CAPPING OFF OPEN BOOKS AND THE OZSVÁTH–SZABÓ CONTACT INVARIANT

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If (S, ϕ) is an open book with disconnected binding, then we can form a new open book (S', ϕ') by capping off one of the boundary components of S with a disc. Let $M_{S,\phi}$ denote the 3-manifold with open book decomposition (S, ϕ). We show that there is a U-equivariant map from $HF^+(-M_{S',\phi'})$ to $HF^+(-M_{S,\phi})$ which sends $c^+(S', \phi')$ to $c^+(S, \phi)$, and we discuss various applications. In particular, we determine the support genera of almost all contact structures that are compatible with genus one, one boundary component open books. In addition, we compute $d_3(\xi)$ for every tight contact manifold (M, ξ) supported by a genus one open book with periodic monodromy.

1. Introduction

Giroux's correspondence between contact structures up to isotopy and open books up to positive stabilization allows us to translate questions from contact geometry into questions about diffeomorphisms of compact surfaces with boundary [16]. As a result, one is inclined to wonder about the contact-geometric significance of certain natural operations which can be performed on open books. For instance, let us define the "composition" of two open books (S, ϕ_1) and (S, ϕ_2) to be the open book $(S, \phi_1 \circ \phi_2)$. In [3], we use the Ozsváth–Szabó contact invariant (see [31]) to study the relationship between the contact structures supported by two such open books and the contact structure supported by their composition. There, we prove the following.

Theorem 1.1 ([3]). *If $c(S, \phi_1)$ and $c(S, \phi_2)$ are both non-zero, then so is $c(S, \phi_1 \circ \phi_2)$.*

In particular, Theorem 1.1 implies that if the contact structures supported by (S, ϕ_1) and (S, ϕ_2) are both strongly fillable, then the contact structure supported by $(S, \phi_1 \circ \phi_2)$ is tight. (Work by Baldwin [5] and, independently, by Baker et al. [1] has since greatly strengthened this result.)

In this paper, we use the Ozsváth–Szabó contact invariant to study the geometric effect of another natural operation on open books called "capping off." Consider the open book $(S_{g,r}, \phi)$, where $S_{g,r}$ is a genus g surface with $r > 1$ boundary components and ϕ is a diffeomorphism of $S_{g,r}$, which fixes the boundary pointwise. By capping off one of the boundary components of $S_{g,r}$ with a disc, we obtain an open book $(S_{g,r-1}, \phi')$, where ϕ' is the extension of ϕ to $S_{g,r-1}$ by the identity on this disc. Let $M_{S,\phi}$ denote the 3-manifold with open book decomposition (S, ϕ). There is a natural cobordism W from $M_{S_{g,r},\phi}$ to $M_{S_{g,r-1},\phi'}$ obtained by attaching a 0-framed 2-handle along the binding component in $M_{S_{g,r},\phi}$ corresponding to the capped off boundary component of $S_{g,r}$. Alternatively, we can think of W as a cobordism from $-M_{S_{g,r-1},\phi'}$ to $-M_{S_{g,r},\phi}$. Our main theorem is the following.

Theorem 1.2. *There exists a $Spin^c$ structure $\mathfrak{s}_0$ on W for which the map*

$$F^+_{W,\mathfrak{s}_0} : HF^+(-M_{S_{g,r-1},\phi'}) \to HF^+(-M_{S_{g,r},\phi})$$

sends $c^+(S_{g,r-1}, \phi')$ to $c^+(S_{g,r}, \phi)$.

In some sense, Theorem 1.2 generalizes Ozsváth and Szabó's original definition of the contact invariant. Their definition begins with the fact that every contact structure is supported by an open book of the form $(S_{g,1}, \phi)$ for some $g > 1$. By capping off the one boundary component of $S_{g,1}$, we obtain a closed surface $S_{g,0}$. If $M_{S_{g,0},\phi'}$ is the corresponding fibered 3-manifold and $\mathfrak{t}$ is the $Spin^c$ structure on $M_{S_{g,0},\phi'}$ represented by a vector field transverse to the fibers, then $HF^+(-M_{S_{g,0},\phi'}, \mathfrak{t})$ is generated by a single element $c^+(S_{g,0}, \phi')$ [**30**]. In [**31**], Ozsváth and Szabó define the contact invariant $c^+(S_{g,1}, \phi)$ to be the image of this element $c^+(S_{g,0}, \phi')$ under the map

$$F^+_V : HF^+(-M_{S_{g,0},\phi'}) \to HF^+(-M_{S_{g,1},\phi})$$

induced by the corresponding 2-handle cobordism V.

At first glance, this 2-handle attachment does not seem like a very natural contact-geometric operation, although Eliashberg proves in [**10**] that there is a symplectic form Ω on the cobordism V which is positive on the fibers of the fibration $M_{S_{g,0},\phi'} \to S^1$, and for which the contact 3-manifold supported by $(S_{g,1}, \phi)$ is a weakly concave boundary component of (V, Ω). One expects that a similar construction should produce a symplectic structure on the cobordism W considered in Theorem 1.2. In fact, since this paper first appeared, Gay and Stipsicz have shown that one can find a symplectic form on W for which the contact 3-manifolds supported by $(S_{g,r}, \phi)$ and $(S_{g,r-1}, \phi')$ are strongly concave and strongly convex, respectively, as long as certain homological conditions are met [**15**] (cf. also [**38**]). On the other

hand, the contact invariant in Heegaard Floer homology is not known, in general, to behave naturally with respect to the map induced by a strong symplectic cobordism, and so Theorem 1.2 provides new information in this regard.[1]

Below, we explore some consequences and potential applications of Theorem 1.2, and we discuss some natural questions that arise from this result. To begin with, consider the following immediate corollary of Theorem 1.2.

Corollary 1.3. *If* $c^+(S_{g,r-1}, \phi') = 0$ *then* $c^+(S_{g,r}, \phi) = 0.$

This prompts the question below.

Question 1.4. *Is the contact structure supported by* $(S_{g,r}, \phi)$ *overtwisted whenever the contact structure supported by* $(S_{g,r-1}, \phi')$ *is?*

This question has recently been answered in the negative by the author and Etnyre: in [**6**], we provide an infinite family of genus one, two boundary component open books that support tight contact structures but whose once-capped-off open books support overtwisted contact structures. Furthermore, we show that capping off is equivalent to performing an admissible transverse surgery on the relevant binding component, proving that admissible transverse surgery (the transverse analog of Legendrian surgery) does not preserve tightness. In the same paper, we use Corollary 1.3 to provide the first examples of hyperbolic, universally tight contact manifolds that are not weakly fillable.

It is our hope that Corollary 1.3 will also be useful in understanding tight contact structures supported by genus one open books. For example, the genus one, one boundary component open books which support tight contact structures are classified in [**2, 17, 19**]. Combined with this classification, Corollary 1.3 places concrete restrictions on genus one open books with multiple boundary components, which can support tight contact structures. Similarly, Lekili has since used Corollary 1.3 to find a planar open book with four boundary components which is right-veering and non-destabilizable, but supports an overtwisted contact structure [**22**], answering a question of Honda [**17**].

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[1]By contrast, the contact invariant in monopole Floer homology *is* known to behave naturally with respect to maps induced by exact symplectic cobordisms and certain strong symplectic cobordisms; see [**24**]. See also [**20**].

If W is the 2-handle cobordism in Theorem 1.2, then the map F_W^+ (summing over all Spin^c structures) fits into a surgery exact triangle,

$$HF^+(-M_{S_{g,r-1},\phi'}) \xrightarrow{F_W^+} HF^+(-M_{S_{g,r},\phi})$$
$$F_X^+$$
$$HF^+(-M_{S_{g,r},\phi \cdot t_\gamma^{-1}}),$$

where t_γ^{-1} is a left-handed Dehn twist around a curve γ parallel to the boundary component B that we are capping off [28]. In this triangle, X is the cobordism obtained by attaching a (-1)-framed 2-handle to $-M_{S_{g,r},\phi}$ along the binding component corresponding to B. According to the following theorem, the contact invariant behaves naturally under the map induced by X as well (strictly speaking, the theorem below is only proved in [17] for curves γ, which are non-separating, but as long as $r > 1$, we can stabilize the open book and then apply the result from [17]).

Theorem 1.5 ([17,31]). *The map F_X^+ sends $c^+(S_{g,r}, \phi)$ to $c^+(S_{g,r}, \phi \cdot t_\gamma^{-1})$.*

The surgery exact triangle has proved to be one of the most versatile tools in Heegaard Floer homology. One therefore expects that the triangle above, combined with Theorems 1.2 and 1.5, may be used to provide interesting contact-geometric information in many settings. For example, we obtain the following obstruction to there being a Stein structure on W (it is clear that the 2-handle attachment used to form W cannot locally be done in a Stein way; however, it may sometimes be possible to globally construct such a Stein structure).

Corollary 1.6. *The cobordism $W : M_{S_{g,r},\phi} \to M_{S_{g,r-1},\phi'}$ has a Stein structure compatible with the contact structures on either end only if $c^+(S_{g,r}, \phi \cdot t_\gamma^{-1}) = 0$.*

For, if W has a Stein structure, then the map F_W^+ sends $c^+(S_{g,r-1}, \phi')$ to $c^+(S_{g,r}, \phi)$ (see [31]), and the exactness of the triangle above implies that

$$c^+(S_{g,r}, \phi \cdot t_\gamma^{-1}) = F_X^+(c^+(S_{g,r}, \phi)) = 0.$$

Below, we describe a consequence of Theorem 1.2 for contact surgery on stabilized Legendrian knots. Suppose K is an oriented Legendrian knot in (M, ξ), and let $(M_{\pm 1}(K), \xi_{\pm 1}(K))$ be the contact 3-manifold obtained from (M, ξ) via contact ± 1-surgery on K. We denote by $S_+(K)$ and $S_-(K)$ the positive and negative Legendrian stabilizations of K, as defined in [13]. Let K' be either $S_+(K)$ or $S_-(K)$. As we shall see in Section 3, the following is a special case of Theorem 1.2.

Theorem 1.7. *There is a U-equivariant map $F^+ : HF^+(-M_{\pm 1}(K)) \to HF^+(-M_{\pm 1}(K'))$, which sends $c^+(\xi_{\pm 1}(K))$ to $c^+(\xi_{\pm 1}(K'))$.*

Theorem 1.7 has since been used by Lisca and Stipsicz to study a new transverse invariant, which they define in [**23**].

The operation of capping off is closely related to the operation of gluing two open books together along some proper subset of their binding components. More precisely, suppose that (S, ϕ) and (S', ϕ') are two open books such that either S or S' has more than n boundary components. Let $B_1, \ldots, B_n$ and $B'_1, \ldots, B'_n$ denote boundary components of S and S', respectively. One forms a surface S'' by gluing S to S' by a map that identifies B_i with B'_i for each $i = 1, \ldots, n$. Moreover one can define a diffeomorphism ϕ'' of S'' whose restriction to $S \subset S''$ is ϕ and whose restriction to $S' \subset S''$ is ϕ''. We say that (S'', ϕ'') is an open book obtained by gluing (S, ϕ) to (S', ϕ').

