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Symplectic packings in dimension 4 and singular curves

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The main goal of this paper is to give constructive proofs of several existence results for symplectic embeddings. The strong relation between symplectic packings and singular symplectic curves, which can be derived from McDuff's inflations on the blow-ups, is revisited through a new inflation technique that lives at the level of the manifold. As an application, we explain constructions of maximal symplectic packings of $\mathbb{P}^2$ by 6, 7 or 8 balls.

1. Introduction

In [8], M. Gromov shows a striking relation between curves and symplectic embeddings. Namely, the symplectic curves may give obstructions to symplectic embeddings beyond the volume constraint. For packings (i.e. embeddings of disjoint balls), the basic idea is to produce symplectic curves through the centers of the balls of the packing, within a prescribed homology class. Such a curve automatically gives an obstruction to the size of the balls. In [13], McDuff and Polterovich proved a converse statement for less than nine balls in $\mathbb{P}^2$: in particular, the only symplectic obstructions are given by such curves. This result hints at an even deeper relation between packings and symplectic curves, which has been mostly confirmed by McDuff via her blow-up and inflation techniques [9–11]. For instance, the following theorem follows from these techniques (see also [3]):

Theorem 1. *Let (M^4,ω) be a symplectic manifold with rational class ($[\omega]\in H^2(M,\mathbb{Q})$). Then there is a symplectic packing by closed disjoint balls*

$$\coprod_{i=1}^{p}\overline{B^4(a_i)}\overset{\omega}{\hookrightarrow} M \qquad (a_i\in\mathbb{Q})$$

if and only if there exists an irreducible symplectic curve Σ Poincaré dual to $k[\omega]$, with p nodal singularities of multiplicities $(ka_1+1,\dots,ka_p+1)$.

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It will be clear through this text that these "+1" can mostly be disregarded in a first approximation, the important point being the linear growth rate (see also Theorem 4). The rough idea is that symplectic blow-up and blow-down establish a correspondence between symplectic forms on blow-ups and existence of packings, while the inflation technique - based on curves of the blow-up, hence on singular curves of M - produces symplectic forms on the blow-up. This strategy works well for theoretic result, but not for constructions. The main lack for effectiveness lies in the above mentioned correspondence, which is far from being explicit. Several works have therefore been concerned with explicit constructions of symplectic packings, with completely independent techniques coming from toric geometry [21, 23, 24]. On this side of the story, the "obstruction curves" for the packings - the symplectic curves responsible for the obstruction - are completely discarded. In short, inflation provides existence results from the obstruction curves but fails to be effective, while the toric constructions hide one symplectic aspect of the packings, since they do not describe the obstruction curves at all. The main object of this paper is to reprove Theorem 1 in an effective way: given a symplectic curve with prescribed singularity, we construct the desired symplectic packing. The idea is to use Liouville vector fields in order to "inflate" directly in the manifold rather than on the blow-up. This more direct approach leads to a more accurate version of Theorem 1 (see Section 2) and allows for instance to construct maximal symplectic packings of $\mathbb{P}^2$ by six, seven and eight balls (up to five balls, the constructions are already available, see [17]).

Theorem 2. *There exist symplectic packings of* $\mathbb{P}^2$ *by six, seven and eight open balls of capacity* $2/5$, $3/8$ *and* $6/17$ *respectively (the maximal capacities). Each of these open balls intersects the predicted obstruction curve along a finite number of Hopf discs.*

Of course, we will not only prove this theorem but also explain the constructions of these maximal packings. The boundary regularity of these packings will be also briefly addressed.

Close to the packing problem, the question whether an ellipsoid embeds into a symplectic manifold has also been thoroughly considered. The question is really about flexibility of symplectic embeddings: how much can such a map fold a Euclidean domain such as an ellipsoid [15, 20]? Again, McDuff used the inflation process, this time in successive blow-ups to get the optimal results [12]: in a certain class of symplectic manifolds (containing $\mathbb{P}^2$ or the 4-ball), embedding an ellipsoid is equivalent to some specific ball packing problem. Again, this answer is of theoretical nature, and does not provide constructions, even from ball packings. The next result is in the wake of Theorem 1 and mostly follows from [12].

Theorem 3. *Let (M^4, ω) be a symplectic manifold with rational class. Then the closed ellipsoid*

$$\overline{E(a,b)} := \tau \overline{E(p,q)} \quad (p, q \in \mathbb{Z},\ \tau \in \mathbb{Q},\ \gcd(p,q) = 1)$$

symplectically embeds into M if and only if there exists an irreducible symplectic curve Σ, Poincaré dual to $k[\omega]$, with an ordinary singular point, with local symplectic model

$$\prod_{j=1}^{k\tau} (z_1^q - \alpha_j z_2^p), \qquad (\alpha_j \in \mathbb{C}).$$

The existence of an ellipsoid embedding also depends on the existence of a singular symplectic curve, now with a multi-cusp. The number of branches and the singularity type of the cusps are, respectively, responsible for the size and the shape of the ellipsoid. Again, our aim is to give an effective proof. It will also be clear that ellipsoid packings can be considered, and simply need curves with several singularities.

The paper is organized as follows. First, we give more precise versions of Theorem 1, which will be useful in practice, for instance to prove Theorem 2. Then we explain the main idea of the paper by sketching a proof for Theorem 1. In Section 4, we explain some properties and constructions of Liouville forms that will be needed to perform our inflations. Sections 5 and 6 are devoted to the proofs of the theorems on balls packings and ellipsoid embeddings, respectively, letting aside technical assertions about Donaldson's method. We deal with this last point in Section 7 and end the paper with some remarks and questions.

Notations. We adopt the following conventions throughout this paper:

- *All* angles will take values in $\mathbb{R}/\mathbb{Z}$. In other terms an angle 1 is a full turn in the plane, and the integral of the form $d\theta$ over a circle around the origin in the plane is 1.
- The standard symplectic form on $\mathbb{C}^2 = \mathbb{R}^4$ is $\omega_{\text{st}} := \sum dr_i^2 \wedge d\theta_i$, where (r_i, θ_i) are polar coordinates on the plane factors. With this convention, the euclidean ball of radius 1 has capacity 1.
- A Liouville form λ of a symplectic form ω is a one-form satisfying $\omega = -d\lambda$. The standard Liouville form on the plane is $\lambda_{\text{st}} := -r^2 d\theta$.

- A symplectic ball or ellipsoid is the image of a Euclidean ball or ellipsoid in $\mathbb{C}^n$ by a symplectic embedding.
- The Hopf discs of a Euclidean ball in $\mathbb{C}^2$ are its intersections with complex lines.
- $\mathcal{E}(a,b)$ denotes the 4-dimensional ellipsoid $\{a^{-1}|z|^2 + b^{-1}|w|^2 < 1\} \subset \mathbb{C}^2(z,w)$. Because of our normalizations, its Gromov width is $\min(a,b)$.
- When there is no ambiguity for the symplectic form ω on a manifold M, we often abbreviate $(M, \tau\omega)$ by τM.

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2. More precise statements

In this paragraph, we present three variations on Theorem 1, where we either remove a hypothesis (such as the irreducibility) or get slightly more precise results (constructing open objects of maximal size, for instance). These precisions may not be of fundamental importance, but they will be useful in practice. The first remark is that irreducibility can be omitted for cheap. In the theorem below, an n-cross of size a refers to a figure composed of n symplectic discs of area a which all intersect transversally at exactly one point. A symplectic model is simply the intersection of $B^4(a)$ with n complex lines in $\mathbb{C}^2$, which we call a standard n-cross.

Theorem 4. *Let (M^4, ω) be a rational symplectic manifold.*

i) *If there exists a symplectic packing by closed balls*

$$\overline{B(a_1)} \sqcup \cdots \sqcup \overline{B(a_p)} \hookrightarrow M, \qquad a_i \in \mathbb{Q},$$

then for all sufficiently large k for which the ka_i are integers, there exists a symplectic irreducible curve Poincaré dual to $k[\omega]$ in M, with exactly p nodes of multiplicities $(ka_1, \ldots, ka_p)$.

ii) *Given a symplectic curve Σ Poincaré dual to $k[\omega]$ with p nodes of multiplicities $(a_1k_1, \ldots, a_pk_p)$, one can construct a symplectic packing by open balls*

$$B(a_1) \sqcup \cdots \sqcup B(a_p) \hookrightarrow M$$

provided that $k_i \geq k$ *for each* i *and that the curve* Σ *contains* p *disjoint closed* k_ia_i*-crosses of size* a_i *around each node. When* Σ *is irreducible and* $k_i = k$*, this last condition is equivalent to the volume obstruction.*

Theorem 4 obviously implies Theorem 1 (apply Theorem 4 to a slightly larger ball to get the "+1"). Notice that when only the interior of the crosses are disjoint, Theorem 4 ii) allows to construct balls of size $a_i - \varepsilon$ for arbitrarily small ε. The next variation is concerned with the construction part ii) only, and states that we can even take $\varepsilon = 0$. This apparently innocent precision is rather expensive for the proof: it needs the introduction of the notion of a tame Liouville form and some analysis (not very difficult however). This is the reason why we state it separately: the proof will be given independently, so that the reader can discard this technical part at first. The main purpose of this precision is to allow for constructions of maximal packings by open balls of maximal size, and not only by balls of approximately optimal size. In my opinion, the importance of this point is of conceptual nature. The impossibility of reaching the limit size would mean that a deep rigidity phenomenon appears, which would need an explanation - recall that prior to the present paper, the only available proof for the *existence* of maximal packings by open balls in $\mathbb{P}^2$ relied on a deep result on symplectic isotopies [11]. Theorem 5 ensures that such a phenomenon does not happen in general, and that nothing deep hides here.

Theorem 5. *Let* (M, ω) *be a rational symplectic manifold. From a symplectic curve* Σ *Poincaré dual to* $k[\omega]$ *with* p *nodes of multiplicities* $(k_1a_1, \dots, k_pa_p)$ *one can construct a* $\mathcal{C}^1$*-smooth symplectic packing by* p *open balls of capacity* a_i*, provided that* $k_i \geq k$ *for all* i *and that* Σ *contains* p *disjoint* **open** k_ia_i*-crosses of size* a_i *around each node.*

The last variation we give concerns the rationality hypothesis. Classically, the main importance of this hypothesis is the existence of symplectic polarizations in the sense of Biran [4] (curves Poincaré dual to a multiple of the symplectic form). Of course, no such curve exists if the symplectic form is irrational, but a singular notion of polarization was defined in [18].

Definition 2.1. A polarization of a symplectic manifold (M, ω) is a union of weighted symplectic curves $\mathbf{\Sigma} := \{(\Sigma_l, \tau_l),\ l = 1, \dots, n\}$ $(\tau_l \in \mathbb{R}^+)$, such that

$$\sum_{l=1}^{n} \tau_l \mathrm{PD}(\Sigma_l) = [\omega],$$

the intersections between different curves are positive, and the singularities of each curve have the symplectic type of a complex singularity (these last two conditions can be replaced by assuming J-holomorphicity for Σ, but almost complex curves play no role here). The curve $\Sigma := \cup\Sigma_i$ will be called the total curve of the polarization.

It might be worth making clear from the begining that we will be only interested in very specific singularities: crosses and multi-cusps. Although the theorems of this paper are easier to state and understand in the rational setting, this definition makes the rationality hypothesis completely irrelevant.

