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THE CLOSURE OF THE SYMPLECTIC CONE OF ELLIPTIC SURFACES

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The symplectic cone of a closed oriented 4-manifold is the set of cohomology classes represented by symplectic forms. A well-known conjecture describes this cone for every minimal Kähler surface. We consider the case of the elliptic surfaces $E(n)$ and focus on a slightly weaker conjecture for the closure of the symplectic cone. We prove this conjecture in the case of the spin surfaces $E(2m)$ using inflation and the action of self-diffeomorphisms of the elliptic surface. An additional obstruction appears in the non-spin case.

1. Introduction

Let M be a closed oriented 4-manifold. We are interested in the set $\mathcal{C}_M$ of real cohomology classes represented by symplectic forms on M, called the *symplectic cone* of M. It is indeed a cone because a non-zero multiple of any symplectic form is again symplectic. We only consider symplectic forms ω compatible with the orientation, so that $\omega \wedge \omega$ is everywhere positive. It follows that the symplectic cone is a subset of the *positive cone* $\mathcal{P}$, given by the set of elements in $H^2(M;\mathbb{R})$ which have positive square. In fact, according to the proof of Observation 4.3 in [**9**], the symplectic cone is always an open subset of the positive cone. If the 4-manifold M does not admit a symplectic form then the set $\mathcal{C}_M$ is empty. It is also useful to denote by $\mathcal{P}^A$ for a non-zero cohomology class $A \in H^2(M;\mathbb{R})$ the set of elements in $\mathcal{P}$ which have positive cup product with A. Clearly, $\mathcal{P}^A \cup \mathcal{P}^{-A}$ is a cone. In addition, we set $\mathcal{P}^0 = \mathcal{P}$.

The symplectic cone has been determined in the following cases:

(a) S^2-bundles over surfaces [**17**].
(b) T^2-bundles over T^2 [**8**].
(c) All 4-manifolds with a fixed point free circle action [**3, 5, 6**].
(d) All symplectic 4-manifolds with $b_2^+ = 1$ [**14**].
(e) The $K3$ surface [**13**].

(f) Fibre sums along tori of $T^2 \times \Sigma_g$ and minimal elliptic Kähler surfaces with $b_2^+ = 1$, for example Enriques or Dolgachev surfaces [**4**].

The simply-connected 4-manifolds among these cases either have $b_2^+ = 1$ or are diffeomorphic to the $K3$ surface, because the 4-manifolds in (c) have zero Euler characteristic.

From now on we denote by M a simply connected elliptic surface $E(n)$ without multiple fibres and with the complex orientation. Since by the results mentioned above the symplectic cone is known for $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ and the $K3$ surface $E(2)$ we assume that $n \geq 3$. Let F denote the class of the fibre in an elliptic fibration on M. We set

$$c_1(M) = -(n-2)PD(F),$$

where PD denotes the Poincaré dual of the homology class. Note that symplectic forms have well-defined Chern classes, defined by considering any compatible almost complex structure. If ω is a symplectic form with first Chern class $c_1(M,\omega)$, then $-\omega$ is a symplectic form with first Chern class $-c_1(M,\omega)$. It is known from Seiberg–Witten theory [**10**] that every symplectic form on $E(n)$ has up to sign first Chern class equal to $c_1(M)$. It is also known from the theorems of Taubes [**18**] that for every symplectic structure ω the Poincaré dual of the class $-c_1(M,\omega)$ is represented by an embedded symplectic surface. This implies that $\omega \cdot c_1(M)$ is non-zero, hence the symplectic cone satisfies

$$\mathcal{C}_M \subset \left(\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}\right).$$

A well-known conjecture due to Tian–Jun Li [**13**] says that the following holds:

Conjecture 1 (Strong conjecture). *We have*

$$\left(\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}\right) \subset \mathcal{C}_M.$$

Hence, every class of positive square whose cup product with the first Chern class of M is non-zero is represented by a symplectic form.