Remark 1.8. When $n = 1$, the 3-manifold $M_{S'', \phi''}$ is homeomorphic to that corresponding to the *contact fiber sum* of the open books (S, ϕ) and (S', ϕ') (see [**37**] for a recent application of contact fiber sum). In contrast, the contact structure supported by the glued open book (S'', ϕ'') is generally different from that associated to the contact fiber sum.

We discuss the relationship between capping off and gluing in more detail in Section 4; in particular, we prove the following consequence of Theorem 1.2.

Theorem 1.9. *Suppose that (S'', ϕ'') is an open book obtained by gluing (S, ϕ) to (S', ϕ'). If $c(S, \phi)$ and $c(S', \phi')$ are both non-zero, then so is $c(S'', \phi'')$.*

Our study of the effect of capping off on the Ozsváth–Szabó contact invariant began as an attempt to better understand the *support genus* of contact structures, as defined by Etnyre and Ozbagci in [**14**]. The support genus of a contact structure ξ is defined to be the minimum, over all open books (S, ϕ) compatible with ξ, of the genus of S; we denote this invariant by $sg(\xi)$. In 2004, Etnyre showed that all overtwisted contact structures have support genus zero, while there are fillable contact structures with $sg(\xi) > 0$ [**12**]. More recently, Ozsváth et al. have found a Heegaard–Floer homology obstruction to $sg(\xi) = 0$. Their main result is the following.

Theorem 1.10 (**[34]**)**.** *Suppose that ξ is a contact structure on the 3-manifold M. If $sg(\xi) = 0$, then $c^+(\xi) \in U^d \cdot HF^+(-M)$ for all $d \in \mathbb{N}$.*

Note that Theorem 1.10 follows as an immediate corollary of our Theorem 1.2. For, if $sg(\xi) = 0$, then ξ is supported by an open book of the form $(S_{0,r}, \phi)$, and we may cap off all but one of the boundary components of $S_{0,r}$ to obtain an open book $(S_{0,1}, \phi')$. Since all diffeomorphisms

of the disc $S_{0,1}$ are isotopic to the identity, $(S_{0,1}, \phi')$ is an open book for S^3. And, because every element of $HF^+(S^3)$ is in the image of U^d for all $d \in \mathbb{N}$, Theorem 1.2 implies that the same is true of the contact invariant $c^+(S_{0,r}, \phi) \in HF^+(-M)$ (since the maps $F^+_{W,\mathfrak{s}_0}$ are U-equivariant).

In order to use Theorem 1.10 to prove that $sg(\xi) > 0$, one must be able to show that $c^+(\xi)$ is not in the image of U^d for some $d \in \mathbb{N}$. In practice, this can be very difficult, although the proposition below is sometimes helpful in this regard.

Proposition 1.11 ([**30, 34**]). *Suppose that ξ is a contact structure on M, and let $\mathfrak{t}_\xi$ be the Spin^c structure associated to ξ. If $c^+(\xi) \neq 0$ and the first Chern class $c_1(\mathfrak{t}_\xi)$ is non-torsion, then there is some $d \in \mathbb{N}$ for which $c^+(\xi) \notin U^d \cdot HF^+(-M)$; hence $sg(\xi) > 0$.*

Theorem 1.2 may be used to extend the reach of this proposition a bit further. This is illustrated by the following example in which we use a genus one, two boundary component open book to prove that a certain genus one, one boundary component open book (obtained from the former by capping off) supports a contact structure (denote below by ξ') with support genus one.

Example 1.12. Let a, b, c and γ be the curves on $S_{1,2}$ shown on the left in Figure 1. If (M, ξ) is the contact 3-manifold supported by the open book $(S_{1,2}, (t_a t_b)^5 t_\gamma^2 t_c^2)$, then $c_1(\mathfrak{t}_\xi)$ is twice a generator of $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ (see the proof of [**14,** Theorem 6.2]). Moreover, ξ is Stein fillable since the diffeomorphism $(t_a t_b)^5 t_\gamma^2 t_c^2$ is a product of right-handed Dehn twists; hence, $c^+(\xi) \neq 0$. By Proposition 1.11, there is some $d \in \mathbb{N}$ for which $c^+(\xi) \notin U^d \cdot HF^+(-M)$.

After capping off the boundary component of $S_{1,2}$ labeled B, the curve c becomes null-homotopic, and γ becomes isotopic to the curve b on $S_{1,1}$. The capped off open book is therefore $(S_{1,1}, (t_a t_b)^5 t_b^2)$. If (M', ξ') is the contact 3-manifold compatible with this open book, then Theorem 1.2 implies that there is some $d \in \mathbb{N}$ for which $c^+(\xi') \notin U^d \cdot HF^+(-M')$ since the same is
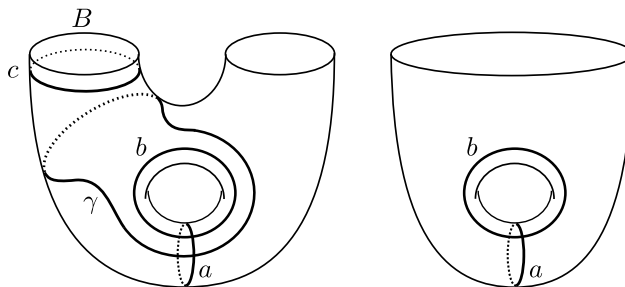

Figure 1. Surfaces $S_{1,2}$ and $S_{1,1}$, and the curves a, b, c and γ.

true for $c^+(\xi)$; hence, $sg(\xi') > 0$. Note that we could *not* have drawn this conclusion directly from Proposition 1.11 since $c_1(\mathfrak{t}_{\xi'}) = 0$; indeed, the $Spin^c$ structure associated to any contact structure compatible with a genus one, one boundary component open book has trivial first Chern class [**14**].

In Section 7, we use Example 1.12 to determine the support genera of almost all contact structures compatible with genus one, one boundary component open books whose monodromies are pseudo-Anosov.

The support genus is not well understood in general, revealing a fundamental gap in our understanding of the link between open books and contact geometry. To begin with, it is not known whether there exist contact structures with $sg(\xi) > 1$. Moreover, all of the contact structures that we know of with $sg(\xi) > 0$ are at least weakly fillable. It is our hope that Theorem 1.2 may be helpful in addressing the first problem. Suppose we wished to find an obstruction to $sg(\xi) = 1$. Every genus one open book can be reduced, via capping off, to a genus one open book with one binding component, and much is known about the contact structures compatible with (and the contact invariants associated to) the latter sort of open book [**2, 17, 19**]. Any property shared by the Ozsváth–Szabó invariants for such contact structures, which is preserved by the map induced by capping off, will provide an obstruction to $sg(\xi) = 1$ (we used this principle above to rederive the obstruction in Theorem 1.10 to $sg(\xi) = 0$).

So far, this approach has borne a very modest amount of fruit. A diffeomorphism ϕ of $S_{g,r}$ is called *reducible* if ϕ is freely isotopic to a diffeomorphism which fixes an essential multi-curve on $S_{g,r}$ (a *free* isotopy is not required to fix points on $\partial S_{g,r}$). We say that ϕ is *periodic* if ϕ^m is freely isotopic to the identity for some $m \in \mathbb{N}$ and ϕ is not reducible. Using the strategy outlined above, we prove the following theorem in Section 5.

Theorem 1.13. *Suppose that the contact 3-manifold (M, ξ) is supported by a genus one open book with r binding components and periodic monodromy. If ξ is tight, then $r \geq -1 - 4d_3(\xi)$.*

Here, $d_3(\xi)$ is the "3-dimensional" invariant associated to ξ, which is well-defined in $\mathbb{Q}$ as long as $c_1(\mathfrak{t}_\xi)$ is a torsion class. We strengthen Theorem 1.13 at the end of Section 5, giving an explicit formula for $d_3(\xi)$ whenever ξ is supported by a genus one open book with periodic monodromy.

Related to the notion of support genus (and equally mysterious) is that of *binding number* [**14**]. If $sg(\xi) = g$, then the binding number of ξ is defined to be the minimum, over all open books $(S_{g,r}, \phi)$ compatible with ξ, of the number of binding components of the open book, r; we denote this invariant by $bn(\xi)$. If $sg(\xi) > 0$ and ξ is supported by a genus one open book with periodic monodromy, then Theorem 1.13 implies that $bn(\xi) \geq -1 - 4d_3(\xi)$. Note that this inequality is sharp for the tight contact structure ξ_{std} on S^3, as $bn(\xi_{std}) = 1$ and $d_3(\xi_{std}) = -1/2$.

If ϕ is neither reducible nor periodic, then ϕ is called *pseudo-Anosov*; these are the most abundant sort. In Section 6, we give a more intrinsic definition of pseudo-Anosov diffeomorphisms, and we discuss properties of these maps that are preserved under capping off.

2. Proof of Theorem 1.2

In this section, we construct a pointed Heegaard triple-diagram corresponding to the capping off cobordism W and show explicitly that, for a certain $Spin^c$ structure $\mathfrak{s}_0$ on W, the induced map on Heegaard Floer homology $F_{W,\mathfrak{s}_0}$, obtained by counting holomorphic triangles, sends one contact invariant to the other as prescribed by Theorem 1.2.

2.1. Heegaard diagrams and the contact class. Let S be a compact surface with boundary, and suppose that ϕ is a diffeomorphism of S, which restricts to the identity on ∂S. Recall that the open book (S, ϕ) specifies a closed, oriented 3-manifold $M_{S,\phi} = S \times [0,1]/\sim$, where $\sim$ is the identification given by

$$(x, 1) \sim (\phi(x), 0), \quad x \in S$$
$$(x, t) \sim (x, s), \quad x \in \partial S, \quad t, s \in [0, 1].$$

$M_{S,\phi}$ has a Heegaard splitting $M_{S,\phi} = H_1 \cup H_2$, where H_1 is the handlebody $S \times [0, 1/2]$ and H_2 is the handlebody $S \times [1/2, 1]$. Let S_t denote the page $S \times \{t\}$. The Heegaard surface in this splitting is

$$\Sigma := \partial H_1 = S_{1/2} \cup -S_0.$$

If $S = S_{g,r}$ then Σ has genus $n = 2g + r - 1$. To give a pointed Heegaard diagram for $M_{S,\phi}$, it remains to describe the α- and β- attaching curves and the placement of a basepoint z; the following construction is due to Honda, Kazez and Matić [**17**].