Theorem 6. *Let (M, ω) be a symplectic 4-manifold. Then there is a symplectic packing by closed disjoint balls*

$$\coprod_{i=1}^{p} \overline{B^4(a_i)} \overset{\omega}{\hookrightarrow} M$$

if and only if there exists a polarization $(\Sigma_l, \tau_l)_{l=1,\dots,n}$ of M with the following properties:

- *Its total curve Σ has p nodes $(x_1, \dots, x_p)$ of multiplicities k^i ($i \in [1, \dots, p]$). We also denote by k_l^i the number of branches of Σ_l through x_i (so $\sum k_l^i = k^i$).*
- *Σ possesses p disjoint closed k^i-crosses of size a_i centered at x_i.*
- *For each i, $\sum k_l^i \tau_l > a_i$.*

It will be clear from the proof that the same precision for the construction of open balls from open crosses will hold also in this situation. Moreover, all these variations also hold for the problem of ellipsoid embeddings.

3. Sketch of the proof

Before going into proofs, let us illustrate the basic idea of the paper by explaining (not proving) theorem 1 for one ball.

i). In one direction, the argument is the following. Assume you have a ball B of capacity a in M. We wish to prove that for k sufficiently large, there is a symplectic curve, Poincaré dual to $k[\omega]$ whose intersection with B is exactly k Hopf discs. Now by Donaldson's result, there are curves Poincaré dual to $k\omega$ for k large. The observation is that the work is almost done: These curves naturally pass through ∂B along ka Hopf circles, because we know (from the McDuff blow-up process) that we can think of ∂B as a curve of symplectic area a, whose points correspond to the Hopf circles of ∂B. Now cutting out the part of this curve inside the ball, and pasting in the ka corresponding Hopf discs gives the desired singular curve.

ii). Conversely, given an irreducible curve Σ Poincaré dual to $k[\omega]$ with exactly one singularity, composed of ka transverse branches, we need to construct a ball of capacity a. The construction relies on the three following observations.

- First, provided that a satisfies the volume constraint $a^2/2 < \mathrm{Vol}\, M$, we can find a ka-cross X of size a in Σ (ka discs in Σ of area a, which intersect one another exactly at the singular point). Indeed,

$$\mathcal{A}_\omega(\Sigma) = \int_\Sigma \omega \overset{\text{PD}}{=} \int_M \omega \wedge k\omega = 2k\mathrm{Vol}\, M > ka^2 = ka \cdot a.$$

  By a standard neighbourhood theorem, there exists an embedding φ of a neighbourhood of the cross $X_{ka} := \cup \Delta_i := \cup\{z = \alpha_i w\} \cap B^4(a)$ into a neighbourhood of the cross X in M. The desired ball will be obtained by inflating this standard neighbourhood of the cross in M through well-chosen Liouville vector fields.

- Next, there is a (contracting) Liouville vector field on $B^4(a)\backslash X_{ka}$, which points away from the Δ_i and whose flow, restricted to any neighbourhood of X_{ka}, but considered for all positive times (when it is defined), recovers the whole of $B^4(a)$.

- Finally, there is a Liouville vector field on $M\backslash\Sigma$, also pointing outwards along Σ. Since M is assumed to be closed, this vector field is defined for all positive times.

The construction of the embedding then goes as follows. Take any point in $B^4(a)$. From this point, follow negatively the flow of the Liouville vector field, until you reach the domain of definition of the local embedding φ. Note τ the amount of time you had to flow. Then use φ to send the point to M, and flow positively along the Liouville vector field in M for time τ. Provided this map is well-defined, it is defined on $B^4(a)$ by the second point above, and it is symplectic. For this map to be well-defined, we need that the Liouville vector field in M is an extension of the push-forward of the Liouville vector field in $B^4(a)$ by φ. This is a cohomological constraint which involves the residue of the corresponding Liouville forms, and which is easy to deal with.

4. Generalities on Liouville forms

As we explained in the previous paragraph, the central objects of this paper are Liouville forms, on the manifolds and on the objects we wish to embed. The aim of the present section is to present the main features and examples of these Liouville forms which we will need subsequently.

4.1. Residues of a Liouville form

Consider a symplectic manifold (M, ω) equipped with a polarization $\mathbf{\Sigma} = \cup(\Sigma_i, \tau_i)_{i=1,\dots,n}$ (see Definition 2.1). Thus

$$[\omega] = \sum_{i=1,\dots,n} \tau_i \mathrm{PD}(\Sigma_i),$$

where τ_i are positive numbers and Σ_i are possibly singular symplectic curves. Note that the symplectic form is exact on $M\backslash\Sigma$, so there are Liouville forms on $M\backslash\Sigma$. Fix also a smooth, local disc fibration above the regular part Σ_i^{reg} of Σ_i, with the additional property that each disc must intersect Σ_i orthogonally with respect to the symplectic form. We mostly view these fibrations as families $D_i(p)$ of symplectic discs, smoothly parameterized by $p \in \Sigma_i^{\mathrm{reg}}$. In each such disc, we choose a collection of loops $\gamma_i^\varepsilon(p)$ which approach p as ε goes to 0.

Lemma 4.1. *Let $\mathcal{V}$ be a tubular neighbourhood of Σ in M, and β a Liouville form on $\mathcal{V}\backslash\Sigma$. Then:*

i) *If $p \in \Sigma_i$, the numbers*

$$\beta(\gamma_i^\varepsilon(p)) := \int_{\gamma_i^\varepsilon(p)} \beta$$

have a limit $b_i(p)$ when ε goes to zero.

ii) *The numbers $b_i(p)$ depend neither on $p \in \Sigma_i$, nor on the chosen fibration. The number b_i will be called the residue of β at Σ_i.*

iii) *If β has residues (τ_i) at Σ, then the form β extends to a Liouville form on $M\backslash\Sigma$.*

Similarly, if $\mathcal{V}$ is a regular neighbourhood in M of any simply connected open subset U of Σ, and β is a Liouville form on $\mathcal{V}\backslash\Sigma$, then:

iv) *The restriction of* β *to any proper open subset of* $\mathcal{V}$ *extends to a Liouville form on* $M\backslash\Sigma$*, as long as* β *has residues* (τ_i) *at* Σ*.*

Here are some remarks about this lemma.

1) In iv), if U meets only some components of Σ (say Σ_{i_j}, $j = 1, \dots, p$), the condition on β is $\mathrm{Res}\,(\beta, \Sigma_{i_j}) = \tau_{i_j}$.

2) If Σ is an irreducible polarization, any Liouville form β on $M\backslash\Sigma$ verifies

$$\lim_{\varepsilon\to 0} \beta(\gamma_i^\varepsilon(p)) = \tau, \qquad \text{where } [\omega] = \tau\mathrm{PD}(\Sigma).$$

For instance, the residue of a Liouville form on $\mathbb{P}^2$ minus an irreducible curve of degree k is $1/k$.

3) By contrast, when there is a linear dependence between the Poincaré dual classes of the Σ_i, a lot of residues are allowed. If Σ_i are (possibly singular) curves of $\mathbb{P}^2$ of degree d_i, there is a Liouville form on $M\backslash\cup\Sigma_i$ with residues ${}^{p_i}/_{d_i}$ at Σ_i as soon as $\sum p_i = 1$ (simply average the Liouville forms on $M\backslash\Sigma_i$). In particular, there is a Liouville form on $\mathbb{P}^2\backslash\cup\Sigma_i$ whose residues at any branch of Σ is ${}^1/_{d_1+\dots+d_n}$.

Proof of Lemma 4.1. The point i) is obvious: $\beta(\gamma_i^\varepsilon(p))$ converges to $\mathcal{A}_\omega(D_i(p)) - \beta(\partial D_i(p))$. In order to prove ii), observe that the symplectic orthogonals to $D(p)$ induce a connection on our local fibration above Σ_i^{reg}. For two points $p, q \in \Sigma_i^{\mathrm{reg}}$, consider a path $\alpha : [0,1] \to \Sigma_i^{\mathrm{reg}}$ between p and q, and parallel transport a circle $C_p \subset D_i(p)$ along α (this is possible provided C_p is close enough to p). Call S the surface obtained by gluing the three following pieces:

- the cylinder $A := \bigcup_{t\in[0,1]} P_\alpha^t(C_p)$, where P_α^t denotes the parallel transport,
- the disc D_1 bounded by C_p in $D_i(p)$,
- the disc D_2 bounded by $C_q := P^1(C_p)$ in $D_i(q)$.

Now by definition of our connection, A is a lagrangian cylinder. This has two consequences. On one hand, since S is null-homologous, the symplectic area of D_1 and D_2 are the same. On the other hand, since A lies in $\mathcal{V}\backslash\Sigma$, on which β is a well-defined Liouville form for ω, we get

$$\beta(C_p) = \beta(C_q) + \mathcal{A}_\omega(A) = \beta(C_q).$$

We therefore see that $b_i(p) = b_i(q)$, since

$$b_i(p) = \beta(C_p) - \mathcal{A}_\omega(D_1) = \beta(C_q) - \mathcal{A}_\omega(D_2) = b_i(q).$$

In order to see that the residue does not depend on the chosen fibration, consider two fibrations $D_i^1(p), D_i^2(p)$, construct a third one which contains the discs $D_i^1(p)$ and $D_i^2(q)$ and apply the last result to this new fibration.

In order to prove iii), consider a Liouville form λ on $M\backslash\Sigma$. By ii), this form has residues $\tau_i + f_i$ at Σ_i. When the residues of λ are the τ_i, the extension of β is obvious. Indeed, $\beta - \lambda$ is closed on $\mathcal{V}\backslash\Sigma$ and has no period by hypothesis, so it is even exact. Writing $\beta - \lambda = dh$, any extension $\widetilde{h}$ of h to $M\backslash\Sigma$ provides an extension $\widetilde{\beta} := \lambda + d\widetilde{h}$ of β. Assume now that the f_i do not vanish. The first observation is that

$$\sum f_i \mathrm{PD}(\Sigma_i) = 0. \tag{1}$$

Indeed, if C is any surface in M,

$$\begin{aligned}\int_C \omega &= \sum_{i=1}^n \mathrm{Res}\,(\lambda, \Sigma_i) C \cdot \Sigma_i = \sum_{i=1}^n (\tau_i + f_i) C \cdot \Sigma_i \\ &= \sum_{i=1}^n \tau_i \mathrm{PD}(\Sigma_i) \cdot [C] = \sum_{i=1}^n \tau_i C \cdot \Sigma_i,\end{aligned}$$

so $\sum f_i \mathrm{PD}(\Sigma_i) \cdot C = 0$.

Consider now a collection of closed 2-forms σ_i representing the class $\mathrm{PD}(\Sigma_i)$. They are obviously exact on $M\backslash\Sigma_i$, so there exist 1-forms α_i on $M\backslash\Sigma_i$ such that $\sigma_i = -d\alpha_i$. It is easy to check that α_i also has a well-defined residue at Σ_i, and even that $\mathrm{Res}\,(\alpha_i, \Sigma_i) = 1$. Since by (1), the form $\sum f_i \sigma_i$ is exact on M, we can write

$$d\left(\sum f_i \alpha_i\right) = d\theta,$$

where θ is a 1-form defined on the whole of M. The form $\alpha := \sum f_i \alpha_i - \theta$ is therefore closed on $M\backslash\Sigma$, with residues f_i at Σ_i. The form $\lambda - \alpha$ is thus a Liouville form on $M\backslash\Sigma$, with residues τ_i on Σ_i, and we are in the previous situation. The proof of iv) is completely similar. □

4.2. Liouville forms on balls

The relevant Liouville forms on the balls are restrictions of global Liouville forms on $\mathbb{C}^2$ obtained in the following way. The symplectic form on $\mathbb{C}^2$ is

$\omega_{\rm st} = dR_1 \wedge d\theta_1 + dR_2 \wedge d\theta_2$, where $(R_1 := |z_1|^2, \theta_1, R_2 := |z_2|^2, \theta_2)$ are polar coordinates on $\mathbb{C}^2(z_1, z_2)$, so for $d_0 := \{z_2 = 0\} \subset \mathbb{C}^2$, the 1-form

$$\lambda_\tau := (\tau - R_2)d\theta_2 - R_1 d\theta_1$$

is Liouville, has residue τ at d_0. Any convex combination of pull backs of such Liouville forms by unitary maps gives a Liouville form on the complement of lines in $\mathbb{C}^2$ with computable residues.