The conjecture should even hold for any closed 4-manifold underlying a minimal Kähler surface, but we only consider the case of elliptic surfaces. There is also a slightly weaker form of the conjecture. We denote by $\overline{\mathcal{C}}_M$ the closure of the symplectic cone in the vector space $H^2(M;\mathbb{R})$.

Conjecture 2 (Weak conjecture). *We have*

$$\left(\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}\right) \subset \overline{\mathcal{C}}_M.$$

Hence, every class of positive square whose cup product with the first Chern class of M is non-zero is the limit of a sequence of symplectic classes. Equivalently, the symplectic cone $\mathcal{C}_M$ is dense in $\left(\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}\right)$.

In the following we only consider the weak conjecture. To state the theorem we want to prove, consider the following definition:

Definition 1. We define $\mathcal{P}^{>} \subset \mathcal{P}$ to be the subcone of elements ω with

$$\omega^2 > (\omega \cdot PD(F))^2.$$

For a non-zero class $A \in H^2(M;\mathbb{R})$ we set

$$\mathcal{P}^{A>} = \mathcal{P}^{>} \cap \mathcal{P}^{A}.$$

In particular, this applies to $A = PD(F)$ and $A = \pm c_1(M)$. Note that $\mathcal{P}^{A>} \cup \mathcal{P}^{-A>}$ is a subcone of $\mathcal{P}^{A} \cup \mathcal{P}^{-A}$.

Then we have:

Theorem 2. *Let $m \geq 2$ be an integer. If M is the spin surface $E(2m)$ then*

$$\left(\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}\right) \subset \overline{\mathcal{C}}_M.$$

If M is the non-spin surface $E(2m-1)$ then

$$\left(\mathcal{P}^{c_1(M)>} \cup \mathcal{P}^{-c_1(M)>}\right) \subset \overline{\mathcal{C}}_M.$$

This proves Conjecture 2 in the case of the spin elliptic surfaces $E(2m)$. At the moment we do not know how to prove the full Conjecture 2 in the non-spin case. One can view these results as evidence that the strong Conjecture 1 is indeed true. The sequences of symplectic forms in the theorem are all obtained from a single symplectic form by inflation along certain symplectic surfaces and the action of the orientation preserving self-diffeomorphisms of the elliptic surface M.

2. Some notation

We follow the notation from [**11**]. In particular, all self-diffeomorphisms of M are orientation preserving. We often denote a symplectic form and its class by the same symbol. Note that considering minus a given symplectic form we see that to prove the weak conjecture it suffices to prove that

$$\mathcal{P}^{PD(F)} \subset \overline{\mathcal{C}}_M.$$

We want to prove the following theorem, which is equivalent to Theorem 2:

Theorem 3. *Let $m \geq 2$ be an integer. If M is the spin surface $E(2m)$ then*

$$\mathcal{P}^{PD(F)} \subset \overline{\mathcal{C}}_M.$$

If M is the non-spin surface $E(2m-1)$ then

$$\mathcal{P}^{PD(F)>} \subset \overline{\mathcal{C}}_M.$$

We will first prove a special case of Theorem 3 since this is easier and uses the same method as in the general case. We need some notation. Consider the manifold $M = E(n)$ with $n \geq 3$ and define an integer m by $n = 2m$ if n is even and $n = 2m - 1$ if n is odd.

Definition 4. Let W be the embedded surface obtained by smoothing the intersections of a section V of the elliptic surface M of square $-n$ and m parallel copies of the fibre F. Let R denote a rim torus of square zero and S a dual vanishing sphere of square -2 in M. Both F, W and R, S intersect in a single transverse positive point. Otherwise the surfaces are disjoint.