Let $a_1, \ldots, a_n$ be pairwise disjoint, properly embedded arcs in S for which the complement $S \setminus \cup a_i$ is a disc. For each $i = 1, \ldots, n$, let b_i be an arc obtained by changing a_i via a small isotopy that moves the endpoints of a_i along ∂S in the direction specified by the orientation of ∂S, so that a_i intersects b_i transversely in one point and with positive sign (where b_i inherits its orientation from a_i). For $i = 1, \ldots, n$, let α_i and β_i be the curves on Σ defined by

$$\alpha_i = a_i \times \{1/2\} \cup a_i \times \{0\},$$
$$\beta_i = b_i \times \{1/2\} \cup \phi(b_i) \times \{0\}.$$

Place a basepoint z in the "big" region of $S_{1/2} \setminus \cup \alpha_i \setminus \cup \beta_i$ (that is, outside of the thin strip regions), and let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $\beta = \{\beta_1, \ldots, \beta_n\}$. We say that $(\Sigma, \alpha, \beta, z)$ is a *standard* pointed Heegaard diagram for the open book (S, ϕ). See Figure 2 for an example.
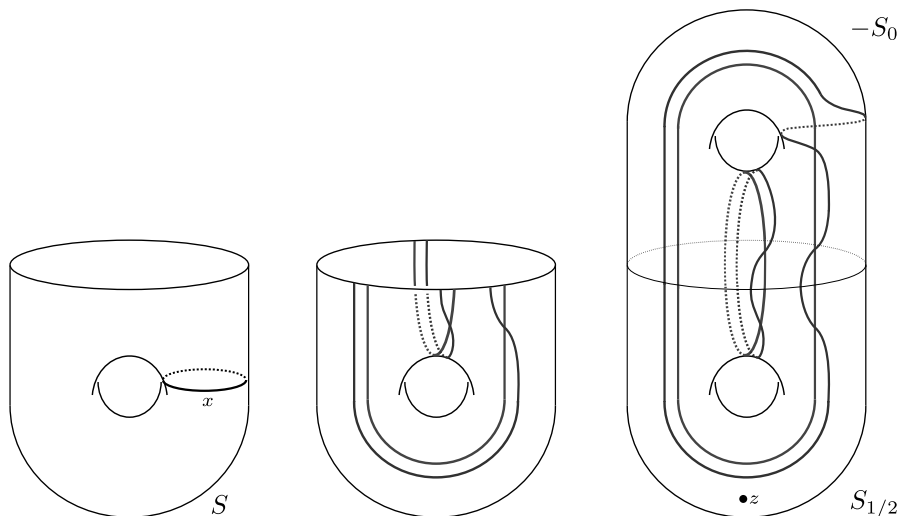
Figure 2. On the left is the surface $S = S_{1,1}$. The figure in the middle shows the arcs a_i (in red) and b_i (in blue). On the right is a standard pointed Heegaard diagram for the open book (S, D_x), where D_x is a right-handed Dehn twist around the curve x.
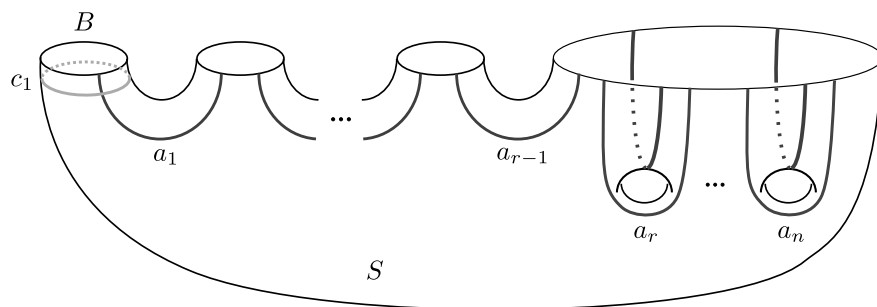

Figure 3. The surface $S = S_{g,r}$ and the curve c_1. The arcs $a_1, \ldots, a_n$ are drawn in red.

For each $i = 1, \ldots, n$, let y_i be the intersection point on $S_{1/2}$ between α_i and β_i. Then $\mathbf{y} = \{y_1, \ldots, y_n\}$ represents an intersection point between $\mathbb{T}_\beta$ and $\mathbb{T}_\alpha$ in $\mathrm{Sym}^n(\Sigma)$, and we may think of $[\mathbf{y}, 0]$ as a cycle in $CF^+(\Sigma, \beta, \alpha, z) = CF^+(-M_{S,\phi})$.

Theorem 2.1 ([**17**, Theorem 3.1]). *The image of $[\mathbf{y}, 0]$ in $HF^+(-M_{S,\phi})$ is the Ozsváth–Szabó contact class $c^+(S, \phi)$.*

Now suppose that $S = S_{g,r}$, and let B denote the boundary component of S that we wish to cap off. Let $a_1, \ldots, a_n$ (where $n = 2g + r - 1$) be pairwise

disjoint, properly embedded arcs on S so that $S \setminus \cup a_i$ is a disc and only a_1 intersects B, as shown in Figure 3. For each $i = 1, \ldots, n$, let b_i be an arc obtained by changing a_i via a small isotopy as described above. For each $i = 2, \ldots, n$, let c_i be an arc obtained by changing b_i via a similar isotopy (so that c_i intersects each of a_i and b_i transversely in one point and with positive sign; we also require that c_i intersects b_i first when moving away from ∂S — this is essentially the picture in Figure 5), and let c_1 be a curve on S parallel to the boundary component B. See Figure 3 for an illustration of the curve c_1 and the arcs a_i.

For $i = 1, \ldots, n$, let α_i and β_i be the curves on $\Sigma = S_{1/2} \cup -S_0$ defined by

$$\alpha_i = a_i \times \{1/2\} \cup a_i \times \{0\},$$
$$\beta_i = b_i \times \{1/2\} \cup \phi(b_i) \times \{0\}$$

as above. In addition, define

$$\gamma_1 = c_1 \times \{1/2\},$$

and let

$$\gamma_i = c_i \times \{1/2\} \cup \phi(c_i) \times \{0\}$$

for $i = 2, \ldots, n$. Finally, place a basepoint z in the "big" region of $S_{1/2} \setminus \cup \alpha_i \setminus \cup \beta_i \setminus \cup \gamma_i$ (that is, neither in one of the thin strip regions nor in the region between B and γ_1), and let α, β and γ denote the sets of attaching curves $\{\alpha_1, \ldots, \alpha_n\}$, $\{\beta_1, \ldots, \beta_n\}$ and $\{\gamma_1, \ldots, \gamma_n\}$. Then $(\Sigma, \alpha, \beta, z)$ is a standard pointed Heegaard diagram for $(S_{g,r}, \phi)$.

Let K_B denote the binding component in $M_{S_{g,r},\phi}$ which corresponds to B. Observe that β_1 is a meridian of K_B, and that the Heegaard diagram $(\Sigma, \alpha, \beta \setminus \beta_1)$ specifies the knot complement $M_{S_{g,r},\phi} \setminus K_B$. Since γ_1 is a 0-framed longitude of K_B and γ_i is isotopic to β_i for $i \geq 2$, it follows that (Σ, α, γ) is a Heegaard diagram for the 3-manifold $M_{S_{g,r-1},\phi'}$ obtained by performing 0-surgery on K_B. In fact, it is easy to see that $(\Sigma, \alpha, \gamma, z)$ is the stabilization of a standard pointed Heegaard diagram for the open book $(S_{g,r-1}, \phi')$.

For $i = 1, \ldots, n$, let θ_i, x_i and y_i be the points in Σ defined by

$$\theta_i = \beta_i \cap \gamma_i \cap S_{1/2},$$
$$x_i = \gamma_i \cap \alpha_i \cap S_{1/2},$$
$$y_i = \beta_i \cap \alpha_i \cap S_{1/2},$$

and let $\boldsymbol{\Theta}$, $\mathbf{x}$ and $\mathbf{y}$ be the corresponding points in $\mathrm{Sym}^n(\Sigma)$ defined by

$$\boldsymbol{\Theta} = \{\theta_1, \ldots, \theta_n\} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma,$$
$$\mathbf{x} = \{x_1, \ldots, x_n\} \in \mathbb{T}_\gamma \cap \mathbb{T}_\alpha,$$
$$\mathbf{y} = \{y_1, \ldots, y_n\} \in \mathbb{T}_\beta \cap \mathbb{T}_\alpha.$$

According to Theorem 2.1, the image of $[\mathbf{y}, 0]$ in

$$HF^+(\Sigma, \beta, \alpha, z) = HF^+(-M_{S_{g,r},\phi})$$

is the contact class $c^+(S_{g,r}, \phi)$; likewise, the image of $[\mathbf{x}, 0]$ in

$$HF^+(\Sigma, \gamma, \alpha, z) = HF^+(-M_{S_{g,r-1},\phi'})$$

is $c^+(S_{g,r-1}, \phi')$. Meanwhile, $[\boldsymbol{\Theta}, 0]$ represents the top-dimensional generator of

$$HF^{\leq 0}(\Sigma, \beta, \gamma, z) = HF^{\leq 0}(\#^{n-1}(S^1 \times S^2)).$$

2.2. The map induced by capping off. Suppose that W is the cobordism from $M_{S_{g,r},\phi}$ to $M_{S_{g,r-1},\phi'}$ obtained by attaching a 0-framed 2-handle to the knot K_B in $M_{S_{g,r},\phi}$. As mentioned in the introduction, W may be viewed as a cobordism from $-M_{S_{g,r-1},\phi'}$ to $-M_{S_{g,r},\phi}$ instead. If $\mathfrak{s}$ is a Spin^c structure on W, then the map

$$F^+_{W,\mathfrak{s}} : HF^+(-M_{S_{g,r-1},\phi'}) \to HF^+(-M_{S_{g,r},\phi})$$

is induced by the chain map,

$$f^+_{W,\mathfrak{s}} : CF^+(-M_{S_{g,r-1},\phi'}) \to CF^+(-M_{S_{g,r},\phi}),$$

which is defined using the pointed triple-diagram $(\Sigma, \beta, \gamma, \alpha, z)$. (Technically, this is a *left-subordinate* triple diagram for the cobordism W, as opposed to the more often used notion of a *right-subordinate* triple-diagram. Right- and left-subordinate diagrams induce the same maps on homology [**32,** Lemma 5.2].) Recall that, for $\mathbf{v} \in \mathbb{T}_\gamma \cap \mathbb{T}_\alpha$,
(2.1)
$$f^+_{W,\mathfrak{s}}([\mathbf{v}, i]) = \sum_{\mathbf{w} \in \mathbb{T}_\beta \cap \mathbb{T}_\alpha} \sum_{\{\psi \in \pi_2(\boldsymbol{\Theta},\mathbf{v},\mathbf{w}) \,|\, \mu(\psi)=0, \, \mathfrak{s}_z(\psi)=\mathfrak{s}\}} (\#\mathcal{M}(\psi)) \cdot [\mathbf{w}, i - n_z(\psi)].$$

In this sum, $\pi_2(\boldsymbol{\Theta}, \mathbf{v}, \mathbf{w})$ is the set of homotopy classes of Whitney triangles connecting $\boldsymbol{\Theta}$, $\mathbf{v}$, and $\mathbf{w}$, where $\boldsymbol{\Theta}$ is as in the previous subsection; $\mu(\psi)$ is the expected dimension of the moduli space, $\mathcal{M}(\psi)$, of holomorphic representatives of ψ; $\mathfrak{s}_z(\psi)$ is the Spin^c structure on W corresponding to ψ; and $n_z(\psi)$ is the algebraic intersection number of ψ with the subvariety $\{z\} \times \mathrm{Sym}^{n-1}(\Sigma) \subset \mathrm{Sym}^n(\Sigma)$. Below, we review some relevant definitions; for more details, see [**29**].

