F\colon M \to \mathbb{R}^2$ is called *non-degenerate* if its singularities are non-degenerate (see Definition 3.1). Throughout this section, F is assumed to be proper. If F is non-degenerate, then Σ_F is the image of a piecewise smooth immersion of a 1-dimensional manifold and of isolated points (Proposition 5.3). We say that a vector in $\mathbb{R}^2$ is *tangent to* Σ_F whenever it is directed along a left limit or a right limit of the differential of the immersion. We say that the curve γ has a *vertical tangency* at a point c if there is a vertical tangent vector at c. Our first main result is the following.

Theorem 1 (Fiber connectivity in the compact case). *Suppose that (M, ω) is a compact connected symplectic four-manifold. Let $F\colon M \to \mathbb{R}^2$ be a non-degenerate integrable system without hyperbolic singularities. Denote by Σ_F the bifurcation set of F. Assume that there exists a diffeomorphism $g\colon F(M) \to \mathbb{R}^2$ onto its image such that $g(\Sigma_F)$ does not have vertical tangencies (see Figure 1). Then F has connected fibers.*

Note that the absence of vertical tangency can be expressed in terms of the map $(J, H) := g \circ F$ by requiring that all critical points of J must be also critical points for H.

Remark 2.1. Although in this paper we put the emphasis on the integrable system F rather than on the manifold M, it should be noticed that the Leung-Symington classification [15, Table 1] implies that a compact connected symplectic 4-manifold (M, ω) that carries an almost toric integrable system must be diffeomorphic to $S^2 \times S^2$, $\mathbb{C}P^2$, $S^2 \times T^2$, $S^2 \tilde{\times} T^2$ (the non-trivial sphere bundle over the torus), or connected sums of any of the last three with a number of $\overline{\mathbb{C}P^2}$ (the bar over $\mathbb{C}P^2$ means that the sign on the standard symplectic form defined by the Fubini-Study form is changed). For more comments see the subsection "Leaf Space" below.

The assumption on vertical tangencies imposes an additional restriction on this list, that can only be satisfied by manifolds diffeomorphic to $S^2 \times S^2$ or the sum manifolds $\mathbb{C}P^2 \# k\, \overline{\mathbb{C}P^2}$. Is it interesting to notice that this restriction can be directly detected on the bifurcation set.

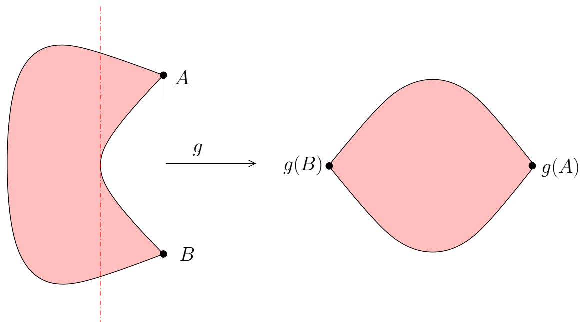

Figure 1: Suppose that the bifurcation set Σ_F of F consists precisely of the boundary points in the left figure (which depicts $F(M)$). The diffeomorphism g transforms $F(M)$ to the region on the right hand side of the figure, in order to remove the original vertical tangencies on Σ_F.

Remark 2.2. If $F\colon M \to \mathbb{R}^2$ in Theorem 2 is the momentum map of a Hamiltonian 2-torus action then $\Sigma_F = \partial(F(M))$. This is no longer true for general integrable systems; the simplest example of such a situation is the spherical pendulum, which has a point in the bifurcation set in the interior of $F(M)$.

It is remarkable that such a simple condition ensures connectivity. However, this is just a sufficient condition which can be weakened (see also Theorem 3 below). Nonetheless, the examples we know that feature non-connected fibers, all violate this hypothesis. For instance, in Example 4.10 we construct an almost toric system with disconnected fibers, where Σ_F is the boundary of an annulus (Figure 2).

If M is not compact, the same assumption can be used under the weak additional hypothesis that the image of the momentum map can be enclosed in a proper cone.

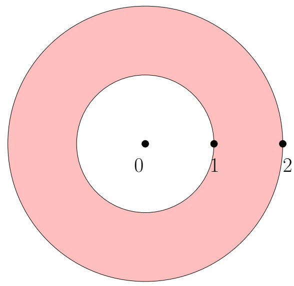

Figure 2: Image $F(M)$ of integrable system with disconnected fibers, on the compact manifold $S^2 \times S^1 \times S^1$.

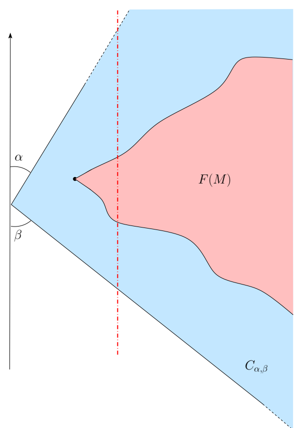

Figure 3: The image $F(M)$ lies in the convex cone $C_{\alpha,\beta}$ and has no vertical tangencies. See condition 1 in Theorem 2.

More precisely, we denote by $C_{\alpha,\beta}$ the cone in Figure 3, i.e., the intersection of the half-planes defined by $y \geqslant (\tan\alpha)\, x$ and $y \leqslant (\tan\beta)\, x$ in the plane $\mathbb{R}^2$. This cone said to be *proper*, if $\alpha > 0$, $\beta > 0$, $\alpha + \beta < \pi$. Theorem 2 can be extended to non-compact manifolds as follows.

Theorem 2 (Fiber connectivity in the non-compact case). *Suppose that (M, ω) is a connected symplectic four-manifold. Let $F\colon M \to \mathbb{R}^2$ be a non-degenerate integrable system without hyperbolic singularities such that F is a proper map. Denote by Σ_F the bifurcation set of F. Assume that there exists a diffeomorphism $g\colon F(M) \to \mathbb{R}^2$ onto its image such that:*

(i) *the image $g(F(M))$ is included in a proper convex cone $C_{\alpha,\beta}$ (see Figure 3);*

(ii) *the image $g(\Sigma_F)$ does not have vertical tangencies (see Figure 1).*

Then F has connected fibers.

Note that Theorem 2 clearly implies Theorem 1.

Non-compact examples are of particular interest to physicists because most mechanical systems have non-compact phase spaces, e.g., they are cotangent bundles. Two physically interesting examples of almost toric system with precisely one focus-focus singularity are the Jaynes-Cummings model, also called coupled spin-oscillator, and the spherical pendulum. Even though the spherical pendulum is an algebraic example, checking fiber connectivity with algebraic tools is far from trivial — see [7]. This example has an S^1 symmetry, but, nevertheless, we cannot apply the semitoric theory because the S^1-momentum map is not proper. Further, in this article, we introduce a weaker transversality condition that allows us to deal with some cases of vertical tangencies. Using this condition and Theorem 2, we will prove the following.

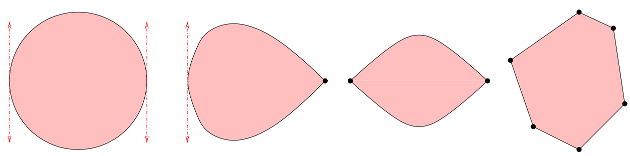

Figure 4: A disk, a disk with a conic point, a disk with two conic points, or a compact convex polygon.

Theorem 3. *Suppose that (M,ω) is a compact connected symplectic four-manifold. Let $F\colon M \to \mathbb{R}^2$ be a non-degenerate integrable system without hyperbolic singularities. Assume that:*

(i) *the interior of $F(M)$ contains a finite number of critical values;*

(ii) *there exists a diffeomorphism g such that $g(F(M))$ is either a disk, a disk with a conic point, a disk with two conic points, or a compact convex polygon (see Figure 4).*

Then the fibers of F are connected.

Here, a neighborhood of a *conic point* is, by definition, locally diffeomorphic to some proper cone $C_{\alpha,\beta}$.

Image of the integrable system. From the point of view of classical (and even quantum) mechanics, the points in the image of the integrable system represent the "observable" part of the system: it is the set of possible values of energy, linear or angular momenta, etc. Therefore, is it important in mathematical physics not only to develop tools that help understanding the inner structure of the system from these observable quantities, as we have done in the previous theorems, but also to search for a theoretical description of what the image of the integrable system may look like; see, for instance, the interesting recent works by Babelon et al. [3] and Dullin [9]. This is the incentive for the next results below (Theorems 4 and 5).

Atiyah proved fiber connectivity of the momentum map of a torus action [2] simultaneously with the so called Convexity Theorem of Atiyah, Guillemin, and Sternberg [2, 13]; it is one of the main results in symplectic geometry and was an eye opening tool for many further developments. The convexity theorem describes the image of the momentum map of a Hamiltonian torus action: it is the convex hull of the images of the fixed point sets of the torus action.

A Hamiltonian torus action is a particular case of an almost-toric system. However, for a general almost-toric system with proper map, the image of the system is no longer a polygon and there is no reason of being convex in the traditional $\mathbb{R}^2$ sense. However, the image admits a singular affine structure (whose regular part is given by local action-angle variables), and it follows from the local normal forms of non-degenerate singularities that the image is locally convex in terms of this affine structure. In fact, the affine structure is the "projection" $\mathcal{F}$ onto $F(M)$ of the intrinsic affine structure on the leaf space of the singular Lagrangian foliation (cf. Equation (1) below). The difficult point is to understand the global structure of the image of $\mathcal{F}$. In order to achieve this goal, we are naturally led to study the case where the fibers are connected, *ie.* $\mathcal{F}$ is injective.

Before stating the result we recall that the *epigraph* $\mathrm{epi}(f) \subseteq \mathbb{R}^{n+1}$ *of a map* $f\colon A \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ consists of the points lying on or above its graph, i.e., the set $\mathrm{epi}(f) := \{(x, y) \in A \times \mathbb{R} \,|\, y \geqslant f(x)\}$. Similarly, the *hypograph* $\mathrm{hyp}(f) \subseteq \mathbb{R}^{n+1}$ *of a map* $f\colon A \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ consists of the points lying on or below its graph, i.e., the set $\mathrm{hyp}(f) := \{(x, y) \in A \times \mathbb{R} \,|\, y \leqslant f(x)\}$.

Theorem 4 (Image of Lagrangian fibration in the compact case). *Suppose that* (M,ω) *is a compact connected symplectic four-manifold. Let* $F\colon M \to \mathbb{R}^2$ *be a non-degenerate integrable system without hyperbolic singularities. Denote by* Σ_F *the bifurcation set of* F*. Assume that there exists* *a diffeomorphism $g\colon F(M) \to \mathbb{R}^2$ onto its image such that $g(\Sigma_F)$ does not have vertical tangencies (see Figure 1). Then:*

(1) *the image $F(M)$ is contractible and the bifurcation set can be described as $\Sigma_F = \partial(F(M)) \sqcup \mathfrak{F}$, where $\mathfrak{F}$ is a finite set of rank 0 singularities which is contained in the interior of $F(M)$;*

(2) *let $(J, H) := g \circ F$ and let $J(M) = [a, b]$. Then the functions H^+, H^- : $[a, b] \to \mathbb{R}$ defined by $H^+(x) := \max_{J^{-1}(x)} H$ and $H^-(x) := \min_{J^{-1}(x)} H$ are continuous and $F(M)$ can be described as $F(M) = g^{-1}(\mathrm{epi}(H^-) \cap \mathrm{hyp}(H^+))$.*

Figure 5 shows a possible image $F(M)$, as described in Theorem 4. In the case of non-compact manifolds we have the following result.

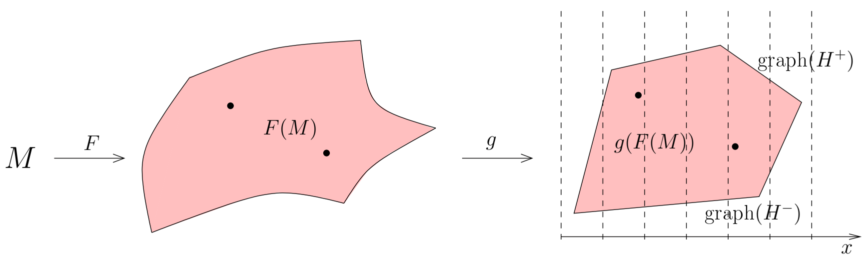

Figure 5: Description of the image of an integrable system. The image is first transformed to remove vertical tangencies; then it can be described as a region bounded by two graphs.