Proposition 4.2. *Let $X := \Delta_1 \cup \cdots \cup \Delta_n$ be a standard n-cross in $B(a) \subset \mathbb{C}^2$. If $\tau_1, \ldots, \tau_n \in \mathbb{R}^+$ are such that $\sum \tau_i \geq a$, there exists a Liouville form λ on $B(a) \backslash X$, with residues*

$$\mathrm{Res}\,(\lambda, \Delta_i) = \tau_i,$$

and whose associated Liouville vector field has the following properties:

i) *It is not defined on X, but it points outwards along the cross.*

ii) *Its flow is radius increasing: $X_\lambda \cdot r > 0$. In particular, the negative flow of any point in $B(a)$ is well-defined until it reaches one of the Δ_i. In other terms, the basin of repulsion of X, defined by*

$$\left\{ p \in \mathring{B}(a) \,|\, \exists \tau \in \mathbb{R}^+ \text{ with } \left| \begin{array}{l} \Phi^{-t}_{X_\lambda}(p) \text{ exists for } t \in [0, \tau[\\ \lim\limits_{t \to \tau} \Phi^{-t}_{X_\lambda}(p) \in X \end{array} \right. \right\}$$

is exactly $\mathring{B}(a)$.

iii) *More basically, if $\mathcal{U}$ is any neighbourhood of X in $B(a)$,*

$$\bigcup_{t \geq 0} \Phi^t_{X_\lambda}(\mathcal{U}) = \mathring{B}(a).$$

Proof. Observe that when $\kappa \geq a$, the 1-form λ_κ defined above is $\omega_{\rm st}$-dual to the vector field

$$X_\kappa := (\kappa - R_2)\frac{\partial}{\partial R_2} - R_1 \frac{\partial}{\partial R_1},$$

with $X_\kappa \cdot (R_1 + R_2) = \kappa - R_1 - R_2$, so $\Phi^t_{X_\kappa}$ increases the radius inside $B(a)$ (it is tangent to $\partial B(a)$ exactly when $\kappa = a$). Moreover, since

$$\frac{\partial}{\partial R_2} = \frac{1}{2r_2}\frac{\partial}{\partial r_2},$$

X_κ explodes and points outwards along d_0. Given now (Δ_i, τ_i) as in Proposition 4.2, consider unitary transformations u_i taking d_0 to $d_i := \langle \Delta_i \rangle$, positive weights μ_i (with $\sum \mu_i = 1$), and real numbers $\kappa_i \geq a$ such that $\mu_i \kappa_i = \tau_i$. Since the u_i are symplectic and radius-preserving, the Liouville form

$$\lambda := \sum \mu_i u_{i*} \lambda_{\kappa_i}$$

has the properties announced. □

4.3. Liouville forms on ellipsoids

Ellipsoids can be presented as quotient of balls by ramified symplectic coverings. Pushing-forward Liouville forms from the balls, we now produce the analogous Liouville forms in ellipsoids. Fix an ellipsoid $E(a,b)$ with $a/b \in \mathbb{Q}$, and put $a := \tau p$, $b := \tau q$, where p, q are relatively prime integers.

Lemma 4.3. *The map*

$$\begin{array}{ccccc} \Phi & : & B(\tau pq) & \longrightarrow & E(a,b) = \tau E(p,q) \\ & & (R_1, \theta_1, R_2, \theta_2) & \longmapsto & (\frac{R_1}{q}, q\theta_1, \frac{R_2}{p}, p\theta_2) \end{array}$$

is a symplectic covering of degree pq, ramified over the coordinate axis $\{R_1 = 0\}$ and $\{R_2 = 0\}$. It is invariant by the diagonal action of $\mathbb{Z}_p \times \mathbb{Z}_q \approx \{(e^{2i\pi k/p}, e^{2i\pi l/q})\}$ on $\mathbb{C}^2$.

Definition 4.4. Let $\alpha := |\alpha| e^{i\varphi}$ be a complex number. We denote by $\Delta_\alpha \subset E(a,b)$ the cone defined by

$$\left\{ \begin{array}{l} R_1 = |\alpha| R_2 \\ p\theta_1 = q\theta_2 + \varphi \end{array} \right. .$$

It is a symplectic surface, smooth except at the origin (the vertex of the cone), and whose intersection with $\partial E(a,b)$ is a characteristic leaf.

A straightforward computation shows that the preimage of Δ_α is a union of pq lines, invariant by the action of $\mathbb{Z}_p \times \mathbb{Z}_q$. Averaging a Liouville form with residue κ around one of these disc (given by Proposition 4.2) by this action, we therefore get a Liouville form on $B(\tau pq)$, which descends to a Liouville form on $E(a,b) \backslash \Delta_\alpha$, with the same nice features as for the Liouville forms on the ball. For instance, if the residue at Δ_α is at least τ, then the residues of the Liouville forms on $B(\tau pq)$ are at least τ, and the Liouville vector field is radius increasing. The Liouville vector field on the ellipsoid therefore increases the function $qR_1 + pR_2$, hence the function ${}^{R_1}/_{a} + {}^{R_2}/_{b}$. Averaging the different Liouville forms associated to different Δ_α, we get the analogue of Proposition 4.2 for ellipsoids.

Proposition 4.5. *Let $\alpha_1, \dots, \alpha_n$ be n complex numbers and $\Delta_{\alpha_1}, \dots, \Delta_{\alpha_n} \subset \tau E(p,q)$ their associated cones. If $\tau_1, \dots, \tau_n \in \mathbb{R}^+$ are such that $\sum \tau_i \geq \tau$, there exists a Liouville form λ on $\tau E(a,b) \backslash \Delta_{\alpha_1} \cup \dots \cup \Delta_{\alpha_n}$, with residues*

$$\operatorname{Res}(\lambda, \Delta_{\alpha_i}) = \tau_i,$$

and whose associated Liouville vector field has the following properties:

i) *It is not defined on $\cup \Delta_{\alpha_i}$, but it points outwards along these curves.*

ii) *Its flow is "radius increasing". Namely, if $R := \frac{R_1}{a} + \frac{R_2}{b}$, $X_\lambda \cdot R > 0$. In particular, the negative flow of any point in $\tau E(a,b)$ is well-defined until it reaches one of the Δ_{α_i}. In other terms, the basin of repulsion of $\cup \Delta_{\alpha_i}$, defined by*

$$\left\{ p \in \tau \overset{\circ}{E}(a,b) \,|\, \exists T \in \mathbb{R}^+ \text{ with } \left| \begin{array}{l} \Phi^{-t}_{X_\lambda}(p) \text{ exists for } t \in [0, T[\\ \lim\limits_{t \to T} \Phi^{-t}_{X_\lambda}(p) \in \Delta_{\alpha_i} \end{array} \right. \right\}$$

is exactly $\tau \overset{\circ}{E}(a,b)$.

iii) *More basically, if $\mathcal{U}$ is any neighbourhood of $\cup \Delta_{\alpha_i}$ in $\tau E(a,b)$,*

$$\bigcup_{t \geq 0} \Phi^t_{X_\lambda}(\mathcal{U}) = \tau \overset{\circ}{E}(a,b).$$

4.4. Tameness

In this paragraph, we define a regularity notion for Liouville forms, which will be central in producing maximal open packings or embeddings.

Definition 4.6 (Angular form). Let Σ be a codimension-2 submanifold in M and $\mathcal{N}_\Sigma$ its normal bundle in M. Endow $\mathcal{N}_\Sigma$ with a hermitian metric and a hermitian connection α. An angular form around Σ is the push-forward of α by any smooth local embedding of $(\mathcal{N}_\Sigma, 0_{\mathcal{N}_\Sigma})$ into (M, Σ). More informally, it is a 1-form which is locally the $d\theta$ of local polar coordinates around Σ.

If Σ is a symplectic submanifold, we further impose a compatibility condition: α must be positive on the small closed loops that turn positively around Σ.

Definition 4.7 (Tameness). Let (M^4, Σ) be a 4-dimensional manifold with a codimension-2 submanifold with isolated singularities. We say that a 1-form λ is tame at $U \subset \Sigma^{\text{reg}}$ if there exists a smooth function κ on some neighbourhood V of U in M, a smooth 1-form μ on $V \backslash U$ which is bounded (i.e. its coefficients are bounded near U), and an angular form α over Σ^{reg} such that

$$\lambda = \kappa\alpha + \mu \qquad \text{on } V \backslash U.$$

For simplicity, we will say that a 1-form is tame at Σ if it is tame at Σ^{reg}. When we need to be more precise, we call α-tame a form which is tame with respect to a specific angular form α. The aim of this section is to produce tame Liouville forms associated to any polarization with reasonable singularities.

Remark 4.8. Tameness is a differentiable notion: if the fibration or the connection is modified, the class of tame forms remains unchanged.

Remark 4.9. The Liouville forms defined in Section 4.2 provide tame Liouville forms with arbitrary residues on the complement of a cross in a ball.

Remark 4.10. If λ is a tame Liouville form with positive residues on $M \backslash \Sigma$, where Σ is a symplectic polarization, the associated vector field X_λ points outwards the regular part of Σ.

Proof. Let $(z, R = r^2, \theta)$ be coordinates in a neighbourhood V of a point $p \in \Sigma$, with

$$V \cap \Sigma = \{R = 0\} \text{ and } \omega = dR \wedge d\theta + \pi^*\tau,$$

where τ is a symplectic form on Σ and $\pi(R, \theta, z) = (0, 0, z)$. Tameness of λ means that

$$\lambda = \kappa d\theta + \mu,$$

and positivity of the residue means that κ takes positive values in V, provided V is small enough. Thus,

$$X_\lambda = \kappa \frac{\partial}{\partial R} + Z = \frac{\kappa}{2r} \frac{\partial}{\partial r} + Z,$$

where Z is a bounded vector field (it is ω-dual to a bounded form). Near $\Sigma = \{r = 0\}$, this vector field is clearly radius increasing. □

Remark 4.11. For a fixed angular form α, α-tame forms can be glued. Namely, let λ_1, λ_2 be two α-tame 1-forms, defined in neighbourhoods V_1, V_2 of $U_1, U_2 \subset \Sigma^{\text{reg}}$, with well-defined residues at Σ (there might be several ones if U_1 or U_2 is not connected). Assume that the residues coincide, and that $\lambda_1 - \lambda_2$ is exact on $(V_1 \backslash U_1) \cap (V_2 \backslash U_2)$. Then there exists a tame 1-form λ in $V_1 \cup V_2$ which coincides with λ_1 on V_1 and with λ_2 on an arbitrary compact subset of $V_2 \backslash (\overline{V_1 \cap V_2})$.