Let Δ denote the 2-simplex with vertices v_β, v_γ and v_α labeled clockwise, and let e_β, e_γ and e_α, respectively, denote the edges opposite these vertices. A *Whitney triangle* connecting points $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbb{T}_\gamma \cap \mathbb{T}_\alpha$ and $\mathbb{T}_\beta \cap \mathbb{T}_\alpha$ is a smooth map

$$u : \Delta \to \mathrm{Sym}^n(\Sigma)$$

with the boundary conditions that $u(v_\alpha) = \mathbf{r}$, $u(v_\beta) = \mathbf{v}$ and $u(v_\gamma) = \mathbf{w}$, and $u(e_\beta) \subset \mathbb{T}_\beta$, $u(e_\gamma) \subset \mathbb{T}_\gamma$ and $u(e_\alpha) \subset \mathbb{T}_\alpha$. See Figure 4 for a schematic depiction of this map.
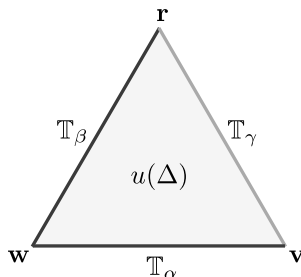
Figure 4. A Whitney triangle.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ denote the connected regions of $\Sigma \setminus \cup \alpha_i \setminus \cup \beta_i \setminus \cup \gamma_i$. A *triply-periodic domain* for the pointed Heegaard diagram $(\Sigma, \beta, \gamma, \alpha, z)$ is a 2-chain $\mathcal{P} = \sum_i a_i \mathcal{D}_i$ in $C(\Sigma; \mathbb{Z})$ whose boundary is a sum of β, γ and α curves, and whose multiplicity at the basepoint z is 0 (the *multiplicity* of a 2-chain at a point refers to the coefficient, in the 2-chain, of the region containing that point). The diagram $(\Sigma, \beta, \gamma, \alpha, z)$ is said to be *weakly-admissible* if every non-trivial triply-periodic domain has both positive and negative multiplicities (this is slightly stronger than the definition of weak-admissibility given in [**29**]). In general, the map $f_{W,\mathfrak{s}}^+$ is not well-defined unless the pointed triple-diagram $(\Sigma, \beta, \gamma, \alpha)$ is weakly-admissible. This is, therefore, our first consideration.

Lemma 2.2. *The pointed triple-diagram* $(\Sigma, \beta, \gamma, \alpha, z)$ *constructed above is weakly-admissible.*

Proof of Lemma 2.2. Figure 5 shows a local picture of Σ near the intersection points θ_i, x_i and y_i for $i \geq 2$. Let $\mathcal{P}$ be a triply-periodic domain whose multiplicities in the regions A, B, C, D, E and F are given by the integers a, b, c, d, e and f, respectively. Note that $c = 0$ since the region C contains the basepoint. Since $\partial \mathcal{P}$ consists of complete β, γ and α curves, it must be that

$$b = d - e = -f,$$
$$a = b - d = -e.$$

Therefore, $\mathcal{P}$ has both positive and negative multiplicities unless

$$a = b = c = d = e = f = 0.$$

We perform this local analysis for each $i = 2, \ldots, n$ and conclude that either $\mathcal{P}$ has both positive and negative multiplicities or $\partial \mathcal{P}$ is a linear combination of the curves β_1, γ_1 and α_1. Let us assume the latter.

Now, consider the regions labeled A, B, and C in Figure 6, and suppose that $\mathcal{P}$ has multiplicities a, b and c in these regions. Again, $c = 0$ since region C contains the basepoint; and $a = -b$. Therefore, $\mathcal{P}$ has both positive and negative multiplicities unless $a = b = 0$, in which case $\partial \mathcal{P}$ is some multiple
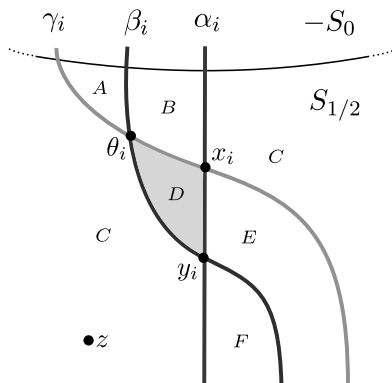
Figure 5. The local picture near the intersection points θ_i, x_i and y_i for $i \geq 2$. The shaded region is Δ_i.

of the curve γ_1. Since γ_1 is not null-homologous in Σ, this multiple must be zero, which implies that $\mathcal{P}$ is the trivial domain. To summarize, we have shown that $\mathcal{P}$ has both positive and negative multiplicities unless $\mathcal{P}$ is trivial. Hence, the diagram $(\Sigma, \beta, \gamma, \alpha, z)$ is weakly-admissible.

$\square$

Recall that a homotopy class ψ of Whitney triangles has an associated domain $\mathcal{D}(\psi) = \sum_i n_{p_i}(\psi)\mathcal{D}_i$, where p_i is a point in $\mathcal{D}_i$. For each $i = 1, \ldots, n$, let $\Delta_i \subset S_{1/2}$ be the shaded triangular region with vertices at θ_i, x_i and y_i shown in Figures 5 and 6. Then the homotopy class $\psi_0 \in \pi_2(\boldsymbol{\Theta}, \mathbf{x}, \mathbf{y})$ with
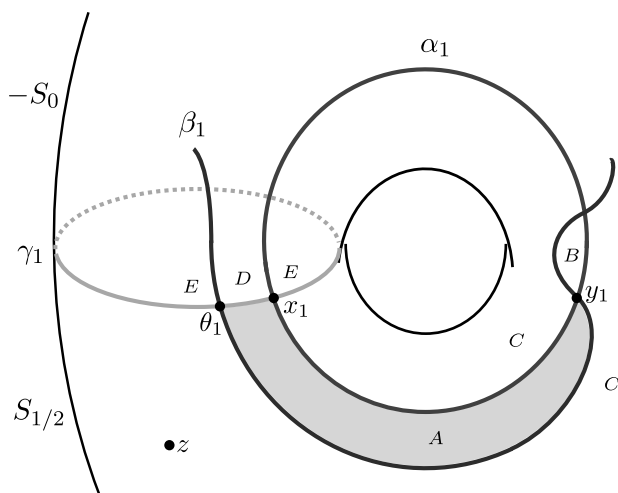

Figure 6. The local picture near the intersection points θ_1, x_1 and y_1. The shaded region is Δ_1.

domain $\mathcal{D}(\psi_0) = \Delta_1 + \cdots + \Delta_n$ has a unique holomorphic representative, by the Riemann Mapping Theorem (in particular, $\mu(\psi_0) = 0$). Let $\mathfrak{s}_0$ denote the Spin^c structure $\mathfrak{s}_z(\psi_0)$, and observe that $n_z(\psi_0) = 0$.

Proposition 2.3. *Suppose that ψ is a homotopy class of Whitney triangles connecting $\mathbf{\Theta}$, $\mathbf{x}$ and any other point $\mathbf{w} \in \mathbb{T}_\beta \cap \mathbb{T}_\alpha$. Let w_i denote the component of $\mathbf{w}$ on γ_i. If ψ has a holomorphic representative and satisfies $n_z(\psi) = 0$, then $w_i = y_i$ for $i = 2, \ldots, n$, and $\mathcal{D}(\psi) = \Delta_1' + \Delta_2 + \cdots + \Delta_n$, where Δ_1' is a (possibly non-embedded) triangle in $\Sigma \setminus \{z\}$ with vertices at θ_1, x_1, and w_1. If, in addition, $\mathfrak{s}_z(\psi) = \mathfrak{s}_0$, then $\psi = \psi_0$ and $\mathbf{w} = \mathbf{y}$.*

This proposition, together with the above remark that ψ_0 has a unique holomorphic representative, implies that the map $f_{W,\mathfrak{s}_0}^+$ sends $[\mathbf{x}, 0]$ to $[\mathbf{y}, 0]$, proving Theorem 1.2.

Proof of Proposition 2.3. Suppose ψ has a holomorphic representative and satisfies $n_z(\psi) = 0$. Then every coefficient in the domain $\mathcal{D}(\psi)$ is non-negative, and $\mathcal{D}(\psi)$ must have multiplicity 0 in the region containing the basepoint z. Moreover, the oriented boundary of $\mathcal{D}(\psi)$ consists of arcs along the β curves from the points $w_1, \ldots, w_n$ to the points $\theta_1, \ldots, \theta_n$; arcs along the γ curves from the points $\theta_1, \ldots, \theta_n$ to the points $x_1, \ldots, x_n$; and arcs along the α curves from the points $x_1, \ldots, x_n$ to the points $w_1, \ldots, w_n$.

Let a, b, c, d, e and f be the multiplicities of $\mathcal{D}(\psi)$ in the regions A, B, C, D, E and F shown in Figure 5. We have already established that $c = 0$. The boundary constraints on $\mathcal{D}$ then imply that

$$(2.2) \qquad\qquad a + d = b + 1,$$
$$d = b + e + 1.$$

Subtracting one equation from the other, we find that $a = -e$. Since all coefficients of $\mathcal{D}(\psi)$ are non-negative, $a = e = 0$. If $w_i \neq y_i$, then the constraints on $\partial\mathcal{D}(\psi)$ force $f + d = 0$, which implies that $f = d = 0$. However, plugging this back into equation (2.2), together with $a = 0$, implies that $0 = b + 1$, which contradicts the fact that b is non-negative. As a result, it must be the case that $w_i = y_i$. Then the constraints on $\partial\mathcal{D}(\psi)$ (together with the fact that $e = c = 0$) require that $d + f = 1$. Combined with equation (2.2), this implies that $d = 1$ and $f = b = 0$. So, we have found that $d = 1$ and $a = b = c = e = f = 0$; that is, the domain $\mathcal{D}(\psi)$ is locally just Δ_i.

We perform this local analysis for each $i = 2, \ldots, n$ and conclude that $w_i = y_i$ for $i = 2, \ldots, n$ and that $\mathcal{D}(\psi) = \Delta_1' + \Delta_2 + \cdots + \Delta_n$, where Δ_1' is a region whose oriented boundary consists of arcs along β_1 from w_1 to θ_1; along γ_1 from θ_1 to x_1; and along α_1 from x_1 to w_1. In fact, since $\Delta_2, \ldots, \Delta_n$ are triangles in Σ and $\mathcal{D}(\psi)$ is the image of a map from the n-fold branched cover of a triangle into Σ (see [**29**]), Δ_1' must be a (possibly non-embedded) triangle in Σ as well which avoids the basepoint z.