Theorem 5 (Image of Lagrangian fibration in the non-compact case). *Suppose that (M, ω) is a connected symplectic four-manifold. Let $F\colon M \to \mathbb{R}^2$ be a non-degenerate integrable system without hyperbolic singularities such that F is a proper map. Denote by Σ_F the bifurcation set of F. Assume that there exists a diffeomorphism $g\colon F(M) \to \mathbb{R}^2$ onto its image such that:*

(i) *the image $g(F(M))$ is included in a proper convex cone $C_{\alpha,\beta}$ (see Figure 3);*

(ii) *the image $g(\Sigma_F)$ does not have vertical tangencies (see Figure 1).*

Equip $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ with the standard topology. Then:

(1) *the image $F(M)$ is contractible and the bifurcation set can be described as $\Sigma_F = \partial(F(M)) \sqcup \mathfrak{F}$, where $\mathfrak{F}$ is a countable set of rank zero singularities which is contained in the interior of $F(M)$;*

(2) *let $(J, H) := g \circ F$. Then the functions $H^+, H^- \colon J(M) \to \mathbb{R}$ defined on the interval $J(M)$ by $H^+(x) := \max_{J^{-1}(x)} H$ and $H^-(x) := \min_{J^{-1}(x)} H$ are continuous and $F(M)$ can be described as $F(M) = g^{-1}(\mathrm{epi}(H^-) \cap \mathrm{hyp}(H^+))$.*

Note that Theorem 5 clearly implies Theorem 4. The rest of this paper is devoted to proving Theorem 2, Theorem 3, and Theorem 5.

The leaf space. An integrable system F on a $2n$-dimensional symplectic manifold M defines a (singular) Lagrangian foliation on M, whose leaves are the connected components of the level sets $F^{-1}(c)$, $c \in \mathbb{R}^n$. In the almost toric case, when F is proper, the set of leaves (commonly referred to as the *leaf space*) may be naturally endowed with the structure of a smooth stratified manifold, where the top stratum has dimension n, and each stratum inherits an integral affine structure. (The stratification we need is the "Whitney B cone stratified differential space", see [22] and [17].) Thus the map F factors through the projection map $\varpi \colon M \to \mathcal{B}$ onto $\mathcal{B}$ as in the following commutative diagram:

(1)
$$\begin{tikzcd} M \arrow[r, "\varpi"] \arrow[dr, "F"'] & \mathcal{B} \arrow[d, "\mathcal{F}"] \\ & \mathbb{R}^n. \end{tikzcd}$$

The fact that F has non–degenerate singularities implies that $\mathcal{F}$ must be a local diffeomorphism [28], in the sense of smooth stratified manifolds ([17]). From this perspective, the question of connectivity of fibers of F reduces to the injectivity of $\mathcal{F}$. For instance, the condition (a) in Theorem 4 can be rephrased as:

(a) The topological boundary $\partial\mathcal{B}$ is mapped by $\mathcal{F}$ into $\partial F(M)$.

However, as we mentioned above, from the mechanics view point, the leaf space is not accessible by observation and is not an easy object to deal with in practice, which explains why we insist on expressing our results in terms of the map F itself.

When $n = 2$, the leaf space $\mathcal{B}$ is a smooth manifold with boundary, corners, and "nodes": smooth boundaries correspond to transversally elliptic singularities, corners to elliptic-elliptic singularities, and nodes are focus-focus singularities. The data $(M, \varpi, \mathcal{B})$ is called an *almost toric manifold.* When the manifold M is compact, Leung and Symington [15, Table 1] have completely classified almost toric symplectic 4-manifolds up to diffeomorphisms (not symplectomorphisms). Examples of singular fibered manifolds that are almost toric include K3 surfaces, semitoric manifolds (classified in [18, 19]), and toric manifolds (classified in [8]).

The proofs in our article are mainly developed for non-compact symplectic manifolds with an eye towards systems which appear in mathematical physics and whose treatment could benefit from theoretical tools; in fact, a strong incentive was to include the famous example of the spherical pendulum, as well as the Jaynes-Cummings model (coupling of a spin and a harmonic oscillator). For this reason we do not rely in this paper on the Leung-Symington results. Our method is more elementary because we don't need a complete classification.

In order to clarify the problem and to emphasize that the main difficulty is hidden in the map $\mathcal{F}$ in diagram (1), we make a last remark. If $\varpi : M \to \mathcal{B}$ is an almost-toric fibration, then *one can always choose a smooth map $\mathcal{F}$ such that the integrable system $F := \mathcal{F} \circ \varpi$ has connected fibers.* In some sense, it means that the lack of connectivity is due to a "wrong choice of first integrals". In other words, if the initial integrals where, say, energy and angular momentum, then there is a locally diffeomorphic combination of energy and angular momentum that will have connected fibers. This fact follows from the Leung-Symington classification [15, Table 1] of possible bases $\mathcal{B}$. Indeed, if M is compact, then the integrable system must have some elliptic singularities, and thus the only possible bases are disks with corners and the annulus. In all cases, there is a natural immersion of these into $\mathbb{R}^2$.

3. Basic properties of almost-toric systems

In this section we prove some basic results that we need in of Section 4 and Section 5. Let (M, ω) be a connected symplectic 4-manifold.

Toric type maps. A smooth map $F\colon M \to \mathbb{R}^2$ is *toric* if there exists an effective (i.e., the intersection of all isotropy groups of the action is the identity element), integrable Hamiltonian $\mathbb{T}^2$-action on M whose momentum map is F. It was proven in [14] that if F is a proper momentum map for a Hamiltonian $\mathbb{T}^2$-action, then the fibers of F are connected and the image of F is a rational convex polygon.

Almost-toric systems. We shall be interested in maps $F : M \to \mathbb{R}^2$ that are not toric yet retain enough useful topological properties. In the analysis carried out in the paper we shall need the concept of non-degeneracy in the sense of Williamson of a smooth map from a 4-dimensional phase space to the plane.

Definition 3.1. Suppose that (M, ω) is a connected symplectic four-manifold. Let $F = (f_1, f_2)$ be an integrable system on (M, ω) and $m \in M$ a critical point of F. If $\mathrm{d}_m F = 0$, then m is called *non-degenerate* if the Hessians $\mathrm{Hess}\, f_j(m)$ span a Cartan subalgebra of the symplectic Lie algebra of quadratic forms on the tangent space $(\mathrm{T}_m M, \omega_m)$. If $\mathrm{rank}\,(\mathrm{d}_m F) = 1$ one can assume that $\mathrm{d}_m f_1 \neq 0$. Let $\iota \colon S \to M$ be an embedded local 2-dimensional symplectic submanifold through m such that $\mathrm{T}_m S \subset \ker(\mathrm{d}_m f_1)$ and $\mathrm{T}_m S$ is transversal to the Hamiltonian vector field $\mathcal{H}_{f_1}$ defined by the function f_1. The critical point m of F is called *transversally non-degenerate* if $\mathrm{Hess}(\iota^* f_2)(m)$ is a non-degenerate symmetric bilinear form on $\mathrm{T}_m S$.

Remark 3.2. One can check that Definition 3.1 does not depend on the choice of S. The existence of S is guaranteed by the classical Hamiltonian Flow Box theorem (see, e.g., [1, Theorem 5.2.19]; this result is also called the Darboux-Carathéodory theorem [20, Theorem 4.1]). It guarantees that the condition $\mathrm{d}_m f_1(m) \neq 0$ ensures the existence of a symplectic chart (x_1, x_2, ξ_1, ξ_2) on M centered at m, i.e., $x_i(m) = 0, \xi_i(m) = 0$, such that $\mathcal{H}_{f_1} = \partial/\partial x_1$ and $\xi_1 = f_1 - f_1(m)$. Therefore, since $\ker\,(\mathrm{d}_m f_1) = \mathrm{span}\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial \xi_2\}$, S can be taken to be the local embedded symplectic submanifold defined by the coordinates (x_2, ξ_2).

Definition 3.1 concerns symplectic four-manifolds, which is the case relevant to the present paper. For the notion of non-degeneracy of a critical point in arbitrary dimension see [24], [25, Section 3]. Non-degenerate critical points can be characterized (see [10, 11, 27]) using the Williamson normal form [29]. The analytic version of the following theorem by Eliasson is due to Vey [24].

Theorem 3.3 (H. Eliasson 1990). *The non-degenerate critical points of a completely integrable system $F \colon M \to \mathbb{R}^n$ are linearizable, i.e., if $m \in M$ is a non-degenerate critical point of the completely integrable system $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$, then there exist local symplectic coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ about m, in which m is represented as $(0, \ldots, 0)$, such that $\{f_i, q_j\} = 0$, for all indices i, j, where we have the following possibilities for*

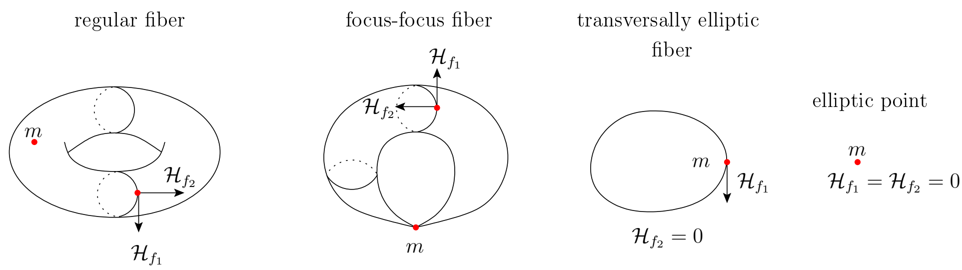

Figure 6: The figures show some possible singularities of a integrable system and their corresponding fibers. In the left most figure, m is a regular point (rank 2); in the second figure, m is a focus-focus point (rank 0); in the third one, m is a transversally elliptic singularity (rank 1); in the right most figure, m is an elliptic-elliptic point (rank 0).

the components $q_1, \ldots, q_n$, each of which is defined on a small neighborhood of $(0, \ldots, 0)$ in $\mathbb{R}^n$:

(i) *Elliptic component: $q_j = (x_j^2 + \xi_j^2)/2$, where j may take any value $1 \leq j \leq n$.*

(ii) *Hyperbolic component: $q_j = x_j \xi_j$, where j may take any value $1 \leq j \leq n$.*

(iii) *Focus-focus component: $q_{j-1} = x_{j-1}\, \xi_j - x_j\, \xi_{j-1}$ and $q_j = x_{j-1}\, \xi_{j-1} + x_j\, \xi_j$ where j may take any value $2 \leq j \leq n-1$ (note that this component appears as "pairs").*

(iv) *Non-singular component: $q_j = \xi_j$, where j may take any value $1 \leq j \leq n$.*

Moreover if m does not have any hyperbolic component, then the system of commuting equations $\{f_i, q_j\} = 0$, for all indices i, j, may be replaced by the single equation

$$(F - F(m)) \circ \varphi = g \circ (q_1, \ldots, q_n),$$

where $\varphi = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)^{-1}$ and g is a diffeomorphism from a small neighborhood of the origin in $\mathbb{R}^n$ into another such neighborhood, such that $g(0, \ldots, 0) = (0, \ldots, 0)$.

If the dimension of M is 4 and F has no hyperbolic singularities — which is the case we treat in this paper — we have the following possibilities for the map $(q_1,\, q_2)$, depending on the rank of the critical point:

(1) if m is a critical point of F of rank zero, then q_j is one of
(i) $q_1 = (x_1^2 + \xi_1^2)/2$ and $q_2 = (x_2^2 + \xi_2^2)/2$.
(ii) $q_1 = x_1\xi_2 - x_2\xi_1$ and $q_2 = x_1\xi_1 + x_2\xi_2$;
on the other hand,

(2) if m is a critical point of F of rank one, then
(iii) $q_1 = (x_1^2 + \xi_1^2)/2$ and $q_2 = \xi_2$.