The vanishing sphere S is obtained by sewing together in the fibre sum

$$E(n) = E(1)\#_{F=F}E(n-1)$$

two vanishing disks coming from singular fibres with the same vanishing cycles. The surface W has self-intersection zero if n is even and one if n is odd. The surfaces F and W span a copy of the standard hyperbolic form H in the intersection form if n is even and a copy of H' if n is odd, where

$$H' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

We will denote both intersection forms by $H(n)$. Since this form is unimodular the total intersection form over the integers looks like

$$Q_M = H(n) \oplus H(n)^{\perp}.$$

We can decompose any class $\omega \in H^2(M; \mathbb{R})$ according to this splitting as

$$\omega = PD(\alpha F + \beta W) + \omega'$$

with $\omega' \in H(n)^{\perp}$ (here we mean the real subspace spanned by this lattice). Note that

$$\beta = \omega \cdot PD(F).$$

Definition 5. We define

$$\mathcal{P}^{PD(F)+} \subset \mathcal{P}^{PD(F)}$$

to be the subset of elements of the form $\omega = PD(\alpha F + \beta W) + \omega'$ where α, β and ω'^2 are positive. We call the classes in this subset *positive.*

Theorem 6. *We have*

$$\mathcal{P}^{PD(F)+} \subset \overline{\mathcal{C}}_M.$$

This theorem describes the first subset of $\mathcal{P}^{PD(F)}$ that we can represent by the limits of symplectic forms. We will extend it later and prove Theorem 3.

3. Symplectic forms and diffeomorphisms

The inflation procedure, introduced by Lalonde and McDuff [**12, 17**], shows that if Σ is a closed connected symplectic surface of non-negative square in a closed symplectic 4-manifold (Y, ω), then the class $[\omega] + tPD(\Sigma)$ is represented by a symplectic form for all $t \geq 0$. We need the following generalized inflation lemma:

Lemma 7. *Let (Y, ω) be a closed symplectic 4-manifold and $\Sigma_1, \Sigma_2 \subset Y$ closed connected symplectic surfaces of non-negative square which intersect transversely in a single positive point. Then for all real numbers $r_1, r_2 \geq 0$ the class*

$$[\omega] + r_1 PD(\Sigma_1) + r_2 PD(\Sigma_2)$$

is represented by a symplectic form.

Proof. By the symplectic neighbourhood theorem Σ_1 has a tubular neighbourhood $\nu\Sigma_1$ with symplectic fibres. According to Lemma 2.3 in [**9**] we can assume that Σ_2 intersects $\nu\Sigma_1$ in one of the disk fibres. If we first do inflation along Σ_1 as in [**17,** Lemma 3.7] then the symplectic form changes only in the tubular neighbourhood $\nu\Sigma_1$ and the fibres stay symplectic. Hence Σ_2 remains symplectic and we can then do inflation along Σ_2. Compare with [**2,** Lemma 2.1.A] and [**15,** Theorem 2.3]. □

Proposition 8. *There exists a symplectic form on M such that F, W, R and S are symplectic surfaces.*

From the Gompf sum construction [**9**] applied to the fibre sum

$$E(n) = E(1)\#_{F=F}E(n-1)$$

it is clear that there exists a symplectic form on M such that F and V are symplectic. Hence the surface W is also symplectic.

Lemma 9. *We can choose the surfaces R and S such that they are Lagrangian for a symplectic form from the Gompf construction.*

Proof. The claim is clear for the rim torus R: in the fibre sum construction it is given by $R = \gamma \times \partial D^2$ where γ is one of the circle factors of the torus $F = S^1 \times S^1$ in a tubular neighbourhood $F \times D^2$ on which the symplectic form is a standard product form. The claim for the vanishing sphere S follows from section 8 in [**1**]. □

Hence Proposition 8 is a consequence of the following theorem that we formulate in a more general way. The proof is very similar to Lemma 1.6 in [**9**] due to Gompf which states the same for disjoint Lagrangians.