Now, suppose that ψ also satisfies $\mathfrak{s}_z(\psi) = \mathfrak{s}_0$. The only thing left to prove is that $w_1 = y_1$ and $\Delta'_1 = \Delta_1$. Since $\mathfrak{s}_z(\psi) = \mathfrak{s}_0 = \mathfrak{s}_z(\psi_0)$, it follows from [**29,** Proposition 8.5] that

$$\mathcal{D}(\psi) - \mathcal{D}(\psi_0) = \mathcal{D}(\phi_1) + \mathcal{D}(\phi_2) + \mathcal{D}(\phi_3),$$

where ϕ_1, ϕ_2 and ϕ_3 are homotopy classes of Whitney discs in $\pi_2(\mathbf{y}, \mathbf{w})$, $\pi_2(\mathbf{\Theta}, \mathbf{\Theta})$ and $\pi_2(\mathbf{x}, \mathbf{x})$, respectively. Since $\mathcal{D}(\psi) - \mathcal{D}(\psi_0) = \Delta'_1 - \Delta_1$, and γ_1 is homologically independent of both α_1 and β_1, $\mathcal{D}(\phi_2) = \mathcal{D}(\phi_3) = 0$, and $\mathcal{D}(\phi_1)$ is a disc in $\Sigma \setminus \{z\}$ whose oriented boundary consists of arcs along α_1 from y_1 to w_1, and arcs along β_1 from w_1 to y_1.

Let a, b, d and e be the multiplicities of Δ'_1 in the regions labeled A, B, D and E in Figure 6 (the multiplicity of Δ'_1 in region C is 0). Since $\Delta'_1 - \Delta$ is the disc $\mathcal{D}(\phi_1)$, the multiplicity of $\Delta'_1 - \Delta$ in the region labeled D must be the same as its multiplicity in the region labeled A; that is,

$$(2.3) \qquad\qquad d = a - 1$$

(the multiplicities of Δ in these regions are 0 and 1, respectively). However, the boundary constraints on $\mathcal{D}(\psi)$ imply that

$$a + e = d + 1.$$

Combining this equation with the former, we find that $e = 0$. If $w_1 \neq y_1$, then the same boundary constraints require that $a + b = 0$. However, combined with equation (2.3), this implies that either a, b or d is negative, which contradicts our assumption that ψ has a holomorphic representative. Therefore, $w_1 = y_1$, and the boundary constraints on $\mathcal{D}(\psi)$ imply that

$$a + b = 1.$$

It follows that $a = 1$ and $b = d = 0$, and, hence, that $\Delta'_1 = \Delta$. Thus, $\psi = \psi_0$, completing the proof of Proposition 2.3. $\qquad\qquad\square$

3. Contact surgery and Legendrian stabilization

In this section, we describe how contact ± 1-surgery on a stabilized Legendrian knot fits into the framework of capping off. Suppose that K is an oriented Legendrian knot in a contact 3-manifold (M, ξ), and let $\mathbb{R}^3_y$ denote the quotient $\mathbb{R}^3/(y \sim y + 1)$. There is a contactomorphism from a neighborhood of K to (N, ξ'), where

$$N = \{(x, y, z) \in \mathbb{R}^3_y \,|\, x^2 + z^2 < \epsilon\},$$

$\xi' = \ker(dz + x\,dy)$, and K is sent to the image of the y-axis in N. There is a natural "front" projection in N defined by the map which sends (x, y, z) to (y, z). In [**13**], Etnyre and Honda define the positive and negative Legendrian stabilizations of K, $S_+(K)$ and $S_-(K)$, to be the Legendrian knots in M corresponding to the curves in N shown in Figure 7. Note that this definition
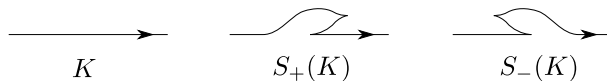
Figure 7. The Legendrian stabilizations $S_+(K)$ and $S_-(K)$, as seen via their front projections in N.

agrees with the usual definition of stabilization for Legendrian knots in the standard tight contact structure on S^3.

By incorporating K into the 1-skeleton of a contact cell decomposition for (M, ξ), we can find an open book $(S_{g,r}, \phi)$ compatible with ξ so that K lies on a page of this open book and the contact framing of K agrees with the framing induced by this page. The lemma below is based upon this idea as well.

Lemma 3.1 ([12, Lemma 3.3]). *Suppose the oriented Legendrian knot K lies on a page of the open book $(S_{g,r}, \phi)$. If we positively stabilize $(S_{g,r}, \phi)$ twice as shown in Figure 8, then we may isotop the page of the stabilized open book so that both $S_+(K)$ and $S_-(K)$ appear on the page as in Figure 8. The contact framings of these stabilized Legendrian knots agree with their page framings.*

Proof of Theorem 1.7. Let $(S_{g,r+2}, \phi')$ refer to the twice stabilized open book in Lemma 3.1, and let K' be either $S_+(K)$ or $S_-(K)$. We think of K' as lying in a page of this open book, per Lemma 3.1. Since the contact framings of K and K' agree with their page framings, the contact 3-manifolds $(M_{\pm 1}(K), \xi_{\pm 1}(K))$ and $(M_{\pm 1}(K'), \xi_{\pm 1}(K'))$ are supported by the open books $(S_{g,r}, \phi \cdot t_K^{\mp 1})$ and $(S_{g,r+2}, \phi' \cdot t_{K'}^{\mp 1})$, respectively. Note that $(S_{g,r}, \phi \cdot t_K^{\mp 1})$ is obtained from $(S_{g,r+2}, \phi' \cdot t_{K'}^{\mp 1})$ by capping off the boundary components
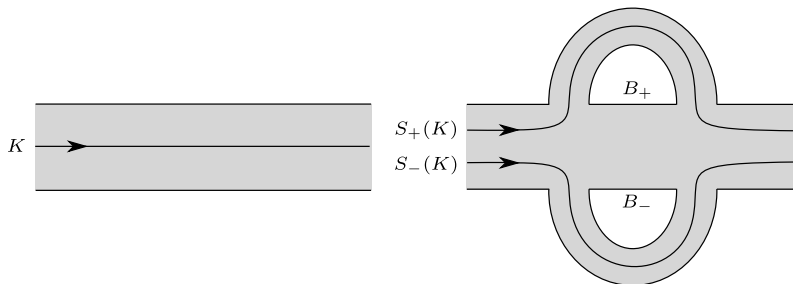

Figure 8. On the left is a neighborhood of a piece of K in $S_{g,r}$. On the right is a portion of the twice stabilized open book with the curves $S_+(K)$ and $S_-(K)$. We have labeled the two new boundary components B_+ and B_-.

B_+ and B_-. Therefore, by two applications of Theorem 1.2, there is a U-equivariant map

$$F^+ : HF^+(-M_{\pm 1}(K)) \to HF^+(-M_{\pm 1}(K'))$$

which sends $c^+(\xi_{\pm 1}(K))$ to $c^+(\xi_{\pm 1}(K'))$.   □

Theorem 1.7 has a nice interpretation in terms of contact surgery. Recall that, for $n \in \mathbb{Z}^{<0}$, contact n-surgery on a Legendrian knot $K \subset M$ may be performed by stabilizing K a total of $-n-1$ times to obtain K', and then performing contact -1-surgery on K' [8, 9]. In particular, such contact surgery is not unique unless $n = -1$; the ambiguity corresponds to the various ways of stabilizing K. By applying Theorem 1.7 to K', we obtain, under the appropriate interpretations of the contact manifolds $(M_n(K), \xi_n(K))$ and $(M_{n-1}(K), \xi_{n-1}(K))$, a map

$$F^+ : HF^+(-M_n(K)) \to HF^+(-M_{n-1}(K)),$$

which sends $c^+(\xi_n(K))$ to $c^+(\xi_{n-1}(K))$.

4. Gluing open books

Let (S'', ϕ'') denote the result of gluing (S, ϕ) to (S', ϕ') along boundary components B and B' of S and S', respectively. The open book (S'', ϕ'') may also be obtained by taking the boundary connected sum of the open books (S, ϕ) and (S', ϕ') along B and B', and then capping off the boundary component $B \# B'$ of the resulting surface, as illustrated in Figure 9. Since the boundary connected sum of these open books supports the contact connected sum $\xi_{S,\phi} \# \xi_{S',\phi'}$, Theorem 1.2 and [31, Proposition 2.1] imply the following.
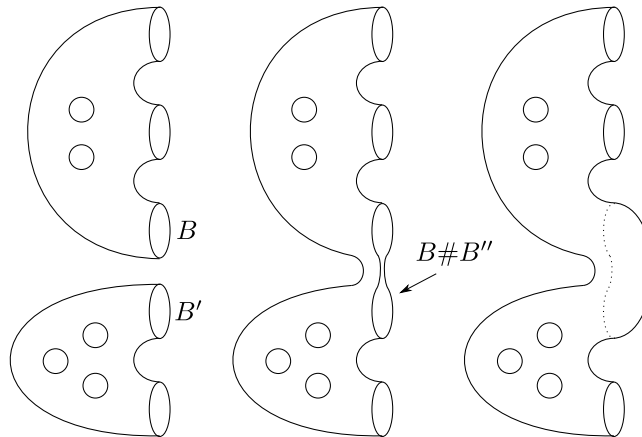

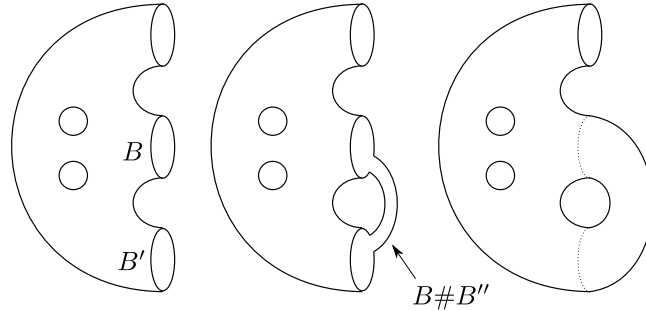**Figure 9.** An example of gluing via boundary connected sum and capping off.

Figure 10. An example of self-gluing via 1-handle attachment and capping off.

Lemma 4.1. *Suppose that (S'', ϕ'') is the open book obtained by gluing (S, ϕ) to (S', ϕ') along boundary components B, B'. If $c(S, \phi)$ and $c(S', \phi')$ are both non-zero, then so is $c(S'', \phi'')$.*

One further observation is needed to complete the proof of Theorem 1.9. Namely, suppose that (S, ϕ) is an open book with at least three boundary components. Let B and B' denote two of them, and consider the open book (S', ϕ') obtained from (S, ϕ) by gluing B to B' (we shall refer to this operation as *self-gluing*). (S', ϕ') may alternatively be obtained by attaching a 1-handle to (S, ϕ) with feet on B and B', and then capping off the boundary component $B \# B'$ of the resulting surface; see Figure 10. Since this 1-handle attachment corresponds to taking a contact connected sum with the Stein fillable contact structure on $S^1 \times S^2$, Theorem 1.2 and [**31**, Proposition 2.1] combine to give the lemma below.