Proof. By assumption, $\lambda_1 - \lambda_2 = dh$, with $h \in \mathcal{C}^\infty(V_1 \cap V_2 \backslash U_1 \cap U_2)$, and since λ_1, λ_2 are α-tame,

$$(\lambda_1 - \lambda_2)(x) = [\kappa_1(x) - \kappa_2(x)]\alpha + \mu,$$

where α is an angular form over $U_1 \cap U_2$, κ_1, κ_2 are smooth functions and μ is a smooth 1-form on $V_1 \cap V_2$. Imposing the same residues implies that $\kappa_1 - \kappa_2 = 0$ on $U_1 \cap U_2$, so $(\kappa_1 - \kappa_2)(x) = rb(x)$, where r is a radial coordinate around Σ and b is a bounded, smooth function on $V \backslash U$. Now the form $r\alpha$ is bounded on a neighbourhood of Σ, so h is a smooth function on $V_1 \cap V_2 \backslash (U_1 \cap U_2)$ with bounded derivatives. This function can be extended by a function with arbitrary compact support in a neighbourhood of $V_1 \cap V_2$, still with bounded derivatives near $U_1 \cap U_2$. The gluing is the form that coincides with λ_1 on V_1 and $\lambda_2 + dh$ on V_2. □

Lemma 4.12. *If Σ is smooth and σ is a 2-form representing the Poincaré dual class to Σ, there exists a tame 1-form λ on $M \backslash \Sigma$ with $d\lambda = \sigma$ and residue 1 at Σ. The same holds when Σ has isolated singularities, provided that each singularity has some neighbourhood B already equipped with a tame 1-form with residue 1, whose differential is a smooth form on B.*

Since tame forms are singular along Σ, it may seem strange to assume that their differential is smooth. Notice however that this assumption automatically holds true for Liouville forms for instance.

Proof. Consider first the case when Σ is non-singular. Let $\mathcal{V}$ be a tubular neighbourhood of Σ, $\pi : \mathcal{V} \longrightarrow \Sigma$ a disc fibration and α an angular 1-form associated to some connection form on this fiber bundle. Inside $\mathcal{V}$, $d\alpha$ represents $\text{PD}(\Sigma)$, so

$$d\alpha - \sigma = d\mu,$$

where μ is a smooth 1-form on $\mathcal{V}$. The form $\alpha' := \alpha - \mu$ is therefore a tame primitive of σ, but is defined only in $\mathcal{V}$. Choose now a 1-form λ on $M \backslash \Sigma$ with

$d\lambda = \sigma$. Let also D be a local disc transverse to Σ at p and γ_ε a family of loops of D converging to p. As in Lemma 4.1 i), $\lambda(\gamma_\varepsilon)$ has a limit, denoted by κ. Considering a deformation of Σ whose intersections with Σ are very close to p, and which coincides with a union of discs very close to D near p, we see that κ must be 1, as soon as the normal bundle is non-trivial. On the other hand, when it is trivial, the 1-form α' previously defined is closed. Hence, the 1-form $\lambda - \kappa\alpha'$, defined on $\mathcal{V}\backslash\Sigma$, is closed. Since

$$\lim_{\varepsilon\to 0} \lambda - \kappa\alpha'(\gamma_\varepsilon) = 0,$$

it follows that $\lambda - \kappa\alpha'(\gamma_\varepsilon) = 0$ for all ε. Now, if γ_i is any basis of $H_1(\Sigma)$, there is a closed 1-form ν on Σ with periods

$$\nu(\gamma_i) = \lambda - \kappa\alpha'(\widetilde{\gamma}_i),$$

where $\widetilde{\gamma}_i$ is a small perturbation of γ_i in $\mathcal{V}\backslash\Sigma$. The form $\lambda - \kappa\alpha' - \pi^*\nu$ is therefore exact on $\mathcal{V}\backslash\Sigma$ so there is a function $h \in \mathcal{C}^\infty(\mathcal{V}\backslash\Sigma)$ such that

$$\lambda - \kappa\alpha' - \pi^*\nu = dh.$$

Any extension $\widetilde{h}$ of h to $M\backslash\Sigma$ defines a primitive of σ on $M\backslash\Sigma$ (by $\lambda - d\widetilde{h}$), which is tame (because it coincides with $\kappa\alpha' - \pi^*\nu$ on $\mathcal{V}$). Notice that it is even α-tame for an angular form α that was choosen arbitrarily at the begining of the argument.

When Σ has isolated singularities (p_i), and when each singularity has a small neighbourhood B_i equipped with a tame 1-form λ_i with residue 1 at $\Sigma \cap B_i$, with $d\lambda_i$ smooth on B_i, we proceed as follows. Consider first an angular form α on Σ^{reg} which extends the one for which the λ_i are tame (which means that each λ_i is α-tame). Notice that the forms λ_i can be assumed to be primitives of σ. Indeed, since $d\lambda_i$ extends to a smooth 2-form on B_i the form $\sigma - d\lambda_i = d\mu$ for a smooth one-form μ defined on B_i. Correcting λ_i by adding μ, we therefore get a primitive of σ on $B_i\backslash\Sigma$, tame at $\Sigma \cap B_i$ and with residue 1.

Consider now a smooth perturbation Σ' of Σ which coincides with Σ outside balls B_i^ε much smaller than B_i. By the previous argument, there exists a primitive λ of σ on $M\backslash\Sigma'$, α-tame and with residue 1 at Σ'. In view of Remark 4.11, we only need to understand that $\lambda - \lambda_i$ is exact in a neighbourhood of $\partial(B_i\backslash\Sigma)$. Since λ and λ_i have the same residues, it amounts to showing that $\lambda - \lambda_i$ vanishes on the knot defined by $\partial B_i \cap \Sigma$. Consider a connected component C of this knot, and fill it inside B_i with a possibly singular disc D, which does not meet B_i^ε, and whose singularities lie outside Σ. Then,

$$\begin{aligned}
\lambda(C) &= \int_D \sigma - D\cdot\Sigma' && \text{(because } \lambda \text{ has residue 1 at } \Sigma)\\
&= \int_D \sigma - D\cdot(\Sigma\cap B_i) && \text{(because } D \text{ avoids } B_i^\varepsilon)\\
&= \lambda_i(C) && \text{(because } \lambda_i \text{ has residue 1 at } \Sigma).
\end{aligned}$$

□

Corollary 4.13. *Let (M,ω) be a symplectic manifold with a singular polarization $\Sigma = (\Sigma_i)$ with nodal singularities only. Then, if*

$$[\omega] = \sum \kappa_i \mathrm{PD}(\Sigma_i),$$

there exists a tame Liouville form on $M\backslash\Sigma$ with residues κ_i at Σ_i.

Proof. Since each singularity consists of several branches intersecting transversally, there exists a tame Liouville form with residue κ_i at Σ_i near each singularity of Σ_i (see Remark 4.9). Moreover, this Liouville form decomposes as a sum of tame 1-forms near each Σ_i, whose differentials are smooth. Consider now closed two-forms σ_i on M Poincaré dual to Σ_i, such that $\sum \kappa_i\sigma_i = \omega$. By Lemma 4.12, there exists a tame 1-form λ_i on $M\backslash\Sigma_i$ with residue 1 such that $d\lambda_i = \sigma_i$. Our tame Liouville form on $M\backslash\Sigma$ is simply $\lambda := \sum \kappa_i\lambda_i$. □

Of course, in view of Proposition 4.5, the same statement holds when the singularities have the form $\Pi_\alpha(z^p - \alpha w^q)$. Let us now explain the relevance of tameness in our discussion. As we observed in Remark 4.10, the Liouville vector field associated to a tame form β on $M\backslash\Sigma$ is radial around the polarization - and it explodes in a controled way. A basic consequence is that there is a well-defined X_β-trajectory emanating from any point of the polarization in each normal direction. In other terms, the flow of X_β is well-defined on pairs $(p\in\Sigma,\theta)$ - where θ is a local angular coordinate around Σ -, in other terms on the blow-up Σ^* of Σ (see [19], appendix A for details). As a result, we get the following (local) statement.

Proposition 4.14. *Let λ be a tame Liouville form on $B\backslash\Delta := B^4(1)\backslash\{z_2 = 0\} \subset \mathbb{C}^2$, and B_λ the basin of attraction of Δ in B:*

$$B_\lambda := \{p \in B \mid \exists\tau\ ,\ \Phi^{-\tau}_{X_\lambda}(p) \in \Delta\}.$$

Let also $(M^4, \omega, \Sigma, \beta)$ be a closed symplectic manifold with symplectic polarization Σ and tame Liouville form on $M\backslash\Sigma$, and $\varphi : \Delta^* \hookrightarrow \Sigma^*$ be a lift to the blow-ups of an area preserving map between Δ and Σ. Assume that the residues of λ and β at Δ and $\varphi(\Delta)$ coincide. Then, there exists a unique $\mathcal{C}^1$ symplectic embedding $\Phi : B_\lambda \hookrightarrow M$, such that $\Phi_{|\Delta} = \varphi$ and $\Phi^*\beta = \lambda$.

5. Ball packings

5.1. Proof of Theorem 4

Proof of i). Assume that M has a symplectic packing by closed balls B_i of capacities $a_i \in \mathbb{Q}$. Consider the symplectic blow-up $(\hat{M}, \hat{\omega})$ of M along all the balls (recall that they are closed and disjoint). The cohomology class of $\hat{\omega}$ is therefore

$$[\hat{\omega}] = [\pi^*\omega] - \sum_{i=1}^{p} a_i e_i,$$

where $\pi : \hat{M} \to M$ is the blow-up map and the e_i are the Poincaré duals of the exceptional divisors E_i corresponding to the balls B_i. Since ω is assumed to be a rational form, and the a_i are rational numbers, the form $\hat{\omega}$ is also rational. By Donaldson's result, for all k sufficiently large and such that $k[\hat{\omega}] \in H^2(\hat{M}, \mathbb{Z})$, there is a symplectic curve $\hat{\Sigma}_k$ Poincaré dual to $k[\hat{\omega}]$. By definition, the homological intersection between $\hat{\Sigma}_k$ and E_i is ka_i, but there may be positive and negative intersections. The following lemma ensures that these intersections may be assumed to be positive. It is an easy, well-known adaptation of Donaldson's proof done in Section 7,

Lemma 5.1. *For any k sufficiently large and such that $k[\hat{\omega}] \in H^2(\hat{M}, \mathbb{Z})$, there exists a symplectic curve $\hat{\Sigma}_k$ Poincaré dual to $k[\hat{\omega}]$ and whose intersections with the E_i are transverse and positive. They therefore consist of ka_i distinct points.*

Now $\Sigma_k := \pi(\hat{\Sigma}_k)$ is a symplectic curve in $M\backslash \cup \overset{\circ}{B}_i$ with boundary. The boundary components are precisely a union of ka_i Hopf circles of each ∂B_i. Gluing to Σ_k the corresponding Hopf discs inside the B_i, we get a symplectic curve which almost convenes. It is obviously Poincaré dual to $k[\omega]$, irreducible (because $\hat{\Sigma}_k$ is by Donaldson's method itself), and it has the desired singularities at the centers of the balls. The only problem is that Σ_k may well be non-smooth along the union of the Hopf circles if we do not care about the smoothness of the gluing (it is *a priori* only topological). However, in [17], lemma 5.2, it is proved that starting with the topological gluing, one can easily smoothen - while keeping the curve symplectic - by a $\mathcal{C}^0$-small perturbation localized inside the B_i and near the Hopf circles. In short, one can find a smooth gluing, so Σ_k is the required curve. □

Remark. Lemma 5.1 can be easily proved using pseudo-holomorphic methods, as in [13]. Indeed, the curves $\hat{\Sigma}_k$ given by Donaldson's method are J_k holomorphic for some almost-complex structure J_k, which are close to an almost-complex structure J compatible with $\hat{\omega}$. For instance, J can be a structure inherited from a compatible almost complex structure on M such that the E_i are J-holomorphic (see [14]). Then using automatic regularity of the exceptional J-spheres and their uniqueness (in a fixed homology class), we can deform the E_i to exceptional J_k-spheres E_i', which therefore intersect Σ_k positively and transversally. By positivity of intersections, the E_i' do not intersect, and these deformations come from a global Hamiltonian isotopy. Applying the inverse Hamiltonian flow to $\hat{\Sigma}_k$, we get the needed symplectic curve, which proves Lemma 5.1. We present however a different proof in Section 7, which only relies on Donaldson's techniques, and which is more suited to treat the singular situation raised by ellipsoid embeddings.