Theorem 10. *Let (X, ω) be a closed symplectic 4-manifold and $L_1, \dots, L_n$ closed connected embedded oriented Lagrangian surfaces in X which intersect each other transversely so that at most two surfaces intersect in any given* *point of X. Suppose that the classes of these surfaces are linearly independent in $H_2(X;\mathbb{R})$. Then there exists a symplectic structure ω' on X, deformation equivalent to ω, such that all of these Lagrangian surfaces become symplectic. We can choose the symplectic structure ω' such that the induced volume forms on the Lagrangians have any given sign. We can also assume that any symplectic surface disjoint from the Lagrangians stays symplectic.*

Proof. Let $a_1,\dots,a_n$ be any real numbers. Since $H^2(X;\mathbb{R})$ is the dual space of second real homology there exists a closed 2-form η on X such that

$$\int_{L_i}\eta = a_i, \quad i=1,\dots,n.$$

Choose volume forms ω_i on L_i for each i such that

$$\int_{L_i}\omega_i = \int_{L_i}\eta.$$

Let j_i denote the embedding of L_i into X. There exist 1-forms α_i on L_i such that

$$\omega_i - j_i^*\eta = d\alpha_i.$$

Let $\pi_i\colon \nu L_i \to L_i$ denote tubular neighbourhoods and choose cut-off functions $\rho_i(r)$ with support on the tubular neighbourhoods which depend only on the radius r and are 1 on the zero section. Define 1-forms

$$\overline{\alpha}_i = \rho_i\pi_i^*\alpha_i$$

on the tubular neighbourhoods. Extend them by zero outside of the neighbourhood and set

$$\eta' = \eta + \sum_i d\overline{\alpha}_i.$$

We claim that $j_i^*\eta' = \omega_i$. This follows if we can show that

$$j_i^* d\overline{\alpha}_k = 0 \quad \text{for } k \neq i.$$

This is clear if L_k does not intersect L_i by making the tubular neighbourhood of L_k small enough so that it does not intersect L_i. Suppose that L_k and L_i intersect in a point p. We can assume that L_i intersects νL_k in a disk fibre of the tubular neighbourhood. We have

$$d\overline{\alpha}_k = \rho_k' dr \wedge \pi_k^*\alpha_k + \rho_k\pi_k^* d\alpha_k.$$

By assumption, π_{k*} is the zero map on T_qL_i for each point q on the disk fibre $L_i\cap\nu L_k$. Therefore $d\overline{\alpha}_k$ is zero on any two vectors in T_qL_i. Hence $j_i^*d\overline{\alpha}_k = 0$.

Consider the closed 2-form

$$\omega' = \omega + t\eta'.$$

For small positive t the form ω' is symplectic. Since the L_i are Lagrangian for ω we have $j_i^*\omega' = t\omega_i$. Hence the L_i are now symplectic surfaces with (small) positive or negative volume, depending on the sign of a_i. □

Definition 11. Let ω_0 denote a symplectic form on M given by Proposition 8. We can assume that the symplectic form has the same sign on both R and S. Let T denote the symplectic torus of square 0 obtained by smoothing the intersection between R and S. The tori R and T intersect in a single positive transverse point.

The surfaces R and T together span a copy of H in the intersection form, which we denote by H_{RT}. In summary the intersection form of M is equal to

$$Q_M = H(n) \oplus H_{RT} \oplus aH \oplus b(-E_8)$$

with certain integers $a, b \geq 1$.

Definition 12. We say that a self-diffeomorphism of M satisfies $(*)$ if it is the identity on the first summand of $H(n) \oplus H(n)^{\perp}$. It then preserves the splitting $H(n) \oplus H(n)^{\perp}$.

We will frequently use the following proposition that was proved in [**11**].

Proposition 13. *Every integral class in $H(n)^{\perp}$ can be mapped to any integral linear combination of R and T of the same square and divisibility by a self-diffeomorphism of the elliptic surface M that satisfies $(*)$. Taking a multiple we see that we can map in this way any rational class in $H(n)^{\perp}$ to a rational linear combination of R and T.*

The following is clear:

Lemma 14. *Let $f\colon M \to M$ be an orientation preserving diffeomorphism. If C and D are homology classes on M with $f_*C = D$, then $(f^{-1})^*PD(C) = PD(D)$.*