In this case, a non-degenerate critical point is respectively called *elliptic-elliptic, focus-focus*, or *transversally-elliptic* if both components q_1, q_2 are of elliptic type, q_1, q_2 together correspond to a focus-focus component, or one component is of elliptic type and the other component is ξ_1 or ξ_2, respectively.

Similar definitions hold for *transversally-hyperbolic, hyperbolic-elliptic* and *hyperbolic-hyperbolic* non-degenerate critical points.

Definition 3.4. Suppose that (M, ω) is a connected symplectic four-manifold. An integrable system $F \colon M \to \mathbb{R}^2$ is called *almost-toric* if all the singularities are non-degenerate without hyperbolic components.

Remark 3.5. Suppose that (M, ω) is a connected symplectic four-manifold. Let $F \colon M \to \mathbb{R}^2$ be an integrable system. If F is a toric integrable system, then F is almost-toric, with only elliptic singularities. This follows from the fact that a torus action is linearizable near a fixed point; see, for instance [8].

A version of the following result is proven in [26] for almost-toric systems for which the map F is proper. Here we replace the condition of F being proper by the condition that $F(M)$ is a closed subset of $\mathbb{R}^2$; this introduces additional subtleties. Our proof here is independent of the argument in [26].

Theorem 3.6. *Suppose that (M, ω) is a connected symplectic four-manifold. Assume that $F \colon M \to \mathbb{R}^2$ is an almost-toric integrable system with $B := F(M)$ closed. Then the set of focus-focus critical values is countable, i.e., we may write it as $\{c_i \mid i \in I\}$, where $I \subset \mathbb{N}$. Consider the following statements:*

(i) *the fibers of F are connected;*

(ii) *the set B_r of regular values of F is connected;*

(iii) *for any value c of F, for any sufficiently small disc D centered at c, $B_r \cap D$ is connected;*

(iv) *the set of regular values is* $B_r = \mathring{B} \setminus \{c_i \mid i \in I\}$. *Moreover, the topological boundary* ∂B *of* B *consists precisely of the values* $F(m)$, *where* m *is a critical point of elliptic-elliptic or transversally elliptic type.*

Then statement (i) *implies statement* (ii), *statement* (iii) *implies statement* (iv), *and statement* (iv) *implies statement* (ii).

If, in addition, F *is proper, then statement* (i) *implies statement* (iv).

It is interesting to note that the statement is optimal in that no other implication is true (except (iii)⇒(ii) which is a consequence of the stated implications). This gives an idea of the various pathologies that can occur for an almost-toric system.

Proof of Theorem 3.6. From the local normal form 3.3, focus-focus critical points are isolated, and hence the set of focus-focus critical points is countable (remember that all our manifolds are second countable). Moreover, the image of a focus-focus point is necessarily in the interior of B.

Let us show that

$$B_r \subset \mathring{B} \setminus \{c_i \mid i \in I\}. \tag{2}$$

This is equivalent to showing that any value in ∂B is a critical value of F. Since B is closed, $\partial B \subset B$, so for every $c \in \partial B$ we have that $F^{-1}(c)$ is nonempty. By the Darboux-Carathéodory theorem, the image of a regular point must be in the interior of B, therefore $F^{-1}(c)$ cannot contain any regular point: the boundary can contain only singular values.

Since a point in ∂B cannot be the image of a focus-focus singularity, it has to be the image of a transversally elliptic or an elliptic-elliptic singularity.

We now prove the implications stated in the theorem.

(i) ⇒ (ii): Since F is almost-toric with connected fibers, the singular fibers are either points (elliptic-elliptic), one-dimensional submanifolds (codimension 1 elliptic), or a stratified manifold of maximal dimension 2 (focus-focus).The only critical values that can appear in one-dimensional families are transversally elliptic and elliptic-elliptic critical values (see Figure 7); the elliptic-elliptic critical values are isolated from each other but they appear inside of a family where every other critical value is transversally elliptic. The focus-focus singularities are isolated. Therefore, the union of all critical fibers is a locally finite union of stratified manifolds of codimension at least 2; hence this union has codimension at least 2. Thus the complement is connected and therefore its image by F is also connected.

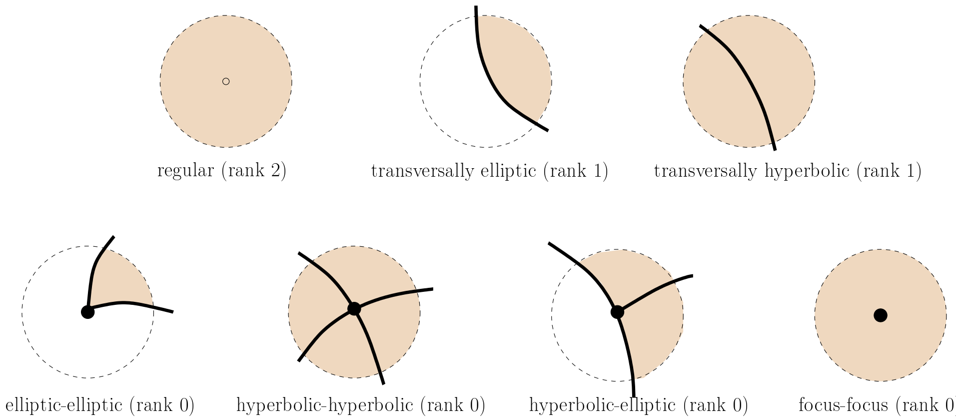

Figure 7: The local classification of the range of F in an open disk.

(iii) $\Rightarrow$ (iv): There is no embedded line segment of critical values in the interior of B (which would come from codimension 1 elliptic singularities) because this is in contradiction with the hypothesis of local connectedness (iii). Therefore $\mathring{B} \setminus \{c_i \mid i \in I\} \subset B_r$. Hence by (2),

$$\mathring{B} \setminus \{c_i \mid i \in I\} = B_r,$$

as desired, and all the elliptic critical values must lie in ∂B.
(iv) $\Rightarrow$ (ii): As we saw above, $F^{-1}(\partial B)$ contains only critical points, of elliptic type. Because of the local normal form, the set of rank 1 elliptic critical points in M form a 2-dimensional symplectic submanifold. Its topological closure in M is obtained by adding the discrete set of rank 0 elliptic points. Therefore, $M \setminus F^{-1}(\partial B)$ is connected. This set is equal to $F^{-1}(\mathring{B})$, which in turn implies that $\mathring{B}$ is connected. By hypothesis (iv), this ensures that B_r is connected.

Assume for the rest of the proof that F is proper.
(i) $\Rightarrow$ (iv): Assume that (iv) does not hold. In view of (2), there exists an elliptic singularity (of rank 0 or 1) c in the interior of B. Let Λ be the corresponding fiber. Since it is connected, it must entirely consist of elliptic points (this comes from the normal form Theorem 3.3). The normal form also implies that c must be contained in an embedded line segment of elliptic singularities, and the points in a open neighborhood Ω of Λ are sent by F in only one side of this segment. Since c is in the interior of B, there is a sequence $c_k \in B$ on the other side of the line segment that converges to c as $k \to \infty$. Hence there is a sequence $m_k \in M \setminus \Omega$ such that $F(m_k) = c_k$. Since

F is proper, one can assume that m_k converges to a point m (necessarily in $M \setminus \Omega$). By continuity of F, m belongs to the fiber over c, and thus to Ω, which is a contradiction with the hypothesis (i). $\square$

4. The fibers of an almost-toric system

In this section we study the structure of the fibers of an almost-toric system.

We shall need below the definition and basic properties of the bifurcation set of a smooth map.

Definition 4.1. Let M and N be smooth manifolds. A smooth map $f : M \to N$ is said to be *locally trivial* at $n_0 \in f(M)$, if there is an open neighborhood $U \subset N$ of n_0 such that $f^{-1}(n)$ is a smooth submanifold of M for each $n \in U$ and there is a smooth map $h : f^{-1}(U) \to f^{-1}(n_0)$ such that $f \times h : f^{-1}(U) \to U \times f^{-1}(n_0)$ is a diffeomorphism. The *bifurcation set* Σ_f consists of all the points of N where f is not locally trivial.

Note, in particular, that $h|_{f^{-1}(n)} : f^{-1}(n) \to f^{-1}(n_0)$ is a diffeomorphism for every $n \in U$. Also, the set of points where f is locally trivial is open in N.

Remark 4.2. Recall that Σ_f is a closed subset of N. It is well known that the set of critical values of f is included in the bifurcation set (see [1, Proposition 4.5.1]). In general, the bifurcation set strictly includes the set of critical values. This is the case for the momentum-energy map for the two-body problem [1, §9.8]. However (see [1, Page 340]), if $f\colon M \to N$ is a smooth *proper* map, then the bifurcation set of f is equal to the set of critical values of f.

Recall that a smooth map $f : M \to \mathbb{R}$ is *Morse* if all its critical points are non-degenerate. The smooth map f is *Morse-Bott* if the critical set of f is a disjoint union of connected submanifolds $\mathcal{C}_i$ of M, on which the Hessian of f is non-degenerate in the transverse direction, i.e.,

$$\ker(\mathrm{Hess}_m f) = \mathrm{T}_m\mathcal{C}_i, \ \text{ for all } i, \ \text{ for all } m \in \mathcal{C}_i.$$

The *index of* m is the number of negative eigenvalues of $(\mathrm{Hess}\, f)(m)$.

The goal of this section is to prove a result, Theorem 4.7 below, which we believe is of independent interest, and that will ultimately imply the connectedness of the fibers of an integrable system. Here we do not rely on Fomenko's Morse theory [5, 12], because we do not want to select a nonsingular energy surface. Instead, the model is [16, Lemma 5.51]; however, the proof given there does not extend to the non-compact case, as far as we can tell. We thank Helmut Hofer and Thomas Baird for sharing their insights on Morse theory with us that helped us in the proof of the following result.

Lemma 4.3. *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a connected manifold M. Assume f is proper and bounded from below and has no critical manifold of index 1. Then the set of critical points of index 0 is connected.*

Proof. We endow M with a Riemannian metric. The negative gradient flow of f is complete. Indeed, along the the flow the function f cannot increase and, by hypothesis, f is bounded from below. Therefore, the values of f remain bounded along the flow. By properness of f, the flow remains in a compact subset of M and hence it is complete.

Let us show, using standard Morse-Bott theory, that the integral curve of $-\nabla f$ starting at any point $m \in M$ tends to a critical manifold of f. The compact set in $\{x \in M \mid f(x) \leqslant f(m)\}$, must contain a finite number of critical manifolds. If the integral curve avoids a neighborhood of these critical manifolds, by compactness, it has a limit point, and by continuity the vector field at the limit point must vanish; we get a contradiction, thus proving the claim.

Therefore, we have the disjoint union $M = \bigsqcup_{k=0}^{n} W^s(C_k)$, where C_k is the set of critical points of index k, and $W^s(C_k)$ is its stable manifold:

$$W^s(C_k) := \{m \in M \mid d(\varphi^t_{-\nabla f}(m), C_k) \to 0 \text{ as } t \to +\infty\},$$

where d is any distance compatible with the topology of M (for example, the one induced by the given Riemannian metric on M) and $t \mapsto \varphi^t_{-\nabla f}$ is the flow of the vector field $-\nabla f$. Since f has no critical manifold of index 1, we have

$$C_0 = W^s(C_0) = M \setminus \bigsqcup_{k=2}^{n} W^s(C_k).$$

The local structure of Morse-Bott singularities given by the Morse-Bott lemma [4, 6]) implies that $W^s(C_k)$ is a submanifold of codimension k in M. Hence $\bigsqcup_{k=2}^{n} W^s(C_k)$ cannot disconnect M. □

Remark 4.4. Since all local minima of f are in C_0, we see that C_0 must be the set of global minima of f; thus C_0 must be equal to the level set $f^{-1}(f(C_0))$.