Lemma 4.2. *Suppose that (S', ϕ') is the open book obtained from (S, ϕ) by self-gluing along B, B'. If $c(S, \phi)$ is non-zero, then so is $c(S', \phi')$.*

Now, suppose that the open book (S'', ϕ'') is obtained by gluing (S, ϕ) to (S', ϕ') along boundary components $B_1, \dots, B_n$ of S and $B_1', \dots, B_n'$ of S', as in the introduction. Note that (S'', ϕ'') is result of gluing (S, ϕ) to (S', ϕ') along B_1, B_1', followed by $n - 1$ self-gluings along the other B_i, B_i'. Theorem 1.9 therefore follows from Lemmas 4.1 and 4.2.

Remark 4.3. Gluing has an inverse operation called *splitting*. More precisely, suppose that ϕ is a reducible diffeomorphism of S which fixes disjoint simple closed curves $C_1, \dots, C_n$ pointwise. Splitting S along the C_i, one obtains open books $(S^{(1)}, \phi^{(1)}), \dots, (S^{(m)}, \phi^{(m)})$; conversely, we can recover (S, ϕ) from the $(S^{(j)}, \phi^{(j)})$ via a combination of gluings and self-gluings. Lemmas 4.1 and 4.2 then tell us that $c(S, \phi)$ is non-zero as long as all of the $c(S^{(j)}, \phi^{(j)})$ are.

5. Capping off and periodic open books

In this section, we study the 3-dimensional invariants associated to contact structures supported by genus one open books with periodic monodromy.

5.1. Periodic diffeomorphisms and the first Chern class. Recall that $HF^+(M, \mathfrak{t})$ comes equipped with a $\mathbb{Q}$-grading whenever $c_1(\mathfrak{t})$ is a torsion class. We denote the grading of a homogeneous element $x \in HF^+(M, \mathfrak{t})$ by $gr(x)$. The proposition below appears in a slightly different form in [**31**].

Proposition 5.1 ([**31**, Proposition 4.6]). *If (M, ξ) is a contact 3-manifold for which $c_1(\mathfrak{t}_\xi)$ is torsion, then $d_3(\xi) = -gr(c^+(\xi)) - 1/2$.*

Suppose that ϕ is a diffeomorphism of $S_{g,r}$ such that ϕ^m is freely isotopic to the identity for some $m \in \mathbb{N}^{\geq 0}$. Let $B_1, \ldots, B_r$ denote the boundary components of $S_{g,r}$, and let c_i be a curve on $S_{g,r}$ parallel to B_i for each $i = 1, \ldots, r$. It follows that ϕ^m is isotopic to a product of Dehn twists of the form $t_{c_1}^{k_1} \cdots t_{c_r}^{k_r}$. For each $i = 1, \ldots, r$, we define the *fractional Dehn twist coefficient* (FDTC) of ϕ around B_i to be k_i/m (see [**7**]). If ϕ is periodic, Colin and Honda show that the contact structure compatible with the open book $(S_{g,r}, \phi)$ is tight if and only if the FDTC of ϕ around every boundary component is non-negative [**7**]. In this case, the contact structure is also Stein fillable [**7**]. So, in particular, if (M, ξ) is supported by an open book with periodic monodromy, then ξ is tight if and only if $c^+(\xi) \neq 0$. Therefore, Theorem 1.13 may be reformulated as follows.

Theorem 5.2. *Suppose that (M, ξ) is supported by a genus one open book with r binding components and periodic monodromy. If $c^+(\xi) \neq 0$, then $r \geq 1 + 4gr(c^+(\xi))$.*

To prove this theorem, we bound the grading shifts associated to the maps induced by capping off, and we use the fact that $gr(c^+(\xi)) \leq 0$ whenever ξ is tight and is supported by a genus one open book with *one* boundary component and periodic monodromy (see Table 1). Before we compute these grading shifts, we must know that they are well-defined. To this end, we establish the following.

Proposition 5.3. *Suppose the contact 3-manifold (M, ξ) is supported by an open book (S, ϕ) for which ϕ^m is freely isotopic to the identity. If the FDTCs of ϕ are non-negative, then $c_1(\mathfrak{t}_\xi)$ is a torsion class.*

Proof of Proposition 5.3. It suffices to show that $\langle c_1(\mathfrak{t}_\xi), h \rangle = 0$ for every $h \in H_2(M, \mathbb{Z})$. Let $(\Sigma, \alpha, \beta, z)$ be a standard pointed Heegaard diagram for the open book $(S_{g,r}, \phi)$, and let $\mathcal{D}_1, \ldots, \mathcal{D}_k$ denote the connected regions of $\Sigma \setminus \cup \alpha_i \setminus \cup \beta_i$. Recall that a doubly-periodic domain for this pointed Heegaard diagram is a 2-chain $\mathcal{P} = \sum_i a_i \mathcal{D}_i$ whose boundary is a sum of α and β curves, and whose multiplicity at the basepoint z is 0. It is often

Table 1. Grading versus fractional Dehn twist coefficient. In this table, $k \geq 0$; otherwise, ξ is overtwisted.

$FDTC$	$gr(c^+(\xi))$
$(6k+5)/6$	-2
$(4k+3)/4$	$-7/4$
$(3k+2)/3$	$-3/2$
$(6k+1)/6$	$-1/2$
$(4k+1)/4$	$-1/4$
$(3k+1)/3$	0
$(2k-1)/2$	-1
k	-1

convenient to think of a periodic domain as a linear relation in $H_1(\Sigma; \mathbb{Z})$ among the α and β curves. Doubly-periodic domains are in one-to-one correspondence with elements of $H_2(M; \mathbb{Z})$; we denote by $H(\mathcal{P})$ the homology element corresponding to $\mathcal{P}$. Suppose that $\mathbf{y} = \{y_1, \ldots, y_n\}$ is the intersection point between $\mathbb{T}_\beta$ and $\mathbb{T}_\alpha$ described in Section 2.1 for which $[\mathbf{y}, 0] \in CF^+(\Sigma, \beta, \alpha, z)$ represents $c^+(\xi)$ (here, $n = 2g + r - 1$). Then $\mathfrak{t}_\xi$ is the $Spin^c$ structure associated to $\mathbf{y}$; that is, $\mathfrak{t}_\xi = \mathfrak{s}_z(\mathbf{y})$ [**31**]. So, our goal is to show that $\langle c_1(\mathfrak{s}_z(\mathbf{y})), H(\mathcal{P}) \rangle = 0$ for every doubly periodic domain $\mathcal{P}$.

The *Euler measure* of a region $\mathcal{D}_i$ is the quantity

$$\widehat{\chi}(\mathcal{D}_i) = \chi(\text{int } \mathcal{D}_i) - \frac{1}{4}(\#\text{corner points of } \mathcal{D}_i),$$

where corner points are to be counted with multiplicity [**32**]. We extend the definition of Euler measure to 2-chains linearly. Let $n_{\mathbf{y}}(\mathcal{P})$ be the sum of the local multiplicities of $\mathcal{P}$ at the points $y_i \in \mathbf{y}$. By [**28,** Proposition 7.5],

$$\langle c_1(\mathfrak{s}_z(\mathbf{y})), H(\mathcal{P}) \rangle = \widehat{\chi}(\mathcal{P}) + 2n_{\mathbf{y}}(\mathcal{P}).$$

Below, we prove that both $\widehat{\chi}(\mathcal{P})$ and $n_{\mathbf{y}}(\mathcal{P})$ vanish for every doubly-periodic domain $\mathcal{P}$.

Suppose that the arcs $a_1, \ldots, a_n$ on $S = S_{g,r}$, used to form the α and β curves, are those shown in Figure 3. Let B_i be the boundary component of S which intersects the arcs a_i and a_{i-1} (unless $i = 1$, in which case B_1 is the boundary component that intersects only a_1). For $i = 1, \ldots, r$, let d_i be the oriented curve on $\Sigma = S_{1/2} \cup -S_0$ defined by $d_i = B_i \times \{1/2\}$, where d_i inherits its orientation from the boundary orientation on B_i. We orient the α and β curves, so that the orientation of the arc $\alpha_i \cap S_{1/2}$ agrees with that of $\beta_i \cap S_{1/2}$. Furthermore, we require that $\alpha_i \cdot d_i = +1$ for $i = 1, \ldots, r - 1$.

We may assume that for some fixed integer m, the FDTC of ϕ around each B_i is given by k_i/m for some integer $k_i \geq 0$. Then, ϕ^m is isotopic to a product of Dehn twists $t_{c_1}^{k_1} \cdots t_{c_r}^{k_r}$, as discussed at the beginning of this section. Recall that the arc b_i on S is obtained from a_i via a small isotopy,

as described in Subsection 2.1. Let $b_i^{(1)}$ denote the arc b_i, and let $b_i^{(j)}$ be the arc on S obtained from $b_i^{(j-1)}$ via a similar isotopy for each $j = 2, \ldots, m$ (so that $b_i^{(j)}$ intersects $b_i^{(j-1)}$ transversely in one point and with positive sign). Recall that α_i and β_i are defined by

$$\alpha_i = a_i \times \{1/2\} \cup a_i \times \{0\},$$
$$\beta_i = b_i \times \{1/2\} \cup \phi(b_i) \times \{0\}.$$

For $j = 2, \ldots, m$, we define

$$\beta_i^{(j)} = b_i^{(j)} \times \{1/2\} \cup \phi^j(b_i^{(j)}) \times \{0\}.$$

Suppose $\mathcal{P}$ is a doubly periodic domain specified by the relation

$$(5.1) \qquad \sum_i s_i \alpha_i + \sum_i t_i \beta_i = 0$$

in $H_1(\Sigma; \mathbb{Z})$. Let δ_i be a curve on S which intersects the arc a_i exactly once (and does not intersect the other a_j). The curve $\delta_i \times \{1/2\} \subset \Sigma$ must algebraically intersect $\partial\mathcal{P}$ zero times; that is, $(\delta_i \times \{1/2\}) \cdot \partial\mathcal{P} = \pm(s_i + t_i) = 0$. We can therefore express the relation in equation (5.1) by

$$(5.2) \qquad \sum_i s_i(\alpha_i - \beta_i) = 0.$$

However this implies that $n_{\mathbf{y}}(\mathcal{P}) = 0$ (see figure 11 for the local picture of $\mathcal{P}$ near the intersection point y_i). To see that $\widehat{\chi}(\mathcal{P}) = 0$ as well, we consider the pointed Heegaard multi-diagram $(\Sigma, \alpha, \beta, \beta^{(2)}, \ldots, \beta^{(m)}, z)$.