We will now cover a large part of the positive cone by symplectic forms in the following way: we have a symplectic form ω_0, so that the surfaces F, W, R and T are symplectic. The class of ω_0 can be written as

$$\omega_0 = PD(\alpha_0 F + \beta_0 W + \gamma_0 R + \delta_0 T) + Z_0,$$

where Z_0 is a class in the real span of $aH \oplus b(-E_8)$. Using inflation with very large parameters and then dividing by a large number it follows that the class

$$\omega = PD(\alpha F + \beta W + \gamma R + \delta T)$$

plus some arbitrarily small rest is represented by a symplectic form for all positive coefficients $\alpha, \beta, \gamma, \delta$. The second method we use are the actions of self-diffeomorphisms on cohomology. In particular, we can map according to Proposition 13 any rational class in $H^2(M;\mathbb{R})$ using a self-diffeomorphism to a rational linear combination of the Poincaré duals of F, W, R and T. This will suffice to prove Theorem 6 in Section 4, because in this situation all coefficients are positive. To prove the more general Theorem 3 in Section 5 we will introduce in Lemma 16 another diffeomorphism that allows in some situations to change a negative coefficient in the expansion of ω into a positive one.

4. Proof of Theorem 6 on positive classes

We have the following lemma that proves one of the steps outlined above.

Lemma 15. *Let ω be a class in $\mathcal{P}^{PD(F)}$. Then there exist a sequence of self-diffeomorphisms ϕ_k of the elliptic surface M and classes σ_k of the form*

$$\sigma_k = PD(\alpha F + \beta W + \gamma_k R + \delta_k T)$$

with $\beta > 0$ such that $\phi_k^* \sigma_k$ converges to the class ω. The diffeomorphisms ϕ_k satisfy $(*)$. If ω is a class in the subset $\mathcal{P}^{PD(F)+}$ then we can assume that all coefficients of σ_k are positive.

Proof. We decompose the class ω as

$$\omega = PD(\alpha F + \beta W) + \omega',$$

where $\omega' \in H(n)^\perp$ and $\beta > 0$. There exists a sequence ω'_k of rational classes in $H(n)^\perp$ converging to the class ω'. Using the second part of Proposition 13 there exist self-diffeomorphisms ϕ_k that satisfy $(*)$ and map

$$\phi_k^* PD(\gamma_k R + \delta_k T) = \omega'_k$$

for certain rational numbers γ_k, δ_k. Setting

$$\sigma_k = PD(\alpha F + \beta W + \gamma_k R + \delta_k T)$$

we get the first claim. If ω is a class in $\mathcal{P}^{PD(F)+}$ we can assume that all $\omega_k'^2$ are positive. Hence, we can assume that γ_k and δ_k are positive. □

Recall that we have a symplectic form ω_0. As above, the class of this form can be written as

$$\omega_0 = PD(\alpha_0 F + \beta_0 W + \gamma_0 R + \delta_0 T) + Z_0,$$

where Z_0 is a class in the real span of $aH \oplus b(-E_8)$. We now prove Theorem 6.

Proof. Let ω be a class in $\mathcal{P}^{PD(F)+}$. Choose a sequence σ_k as in Lemma 15. Then

$$\sigma_k = PD(\alpha F + \beta W + \gamma_k R + \delta_k T),$$

where all coefficients are positive. Consider the symplectic form ω_0 with the symplectic surfaces F, W, R, T. We apply the inflation Lemma 7 to the form

ω_0, which means that we can add to ω_0 any linear combination of the classes F, W, R, T with positive coefficients. This shows that the class

$$N_k\sigma_k + Z_0$$

is represented by a symplectic form for any sufficiently large positive number N_k. Hence also the classes

$$\eta_k = \sigma_k + \frac{1}{N_k}Z_0$$

are represented by symplectic forms. We know that $\phi_k^*\sigma_k$ converges to ω. We can choose the numbers N_k large enough so that $\frac{1}{N_k}\phi_k^* Z_0$ converges to 0. Then $\phi_k^*\eta_k$ converges to ω, hence $\omega \in \overline{\mathcal{C}}_M$. □

5. Proof of the main Theorem 3

We will use the following lemma, which shows that certain automorphisms of the intersection form are realized by self-diffeomorphisms.