Proposition 4.5. *Let M be a connected smooth manifold and $f : M \to \mathbb{R}$ a proper Morse-Bott function whose indices and co-indices are always different from 1. Then the level sets of f are connected.*

Proof. Let c be a regular value of f (such a value exists by Sard's theorem). Then $g := (f-c)^2$ is a Morse-Bott function. On the set $\{x \in M \mid f(x) > c\}$, the critical points of g coincide with the critical points of f and they have the same index. On the set $\{x \in M \mid f(x) < c\}$, the critical points of g also coincide with the critical points of f and they have the same coindex. The level set $\{x \in M \mid f(x) = c\}$ is clearly a set of critical points of index 0 of g. Of course, g is bounded from below. Thus, by Lemma 4.3, the set of critical points of index 0 of g is connected (it may be empty) and hence equal to $g^{-1}(0)$. Therefore $f^{-1}(c)$ is connected. This shows that all regular level sets of f are connected. (As usual, a regular level set — or regular fiber — is a level set that contains only regular points, i.e., the preimage of a regular value.)

Finally let c_i be a critical value of f (if any). Since f is proper and has isolated critical manifolds, the set of critical values is discrete. Let $\epsilon_0 > 0$ be such that the interval $[c_i - \epsilon_0, c_i + \epsilon_0]$ does not contain any other critical value. Consider the manifold $N := M \times S^2$, and, for any $\epsilon \in (0, \epsilon_0)$, let

$$h_\epsilon := f - c_i + \epsilon z : N \to \mathbb{R},$$

where z is the vertical component on the sphere $S^2 \subset \mathbb{R}^3$. Notice that $z : S^2 \to \mathbb{R}$ is a Morse function with indices 0 and 2. Thus h_ϵ is a Morse-Bott function on N with indices and coindices of the same parity as those of f. Thus, no index nor coindex of h can be equal to 1. By the first part of the proof, the regular level sets of h_ϵ must be connected. The definition of ϵ_0 implies that 0 is a regular value of h_ϵ. Thus

$$F_\epsilon := \pi_M(h_\epsilon^{-1}(0)) = \{m \in M \mid \ |f(m) - c_i| \leqslant \epsilon\}$$

is connected. Since f is proper, F_ϵ is compact. Because a non-increasing intersection of compact connected sets is connected, we see that $f^{-1}(c_i) = \cap_{0<\epsilon\leqslant\epsilon_0} F_\epsilon$ is connected. □

There is no a priori reason why the fibers $F^{-1}(x, y) = J^{-1}(x) \cap H^{-1}(y)$ of F should be connected even if J and H have connected fibers (let alone if just one of J or H has connected fibers). However, the following result shows that this conclusion holds. To prove it, we need a preparatory lemma which is interesting on its own.

Lemma 4.6. *Let $f : M \to \mathbb{R}^n$ be a smooth map from a smooth connected manifold X to $\mathbb{R}^n$. Ler B_r be the set of regular values of f. Suppose that f has the following properties.*

(1) *f is a proper map.*

(2) *For every sufficiently small neighborhood D of any critical value of f, $B_r \cap D$ is connected.*

(3) *The regular fibers of f are connected.*

(4) *The set $\mathrm{Crit}(f)$ of critical points of f has empty interior.*

Then the fibers of f are connected.

Proof. We use the following "fiber continuity" fact: if Ω is a neighborhood of a fiber $f^{-1}(c)$ of a continuous proper map f, then the fibers $f^{-1}(q)$ with q close to c also lie inside Ω. Indeed, if this statement were not true, then there would exist a sequence $q_n \to c$ and a sequence of points $x_n \in f^{-1}(q_n)$, $x_n \notin \Omega$, such that there is a subsequence $x_{n_k} \to x \notin \Omega$. However, by continuity $x \in f^{-1}(c)$ which is a contradiction.

Assume a fiber $f^{-1}(p)$ of f is not connected. Then there are disjoint open sets U and V in M such that $f^{-1}(p)$ lies in $U \cup V$ but is not contained in either U or V.

By fiber continuity, there exists a small open disk D about p such that $f^{-1}(D) \subset U \cup V$.

Since the regular fibers are connected, we can define a map $\psi : D \cap B_r \to \{0; 1\}$ which for $c \in D \cap B_r$ is equal to 1 if $f^{-1}(c) \subset U$, and is equal to 0 if $f^{-1}(c) \subset V$. Fiber continuity says that the sets $\psi^{-1}(0)$ and $\psi^{-1}(1)$ are open, thus proving that ψ is continuous. By (2), the image of ψ must be connected, and therefore ψ is constant. We can hence assume, without loss of generality, that all regular fibers above D are contained in U.

Now consider the restriction $\tilde{f}$ of f on the open set $V \cap f^{-1}(D)$. Because of the above argument, it cannot take any value in $B_r \cap D$. Thus this map takes values in the set of critical values of f, which has measure zero by Sard's theorem. This requires that on $V \cap f^{-1}(D)$, the rank of $\mathrm{d}f$ is strictly less than n, which contradicts (4), and hence proves the lemma: $f^{-1}(p)$ has to be connected. $\square$

Now we are ready to prove one of our main theorems.

Theorem 4.7. *Suppose that (M, ω) is a connected symplectic four-manifold. Let $F = (J, H) \colon M \to \mathbb{R}^2$ be an almost-toric integrable system such that F is* *a proper map. Suppose that J has connected fibers, or that H has connected fibers. Then the fibers of F are connected.*

Proof. Without loss of generality, we may assume that J has connected fibers.

Step 1. We shall prove first that for every *regular value* (x, y) of F, the fiber $F^{-1}(x, y)$ is connected. To do this, we divide the proof into two cases.

Case 1A. Assume x is a regular value of J. Then the fiber $J^{-1}(x)$ is a smooth manifold. Let us show first that the non-degeneracy of the critical points of F and the definition of almost-toric systems implies that the function $H_x := H|_{J^{-1}(x)} \colon J^{-1}(x) \to \mathbb{R}$ is Morse-Bott. Let B_r be the set of regular values of F.

Let m_0 be a critical point of H_x. Then there exists $\lambda \in \mathbb{R}$ such that $\mathrm{d}H(m_0) = \lambda \mathrm{d}J(m_0)$. Thus m_0 is a critical point of F; it must be of rank 1 since $\mathrm{d}J$ never vanishes on $J^{-1}(x)$. Since F is an almost-toric system, the only possible rank 1 singularities are transversally elliptic singularities, i.e., singularities with one elliptic component and one non-singular component in Theorem 3.3; see Figure 7. Thus, by Theorem 3.3, there exist local canonical coordinates (x_1, x_2, ξ_1, ξ_2) such that $F = g(x_1^2 + \xi_1^2, \xi_2)$ for some local diffeomorphism g of $\mathbb{R}^2$ about the origin and fixing the origin; thus the derivative

$$\mathrm{D}g(0,0) =: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}).$$

Note $\mathrm{d}J(m_0) \neq 0$ implies that $d \neq 0$. Therefore, by the implicit function theorem, the submanifold $J^{-1}(x)$ is locally parametrized by the variables (x_1, x_2, ξ_1) and, within it, the critical set of H_x is given by the equation $x_1 = \xi_1 = 0$; this is a submanifold of dimension 1. The Taylor expansion of H_x is easily computed to be

$$H_x = a(x_1^2 + \xi_1^2) - \frac{bc}{d}(x_1^2 + \xi_1^2) + \frac{b}{d}x + \mathcal{O}(x_1, \xi_1, x_2)^3. \tag{3}$$

Thus, the coefficient of $(x_1^2 + \xi_1^2)$ is $(a - \frac{bc}{d})$ which is non-zero and hence the Hessian of H_x is transversally non-degenerate. This proves that H_x is Morse-Bott, as claimed.

Second, we prove, in this case, that the fibers of F are connected. At m_0, the transversal Hessian of H_x has either no or two negative eigenvalues, depending on the sign of $(a - \frac{bc}{d})$. This implies that each critical manifold has index 0 or index 2. If this coefficient is negative, the sum of the two corresponding eigenspaces is the full 2-dimensional (x_1, ξ_1)-space.

Note that $H_x\colon J^{-1}(x) \to \mathbb{R}$ is a proper map: indeed, if $K \subset \mathbb{R}$ is compact, then $H_x^{-1}(K) = F^{-1}(\{x\} \times K)$ is compact because F is proper. Thus $H_x\colon J^{-1}(x) \to \mathbb{R}$ is a smooth Morse-Bott function on the connected manifold $J^{-1}(x)$ and both H_x and $-H_x$ have only critical points of index 0 or 2. We are in the hypothesis of Proposition 4.5 and so we can conclude that the fibers of $H_x\colon J^{-1}(x) \to \mathbb{R}$ are connected.

Now, since $(H_x)^{-1}(y) = F^{-1}(x,\, y)$, it follows that $F^{-1}(x,\, y)$ is connected for all $(x,\, y) \in F(M) \subset \mathbb{R}^2$ whenever x is a regular value of J.

Case 1B. Assume that x is not a regular value of J. Note that there exists a point $(a,\, b)$ in every connected component C_r of B_r such that a is a regular value for J; otherwise $\mathrm{d}J$ would vanish on $F^{-1}(C_r)$, which violates the definition of B_r. The restriction $F|_{F^{-1}(C_r)}\colon F^{-1}(C_r) \to C_r$ is a locally trivial fibration since, by assumption, F is proper and thus the bifurcation set is equal to the critical set. Thus all fibers of $F|_{F^{-1}(C_r)}$ are diffeomorphic. It follows that $F^{-1}(x,\, y)$ is connected for all $(x,\, y) \in C_r$.

This shows that all inverse images of regular values of F are connected.

Step 2. We need to show that $F^{-1}(x, y)$ is connected if (x, y) is not a regular value of F. We claim that there is *no* critical value of F in the interior of the image $F(M)$, except for the critical values that are images of focus-focus critical points of F. Indeed, if there was such a critical value $(x_0,\, y_0)$, then there must exist a small segment line ℓ of critical values (by the local normal form described in Theorem 3.3 and Figure 7). Now we distinguish two cases.

Case 2A. First assume that ℓ is not a vertical segment (i.e., ℓ is not contained in a line of the form $x =$ constant) and let $\hat{\ell} := F^{-1}(\ell)$. Then $J(\hat{\ell})$ contains a small interval around x, so by Sard's theorem, it must contain a regular value x_0 for the map J. Then $J^{-1}(x_0)$ is a smooth manifold which is connected, by hypothesis. By the argument earlier in the proof (see Step 1, Case A), both H and $-H$ restricted to $J^{-1}(x_0)$ are proper Morse-Bott functions with indices 0 and 2. So, if there is a local maximum or local minimum, it must be unique. However, the existence of this line of rank 1 elliptic singularities implies that there is a local maximum/minimum of H_{x_0} (see Formula (3)). Since the corresponding critical value lies in the interior of the image of H_{x_0}, it cannot be a global extremum; we arrived at a contradiction. Thus the small line segment ℓ must be vertical.

Case 2B. Second, suppose that ℓ is a vertical segment (i.e., ℓ is contained contained in a line of the form $x =$ constant) and let $\hat{\ell} := F^{-1}(\ell)$. We can assume, without loss of generality, that the connected component of ℓ in the bifurcation set is vertical in the interior of $F(M)$; indeed, if not, apply Case 2A a above.