Proof of ii). Here is what we call the inflation procedure. Consider a curve Σ Poincaré dual to $k[\omega]$ with p nodes, of mutliplicities $k_1a_1,\dots,k_pa_p$, with $k\leq k_i$ for all i. Assume that there are p disjoint closed k_ia_i-crosses X_i of size a_i in Σ, hence also disjoint crosses X_i^ε of size $a_i+\varepsilon$. Each cross X_i^ε has a simply connected neighbourhood $\mathcal{V}_i^\varepsilon$ symplectomorphic (by a map φ_i) to a neighbourhood $\mathcal{U}_i^\varepsilon$ of a union of k_ia_i Hopf discs in $B(a_i+\varepsilon)$. Consider the Liouville form λ_i on $B(a_i)$ associated by Proposition 4.2 to these k_ia_i Hopf discs, with residues ${}^1/_k$. It is also defined on $B(a_i+\varepsilon)$, so since the closed crosses X_i^ε are disjoint, the form

$$\beta_{|\mathcal{V}_i^\varepsilon} := \varphi_{i*}\lambda_{i|\mathcal{U}_i^\varepsilon}$$

defines a Liouville form on $\cup\mathcal{V}_i^\varepsilon$ with residue ${}^1/_k$. By Corollary 4.13, the restriction of β to $\mathcal{V}:=\cup\mathcal{V}_i^\varepsilon$ extends to a tame Liouville form on $M\backslash\Sigma$. Since ${}^{k_ia_i}/_k\geq a_i$ for all i, Proposition 4.2 iii) ensures that the (obviously disjoint) basins of attraction of the sets $\mathcal{V}_i$, defined by

$$B_i := \bigcup_{t\geq 0} \Phi^t_{X_\beta}(\mathcal{V}_i),$$

each contain an open symplectic ball of capacity a_i. The embedding is given by

$$\begin{array}{cccll} \Phi_i & : & B(a_i) & \longrightarrow & B_i \\ & & x & \longmapsto & \Phi^\tau_{X_\beta} \circ \varphi_i \circ \Phi^{-\tau}_{X_{\lambda_i}}(x), \qquad \text{where } \Phi^{-\tau}_{X_{\lambda_i}}(x) \in \mathcal{U}_i. \end{array}$$

Figure 1 is meant at illustrating this inflation process. □

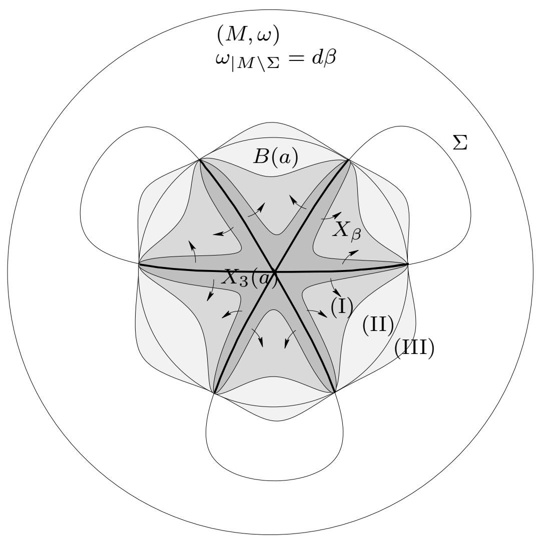

Figure 1: Inflation of a neighbourhood of a cross.

(I) $k_i < k$: inflating the cross does not lead to the embedding of the whole ball.
(II) $k_i = k$: inflating the cross allows the embedding of the whole *open* ball. Singularities on the boundaries are due to the explosion of the Hopf circles at the boundary of the basin of attraction of the cross in the standard ball.
(III) $k_i > k$: inflating leads to an embedding of more than the ball. Then the closed ball embeds smoothly, provided the crosses themselves are smooth.

5.2. Proof of Theorem 5

Compared with the previous paragraph, we need to understand what happens when the closures of the crosses are not disjoint anymore. The argument is roughly the same, but is based on Proposition 4.14, i.e. on the analysis performed in the appendix to [19]. We first fix the notation and describe the geometric picture. In each $B(a_i)$, consider a cross X_i' which is symplectomorphic to X_i. Consider also disjoint symplectic balls B_i^ε in M, of capacity ε, centered at the singular points p_i, and seen as the embeddings of the closed balls $B(\varepsilon) \subset B(a_i)$ by a map φ_i. We further assume that these embeddings send $X_i' \cap B(\varepsilon)$ to X_i. Denote by λ_i the Liouville form on $B(a_i)\backslash X_i$ with residues $1/k$ given by Proposition 4.2. By Corollary 4.13, we can extend the Liouville forms $\varphi_*\lambda_i$ defined on $\cup B_i^\varepsilon$ to a tame Liouville form β on $M\backslash\Sigma$. Our goal is now to extend the φ_i to embeddings of the open balls.

Extension to $X_i'^*$. Fix angular coordinates θ_i and θ_i' around $X_i\backslash\{p_i\}$ and $X_i'\backslash\{0\}$ respectively, and denote by $X_i^*, X_i'^*$ the blow-ups:

$$X_i^* := \{(p,\theta_i), p \in X_i\backslash\{p_i\}, \theta_i \in \mathbb{R}\backslash\mathbb{Z}\}.$$

We can freely assume at this point that the restriction of φ_i to $X_i'^* \cap B(\varepsilon)$ verify $\varphi_i(p',\theta_i') = (\varphi_i(p'),\theta_i')$. Consider now any extension $\widetilde{\varphi}_i$ of $\varphi_{i|X_i'\cap B(\varepsilon)}$ to an area-preserving map of X_i' into X_i. Then the formula

$$\varphi_i(p',\theta') := (\widetilde{\varphi}_i(p'),\theta_i) \qquad \text{on } X_i'^*$$

defines an extension of φ_i to $X_i'^*$ which is smooth on $X_i'^*$.

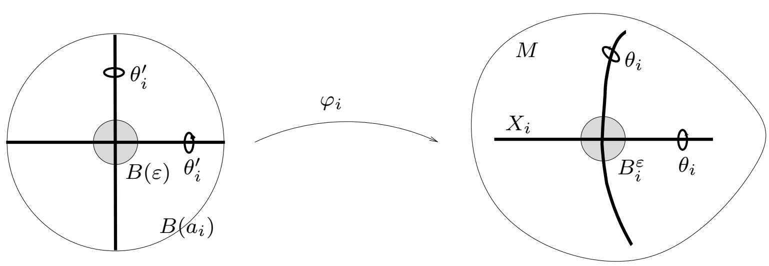

Figure 2: Blowing-up φ_i.

Inflation. As we explained already, the tameness hypothesis implies that the positive flow of X_β is well-defined on X_i^*. Similarly, the backward flow of a point $x \in B(a_i)$ either reaches $B(\varepsilon)$ or a point on X_i' (by Proposition 4.2, because $k_i \geq k$), along a certain direction, whose angle will be denoted $\theta_i'(x)$.

We now inflate each φ_i to an embedding of the open ball $B(a_i)$ in the following way:

$$\begin{array}{rcl}\Phi_i \ : \ B(a_i) & \longrightarrow & B_i \\ x & \longmapsto & \left\{\begin{array}{ll}\Phi^{\tau}_{X_\beta}\circ\varphi_i\circ\Phi^{-\tau}_{X_{\lambda_i}}(x), & \text{when } \Phi^{-\tau}_{X_{\lambda_i}}(x)\in B(\varepsilon),\\ \Phi^{\tau}_{X_\beta}\circ\varphi_i\big(\Phi^{-\tau}_{X_{\lambda_i}}(x),\theta_i'(x)\big), & \text{when } \Phi^{-\tau}_{X_{\lambda_i}}(x)\in X_i'\backslash B(\varepsilon).\end{array}\right.\end{array}$$

The map is well-defined but this formula raises several questions. The first one concerns regularity: Φ_i is not obviously smooth, nor even continuous. However, Φ_i is clearly $\mathcal{C}^1$-smooth on the basin of attraction of the open ball $B(\varepsilon)$ and of the open annulus $X_i'\backslash B(\varepsilon)$ by Proposition 4.14. In order to see that Φ_i is actually $\mathcal{C}^1$ on $B(a_i)$, define Φ_i^2 by the same formula, but replacing $B(\varepsilon)$ by $B(\varepsilon/2)$. The map Φ_i^2 is smooth on the basin of attraction of $X_i'\backslash B(\varepsilon/2)$ - which contains the locus where Φ_i is not known to be smooth. On the other hand, Φ_i^2 coincides with Φ_i by the uniqueness part of Proposition 4.14. Thus Φ_i is $\mathcal{C}^1$-smooth on $B(a_i)$. The same proof shows that Φ_i is symplectic. Finally, the different embeddings are disjoint because they lie in the basins of attraction of disjoint subsets of the polarization. □

5.3. Different shapes for efficient packings

The previous proof shows two important things. First, the natural objects that can be embedded are the basins of attraction in $\mathbb{C}^2$ of crosses under the Liouville vector fields associated to

$$\left\{\begin{array}{l}\lambda := \frac{1}{n}\sum u_j^*\lambda_j, \text{ where}\\ \lambda_j := (n\kappa_j - R_1)d\theta_1 - R_2 d\theta_2,\ u_j\in U(2).\end{array}\right. \tag{2}$$

Moreover, the "vertical part of the embedding", in other terms the inflation process is smooth. If the discs of the crosses are smooth up to the boundary, then the embedding of the corresponding basin of attraction is $\mathcal{C}^1$-smooth up to the boundary, locally near the boundary of the cross. Combining these two observations, we get the following general theorem. The notation $B\big((\kappa_j)_{j=1\dots m};(a_j)_{j=1\dots m};(u_j)_{j=1\dots m}\big)$ stands for the *open* basin of attraction in $\mathbb{C}^2$ of the "anisotropic" standard m-cross composed of standard open discs of sizes (a_j) inside complex lines $d_j := u_j(\{z_2=0\})$ $(u_j\in U(2))$, under the flow of the Liouville vector field associated to the form given by (2).

Theorem 7. *Let* $\mathbf{\Sigma}$ *be a polarization of* M

$$[\omega] = \sum_{l=1}^{N} \tau_l \mathrm{PD}(\Sigma_l).$$

Assume that Σ *is covered by* r m_i*-crosses each composed of* m_i^l *discs of* Σ_l*, of sizes* $(a_i^j)_{j\leq m_i}$*, with respective positions described by* $(u_i^j)_{j\leq m_i}$*. Then* M *has a full packing by*

$$\coprod_{i=1}^{r} B\big((\underbrace{\tau_1,\ldots,\tau_1}_{m_i^1},\ldots,\underbrace{\tau_N,\ldots,\tau_N}_{m_i^N});(a_i^1,\ldots,a_i^{m_i});(u_i^j)\big).$$

Moreover, around each smooth boundary point of the crosses in Σ*, the corresponding embeddings are* $\mathcal{C}^1$*-smooth up to the boundary.*

This theorem may look a bit surprising, in that it seems to give too many different full packings. It is well-known and understandable that curves of different degrees or different kind of singularities, which have no obvious symplectic relations, give different packings. But the parameters (u_i) are too much. Indeed, easy local perturbations of the curves allow to modify freely these parameters. Why should it allow different embeddings? It turns out that the basins of attraction do not really depend on these extra-parameters.