Lemma 16. *For an integer i let f_i denote the map which is the identity on all summands of the intersection form except on $H(n) \oplus H_{RT}$, where it is given by*

$$\begin{aligned} F &\mapsto F, \\ W &\mapsto W + iT, \\ R &\mapsto R - iF, \\ T &\mapsto T. \end{aligned}$$

Then f_i is induced by a self-diffeomorphism of M.

Proof. It is easy to check that f_i is an automorphism of the intersection form. The map f_i leaves F and hence $c_1(M)$ invariant. Letting i be a real number and taking $i \to 0$ we see that f_i has spinor norm one. This implies the claim by the work of Friedman and Morgan [**7**]; see also [**16**]. □

We denote a diffeomorphism that induces f_i by the same symbol. The induced automorphism f_i^* on cohomology maps

$$\omega = PD((\alpha - i\gamma)F + \beta W + \gamma R + (\delta + i\beta)T),$$

to

$$f_i^*\omega = PD(\alpha F + \beta W + \gamma R + \delta T).$$

Note that the class ω can be positive even if $f_i^*\omega$ is not positive. The main difficulty in the case of Theorem 3 is that we have to approximate classes, which are no longer positive, by symplectic forms. However, the automorphism f_i^* allows us in some cases to map a positive class to such a non-positive class. The positive class can then be reached by inflation. Hence, we have to show that under our assumptions we can always find an integer i such that f_i^* maps a positive class to our given class.

Suppose for example that we want the class ω as above to be positive. We can assume that $\beta, \gamma > 0$. Then ω is positive if and only if $\alpha - i\gamma > 0$ and $\delta + i\beta > 0$. This is possible only if

$$\alpha\beta + \gamma\delta > 0,$$

which is equivalent to $\omega^2 > 0$ if M is spin and $\omega^2 > \beta^2$ if M is non-spin. Note that $\beta = \omega \cdot PD(F)$. This is the reason why we have to restrict to the subset $\mathcal{P}^{PD(F)>}$ in the non-spin case.

We now begin with the proof of Theorem 3. Fix a cohomology class ω in $H^2(M;\mathbb{R})$. If M is the elliptic surface $E(2m)$ assume that ω is in the subset $\mathcal{P}^{PD(F)}$ and if M is the surface $E(2m-1)$ assume that ω is in the subset $\mathcal{P}^{PD(F)>}$. We want to approximate ω by symplectic classes. Write

$$\omega = PD(\alpha F + \beta W) + \omega',$$

where ω' is an element of the real span of $H(n)^{\perp}$. The following inequality for the coefficients of the class ω is a consequence of our assumptions.

Lemma 17. *We have*

$$\alpha > -\frac{\omega'^2}{2\beta}.$$

Proof. In both cases $\beta > 0$ and

$$\begin{aligned} 0 < \omega^2 &= 2\alpha\beta + \beta^2 W^2 + \omega'^2 \\ &= 2\alpha\beta + \epsilon(n)\beta^2 + \omega'^2, \end{aligned}$$

where $\epsilon(n) = 0$ if n is even and $\epsilon(n) = 1$ if n is odd. If $n = 2m$ is even we get

$$2\alpha\beta > -\omega'^2$$

hence

$$\alpha > -\frac{\omega'^2}{2\beta}.$$

If $n = 2m-1$ is odd we get by the assumption that ω is in $\mathcal{P}^{PD(F)>}$

$$\beta^2 < \omega^2 = 2\alpha\beta + \beta^2 + \omega'^2.$$

This again implies the claim. □

We now prove a slightly technical lemma. The estimate in (b) will be used in Lemma 19 to show that we can find integers i_k such that the automorphisms $f_{i_k}^*$ map a sequence of positive classes to another sequence, which can then by mapped by diffeomorphisms to a sequence converging to our given class ω.