From Figure 7 we see that $\hat{\ell}$ must contain at least one critical point A of transversally elliptic type together with another point B either regular or of transversally elliptic type. By the normal form of non-degenerate singularities, $J^{-1}(x_0)$ must be locally path connected. Since it is connected by assumption, it must be path connected. So we have a path $\gamma : [0,1] \mapsto J^{-1}(x_0)$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Near A we have canonical coordinates $(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4$ and a local diffeomorphism g defined in a neighborhood of the origin of $\mathbb{R}^2$ and preserving it, such that

$$F = g(x_1^2 + \xi_1^2, \xi_2)$$

and $A = (0,0,0,0)$. Write $g = (g_1, g_2)$. The critical set is defined by the equations $x_1 = \xi_1 = 0$ and, by assumption, is mapped by F to a vertical line. Hence $g_1(0, \xi_2)$ is constant, so

$$\partial_2 g_1(0, \xi_2) = 0.$$

Since g is a local diffeomorphism, $\mathrm{d}g_1 \neq 0$, so we must have $\partial_1 g_1 \neq 0$. Thus, by the implicit function theorem, any path starting at A and satisfying $g_1(x_1^2 + \xi_1^2, \xi_2) = \text{constant}$ must also satisfy $x_1^2 + \xi_1^2 = 0$. Therefore, γ has to stay in the critical set $x_1 = \xi_1 = 0$.

Assume first that $\gamma([0,1])$ does not touch the boundary of $F(M)$. Then this argument shows that the set of $t \in [0,1]$ such that $\gamma(t)$ belongs to the critical set of F is open. It is also closed by continuity of $\mathrm{d}F$. Hence it is equal to the whole interval $[0,1]$. Thus B must be in the critical set; this rules out the possibility for B to be regular. Thus B must be a rank-1 elliptic singularity. Notice that the sign of $\partial_1 g_1$ indicates on which side of ℓ (left or right) lie the values of F near A.

Thus, even if g_1 itself is not globally defined along the path γ, this sign is locally constant and thus globally defined along γ. Therefore, all points near $\hat{\ell}$ are mapped by F to the same side of ℓ, which says that ℓ belongs to the boundary of $F(M)$; this is a contradiction.

Finally, assume that $\gamma([0,1])$ touches the boundary of $F(M)$. From the normal form theorem, this can only happen when the fiber over the contact point contains an elliptic-elliptic point C. Thus there are local canonical coordinates $(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4$ and a local diffeomorphism g defined in a neighborhood of the origin of $\mathbb{R}^2$ and preserving it, such that

$$F = g(x_1^2 + \xi_1^2, x_2^2 + \xi_2^2)$$

near $C = (0,0,0,0)$. We note that the same argument as above applies: simply replace the ξ_2 component by $x_2^2 + \xi_2^2$. Thus we get another contradiction. Therefore there are no *critical* values c in the interior of the image $F(M)$ other than focus-focus values (i.e., images of focus-focus points).

Step 3. We claim that for any critical value c of F and for any sufficiently small disk D centered at c, $B_r \cap D$ is connected.

First we remark that Step 2 implies that item (iv) in Theorem 3.6 holds, and hence item (ii) must hold : the set of regular values of F is connected.

If c is a focus-focus value, it must be contained in the interior of $F(M)$. Therefore, it follows from Step 2 that it is isolated: there exists a neighborhood of c in which c is the only critical value, which proves the claim in this case.

We assume in the rest of the proof that $c = (x_c, y_c)$ is an elliptic (of rank 0 or 1) critical value of F. Since we have just proved in Step 2 that there are no critical values in the interior of $F(M)$ other than focus-focus values, we conclude that $c \in \partial(F(M))$. Moreover, the fiber cannot contain a regular Liouville torus. Then, again by Theorem 3.3, the only possibilities for a neighborhood of c in $F(M)$ are superpositions of elliptic local normal forms of rank 0 or 1 (given by Theorem 3.3) in such a way that $c \in \partial(F(M))$. If only one local model appears, then the claim is immediate.

Let us show that a neighborhood U of c cannot contain several different images of local models. Indeed, consider the possible configurations for two different local images C_1 and C_2: either both C_1 and C_2 are elliptic-elliptic images, or both are transversally elliptic images, or C_1 is an elliptic-elliptic image and C_2 is a transversally elliptic image. Step 2 implies that the critical values of F in C_1 and C_2 can only intersect at a point, provided the neighborhood U is taken to be small enough. Let us consider a vertical line ℓ through C_1 which corresponds to a regular value of J. Any crossing of ℓ with a non-vertical boundary of $F(M)$ must correspond to a local extremum of $H|_{F^{-1}(\ell)}$ and, by Step 2, this local extremum has to be a global one. Since only one global maximum and one global minimum are possible, the only allowed configurations for C_1 and C_2 are such that the vertical line ℓ_c through c separates the regular values of C_1 from the regular values of C_2 (see Figure 8).

Since the critical leaves have codimension at least two in M, the set $M_r \subset M$ of regular points is path connected, so we can find a path $\nu\colon [0,\,1] \to \operatorname{int} F(M)$ connecting a point in $B_r \cap C_1$ to a point in $B_r \cap C_2$ by finding a path in M_r and taking its image under F (see Figure 8). By continuity, the path ν needs to cross ℓ_c, and the intersection point $(x_0 = x_c, y_0)$ must lie

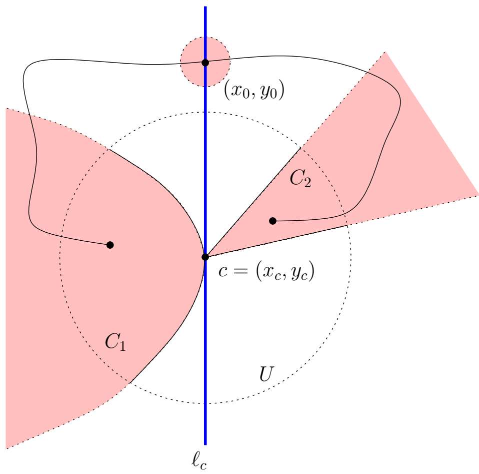

Figure 8: Point overlapping of images near singularities. Such a situation cannot occur if the fibers of J are connected.

outside U. Therefore, there exists an open ball $B_0 \subset B_r$ centered at (x_0, y_0). Suppose, for instance, that $y_0 > y_c$. Then for $(x, y) \in U$, y cannot be a maximal value of $H|_{J^{-1}(x)}$, which means that for each of C_1 and C_2, only local minima for $H|_{J^{-1}(x)}$ are allowed. This cannot be achieved by any of the local models, thus finishing the proof of our claim.

The statement of the theorem now follows from Lemma 4.6. □

Remark 4.8. It is not true that an almost-toric integrable system with connected regular fibers has also connected singular fibers. See Example 4.9 below.

The following are examples of almost-toric systems in which the fibers of F are not connected. In the next section we will combine Theorem 4.7 with an upcoming result on contact theory for singularities (which we will prove too) in order to obtain Theorem 2 of Section 1.

Example 4.9. This example appeared in [25, Chapter 5, Figure 29]. It is an example of an almost toric system $F := (J, H)\colon M \to \mathbb{R}^2$ on a compact manifold for which J and H have some disconnected fibers (the number of connected components of the fibers also changes). Because this example is constructed from the standard toric system $S^2 \times S^2$ by precomposing with a local diffeomorphism, the singularities are non-degenerate. In this case, the fundamental group $\pi_1(F(M))$ has one generator, so $F(M)$ is not simply

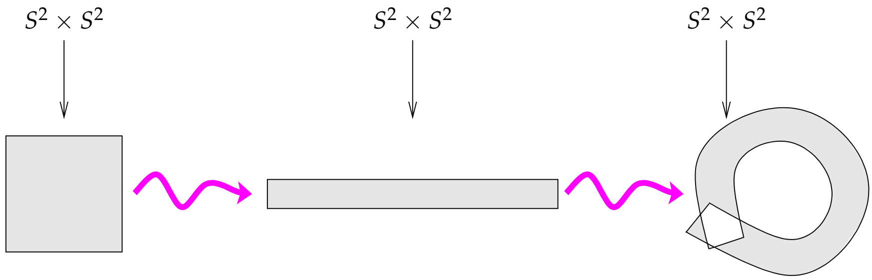

Figure 9: An almost-toric system with some disconnected fibers, constructed from the standard toric system on $S^2 \times S^2$ by precomposing with a local diffeomorphism.

connected, and hence not contractible. See Figure 9. An extreme case of this example can be obtained by letting only two corners overlap (Figure 10). We get then an almost-toric system where all regular fibers are connected, but one singular fiber is not connected.

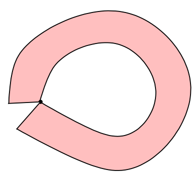

Figure 10: Image of an integrable system constructed from $S^2 \times S^2$ by a one-point identification, and which has one disconnected fiber. All the other fibers are connected.

Example 4.10. The leaf space of Example 4.9 is a square. According to Zung [31, Proposition 3.3], "if C is a topological 2-stratum of the base O of an integrable system on a compact 4-dimensional symplectic manifold X (maybe with boundary) with only non-degenerate singularities, and the image of the boundary of X under the projection to O does not intersect with the closure of C (if X is compact and with no boundary then this condition is satisfied automatically), then C is homeomorphic to either an annulus, a Möbius band, a Klein bottle, a torus, a disk, a projective space, or a sphere (in case of sphere or projective space, C must contain focus-focus points)". We learned from Zung that he had previously written a proof of this statement in [30]. The system presented in Figure 2 is an example of the case where the leaf space is an annulus. The construction is as follows.

Consider the manifold $M = S^2 \times S^1 \times S^1$. Choose the following coordinates on S^2: $h \in [1,\, 2]$ and $a \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Choose coordinates $b, c \in \mathbb{R}/2\pi\mathbb{Z}$ on $S^1 \times S^1$ and let $\mathrm{d}h \wedge \mathrm{d}a + n\mathrm{d}b \wedge \mathrm{d}c$ be the symplectic form on M, where n is a positive integer. The map

$$F(h,\, a,\, b,\, c) = (h\cos(nb),\, h\sin(nb)),$$

defines an integrable system with non-degenerate singularities. The fiber over any regular value of F is n copies of $S^1 \times S^1$ (see Example 4.10). Thus we have an elementary construction of an almost-toric system on a compact manifold which has only transversally elliptic singularities, and has a non-simply connected leaf space, a connected set of regular values, and non-connected fibers. The image $F(M)$ itself is an annulus, and the bifurcation set Σ_F is its boundary; hence vertical tangencies cannot be avoided by deformation, in agreement with Theorem 1.

We thank Thomas Baird for this example.

5. The image of an almost-toric system

In this section we study the structure of the image of an almost-toric system.

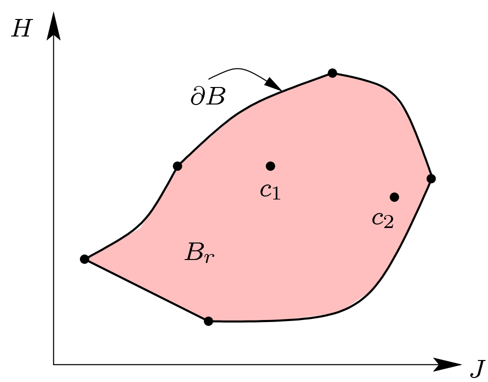

Figure 11: The image $B := F(M)$ of an almost-toric system $F = (J, H) : M \to \mathbb{R}^2$. The set of regular values of F is denoted by B_r. The marked dots $c_1,\, c_2$ inside of B represent singular values, corresponding to the focus-focus fibers. The set B_r is equal to B minus $\partial B \cup \{c_1,\, c_2\}$.

5.1. Images bounded by lower/upper semicontinuous graphs

We start with the following observation.

Lemma 5.1. *Let M be a connected smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse-Bott function with connected fibers. Then the set C_0 of index zero critical points of f is connected. Moreover, if $\lambda_0 := \inf f \geqslant -\infty$, the following hold:*

(1) *If $\lambda_0 > -\infty$ then $C_0 = f^{-1}(\lambda_0)$.*

(2) *If $\lambda_0 = -\infty$ then $C_0 = \varnothing$.*

Proof. The fiber over a point is locally path connected. If the point is critical this follows from the Morse-Bott Lemma and if the point is regular, this follows from the submersion theorem.