Proposition 5.2. *The basins* $B\big((\kappa_j);(a_j);(u_j)\big)$ *and* $B\big((\kappa_j);(a_j);(v_j)\big)$ *are domains of* $\mathbb{C}^2$ *with symplectomorphic boundaries.*

Proof. Consider the standard anisotropic crosses X and Y (associated to $\big((\kappa_j);(a_j);(u_j)\big)$ and $\big((\kappa_j);(a_j);(v_j)\big)$ respectively), and a non-standard cross $Z\subset\mathbb{C}^2$ which coincides with Y in $B(\varepsilon)$ and with X in the complement of $B(2\varepsilon)$. Denote by $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ the infinite crosses in which X, Y and Z lie (just extend by complex lines). Let λ_X, λ_Y be the standard Liouville forms on $\mathbb{C}^2\backslash\widetilde{X}$, $\mathbb{C}^2\backslash\widetilde{Y}$. By Remark 4.11, there is a tame Liouville form λ_Z on $\mathbb{C}^2\backslash\widetilde{Z}$ which coincides with λ_Y inside $B(\varepsilon)$ and with λ_X outside $B(2\varepsilon)$. Following now step by step the proof of Theorem 5, we see that the basins of attraction of Y and Z are symplectomorphic. On the other hand, the basin of attraction of $Z\backslash B(2\varepsilon)$ coincides with that of $X\backslash B(2\varepsilon)$. □

5.4. Some explicit packings

We now explain how to construct geometrically maximal packings of $\mathbb{P}^2$ by 6, 7 and 8 balls, proving Theorem 2.

Six balls. Recall that the capacity of the optimal packing is $2/5$, and that the obstruction comes from 6 conics passing each through five of the six centers of the balls (on the blow-up, these curves lie in classes $2L - E_1 - \cdots - E_5$, $2L - E_1 - \cdots - E_4 - E_6, \ldots, 2L - E_2 - \cdots - E_6$). In fact, the same configuration of curves allows to construct the maximal packing. Indeed, fix six generic points $p_1, \ldots, p_6$ in $\mathbb{P}^2$, and consider a conic through any five points of them. The curve formed by the union of these conics has degree 12, passes five times through each p_i, and it can be split into six crosses centered on the p_i, each disc of the cross having area $2/5$ (simply divide each conic in five such discs). Moreover, the condition on the degree of this curve is verified: $2/5 \cdot 12 \leq 5$. Thus, the six crosses can be inflated to a ball packing of maximal capacity. It may be worth noticing that the produced packing is not very singular on its boundary: it can be chosen for instance to have exactly 10 singularities at each ball, precisely along the polarization, each singularity being rather nice (see the discussion and Figure 2 in [17] for a more precise description).

Seven balls. In this case, the best capacity is $3/8$ and the obstruction comes from a singular cubic passing once through six of the centers of the balls and twice through the last one (on the blow-up, this curve lies in class $3L - 2E_1 - E_2 - \cdots - E_7$). This particular cubic is not enough for our purpose of constructing seven balls of capacity $3/8$, but almost. Consider seven generic points $(p_1, \ldots, p_7)$ in $\mathbb{P}^2$. Consider one singular cubic through $(2p_1, p_2, \ldots, p_7)$, one through $(p_1, 2p_2, \ldots, p_7)$ and one conic through $(p_3, \ldots, p_7)$. The union of these curves is a curve of degree 8 that passes exactly 3 times through each p_i. Split each cubic into 8 discs of area $3/8$ centered on the p_i and consider also discs of area $3/8$ around $p_3, \ldots, p_7$ in the conic (it is possible because $5 \cdot 3/8 \leq 2$). We thus have found a cross of area $3/8$ with 3 branches around each p_i inscribed in a curve of degree 8. That is what is needed for getting the optimal packing. Regularity on the boundary is however more difficult to handle, precisely because in this situation, the capacity of the balls ($3/8$) is really the same as the quantity multiplicity/degree. Thus the singularities that appear in the partition of the curves into discs propagate along the Liouville vector field (hence giving rise to ”nice” discs of singularities), and new ones appear (which we cannot control at all), at the Hopf discs that are not in the basin of attraction of the cross.

Eight balls. The best capacity is $6/17$. In order to produce the packing, we therefore need a curve of degree 17 passing six times through each points, or a curve of degree 34 passing 12 times through each point and so on. In fact we will produce a curve of degree 51 passing 18 times through each point. The obstruction comes from a sextic, passing three times through one point, and twice through all other points (on the blow-up, the class of this curve is $6L - 3E_1 - 2E_2 - \cdots - 2E_8$). As before, the curve that allows the construction will be obtained from this "obstruction" curve. Namely, the curve we consider is composed of eight sextics, each passing twice through each point but one (each time different), where it passes three times. This curve has degree 48 and passes exactly 17 times through each point. Add to this curve a cubic passing once through each of the eight points, and you get the announced curve of degree 51.

Remarks. i) The description of the maximal packings by 7 or 8 balls above involves reducible curves. However, one can check that for both cases, there are extra-intersections for the total curves that allow to produce irreducible ones passing through the same set of points, with the same multiplicities. These curves are less explicit on one hand, but the discs that compose the crosses can be choosen to be disjoint, which allows for less singular packings.

ii) It might be worth noticing that this method to construct explicit embeddings really works *when there is an obstruction curve*. When the obstruction is due to the volume constraint, it usually does not provide a maximal packing. In order to illustrate this point, consider for example ten balls in $\mathbb{P}^2$. By Biran's result on packing stability [2], combined with McDuff's isotopies [11], there is a packing of $\mathbb{P}^2$ by ten open balls of capacities $1/\sqrt{10}$. If we try to construct this packing by a method similar to that explained above, we need to consider approximations of $1/\sqrt{10}$ by rational numbers a_n, then find a curve of large degree $k(n)$ with ten nodal singularities with multiplicities $k(n)a_n$ (note that $k(n)$ must be large since a_n has large denominator), and then inflate to get the balls. But this would give ten balls of size a_n, not $1/\sqrt{10}$. Even constructing a sequence of balls with capacities growing to the right size requires to be able to produce an infinite sequence of curves, with appropriate nodal singularities.

5.5. Proof of Theorem 6

First assume that there are p disjoint closed balls B_i of size $(a_i)_{i=1\dots p}$ in M. Consider rational symplectic forms $\omega_1, \dots, \omega_N$ very close to ω, such that each ω_l coincides with ω in a neighbourhood of the balls, and such that $[\omega]$ lies in the convex hull of the $[\omega_l]$ in $H^2(M, \mathbb{R})$:

$$[\omega] = \sum_{l=1}^{N} \mu_l [\omega_l], \qquad \sum \mu_l = 1, \ \mu_l > 0.$$

Since the ω_l coincide in a neighbourhood of the balls, the symplectic blow-ups of (M, ω_l) along the p balls provide different symplectic forms $\hat{\omega}_l$ on the same manifold $\hat{M}$, with the same exceptional divisors E_i. Now, applying Lemma 5.1, we find $\hat{\omega}_l$-symplectic curves $\hat{\Sigma}_l$ Poincaré dual to $k_l\hat{\omega}_l$ and whose intersections with the E_i are transverse and positive. It will also follow from the proof of this lemma that the intersections of E_i with the different $\hat{\Sigma}_l$ can be assumed to be disjoint (see Remark 7.3). In fact, we can even require that the $\hat{\Sigma}_l$ are $\hat{\omega}$-symplectic, and intersect positively and transversally in $\hat{M}$ (see [18], Theorem 2). Projecting these curves down to M, and gluing smoothly the convenient Hopf discs, we get curves Σ_l Poincaré dual to $k_l\omega_l$, and whose intersections with B_i consists of $k_l^i = k_l a_i$ Hopf discs. Now, $\mathbf{\Sigma} := (\Sigma_l, \mu_l/k_l)$ is a polarization of M with p nodes of multiplicities $k^i := \sum k_l^i$, which contains the p disjoint k^i-crosses of size a_i by construction. Finally,

$$\sum_l k_l^i \tau_l = \sum_l k_l^i \frac{\mu_l}{k_l} = \sum_l k_l a_i \frac{\mu_l}{k_l} = \sum_l \mu_l a_i = a_i.$$

If you wish to verify that the left hand side above may even be strictly larger than a_i, simply notice that M contains also disjoint closed balls of radii $a_i + \varepsilon$.

Conversely, the proof that a polarization with the properties listed in Theorem 6 allows to construct the ball packing is exactly the same as in Theorem 4 ii). □

6. Ellipsoid embeddings

As for balls, it will be convenient to be able to blow up the ellipsoids. The result is a symplectic orbifold, that we describe now. The reader can also consult [6, 16].

6.1. Blowing-up an ellipsoid

Weighted projective space. In this paragraph, we fix two relatively prime integers p, q. It is well-known that $\mathbb{P}^2$ is symplectically a closed ball, whose boundary has undergone a symplectic reduction, i.e. has been collapsed along its characteristic foliation. We define similarly $\mathbb{P}^2(1, p, q)$ as the ellipsoid $E(p, q)$ whose boundary has been collapsed along its characteristic foliation. It is easy to see that this definition gives the same topology to $\mathbb{P}^2(1, p, q)$ as the more classical one

$$\mathbb{P}^2(1, p, q) := \mathbb{C}^3_{/\mathbb{C}}, \text{ with action } \xi \cdot z := (\xi z_1, \xi^p z_2, \xi^q z_3),$$

but it has the advantage of giving an easier geometric model for the symplectic structure. Notice that it is easy to see (using any of these two models) that the divisor at infinity $\{z_1 = 0\}$, which corresponds to the boundary of the ellipsoid after the collapsing process, is topologically the weighted projective space $\mathbb{P}^1(p, q)$ - we determine its symplectic area below.

Let $\Phi : B(pq) \longrightarrow E(p, q)$ be the ramified pq-covering of the ellipsoid by the ball described in the previous paragraph. It obviously sends the characteristic foliation of ∂B to the one of $\partial E(p, q)$, so it descends to a symplectic (ramified) covering of $\mathbb{P}^2(1, p, q)$ by $\mathbb{P}^2(pq)$ (meaning that the projective line has area pq). The importance of this covering is two-fold. First, it explicitly presents $\mathbb{P}^2(1, p, q)$ as a symplectic orbifold, with an explicit desingularizing map. Notice that the divisor at infinity, which is itself singular, is also desingularized by this map, since it corresponds to a projective line. The second point is that it allows to complete the description of our weighted blow-up by giving us the symplectic area of the divisor at infinity. It is precisely 1, since it is a quotient of a symplectic line of area pq by a symplectic group covering of degree pq. Let us sum up this discussion:

Proposition 6.1. *Let p, q be two relatively prime integers. The weighted projective space $\mathbb{P}^2(1, p, q)$ is obtained from the ellipsoid $E(p, q)$ by collapsing the Hopf fibration of $\partial E(p, q)$. It is a symplectic orbifold, with group $\mathbb{Z}_p \times \mathbb{Z}_q$, and a desingularization map $\Phi : \mathbb{P}^2(pq) \longrightarrow \mathbb{P}^2(1, p, q)$. Finally, it has three "distinguished" curves: the horizontal one of area p, the vertical one of area q, and the divisor at infinity, of area* 1*.*

Blowing-up ellipsoids. Assume now that a closed ellipsoid $E(a, b) = \tau E(p, q)$ embeds into a symplectic manifold (M, ω).