The relation in equation (5.2) implies that

$$(5.3) \qquad \sum_i s_i(\beta_i^{(j-1)} - \beta_i^{(j)}) = 0$$

in $H_1(\Sigma; \mathbb{Z})$ as well, for each $j = 2, \ldots, m$. Let $\mathcal{P}_j$ be the doubly-periodic domain specified by the relation in equation (5.3). The doubly-periodic domain $\mathcal{P}_{\mathrm{sum}} = \mathcal{P} + \mathcal{P}_2 + \cdots + \mathcal{P}_m$ is therefore specified by the relation

$$(5.4) \qquad \sum_i s_i(\alpha_i - \beta_i^{(m)}) = 0$$

obtained by summing the relation in equation (5.2) with those in equation (5.3). Since $\widehat{\chi}(\mathcal{P}_j) = \widehat{\chi}(\mathcal{P})$ for each $j = 2, \ldots, m$, and Euler measure is additive,

$$(5.5) \qquad \widehat{\chi}(\mathcal{P}_{\mathrm{sum}}) = m\widehat{\chi}(\mathcal{P}).$$

Observe that $(\Sigma, \alpha, \beta^{(m)}, z)$ is a standard pointed Heegaard diagram for $(S_{g,r}, \phi^m)$, and recall that ϕ^m is isotopic to $t_{c_1}^{k_1} \cdots t_{c_r}^{k_r}$. Then, in $H_1(\Sigma; \mathbb{Z})$,

$$(5.6) \qquad \alpha_i - \beta_i^{(m)} = \begin{cases} -k_i d_i + k_{i+1} d_{i+1}, & 1 \le i < r, \\ 0, & r \le i \le n. \end{cases}$$
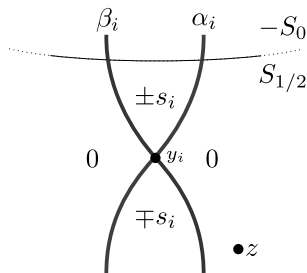
Figure 11. The coefficients of $\mathcal{P}$ near the intersection point y_i.

So, if the relation in equation (5.4) holds, then $\sum_{i<r} s_i(-k_i d_i + k_{i+1} d_{i+1}) = 0$ in $H_1(\Sigma; \mathbb{Z})$ as well. However any relation in $H_1(\Sigma; \mathbb{Z})$ among the curves $d_1, \ldots, d_r$ is of the form $t(d_1 + \cdots + d_r) = 0$. Hence,

$$\sum_{i<r} s_i(-k_i d_i + k_{i+1} d_{i+1}) = t(d_1 + \cdots + d_r).$$

On the other hand, since all of the k_i are non-negative, this can only happen if $t = 0$. It follows that $s_i = s_{i-1}$ if $k_i \neq 0$ (unless $i = 1$, in which case $k_1 \neq 0$ implies that $s_1 = 0$).

Therefore, the relation in equation (5.4) breaks up into smaller relations of the form

(5.7) $$s_i(\alpha_i - \beta_i^{(m)}) = 0,$$

for $i \geq r$, and

(5.8) $$\sum_{i_1 \leq i \leq i_2 < r} s_i(\alpha_i - \beta_i^{(m)}) = 0,$$

where $s_i = s_j$ for i and j between i_1 and i_2. It is not hard to see directly that the doubly periodic domains given by the relations in equations (5.7) and (5.8) have vanishing Euler measure. In either case, these periodic domains, thought of as linear combinations of regions in $\Sigma \setminus \cup \alpha_i \setminus \cup \beta_i^{(m)}$, each consist of two canceling bigon regions together with square regions (whose Euler measures are zero). See Figures 12 and 13 for reference. It follows that $\widehat{\chi}(\mathcal{P}_{\text{sum}}) = 0$, which, in turn, implies that $\widehat{\chi}(\mathcal{P}) = 0$, by equation (5.5). This completes the proof of Proposition 5.3.

$\square$

5.2. Grading shifts and the proof of Theorem 1.13. Below, we study the grading shifts associated to the maps induced by capping off. Suppose $(S_{g,r-1}, \phi')$ is the open book obtained from $(S_{g,r}, \phi)$ by capping off one of the boundary components of $S_{g,r}$. Let W be the corresponding 2-handle cobordism from $-M_{S_{g,r-1}, \phi'}$ to $-M_{S_{g,r}, \phi}$. If ϕ is periodic with non-negative FDTCs, then the same is true of ϕ', and it follows from Proposition 5.3
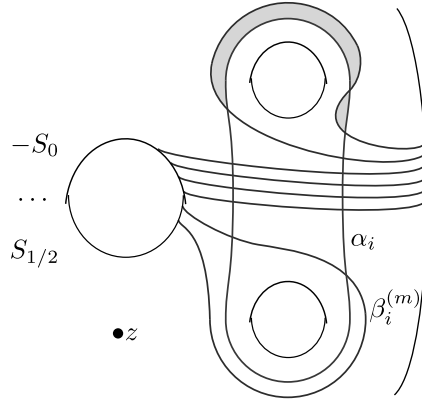
Figure 12. Shown here are α_i and $\beta_i^{(m)}$ for some $i \geq r$. The region bounded by these curves is a periodic domain corresponding to a relation as in equation (5.7). Note that it consists of square regions and two canceling bigon regions (which we have shaded). In this figure, $k_r = 3$.
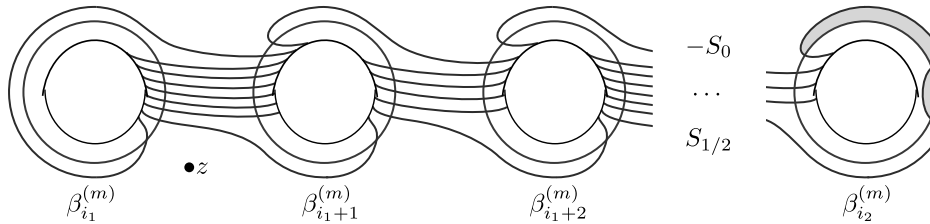

Figure 13. The region bounded by these α and $\beta^{(m)}$ curves is a periodic domain corresponding to a relation as in equation (5.8). It consists of square regions and two canceling bigon regions (which we have shaded). In this figure, $k_{i_1} = 0$, $k_{i_1+1} = 4$, $k_{i_1+2} = 3$, $k_{i_1+3} = 4$, $k_{i_2} = 2$ and $k_{i_2+1} = 0$.

that the contact invariants $c^+(S_{g,r-1}, \phi')$ and $c^+(S_{g,r}, \phi)$ have well-defined $\mathbb{Q}$-gradings. Since $F_{W,\mathfrak{s}_0}^+$ sends $c^+(S_{g,r-1}, \phi')$ to $c^+(S_{g,r}, \phi)$, by Theorem 1.2, the grading shift formula in [**32**] gives

$$(5.9) \quad gr(c^+(S_{g,r}, \phi)) - gr(c^+(S_{g,r-1}, \phi')) = \frac{c_1(\mathfrak{s}_0)^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

Lemma 5.4. *The cobordism W either has trivial intersection form, or $b_2^+(W) = 0$ and $\sigma(W) = -1$.*

Since W is obtained from a single 2-handle attachment, $\chi(W) = 1$. Together with Lemma 5.4, this implies that

$$(5.10) \qquad\qquad gr(c^+(S_{g,r}, \phi)) - gr(c^+(S_{g,r-1}, \phi')) \leq 1/4.$$

Suppose that after capping off all but one of the boundary components of $S_{g,r}$, we are left with an open book $(S_{g,1}, \phi'')$. It follows from Inequality (5.10) that

$$gr(c^+(S_{g,r}, \phi)) \leq (r-1)/4 + gr(c^+(S_{g,1}, \phi'')).$$

Now, suppose $g = 1$. As mentioned in the beginning of this section, $gr(c^+(S_{1,1}, \phi'')) \leq 0$ (see table 1). As a result,

$$gr(c^+(S_{1,r}, \phi)) \leq (r-1)/4,$$

which is equivalent to the statement of Theorem 5.2. All that remains is to prove Lemma 5.4. We assume below that it is the boundary component B_1 that is getting capped off.

Proof of Lemma 5.4. Let Δ denote the 2-simplex with edges, e_β, e_γ and e_α, described in Section 2.2. The pointed Heegaard triple-diagram $(\Sigma, \beta, \gamma, \alpha, z)$ associated to the capping off operation above (and defined in Section 2.1) specifies an identification space

$$X_{\beta,\gamma,\alpha} = \frac{(\Delta \times \Sigma) \amalg (e_\beta \times H_\beta) \amalg (e_\gamma \times H_\gamma) \amalg (e_\alpha \times H_\alpha)}{\substack{(e_\beta \times \Sigma) \sim (e_\beta \times \partial H_\beta), (e_\gamma \times \Sigma) \sim (e_\gamma \times \partial H_\gamma), \\ (e_\alpha \times \Sigma) \sim (e_\alpha \times \partial H_\alpha)}},$$

where H_β, H_γ and H_α are the handlebodies corresponding to the β, γ and α curves (see the diagram on the left of figure 14 for a schematic picture of $X_{\beta,\gamma,\alpha}$). After rounding corners, $X_{\beta,\gamma,\alpha}$ is a smooth cobordism with boundary $-M_{\beta,\gamma} - M_{\gamma,\alpha} + M_{\beta,\alpha}$. (Here, $M_{\beta,\gamma}$ is the 3-manifold specified by the Heegaard diagram (Σ, β, α), and similarly for $M_{\gamma,\alpha}$ and $M_{\beta,\alpha}$.) In fact, $X_{\beta,\gamma,\alpha}$ is just the complement of a neighborhood of a 1-complex in the cobordism W, so the intersection form of $X_{\beta,\gamma,\alpha}$ is the same as that of W (refer to [**29, 32**] for more details).

Elements of $H_2(X_{\beta,\gamma,\alpha}; \mathbb{Z})$ are in one-to-one correspondence with triply-periodic domains for the Heegaard diagram $(\Sigma, \beta, \gamma, \alpha, z)$. If $\mathcal{P}$ is a triply-periodic domain, we denote the β, γ and α components of $\partial \mathcal{P}$ by $\partial_\beta \mathcal{P}$, $\partial_\gamma \mathcal{P}$
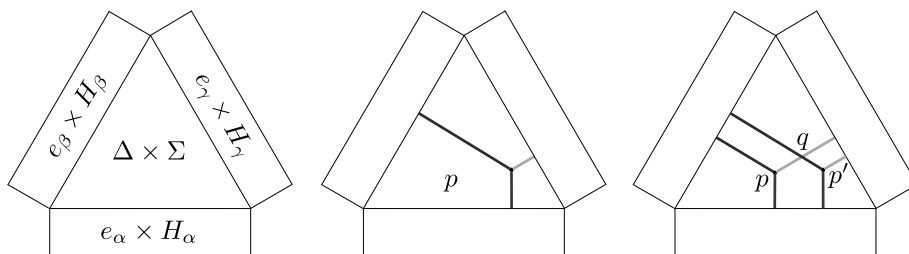

Figure 14. From left to right: the cobordism $X_{\beta,\gamma,\alpha}$, a homology class $H(\mathcal{P})$, and the intersection of two classes, $H(\mathcal{P})$ and $H(\mathcal{P}')$.

and $\partial_\alpha \mathcal{P}$. The homology class corresponding to a triply-periodic domain $\mathcal{P}$ is constructed as follows. Pick a point $p \in \Delta$, and consider the copy of $\mathcal{P}$ contained in $\{p\} \times \Sigma$. Attach cylinders, connecting each component of $\partial_\beta \mathcal{P}$ in $\{p\} \times \Sigma$ with the corresponding component in $\{u\} \times \Sigma$ for some $u \in e_\beta$. Then cap off these boundary components with discs inside $\{u\} \times H_\beta$. Do the same for the components of $\partial_\gamma \mathcal{P}$ and $\partial_\alpha \mathcal{P}$. We denote this homology class by $H(\mathcal{P})$.