Let m be a critical point of index 0 of f, i.e., $m \in C_0$, and let $\lambda := f(m)$, $\Lambda := f^{-1}(\lambda)$. Let $\gamma\colon [0, 1] \to \Lambda$. Since Λ is connected and locally path connected, it is path connected. Let $\gamma\colon [0, 1] \to \Lambda$ be a continuous path starting at m. By the Morse-Bott Lemma, $\mathrm{im}(\gamma) \subset C_0$. Therefore, since Λ is path connected, $\Lambda \subseteq C_0$. Each connected component of C_0 is contained in some fiber of f and hence Λ is the connected component of C_0 that contains m. We shall prove that C_0 has one connected component.

Assume that there is a point m' such that $f(m') < \lambda$ and let $\delta\colon [0, 1] \to M$ be a continuous path from m to m'. Let

$$t_0 := \inf\{t > 0 \,|\, f(\delta(t)) < \lambda\}.$$

Then $f(\delta(t_0)) = \lambda$ and, by definition, for every $\alpha > 0$, there exists $t_\alpha \in [t_0, t_0 + \alpha]$ such that $f(\delta(t_\alpha)) < \lambda$.

Let $m_0 := \delta(t_0)$. Let U be a neighborhood of $\delta(t_0)$ in which we have the Morse-Bott coordinates given by the Morse-Bott Lemma centered at $\delta(t_0)$. For α small enough, $\delta([t_0, t_1]) \subset U$ and therefore for $t \in [t_0, t_1]$ we have that

$$f(\delta(t)) = \sum_{i=1}^{k} y_i^2(\delta(t)) + \lambda \geqslant \lambda,$$

which is a contradiction. $\square$

Note that the following result is strictly Morse-theoretic; it does not involve integrable systems. A version of this result was proven in [26, Theorem 3.4] in the case of integrable systems $F = (J, H)$ for which $J\colon M \to \mathbb{R}$ is both a proper map (hence $F\colon M \to \mathbb{R}^2$ is proper) and a momentum map for a Hamiltonian S^1-action. The version we prove here applies to smooth maps, which are not necessarily integrable systems.

Theorem 5.2. *Let M be a connected smooth four-manifold. Let $F = (J, H)\colon M \to \mathbb{R}^2$ be a smooth map. Equip $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ with the standard topology. Suppose that the component J is a non-constant Morse-Bott function with connected fibers. Let H^+, $H^-\colon J(M) \to \overline{\mathbb{R}}$ be the functions defined by $H^+(x) := \sup_{J^{-1}(x)} H$ and $H^-(x) := \inf_{J^{-1}(x)} H$. The functions H^+, $-H^-$ are lower semicontinuous. Moreover, if $F(M)$ is closed in $\mathbb{R}^2$ then H^+, $-H^-$ are upper semicontinuous (and hence continuous), and $F(M)$ may be described as*

$$F(M) = \mathrm{epi}(H^-) \cap \mathrm{hyp}(H^+). \tag{4}$$

In particular, $F(M)$ is contractible.

Proof. First we consider the case where $F(M)$ is not necessarily closed (Part 1). In Part 2 we prove the stronger result when $F(M)$ is closed.

Part 1. We do not assume that $F(M)$ is closed and prove that H^+ is lower semicontinuous; the proof that $-H^-$ is lower semicontinuous is analogous. Since, by assumption, J is non-constant, the interior set $\mathrm{int}(J(M))$ of $J(M)$ is non-empty. The set $\mathrm{int}(J(M))$ is an open interval (a, b) since M is connected and J is continuous. Lower semicontinuity of H^+ is proved first in the interior of $J(M)$ (case A) and then at the possible boundary (case B).

Case A. Let $x_0 \in \mathrm{int}(J(M))$ and $y_0 := H^+(x_0)$. Let $\epsilon > 0$. By the definition of supremum, there exists $\epsilon' > 0$ with $\epsilon' < \epsilon$ such that if $y_1 := y_0 - \epsilon'$ then $F^{-1}(x_0, y_1) \neq \varnothing$ (see Figure 12). Here we have assumed that $y_0 < +\infty$; if $y_0 = +\infty$, we just need to replace y_1 by an arbitrary large constant. Let $m \in F^{-1}(x_0, y_1)$. Then $J(m) = x_0$. Endow M with a Riemannian metric and, with respect to this metric, consider the gradient vector field ∇J of J. Let $t_0 > 0$ such that the flow $\varphi^t(m)$ of ∇J starting at $\varphi^0(m) = m$ exists for all $t \in (-t_0, t_0)$. Now we distinguish two cases.

A.1. Assume $\mathrm{d}J(m) \neq 0$. Since $\nabla J(m) \neq 0$, the set

$$\Lambda_{t_0} := \{J(\varphi(t)) \,|\, t \in (-t_0, t_0)\}$$

is a neighborhood of x_0.

Let B be the ball of radius ϵ centered at (x_0, y_1). Let $U := F^{-1}(B)$, which contains m. Let $t_0' \leqslant t_0$ be small enough such that $\varphi^t(m) \in U$ for all $|t| < t_0'$. The set $\Lambda_{t_0'}$ is a neighborhood of x_0, so there is $\alpha > 0$ such that $(x_0 - \alpha, x_0 + \alpha) \subseteq \Lambda_{t_0'}$.

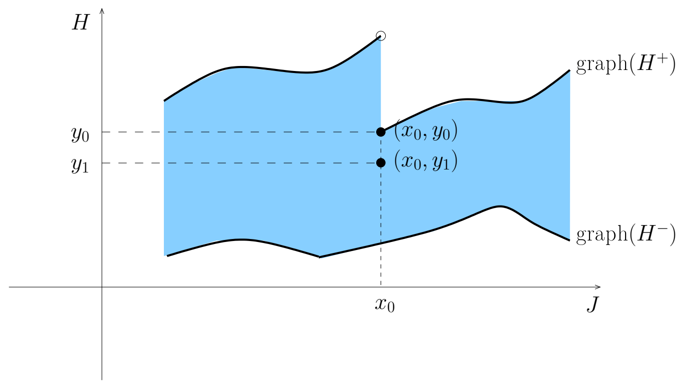

Figure 12: Lower semicontinuity of H^+.

Let $x \in \Lambda_{t_0'}$; there exists $|t| < t_0'$ such that $J(\varphi^t(m)) = x$ by definition of $\Lambda_{t_0'}$. Since $F(\varphi^t(m)) \in B$ we conclude that $y := H(\varphi^t(m)) \in (y_1 - \epsilon,\, y_1 + \epsilon)$, so $H^+(x) \geqslant y \geqslant y_1 - \epsilon$ for all x with $|x - x_0| < \alpha$.

Thus we get $H^+(x) \geqslant y_0 - 2\epsilon$ for all x with $|x - x_0| < \alpha$, which proves the lower semicontinuity.

A.2. Assume $\mathrm{d}J(m) = 0$. By Lemma 5.1, we conclude that m is not of index 0, for otherwise $J(m) = x_0$ would be a global minimum in $J(M)$ which contradicts the fact that, by assumption, $x_0 \in \mathrm{int}(J(M))$.

Thus, the Hessian of J has at least one negative eigenvalue and, therefore, there exists $t_0 > 0$ such that Λ_{t_0} is an open neighborhood of x_0 and we may then proceed as in Case A.1.

Hence H^+ is lower semicontinuous.

Case B. We prove here lower semicontinuity at a point x_0 in the topological boundary of $J(M)$. We may assume that $J(M) = [a,\, b)$ and that $x_0 = a$. By Lemma 5.1, $J^{-1}(a) = C_0$, where C_0 denotes the set of critical points of J of index 0. If $m \in J^{-1}(a)$ and U is a small neighborhood of m, it follows from the Morse-Bott lemma that $J(U)$ is a neighborhood of a in $J(M)$. Hence we may proceed as in Case A to conclude that H^+ is lower semicontinuous.

Part 2. Assuming that $F(M)$ is closed we shall prove now that H^+ is upper semicontinuous.

Suppose that H^+ is not upper semicontinuous at some point $x_0 \in J(M)$ (so we must have $H^+(x_0) < \infty$). Then there exists $\epsilon_0 > 0$ and a sequence $\{x_n\} \subset J(M)$ converging to x_0 such that $H^+(x_n) \geqslant H^+(x_0) + \epsilon_0$. First assume that $H^-(x_0)$ is finite. Since H^- is upper semicontinuous, by Part 1 of the proof, for x close to x_0 we have

$$H^-(x) \leqslant H^-(x_0) + \frac{\epsilon_0}{2}.$$

Thus we have

$$H^-(x_n) \leqslant H^-(x_0) + \frac{\epsilon_0}{2} \leqslant H^+(x_0) + \frac{\epsilon_0}{2} < H^+(x_0) + \epsilon_0 \leqslant H^+(x_n). \tag{5}$$

Let $y_0 \in (H^+(x_0) + \frac{\epsilon_0}{2}, H^+(x_0) + \epsilon_0)$. Because of (5), and since $H(J^{-1}(x_n))$ is an interval whose closure is $[H^-(x_n), H^+(x_n)]$ (remember that J has connected fibers and H is continuous), we must have $y_0 \in H(J^{-1}(x_n))$. Therefore $(x_n, y_0) \in F(M)$. Since $F(M)$ is closed, the limit of this sequence belongs to $F(M)$; thus $(x_0, y_0) \in F(M)$. Hence $y_0 \leqslant H^+(x_0)$, a contradiction.

If $H^-(x_0) = -\infty$, we just replace in the proof $H^-(x_0) + \frac{\epsilon_0}{2}$ by some constant A such that $A \leqslant H^+(x_0) + \frac{\epsilon_0}{2}$.

Hence H^+ is upper semicontinuous. Therefore H^+ is continuous. The same argument applies to H^-.

In order to prove (4), notice that for any $x \in J(M)$, we have the equality

$$\{x\} \times H(J^{-1}(x)) = F(M) \cap \{(x, y) \mid y \in \mathbb{R}\}. \tag{6}$$

Therefore, if $F(M)$ is closed, $\{x\} \times H(J^{-1}(x))$ must be closed and hence equal to $\{x\} \times [H^-(x), H^+(x)]$. The equality (4) follows by taking the union of the identity (6) over all $x \in J(M)$.

Finally, we show that $F(M)$ is contractible. Since $\mathbb{R} \cup \{\pm\infty\}$ is homeomorphic to a compact interval, $F(M)$ is homeomorphic to a closed subset of the strip $\mathbb{R} \times [-1, 1]$ by means of a homeomorphism g that fixes the first coordinate x. Thus, applying to (4) this homeomorphism, we get

$$g(F(M)) = \mathrm{epi}(h^-) \cap \mathrm{hyp}(h^+)$$

for some continuous functions h^+ and h^-. Then the map

$$g(F(M)) \times [0,1] \ni ((x, y), t) \mapsto (x, t(y - h^-(x))) \in \mathbb{R}^2$$

is a homotopy equivalence with the horizontal axis. Since g is a homeomorphism, we conclude that $F(M)$ is contractible. $\square$

5.2. Constructing Morse-Bott functions

We can give a stronger formulation of Theorem 5.2. First, recall that if Σ is a smoothly immersed 1-dimensional manifold in $\mathbb{R}^2$, we say that Σ *has no horizontal tangencies* if there exists a smooth curve $\gamma\colon I \subset \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(I) = \Sigma$ and $\gamma_2'(t) \neq 0$ for every $t \in I$. Note that Σ has no horizontal tangencies if and only if for every $c \in \mathbb{R}$ the 1-manifold Σ is transverse to the horizontal line $y = c$.

We start with the following result, which is of independent interest and its applicability goes far beyond its use in this paper.

We begin with a description of the structure of the set $\Sigma_F := F(\mathrm{Crit}(F))$ of critical values of an integrable systems $F : M \to \mathbb{R}^2$; as usual $\mathrm{Crit}(F)$ denotes the set of critical points of F.