Proposition 6.2. *Removing from M the interior of the ellipsoid $E(a, b)$ and collapsing the characteristic foliation of the resulting boundary (which is $\partial E(a, b)$), we get a symplectic orbifold with one or two singularities located on the exceptional divisor (the projection of $\partial E(a, b)$). This exceptional divisor is a symplectic suborbifold of area τ, which is desingularized by a symplectic desingularization of M.*

The resulting symplectic orbifold will be called the blow-up of M along $E(a, b)$ and denoted $(\hat{M}, \hat{\omega})$. Recall that classical blow-up gives a presentation of a symplectic manifold with some ball inside as a Gompf sum of the classical blow-up and the projective space along the exceptional divisor on one side and a projective line on the other. Similarly, this singular notion of blow-up allows to think of a symplectic manifold with some ellipsoid $E(a, b)$ inside as a Gompf sum of two symplectic orbifolds - the blow-up and the weighted projective space $\tau\mathbb{P}^2(1,p,q)$ - along symplectic suborbifolds - the exceptional divisor and the line at infinity.

Proof of Proposition 6.2. Observe that the map $\Phi : \tau B(pq) \to E(a,b)$ defined above extends to the whole of $\mathbb{C}^2$, hence in a neighbourhood of $\tau B(pq)$. Moreover, if the closed ellipsoid $\overline{E}(a,b)$ symplectically embeds into M, so does a slightly larger ellipsoid. The extension of Φ to a covering of this larger ellipsoid therefore gives a symplectic desingularization of the manifold described in Proposition 6.2. □

6.2. Proof of Theorem 3

Let us first assume that $\mathcal{E} := E(a,b) = \tau E(p,q)$ embeds into (M,ω) where ω is a rational class (p,q are relatively prime integers). Consider the blow-up $\hat{M}$ of M along this ellipsoid, and call E the resulting exceptional divisor. Provided that τ is rational, the symplectic form $\hat{\omega}$ on $\hat{M}$ is rational. Although the manifold is singular and the present framework is that of symplectic orbifolds, the next lemma asserts that Donaldson's techniques can be carried out in this setting.

Lemma 6.3. *For any k sufficiently large and such that $k[\hat{\omega}] \in H^2(\hat{M},\mathbb{Z})$, there exists a smooth symplectic curve $\hat{\Sigma}_k$ Poincaré dual to $k[\hat{\omega}]$, which intersects the exceptional divisor transversally and positively at exactly $k\tau$ regular points.*

Projecting a curve $\hat{\Sigma}_k$ down to M, we get a symplectic curve Σ_k whose boundary is made of $k\tau$ regular characteristic leaves of $\partial\mathcal{E}$. Now each such characteristic bounds a complex curve of equation $z_1^q = \alpha z_2^p$ in $\mathcal{E}$. Gluing these complex curves to Σ_k, we get a closed curve Poincaré dual to $k[\omega]$. As before, this gluing can be assumed to be smooth by [17]. The resulting symplectic curve has therefore all the required properties.

Conversely, assume that M contains a symplectic curve Σ Poincaré dual to $k[\omega]$, with a singularity composed of $k\tau$ branches locally of the form $z_1^q = \alpha z_2^p$. Namely, we assume that a neighbourhood of the singularity has a Darboux chart where Σ restricts to the curve

$$\prod_{i=1}^{k\tau}(z_1^q - \alpha_i z_2^p) = 0.$$

Now, cut the part of this curve that lies inside a very small ellipsoid $\tau' E(p,q)$ centered at the singularity x, and glue back smoothly the cone over the resulting boundary, with center at the origin. The curve Σ' that results from this operation is Poincaré dual to $k[\omega]$ and locally modeled on a cone singularity over a (p,q)-torus knot. Assuming that the volume obstruction is satisfied, we get:

$$k\tau \cdot \tau pq = kab \leq k\mathrm{Vol}\, M = \int_M k\omega \wedge \omega \overset{\mathrm{PD}}{=} \int_\Sigma \omega = \mathcal{A}_\omega(\Sigma),$$

so Σ contains $k\tau$ disjoint (singular) discs of area τpq that meet at x, that is a copy of $\Upsilon := \cup_{i=1,\dots,k\tau} \Delta_{\alpha_i}(p,q)$ (see Definition 4.4). Now, $M\backslash\Sigma$ has a Liouville form with residue $1/k$ at Σ as well as $E(a,b)\backslash\Upsilon$. The associated Liouville vector field on $E(a,b)$ is radius increasing by Proposition 4.5 ii) because $k\tau \cdot 1/k \geq \tau$, so the same construction as for Theorem 4 embeds the ellipsoid $E(a,b)$. □

7. Donaldson curves in blow-ups

7.1. In the blow-up of a ball

We prove now Lemma 5.1 avoiding the use of pseudo-holomorphic curves. The general statement is the following:

Proposition 7.1. *Let (M,ω) be a symplectic manifold with $[\omega] \in H^2(M,\mathbb{Z})$. Let N be a closed symplectic submanifold of arbitrary codimension. Then for all sufficiently large k, there exists a symplectic hypersurface Σ Poincaré-dual to $k[\omega]$ and such that the transverse intersection $\Sigma \cap N$ is a symplectic hypersurface of N Poincaré dual to $k[\omega_{|N}]$.*

Applying this proposition to the blow-up and its exceptional divisor yields a proof of Lemma 5.1. Although it is easy and rather folkloric (a more subtle version for Lagrangian submanifold can be found in [1]), we sketch a proof of Proposition 7.1, because we also need to adapt it to the less usual setting of symplectic orbifolds, see Section 7.2.

Quick review of Donaldson's construction. Let J be a compatible almost complex structure, $\mathcal{L}$ a hermitian line bundle over M with curvature ω. Donaldson's proof of the existence of a symplectic hypersurface consists in producing an approximately holomorphic and η-transverse section of $\mathcal{L}^{\otimes k}$. Recall the following definitions from [5]:

Definition 7.2. A sequence of sections (s_k) of $\mathcal{L}^{\otimes k}$ is said

- approximately holomorphic if $|\overline{\partial} s_k| \leq {}^{C}/\sqrt{k}$,
- η-transverse if $|ds_k(z)| \geq \eta$ for each $z \in M$ such that $|s_k(z)| < \eta$.

The vanishing locus of sections belonging to such a sequence are clearly smooth and symplectic when k is sufficiently large. For convenience, we usually forget the term "sequence" and speak of approximately holomorphic sections.

These sections are obtained as a perturbation of the zero-section by approximately-holomorphic peak-sections very localized around the points of the manifolds. These peak-sections decrease exponentially fast around a point, and have support in balls of radius of order ${}^{1}/\sqrt[3]{k}$. Let us describe a bit more the peak-section σ_p^k around a point $p \in M$. Identify a neighbourhood of p in M with a ball in $\mathbb{C}^n$ by a map χ_p that takes p to the origin, ω to the standard symplectic form on $\mathbb{C}^n$ and J to $i + O(|z|)$. Inside this ball, the bundle $\mathcal{L}^{\otimes k}$ has a connection given by $d + k\sum z_j d\overline{z_j} - \overline{z_j} dz_j$ and in the trivialization of the bundle given by radial parallel transport with respect to this connection, the peak section is $\sigma_p^k = \chi(z)e^{-k|z|^2}$, where $\chi(z)$ is a cut-off function at the right scale (notice that $e^{-k|z|^2}$ is i-holomorphic). Donaldson's method consists in adding inductively to the zero-section some small enough multiple of these peak-functions, centered at points on a regular grid, so that the perturbations gain transversality on the balls where the peak-section are added, while being cautious enough not to destroy the transversality already obtained.

Proof of Proposition 7.1. Take J compatible with ω, but also with $\omega_{|N}$ (in particular, J preserves TN). Notice that the restriction of the bundle $\mathcal{L}^{\otimes k}$ to N has curvature $k\omega_{|N}$. By classical neighbourhood theorems, a neighbourhood of a point $p \in N$ can be identified symplectically with a symplectic ball, where J is again $i + O(z)$ and N is a (complex) linear subspace. In these coordinates, the above mentioned peak-section σ_p^k on M restricts to the peak section $\sigma_{N,p}^k$ on N (associated to $(N, \omega_{|N}, J_{|N})$). Therefore, perturbing the zero section first above N by Donaldson's recipe provides a section which is zero at distance ${}^{1}/\sqrt[3]{k}$ of N, and which is η-transverse to zero over N. If we then continue the process with perturbations sufficiently small, we do not affect the transversality of the section over N, which implies that the intersection of the vanishing locus of the final section with N is transverse and defines a symplectic polarization of degree k of N. $\square$

For the proof of Theorem 6, we need a slightly stronger version of this statement. Namely, we must know that if $\omega_1, \dots, \omega_m$ are rational and close enough to a given symplectic form ω, the intersections of Donaldson hypersurfaces $\Sigma_1^k, \dots, \Sigma_m^k$ with N, which are Donaldson's hypersurfaces of N as we already saw, can be assumed to intersect transversally (in dimension 4, this will imply that these intersections are disjoint). This point is obtained by requiring the η-transversality for the sections of $\mathcal{L}_i^{\otimes k} \oplus \mathcal{L}_j^{\otimes k}$ (where $\mathcal{L}_i$ is Donaldson's bundle associated to ω_i). This analysis is done in detail in [18]. As in the previous proof, getting this transversality first above N and then on M gives the desired result.

Remark 7.3. In dimension 4, the intersections of ω_l-Donaldson hypersurfaces with a symplectic curve N can be assumed to be disjoint, provided the ω_l are close enough to a fixed symplectic form.

7.2. In the blow-up of an ellipsoid

In this paragraph, we explain that Donaldson's techniques generalize to the setting of 4-dimensional orbifolds. Recall that a cyclic orbifold singularity is a quotient of $(\mathbb{C}^2, 0)$ by the action $(e^{2i\pi/p}\cdot, e^{2i\pi/q}\cdot)$ of $\mathbb{Z}_p \times \mathbb{Z}_q$. Lemma 7.1 has the following analogue:

Lemma 7.4. *Let (M, ω) be a 4-dimensional symplectic orbifold with isolated cyclic singularities and $[\omega] \in H^2(M, \mathbb{Z})$. Let N be a closed symplectic suborbifold which is desingularized in the uniformizing charts of M (in particular, N has only isolated cyclic singularities). Then for all sufficiently large k, there exists a symplectic curve Σ Poincaré-dual to $k[\omega]$, which avoids all singular points of M and N, and whose intersection with N is transverse and positive.*

This lemma may well be true in a more general setting but this version is enough for proving Lemma 6.3, which is our purpose. Before proving this lemma, we first need to understand that Donaldson's method works in symplectic orbifolds.

Theorem 8. *Let (M, ω) be a 4-dimensional symplectic orbifold with isolated cyclic singularities, and $[\omega] \in H^2(M, \mathbb{Z})$. Then for all sufficiently large k, there is a smooth symplectic curve Σ_k Poincaré dual to $k[\omega]$ which avoids all singular points of M.*

As explained in the previous paragraph, Donaldson's method is based on several ingredients. One needs:

(1) A compatible almost complex structure J on (M, ω) (call g the induced metric),

(2) Regular grids Γ_k of M at scale $1/\sqrt{k}$,

(3) A line bundle $\mathcal{L}$ on M of curvature ω,

(4) The highly localized approximately holomorphic sections σ_p^k.