Lemma 18. *There exists a sequence ω'_k of rational classes in $H(n)^\perp$ converging to ω' with the following properties:*

(a) $\omega'^2_k > \omega'^2$ *for all indices k.*

(b) *Write $\omega'_k = \frac{1}{A_k}\tau_k$ where A_k is a positive rational number and τ_k is an indivisible integral class in $H(n)^\perp$. Then there exist integers i_k with*

$$\alpha > \frac{i_k}{A_k} > -\frac{\omega'^2}{2\beta}.$$

Proof. Let ω''_k be any rational sequence in $H(n)^\perp$ converging to ω'. We can assume that

$$\omega''^2_k > \omega'^2$$

because every neighbourhood of ω' contains rational elements with this property. Write

$$\omega''_k = \frac{1}{B_k}\mu_k,$$

where B_k is a positive rational number and μ_k is integral and indivisible. For each k we can find an integral basis $e_1, e_2, \ldots, e_r$ of the lattice $H(n)^\perp$ such that $e_1 = \mu_k$. The basis depends on k, but we do not write the index. Let C_k be an arbitrary sequence of positive integers converging to infinity. Consider the rational number $A_k = C_k B_k$ and the integral class $\tau_k = C_k\mu_k + e_2$. Then τ_k is indivisible. Define

$$\omega'_k = \frac{1}{A_k}\tau_k = \frac{1}{B_k}\left(\mu_k + \frac{1}{C_k}e_2\right).$$

If we choose the integers C_k large enough the sequence ω'_k converges to ω' (note that e_2 depends on k). Moreover, we can assume that $\omega'^2_k > \omega'^2$. If C_k and hence A_k is large enough we can find by Lemma 17 an integer i_k such that

$$\alpha > \frac{i_k}{A_k} > -\frac{\omega'^2}{2\beta}.$$

□

Let $\omega'_k = \frac{1}{A_k}\tau_k$ be the sequence from Lemma 18. Since τ_k is an integral indivisible class in $H(n)^\perp$ we can find by Proposition 13 a self-diffeomorphism ϕ_k of the elliptic surface M satisfying $(*)$ such that

$$\tau_k = \phi_k^* PD(R + \delta_k T)$$

for certain integers δ_k. We get

$$\omega'_k = \phi_k^* PD\left(\frac{1}{A_k}R + \frac{\delta_k}{A_k}T\right). \tag{5.1}$$

This implies that the sequence

$$\phi_k^* PD\left(\alpha F + \beta W + \frac{1}{A_k}R + \frac{\delta_k}{A_k}T\right)$$

converges to our given class ω. Consider the automorphism f_i^* from Lemma 16 and apply $(f_i^{-1})^*$ to the sequence

$$PD\left(\alpha F + \beta W + \frac{1}{A_k}R + \frac{\delta_k}{A_k}T\right)$$

where $i = i_k$ for the integer i_k from Lemma 18. This implies that there exist self-diffeomorphisms $\psi_k = f_{i_k} \circ \phi_k$ such that $\psi_k^* \sigma_k$ converges to ω, where

$$\sigma_k = PD\left(\left(\alpha - \frac{i_k}{A_k}\right)F + \beta W + \frac{1}{A_k}R + \left(\frac{\delta_k}{A_k} + i_k\beta\right)T\right).$$

Lemma 19. *The numbers* $\alpha - \frac{i_k}{A_k}$ *and* $\frac{\delta_k}{A_k} + i_k\beta$ *are positive.*

Proof. The first claim is clear by construction in Lemma 18. Note that by formula (5.1) above

$$\omega_k'^2 = \frac{2}{A_k^2}\delta_k$$

and by construction

$$\frac{i_k}{A_k} > -\frac{\omega'^2}{2\beta} > -\frac{\omega_k'^2}{2\beta}.$$

Hence,

$$\frac{\delta_k}{A_k} = \frac{1}{2}\omega_k'^2 A_k$$

and

$$i_k\beta > -\frac{1}{2}\omega_k'^2 A_k.$$

This implies the second claim. □

Note that all coefficients of σ_k are positive. We now argue as in the proof of Theorem 6: There exist classes

$$\eta_k = \sigma_k + \frac{1}{N_k}Z_0$$

represented by symplectic forms such that $\psi_k^*\eta_k$ converges to ω. Hence, $\omega \in \overline{\mathcal{C}}_M$. This proves Theorem 3.

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