Let $c_0 \in \Sigma_F$ and $B \subset \mathbb{R}^2$ a small closed ball centered at c_0. For each point $m \in F^{-1}(B)$ we choose a chart about m in which F has normal form (see Theorem 3.3). There are seven types of normal forms, as depicted in Figure 7. Since $F^{-1}(B)$ is compact, we can select a finite number of such chart domains that still cover $F^{-1}(B)$. For each such chart domain Ω, the set of critical values of $F|_\Omega$ is diffeomorphic to the set of critical values of one of the models described in Figure 7, which is either empty, an isolated point, an open curve, or up to four open curves starting from a common point. Since

$$\Sigma_F \cap B = F(\mathrm{Crit}(F) \cap F^{-1}(B)),$$

it follows that $\Sigma \cap B$ is a finite union of such models. This discussion leads to the following proposition.

Proposition 5.3. *Let (M, ω) be a connected symplectic four-manifold. Let $F : M \to \mathbb{R}^2$ be a non-degenerate integrable system. Suppose that F is a proper map. Then $\Sigma_F := F(\mathrm{Crit}(F))$ is the union of a finite number of stratified manifolds with 0 and 1 dimensional strata. More precisely, Σ_F is a union of isolated points and of smooth images of immersions of closed intervals (since F is proper, a 1-dimensional stratum must either go to infinity or end at a rank-zero critical value of F).*

Definition 5.4. Let $c \in \Sigma_F$. A vector $v \in \mathbb{R}^2$ is called *tangent to* Σ_F if there is a smooth immersion $\iota : \mathbb{R} \supset [0, 1] \to \Sigma_F$ with $\iota(0) = c$ and $\iota'(0) = v$.

Here ι is smooth on $[0, 1]$, when $[0, 1]$ is viewed as a subset of $\mathbb{R}$. Notice that a point c can have several linearly independent tangent vectors.

Definition 5.5. Let γ be a smooth curve in $\mathbb{R}^2$.

- If γ intersects Σ_F at a point c, we say that the intersection is *transversal* if no tangent vector of Σ_F at c is tangent to γ. Otherwise we say that c is a *tangency point*.
- Assume that γ is tangent to a 1-stratum σ of Σ_F at a point $c \in \sigma$. Near c, we may assume that γ is given by some equation $\varphi(x, y) = 0$, where $\mathrm{d}\varphi(c) \neq 0$.

  We say that γ has a *non-degenerate contact* with σ at c if, whenever $\delta : (-1, 1) \to \sigma$ is a smooth local parametrization of σ near c with $\delta(0) = c$, then the map $t \mapsto (\varphi \circ \delta)(t)$ has a non-degenerate critical point at $t = 0$.
- Every tangency point that is not a non-degenerate contact is called *degenerate* (this includes the case where γ is tangent to Σ_F at a point c which is the end point of a 1-dimensional stratum σ).

With this terminology, we can now see how a Morse function on $\mathbb{R}^2$ can give rise to a Morse-Bott function on M.

Theorem 5.6 (Construction of Morse-Bott functions). *Let (M, ω) be a connected symplectic four-manifold. Let $F := (J, H) \colon M \to \mathbb{R}^2$ be an integrable system with non-degenerate singularities (of any type, so this statement applies to hyperbolic singularities too) such that F is proper. Let $\Sigma_F \subset \mathbb{R}^2$ be the set of critical values of F, i.e., $\Sigma_F := F(\mathrm{Crit}(F))$.*

Let $U \subset \mathbb{R}^2$ be open. Suppose that $f : U \to \mathbb{R}$ is a Morse function whose critical set is disjoint from Σ_F and the regular level sets of f intersect Σ_F transversally or with non-degenerate contact.

Then $f \circ F$ is a Morse-Bott function on $F^{-1}(U)$.

Here by regular level set of f we mean a level set corresponding to a regular value of f.

Proof. Let $L := f \circ F$. Writing

$$\mathrm{d}L = (\partial_1 f)\mathrm{d}J + (\partial_2 f)\mathrm{d}H,$$

we see that if m is a critical point of L then either $c = F(m)$ is a critical point of f (so $\partial_1 f(c) = \partial_2 f(c) = 0$), or $\mathrm{d}J(m)$ and $\mathrm{d}H(m)$ are linearly dependent (which means $\mathrm{rank}(\mathrm{T}_m F) < 2$). By assumption, these two cases are disjoint: if $c = F(m)$ is a critical point of f, then $c \notin \Sigma_F$ which means that m is a regular point of F.

Thus $\mathrm{Crit}(L) \subset F^{-1}(\mathrm{Crit}(f)) \sqcup \mathrm{Crit}(F)$ is a disjoint union of two closed sets. Since Hausdorff manifolds are normal, these two closed sets have disjoint open neighborhoods. Thus $\mathrm{Crit}(L)$ is a submanifold if and only if both sets are submanifolds, which we prove next.

Study of $F^{-1}(\mathbf{Crit}(f))$. Let $m_0 \in M$ and $c_0 = F(m_0)$. We assume that c_0 is a critical point of f, i.e., $\mathrm{d}f(c_0) = 0$. By hypothesis, $\operatorname{rank} \mathrm{d}_{m_0} F = 2$. Since the rank is lower semicontinuous, there exists a neighborhood Ω of m_0 in which $\operatorname{rank} \mathrm{d}_m F = 2$ for all $m \in \Omega$. Thus, on Ω, $L = f \circ F$ is critical at a point m if and only if $F(m)$ is critical for f. Since f is a Morse function, its critical points are isolated; therefore we can assume that the critical set of L in Ω is precisely $F^{-1}(c_0) \cap \Omega$.

Since $F^{-1}(c_0)$ is a compact regular fiber (because F is proper), it is a finite union of Liouville tori (this is the statement of the action-angle theorem; the finiteness comes from the fact that each connected component is isolated). In particular, $F^{-1}(\mathrm{Crit}(f))$ is a submanifold and we can analyze the non-degeneracy component-wise.

Given any $m \in F^{-1}(c_0)$, the submersion theorem ensures that J and H can be seen as a set of local coordinates of a transversal section to the fiber $F^{-1}(c_0)$. Thus, using the Taylor expansion of f of order 2, we get the 2-jet of $L - L(m_0)$:

$$
\begin{aligned}
L(m) - L(m_0) = \frac{1}{2} &\operatorname{Hess} f(m_0)(J(m) - J(m_0), H(m) - H(m_0))^2 \\
&+ \text{terms of order } 3,
\end{aligned}
$$

where

$$
\begin{aligned}
\text{(7)} \qquad &\operatorname{Hess} f(m_0)(J(m) - J(m_0), H(m) - H(m_0))^2 \\
&:= (\partial_1^2 f)(m_0)(J(m) - J(m_0))^2 \\
&\quad + 2(\partial_{1,2}^2 f)(m_0)(J(m) - J(m_0)))(H(m) - H(m_0)) \\
&\quad + (\partial_2^2 f)(m_0)(H(m) - H(m_0))^2.
\end{aligned}
$$

Again, since (J, H) are taken as local coordinates, we see that the transversal Hessian of L in the (J, H)-variables is non-degenerate, since $\operatorname{Hess} f(m_0)$ is non-degenerate by assumption (f is Morse).

Thus we have shown that $F^{-1}(\mathrm{Crit}(f))$ is a smooth submanifold (a finite union of Liouville tori), transversally to which the Hessian of L is non-degenerate.

Study of $\mathbf{Crit}(L) \cap \mathbf{Crit}(F)$. Let $m_0 \in M$ be a critical point of F and $c_0 = F(m_0)$. By assumption, c_0 is a regular value of f. Thus, there exists an open neighborhood V of c_0 in $\mathbb{R}^2$ that contains only regular values of f. Therefore, the critical set of L in $F^{-1}(V)$ is included in $\mathrm{Crit}(F) \cap F^{-1}(V)$. In what follows, we choose V with compact closure in $\mathbb{R}^2$ and admitting a neighborhood in the set of regular values of f.

Case 1: rank 1 critical points. There are two types of rank 1 critical points of F: elliptic and hyperbolic. By the Normal Form Theorem 3.3, there are symplectic coordinates (x_1, x_2, ξ_1, ξ_2) in in a chart about m_0 in which F takes the form

$$F = g(\xi_1, q),$$

where q is either $x_2^2 + \xi_2^2$ (elliptic case) or $x_2\xi_2$ (hyperbolic case), and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a local diffeomorphism of a neighborhood of the origin to a neighborhood of $F(m_0)$, $g(0) = F(m_0)$.

We see from this that $\Sigma^V := \mathrm{Crit}\left(F|_{F^{-1}(V)}\right)$ is the 1-dimensional submanifold $\{g(t,0) \mid |t| \text{ small}\}$.

Now consider the case when the level sets of f in V (which are also 1-dimensional submanifolds of $\mathbb{R}^2$) are transversal to this submanifold. We see that the range of $\mathrm{d}_{m_0}F$ is directed along the first basis vector e_1 in $\mathbb{R}^2$, which is precisely tangent to Σ^V. Hence $\mathrm{d}f$ cannot vanish on this vector and hence

$$0 \neq \mathrm{d}_{F(m_0)} f \circ \mathrm{d}_{m_0} F = \mathrm{d}_{m_0} L.$$

This shows that L has no critical points in $F^{-1}(V)$.

Now assume that there is a level set of f in V that is tangent to Σ^V with non-degenerate contact at the point $g(0,0)$. The tangency gives the equation $\mathrm{d}_{F(m_0)} f \cdot (\mathrm{d}_{(0,0)} g(e_1)) = 0$. Since $L = (f \circ g)(\xi_1, q)$, the equation of $\mathrm{Crit}(L)$ is

$$\frac{\partial (f \circ g)}{\partial \xi_1} = 0 \quad \text{and} \quad \frac{\partial (f \circ g)}{\partial q} \mathrm{d}q = 0. \tag{8}$$

Since $\mathrm{d}f \neq 0$ on V and g is a local diffeomorphism, we have $\mathrm{d}(f \circ g) \neq 0$ in a neighborhood of the origin. But the contact equation gives

$$\frac{\partial (f \circ g)}{\partial \xi_1}(0,0) = 0$$

so, taking V small enough, we may assume that $\frac{\partial (f \circ g)}{\partial q}$ does not vanish. Hence the second condition in (8) is equivalent to $\mathrm{d}q = 0$, which means $x_2 = \xi_2 = 0$ (and hence $q = 0$).

By definition, the contact is non-degenerate if and only if the function $t \mapsto f(g(t,0))$ has a non-degenerate critical point at $t = 0$. Therefore, by the implicit function theorem, the first equation

$$\frac{\partial(f \circ g)}{\partial \xi_1}(\xi_1, 0) = 0$$

has a unique solution $\xi_1 = 0$. Thus, the critical set of L is of the form $\{(x_1, \xi_1 = 0, x_2 = 0, \xi_2 = 0)\}$, where x_1 is arbitrary in a small neighborhood of the origin; this shows that the critical set of L is a smooth 1-dimensional submanifold.

It remains to check that the Hessian of L is transversally non-degenerate. Of course, we take (ξ_1, x_2, ξ_2) as transversal variables and we write the Taylor expansion of L, for any $m \in \mathrm{Crit(L)}$:

$$L = L(m) + \mathcal{O}(x_1) + \frac{\partial(f \circ g)}{\partial q} q + \frac{1}{2} \frac{\partial^2(f \circ g)}{\partial \xi_1^2} \xi_1^2 + \mathcal{O}((\xi_1, x_2, \xi_2)^3). \tag{9}$$

We know that $\frac{\partial(f \circ g)}{\partial q} \neq 0$ and, by the non-degeneracy of the contact,

$$\frac{\partial^2(f \circ g)}{\partial \xi_1^2} \neq 0.$$

Recalling that $q = x_2\xi_2$ or $q = x_2^2 + \xi_2^2$, we see that the (ξ_1, x_2, ξ_2)-Hessian of L is indeed non-degenerate.

Case 2: rank 0 critical points. There are four types of rank 0 critical points of F: elliptic-elliptic, focus-focus, hyperbolic-hyperbolic, and elliptic-hyperbolic, giving rise to four subcases. From the normal form of these singularities (see Theorem 3.3), we see that all of them are isolated from each other. Thus, since F is proper, the set of rank 0 critical points of F is finite in $F^{-1}(V)$.