Then the method consists in adding iteratively to the zero-section (or to any approximately holomorphic section) of $\mathcal{L}^{\otimes k}$ some linear combination of the σ_p^k, where p belongs to the grid Γ_k. The first question is to decide whether the ingredients of the recipe are available in the orbiworld. An almost complex structure is easy to produce: in a uniformizing chart near the singularity, choose $J = i$. Since the action of the covering group is by unitary maps, it preserves i so J defines a complex structure on some neighbourhood of the singularities, which we can extend to an almost complex structure by general arguments. Similarly, in the uniformizing chart, the connection $d + k \sum z_j d\overline{z}_j - \overline{z}_j dz_j$ (with curvature $k\omega$) is also unitary-invariant, so it projects to a connection on the trivial line bundle over a neighbourhood of the singularities with curvature $k\omega$. Provided k is a multiple of a convenient number (the product of the orders of the local groups defining the orbifold), this trivial line bundle extends a line bundle over M with curvature $k\omega$. The regular grid is also not a problem. But the fourth object above has no obvious equivalent. Indeed, the controlled decay rate for the peak sections depends heavily on the existence of Darboux charts around each point p of the manifold which depend smoothly on p. But this point fails dramatically on symplectic orbifolds, because Darboux charts around regular points close to a singularity cannot contain this singularity, so they must be very distorted. Hence, this straightforward approach does not go through. Instead we proceed as follows. We first fix a Darboux uniformizing chart around each singularity. Namely, this is a symplectic covering

$$\Phi : B^4(1) \longrightarrow U \subset M$$

ramified along the axes $\{z_i = 0\}$, and invariant under the symplectic action of $G := \mathbb{Z}_p \times \mathbb{Z}_q$ on $B^4(1)$ given by

$$(\xi_1, \xi_2) \cdot (z_1, z_2) = (\xi_1 z_1, \xi_2 z_2) \tag{3}$$

In the complement of the neighbourhood U of the singularities given by the union of these charts, the needed parametric Darboux charts exist. Our approach is simply to produce a Donaldson section over $M\backslash U$ (approximately holomorphic and η-transverse) which naturally extends to M to a non-vanishing section over U. A classical way of summarizing Donaldson's method can be the following:

Proposition 7.5. *Let s_k be a sequence of sections of $\mathcal{L}^{\otimes k}$ such that*

$$\|s_k\|_{\mathcal{C}^0} \leq 1 \tag{4}$$

$$\|\overline{\partial} s_k\|_{\mathcal{C}^0} \leq {}^{c}/\sqrt{k} \qquad \textit{(approximately holomorphic).} \tag{5}$$

Then there exist constants $w_k(p)$, $p \in \Gamma_k$ such that $\widetilde{s}_k := s_k + \sum w_k(p)\sigma_p^k$ is approximately holomorphic and η-transverse for some $\eta > 0$ independent of k.

Starting for instance from the sequence $s_k \equiv 0$, Donaldson gets de existence result of [5]. However, nothing prevents a priori to start with a different section, and that is what we did in paragraph 7.1. As we already explained, the method consists in adding a combination of the peak sections σ_p^k in order to gain transitivity on larger and larger sets. Then the condition (4) seems rather strong and unnecessary since in the region where $|s_k|$ is large, transversality is already achieved and we do not need to modify s_k in this region. The next proposition weakens this hypothesis. The control on $\|s_k\|_{\mathcal{C}^0}$ can not be totally removed, but it is enough to assume that the regions where s_k is large or small lie sufficiently apart one from the other.

Proposition 7.6. *Let Ω be an open subset of M, and denote by $\Omega_{1/\sqrt{k}} \supset \Omega$ the ${}^{1}/\sqrt{k}$-neighbourhood of Ω. Assume that s_k is a sequence of sections of $\mathcal{L}^{\otimes k}$ over M such that:*

$$|s_k| \geq 1 \textit{ on } M\backslash\Omega \tag{6}$$

$$\|s_k\|_{\mathcal{C}^0} \leq 2 \textit{ on } \Omega_{1/\sqrt{k}} \tag{7}$$

$$\|\overline{\partial} s_k\|_{\mathcal{C}^0} \leq {}^{c}/\sqrt{k}. \tag{8}$$

Then there exist constants $w_k(p)$, $p \in \Gamma_k \cap \Omega$ such that $\widetilde{s}_k := s_k + \sum w_k(p)\sigma_p^k$ is approximately holomorphic and η-transverse on M, with $|\widetilde{s}_k| \geq {}^{1}/{2}$ on $M\backslash\Omega$.

The proof goes by checking that Donaldson's method works in this setting. In a few words, the estimates (7) and (8) ensures that s_k is uniformly bounded and approximately holomorphic on the balls $B_k(p)$, $p \in \Gamma_k \cap \Omega$. Then the local analysis ([5], Theorem 20) can be performed because it only involves the boundedness of the section on balls of size ${}^{1}/\sqrt{k}$. Finally, the local-to-global procedure works the same, because all estimates rely on the decay rates of the σ_p^k, which depend themselves only on estimates on the derivatives of the Darboux charts for $p \in \Gamma_k \cap \Omega$, still available in our relative setting.

Corollary 7.7. *If B is a closed ball in M, there exists an approximately holomorphic and η-transverse section of $\mathcal{L}\otimes k$ whose zero set avoids B.*

It might be worth recalling that Donaldson's remark concerning the convergence in the sense of currents of (a renormalization of) the vanishing loci of these sections to ω holds under the condition that the derivatives of the sections are bounded. The apparent contradiction with this corollary is explained by the fact that this condition does not hold for the sections produced below.
Proof: By standard neighbourhoods results, since B is a closed ball, there exists a symplectic embedding of a larger ball that extends B. To fix ideas, assume that $B = \varphi\big(B(1)\big)$ and that φ extends to $B(2)$. Consider the sections

$$s_k := \chi(|z|)e^{k-k|z|^2}$$

of the trivial line bundle over $B(2)$, where $\chi \equiv 1$ on $B(3/2)$ and $\chi \equiv 0$ near $\partial B(2)$. This sequence of sections is *holomorphic* on $B(3/2)$ (for the complex structures on the bundle $\mathcal{L}^{\otimes k}$) and since the derivatives of χ are of order 1,

$$\|\overline{\partial} s_k\|_{\mathcal{C}^0} \leq \|\chi\|_{\mathcal{C}^1} e^{-k/2} \text{ on } B(2),$$

which means that s_k is indeed approximately holomorphic on $B(2)$. Finally, $|s_k| \geq 1$ on $B(1)$, $|s_k| \leq 1$ on $B(2)\backslash B(1)$, and $|s_k| < e$ on $B(1)\backslash B(1 - 1/\sqrt{k})$. Applying Proposition 7.6, we get a section whose zero set obviously avoids $B(1)$.

Proof of Theorem 8. Consider the bundle $\mathcal{L}$, described right after the statement of Theorem 8, which extends the trivial bundle $U \times \mathbb{C}$ over U with curvature $q\omega$ for some $q \in \mathbb{N}$. Near the singularities, this bundle is of the form $(B \times \mathbb{C})_{/G} \approx B_{/G} \times \mathbb{C}$. The local section $s_k := \chi(|z|^2)e^{kq-kq|z|^2}$ over $B(1)$ - already used in the previous corollary - is G-equivariant (because G acts trivially on the fibers and by unitary action on the base). It therefore descends to a section s_k of $\mathcal{L}^{\otimes k}$ supported in a neighbourhood of each singularity. Moreover, this section has exactly the same decreasing properties as the corresponding ones in the ball, so they satisfy the estimates (6), (7) and (8) for $\Omega := M\backslash U$. Now the complement of U is a smooth symplectic manifold, and Proposition 7.6 applies exactly as in Corollary 7.7. This concludes our proof. It is also clear that the same argument as for Proposition 7.1 applies and proves Lemma 7.4. □

8. Remarks and questions

Faithful symplectic charts

This paper is about the relations between symplectic embeddings and curve singularities. As already pointed out in paragraph 5.3, we have no reason to limit ourselves to embeddings of balls or ellipsoids. In fact, any algebraic singularity of a curve gives rise to an embedding of some domain (the basin of attraction of the curve with respect to the Liouville vector field dual to $d^c \ln|f|$, where $f(z) = 0$ is an equation of our singularity). A natural question is the following:

Question. Is there a symplectic topology of curve singularities? For instance, what can be said about the symplectic invariants (like capacities) of the domain $D(S)$ obtained by inflation along a curve with some singularity with prescribed type S?

These domains $D(S)$ might be used to find big symplectic charts. Let me comment a bit on this. A smooth manifold always contains a ball that covers almost everything, in the sense that its complement is of codimension at least one. In symplectic geometry, this is wrong, but there are ellipsoidal charts with this property, at least in rational symplectic manifolds (see [17]). In particular, the volume of its complement vanishes. But this ellipsoid may be very thin and long, so it may not represent well the symplectic features of the ambient manifold.

Question. Given a set of capacities, and a symplectic manifold M, is there a domain $\Omega \subset \mathbb{C}^2$ with the same capacities as M, and which embeds into M? Can the domains $D(S)$ help?

Minimal degree of curves with prescribed singularities [7, 22]

In Section 5.4, we use algebraic geometry to produce symplectic packings. But the reverse direction can also be investigated, and we can ask whether the symplectic point of view produces something new to the algebraic geometry of singularities. One very natural question is the following:

Question. Given a list of singularity types $(S_i)_{i=1\dots r}$, what is the minimal degree $d(S_i)$ of an algebraic curve of $\mathbb{P}^2$ with r singular points of type S_i?

Of course, ”singularity type” can have various meanings (up to analytic, or topological equivalence), which may lead to different answers. We refer for instance to [7, 22] for algebraic approaches to this problem. Now the remark is the following: in view of Theorems 1 and 3, and since the packing problem was completely solved for any configuration of balls and ellipsoids in $\mathbb{P}^2$ by McDuff in [12], the corresponding question in a symplectic setting has a complete solution, at least asymptotically in the degree. But any symplectic lower bound for $d(S_i)$ is also an algebraic one, and it might sometimes be better than what is known. For instance, we know from [12] that the closed ellipsoid $\tau E(1,6)$ embeds into $\mathbb{P}^2$ if and only if $\tau \leq 2/5$. Thus,

Theorem 9. *Let S be a singularity with a symplectic model given by m smooth branches intersecting in one point with tangency order 6:*

$$S := \Big[\{w(w-z^6)\cdots(w-(m-1)z^6)=0\}\Big].$$

Then $d(S) \geq \frac{5}{2}m$.

The universal bound given in [22] $\sqrt{\mu(S)}+1 = \sqrt{6m^2-7m+1}+1 \approx 2.45m$ is smaller that $2.5m$ when m is large.

McDuff’s result on ellipsoid embeddings

Let us recall that McDuff proved in [12] that on some manifolds, the embedding problem for an ellipsoid with fixed shape is equivalent to the packing problem for some balls with given ratio between the different radii. It seems natural to ask whether this result can be directly proved at the level of curve singularities. In some sense, the question is whether when we have a Gromov-Witten invariant for some problem involving a curve in some class with some simple singularities (several nodal points for instance), one can deform this curve in order to produce a new curve with a single but more complicated singularity.

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