Again, let m_0 be a rank 0 critical point of F and $c_0 := F(m_0)$.

(a) *Elliptic-elliptic subcase.*

In the elliptic-elliptic case, the normal form is

$$F = g(q_1, q_2),$$

where $q_i = (x_i^2 + \xi_i^2)/2$. The critical set of F is the union of the planes $\{z_1 = 0\}$ and $\{z_2 = 0\}$ (we use the notation $z_j = (x_j, \xi_j)$). The corresponding critical values in V is the set

$$\Sigma^V := \{g(x = 0, y \geqslant 0)\} \cup \{g(x \geqslant 0, y = 0)\}$$

(the g-image of the boundary of the closed positive quadrant).

The transversality assumption on f amounts here to saying that the level sets of f in a neighborhood of c_0 intersect Σ^V transversally; in other words, the level sets of $h := f \circ g$ intersect the boundary of the positive quadrant transversally. Up to further shrinking of V, this amounts to requiring $\mathrm{d}_z h(e_1) \neq 0$, $\mathrm{d}_z h(e_2) \neq 0$ for all $z \in g^{-1}(V)$, where (e_1, e_2) is the standard $\mathbb{R}^2$-basis.

Any critical point m of F different from m_0 is a rank 1 elliptic critical point. Since the level sets of f don't have any tangency with Σ^V, we know from the rank 1 case above, that m cannot be a critical point of L. Hence m_0 is an isolated critical point for L.

The Hessian of L at m_0 is calculated via the normal form: it has the form $aq_1 + bq_2$, with $a = \mathrm{d}_0 h(e_1)$ and $b = \mathrm{d}_0 h(e_2)$. The Hessian determinant is $a^2 b^2$. The transversality assumption implies that both a and b are non-zero which means that the Hessian is non-degenerate.

(b) *Focus-focus subcase.*

The focus-focus critical point is isolated, so we just need to prove that the Hessian of L is non-degenerate. But the 2-jet of L is

$$\begin{aligned} L(m) - L(m_0) = {} & (\partial_1 f)(F(m_0)(J(m) - J(m_0)) \\ & + (\partial_2 f)(F(m_0)(H(m) - H(m_0)) + \text{terms of order 3}. \end{aligned}$$

Thus, in normal form coordinates (see Theorem 3.3), as in the previous case, it has the form $aq_1 + bq_2$, where this time q_1 and q_2 are the focus-focus quadratic forms given in Theorem 3.3(iii). The Hessian determinant is now $(a^2 + b^2)^2$, which does not vanish.

(c) *Hyperbolic-hyperbolic subcase.*

Here the local model for the foliation is $q_1 = x_1 \xi_1$, $q_2 = x_2 \xi_2$. However, the formulation $F = g(q_1, q_2)$ may not hold; this is a well-known problem for hyperbolic fibers. Nevertheless, on each of the 4 connected components of of $\mathbb{R}^4 \setminus (\{x_1 = 0\} \cup \{x_2 = 0\})$, we have a diffeomorphism g_i, $i = 1, 2, 3, 4$ such that $F = g_i(q_1, q_2)$. These four diffeomorphisms agree up to a flat map at the origin (which means that their Taylor series at $(0, 0)$ are all the same).

Thus, the critical set of F in these local coordinates is the union of the sets $\{q_1 = 0\}$ and $\{q_2 = 0\}$: this is the union of the four coordinate hyperplanes in $\mathbb{R}^4$. The corresponding set of critical values in V is the image of the coordinate axes:

$$\Sigma^V := \bigcup_{i=1,2,3,4} \{g_i(0,y)\} \cup \{g_i(x,0)\},$$

where x and y both vary in a small neighborhood of the origin in $\mathbb{R}$.

For each i we let $h_i := f \circ g_i$. As before, the transversality assumption says that the values $\mathrm{d}_0 h_i(e_1)$ and $\mathrm{d}_0 h_i(e_2)$ (which don't depend on $i = 1,2,3,4$ at the origin of $\mathbb{R}^2$) don't vanish in V. Thus, as in the elliptic-elliptic case, the level sets of f don't have any tangency with Σ^V. Hence, no rank 1 critical point of F can be a critical point of L, which shows that m_0 is thus an isolated critical point of L.

The Hessian determinant of $aq_1 + bq_2$ is again a^2b^2 with $a \neq 0$ and $b \neq 0$; thus the Hessian of L at m_0 is non-degenerate.

(d) *Hyperbolic-elliptic subcase.*

We still argue as above. However, the Hessian determinant in this case is $-a^2b^2 \neq 0$.

Summarizing, we have proved that rank 0 critical points of F correspond to isolated critical points of L; all of them non-degenerate.

Putting together the discussion in the rank 1 and 0 cases, we have shown that the critical set of L consists of isolated non-degenerate critical points and isolated 1-dimensional submanifolds on which the Hessian of L is transversally non-degenerate. This means that L is a Morse-Bott function. $\square$

5.3. Contact points and Morse-Bott indices

Since we have calculated all the possible Hessians, it is easy to compute the various indices that can occur. We shall need a particular case, for which we introduce another condition on f.

Definition 5.7. Let (M, ω) be a connected symplectic four-manifold. Let $F : M \to \mathbb{R}^2$ be an almost-toric system with critical value set Σ_F. A smooth curve γ in $\mathbb{R}^2$ is said to have an *outward contact* with $F(M)$ at a point $c \in F(M)$ when there is a small neighborhood of c in which the point $\{c\}$ is the only intersection of γ with $F(M)$.

In the proof below we give a characterization in local coordinates.

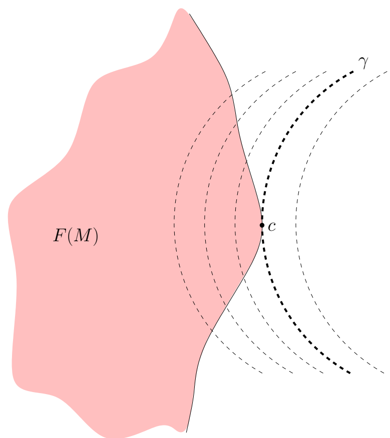

Figure 13: An outward contact point.

Proposition 5.8. *Let (M, ω) be a connected symplectic four-manifold. Let $F : M \to \mathbb{R}^2$ be an almost-toric system with critical value set Σ_F. Let f be a Morse function defined on an open neighborhood of $F(M) \subset \mathbb{R}^2$ such that*

(i) *The critical set of f is disjoint from Σ_F;*

(ii) *f has no saddle points in $F(M)$;*

(iii) *the regular level sets of f intersect Σ_F transversally or have a non-degenerate outward contact with $F(M)$. (See Definitions 5.5 and 5.7.)*

Then $f \circ F : M \to \mathbb{R}$ is a Morse-Bott function with all indices and co-indices equal to 0, 2, or 3.

Proof. Because of Theorem 5.6, we just need to prove the statement about the indices of f. At points of $F^{-1}(\mathrm{Crit}(f))$, we saw in (7) that the transversal Hessian of $f \circ F$ is just the Hessian of f. By assumption, f has no saddle points, so its (co)index is either 0 or 2. We analyze the various possibilities at points of $\mathrm{Crit}(F)$. There are two possible rank 0 cases for an almost-toric system: elliptic-elliptic and focus-focus. At such points, the Hessian determinant is positive (see Theorem 3.3), so the index and co-index are even.

In the rank 1 case, for an almost-toric system, only transversally elliptic singularities are possible. We are interested in the case of a tangency (otherwise $f \circ F$ has no critical point). The Hessian is computed in (9) and we use below the same notations. The level set of f through the tangency point is given by $f(x,y) = f(g(0,0))$. We switch to the coordinates $(\xi_1, q) = g^{-1}(x,y)$, where the local image of F is the half-space $\{q \geqslant 0\}$. Let

$$h = f \circ g - f(g(0,0)).$$

The level set of f is $h(\xi_1, q) = 0$, and h satisfies $\mathrm{d}_0 h \neq 0$, $\frac{\partial h}{\partial \xi_1}(0,0) = 0$, $\frac{\partial^2 h}{\partial \xi_1^2}(0,0) \neq 0$ (this is the non-degeneracy condition in Definition 5.5). By the implicit function theorem, the level set $\{h = 0\}$ near the origin is the graph $\{(\xi_1, q) \mid q = \varphi(\xi_1)\}$ of a function φ, where

$$\varphi'(0) = 0, \qquad \varphi''(0) = \frac{-\frac{\partial^2 h}{\partial \xi_1^2}(0,0)}{\frac{\partial h}{\partial q_2}(0,0)} .$$

This level set has an outward contact if and only if $\varphi''(0) < 0$ or, equivalently, $\frac{\partial^2 h}{\partial \xi_1^2}(0,0)$ and $\frac{\partial h}{\partial q_2}(0,0)$ have the same sign. From (9) we see that the index and coindex can only be 0 or 3. $\square$

5.4. Proof of Theorem 2 and Theorem 5

We conclude by proving the two theorems in the introduction. Both will rely on the following result. In the statement below, we use the stratified structure of the bifurcation set of a non-degenerate integrable system, as given by Proposition 5.3.

Proposition 5.9. *Let (M, ω) be a connected symplectic four-manifold. Let $F\colon M \to \mathbb{R}^2$ be an almost-toric system such that F is proper. Denote by Σ_F the bifurcation set of F. Assume that there exists a diffeomorphism $g\colon F(M) \to \mathbb{R}^2$ onto its image such that:*

(i) *$g(F(M))$ is included in a proper convex cone $C_{\alpha,\beta}$ (see Figure 3).*

(ii) *$g(\Sigma_F)$ does not have vertical tangencies (see Figure 1).*

Write $g \circ F = (J, H)$. Then J is a Morse-Bott function with connected level sets.

Proof. Let $\tilde{F} := g \circ F$. The set of critical values of $\tilde{F}$ is $\tilde{\Sigma} = g(\Sigma_F)$. We wish to apply Proposition 5.8 to this new map $\tilde{F}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate: $f(x,y) = x$, so that $f \circ \tilde{F} = J$. Since f has no critical points, it satisfies the hypotheses (i) and (ii) of Proposition 5.8. The regular levels sets of f are the vertical lines, and the fact that $\tilde{\Sigma}$ has no vertical tangencies means that the regular level sets of f intersect $\tilde{\Sigma}$ transversally. Thus the last hypothesis (iii) of Proposition 5.8 is fulfilled and we conclude that J is a Morse-Bott function whose indices and co-indices are always different from 1.

Now, since $\tilde{F}$ is proper, the fact that $\tilde{F}(M)$ is included in a cone $C_{\alpha,\beta}$ (by hypothesis (i)), easily implies that $f \circ \tilde{F}$ is proper. Thus, using Proposition 4.5, we conclude that J has connected level sets. □

Proof of Theorem 2. From Proposition 5.9, we conclude that J has connected level sets. It is enough to apply Theorem 4.7 to conclude that $\tilde{F}$ (and thus F) has connected fibers. □

Proof of Theorem 5. Using again Proposition 5.9, we conclude that J is a Morse-Bott function with connected level sets. By the definition of an integrable system, J cannot be constant (its differential would vanish everywhere). Thus we can apply Theorem 5.2, which yields the desired conclusion. □

It turns out that even in the compact case, Theorem 2 has quite a striking corollary, which we stated as Theorem 3 in the introduction.

Proof of Theorem 3. The last two cases (a disk with two conic points and a polygon) can be transformed by a diffeomorphism, as in Theorem 1, to remove vertical tangencies, and hence the theorem implies that the fibers of F are connected.

For the first two cases, we follow the line of the proof of Theorem 1. The use of Proposition 5.8 is still valid for the same function $f(x, y) = x$, even though now the level sets of f can be tangent to Σ_F. Indeed, one can check that, in the present case, only non-degenerate outward contacts occur. Then one can bypass Proposition 5.9 and apply directly Proposition 4.5. Therefore the conclusion of the theorem still holds. □

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