The middle diagram in Figure 14 shows a schematic picture of this construction. The point labeled p represents the copy of $\mathcal{P}$ in $\{p\} \times \Sigma$, and the three legs represent the attaching cylinders for the components of $\partial_\beta \mathcal{P}$, $\partial_\gamma \mathcal{P}$ and $\partial_\alpha \mathcal{P}$. The rightmost diagram is meant to represent the intersection of two such homology classes, $H(\mathcal{P})$ and $H(\mathcal{P}')$. The β attaching cylinders of $H(\mathcal{P})$ intersect the γ attaching cylinders of $H(\mathcal{P}')$ at points in $\{q\} \times \Sigma$, and it is not hard to check that the algebraic intersection number

$$(5.11) \qquad H(\mathcal{P}) \cdot H(\mathcal{P}') = (\partial_\beta \mathcal{P}) \cdot (\partial_\gamma \mathcal{P}').$$

In particular, note that if $\mathcal{P}$ is a doubly-periodic domain (by which we mean that $\partial \mathcal{P}$ consists of only two of the three types of attaching curves) then $H(\mathcal{P})$ pairs trivially with every element in $H_2(X_{\beta,\gamma,\alpha}; \mathbb{Z})$.

Suppose that the intersection form of $X_{\beta,\gamma,\alpha}$ is non-trivial. Let $\mathcal{P}$ be a triply-periodic domain given by the relation

$$N\gamma_1 + \sum_i s_i \alpha_i + \sum_i t_i \beta_i = 0$$

in $H_1(\Sigma; \mathbb{Z})$, where $N \neq 0$. Since γ_i is isotopic to β_i for $i = 2, \ldots, n$, every triply-periodic domain differs from some multiple of $\mathcal{P}$ by a sum of doubly-periodic domains. Let $d_1, \ldots, d_r$ be the curves defined in the proof of Proposition 5.3, and orient the α and β curves as before. We orient γ_1 in the same direction as d_1. The same argument used in the proof of Proposition 5.3 shows that $r_i = -s_i$ for each i, so $\mathcal{P}$ is given by the relation

$$(5.12) \qquad N\gamma_1 + \sum_i s_i(\alpha_i - \beta_i) = 0.$$

By equation (5.11),

$$H(\mathcal{P})^2 = (\partial_\beta \mathcal{P}) \cdot (\partial_\gamma \mathcal{P}) = -Ns_1(\beta_1 \cdot \gamma_1) = -Ns_1,$$

so Lemma 5.4 follows if we can show that $Ns_1 > 0$.

As before, we assume that the FDTC of ϕ around B_i is k_i/m, where $k_i \geq 0$. The relation in equation (5.12) implies that

$$(5.13) \qquad N\gamma_1 + \sum_i s_i(\beta_i^{(j-1)} - \beta_i^{(j)}) = 0$$

in $H_1(\Sigma; \mathbb{Z})$ for $j = 2, \ldots, m$. Adding the relations in equations (5.12) and (5.13), we find that

$$(5.14) \qquad mN\gamma_1 + \sum_i s_i(\alpha_i - \beta_i^{(m)}) = 0$$

in $H_1(\Sigma; \mathbb{Z})$. After making the substitutions from equation (5.6), and noting that $\gamma_1 = d_1$, it follows that

$$mNd_1 + \sum_{i<r} s_i(-k_i d_i + k_{i+1} d_{i+1}) = 0$$

in $H_1(\Sigma; \mathbb{Z})$ as well. As in the proof of Proposition 5.3, this implies that

$$(5.15) \qquad mNd_1 + \sum_{i<r} s_i(-k_i d_i + k_{i+1} d_{i+1}) = t(d_1 + \cdots + d_r)$$

for some t. If $t = 0$, then $mN = s_1 k_1$. We are assuming that $X_{\beta,\gamma,\alpha}$ has non-trivial intersection form, so $H(\mathcal{P})^2 = -Ns_1 \neq 0$. Therefore, k_1 is strictly greater than zero, and it follows that $Ns_1 > 0$, as hoped. If $t \neq 0$, we can assume, without loss of generality, that $t > 0$. Then all k_i and s_i are strictly greater than zero, and $mN - s_1 k_1 > 0$, which implies that $Ns_1 > 0$ as well. $\qquad \square$

5.3. A formula for the d_3 invariant. Below, we explicitly compute the grading shift in equation (5.9) in terms of the FDTCs of ϕ. If the intersection form of W is trivial, then this grading shift is simply $-1/2$. Otherwise, $b_2^+(W) = 0$ and $\sigma(W) = -1$ (by Lemma 5.4), and the grading shift is $(c_1(\mathfrak{s}_0)^2 + 1)/4$. According to Proposition 5.3, $c_1(\mathfrak{s}_0)$ is sent to zero by the restriction map

$$H^2(W; \mathbb{Q}) \to H^2(\partial W; \mathbb{Q}).$$

Therefore, $c_1(\mathfrak{s}_0)$ is the image of a class $k \cdot PD(\lambda)$ under the map

$$H^2(W, \partial W; \mathbb{Q}) \to H^2(W; \mathbb{Q}),$$

where λ is a generator of the dimension one subspace of elements $B_2^-(W) \subset H_2(W)$ with negative self-intersection. By definition,

$$c_1(\mathfrak{s}_0)^2 = (k \cdot PD(\lambda))^2 = k^2 \cdot \lambda^2 = \frac{\langle c_1(\mathfrak{s}_0), \lambda \rangle^2}{\lambda^2}.$$

Recall from Section 2.2 that $\mathfrak{s}_0 = \mathfrak{s}_z(\psi_0)$, where ψ_0 is the homotopy class of Whitney triangles whose domain $\mathcal{D}(\psi_0)$ is $\Delta_1 + \cdots + \Delta_n$, and let $H(\mathcal{P})$ be a class, which generates $B_2^-(X_{\beta,\gamma,\alpha})$. Then, the equation above becomes

$$(5.16) \qquad c_1(\mathfrak{s}_0)^2 = \frac{\langle c_1(\mathfrak{s}_z(\psi_0)), H(\mathcal{P}) \rangle^2}{H(\mathcal{P})^2}.$$

To compute $\langle c_1(\mathfrak{s}_z(\psi_0)), H(\mathcal{P}) \rangle$, we recall the *dual spider number* of a Whitney triangle

$$u : \Delta \to \mathrm{Sym}^n(\Sigma)$$

and a triply-periodic domain $\mathcal{P}$, following the exposition in [**32**]. First, note that the orientations on the β, γ and α curves induce "leftward" pointing normal vector fields along the curves. Let β_i', γ_i' and α_i' be copies of the attaching curves β_i, γ_i and α_i, translated slightly in these normal directions, and let $\mathbb{T}_{\beta'}$, $\mathbb{T}_{\gamma'}$ and $\mathbb{T}_{\alpha'}$ denote the corresponding tori in $\mathrm{Sym}^n(\Sigma)$. By construction, $u(e_\beta)$ misses $\mathbb{T}_{\beta'}$, $u(e_\gamma)$ misses $\mathbb{T}_{\gamma'}$ and $u(e_\alpha)$ misses $\mathbb{T}_{\alpha'}$.

Let x be an interior point of Δ so that $u(x)$ misses the β', γ' and α' curves, and choose three oriented paths, b, c and a, from x to e_β, e_γ and e_α, respectively. Let $\partial_{\beta'}(\mathcal{P})$, $\partial_{\gamma'}(\mathcal{P})$ and $\partial_{\alpha'}(\mathcal{P})$ be the 1-chains obtained by translating the corresponding components of $\partial\mathcal{P}$ in the normal directions described above. The dual spider number of u and $\mathcal{P}$ is given by

$$\sigma(u, \mathcal{P}) = n_{u(x)}(\mathcal{P}) + \partial_{\beta'}(\mathcal{P}) \cdot b + \partial_{\gamma'}(\mathcal{P}) \cdot c + \partial_{\alpha'}(\mathcal{P}) \cdot a.$$

In [**32,** Proposition 6.3], Ozsváth and Szabó prove that

$$\langle c_1(\mathfrak{s}_z(\psi_0)), H(\mathcal{P}) \rangle = \widehat{\chi}(\mathcal{P}) + \#(\partial\mathcal{P}) + 2\sigma(u, \mathcal{P})$$

for any Whitney triangle u representing ψ_0.

Suppose ϕ is a periodic diffeomorphism of $S = S_{g,r}$ with FDTCs $0 \leq k_1/m \leq \cdots \leq k_r/m$. We shall cap off the boundary component B_1. Let W denote the associated 2-handle cobordism and suppose that the intersection form of W is non-trivial. Let $\mathcal{P}$ be the triply-periodic domain specified by the relation in equation (5.12). To compute $\widehat{\chi}(\mathcal{P})$, we consider the pointed Heegaard multi-diagram $(\Sigma, \alpha, \beta, \beta^{(2)}, \ldots, \beta^{(m)}, \gamma, z)$. Let $\mathcal{P}_j$ be the triply-periodic domain specified by the relation in equation (5.13) for $j = 2, \ldots, m$. Then $\mathcal{P}_{\mathrm{sum}} = \mathcal{P} + \mathcal{P}_2 + \cdots + \mathcal{P}_m$ is specified by the relation in equation (5.14), and

$$\widehat{\chi}(\mathcal{P}_{\mathrm{sum}}) = m\widehat{\chi}(\mathcal{P}),$$

as before. Per equation (5.6), this relation breaks up into relations in $H_1(\Sigma; \mathbb{Z})$ of the form

$$s_i(\alpha_i - \beta_i^{(m)}) = 0,$$

for $i \geq r$, and

$$(5.17) \qquad mNd_1 + \sum_{i < r} s_i(-k_i d_i + k_{i+1} d_{i+1}) = 0.$$

As noted previously, the doubly periodic domains specified by the former relations have Euler measure zero, and the latter relation implies that

$$mNd_1 + \sum_{i < r} s_i(-k_i d_i + k_{i+1} d_{i+1}) = t(d_1 + \cdots + d_r)$$

for some t. Suppose that $t = 0$. If no k_i is zero, then all of the s_i must vanish. However, this implies that $H(\mathcal{P})^2 = -Ns_1 = 0$, which contradicts our assumption on the intersection form of W. If some k_i vanishes, then k_1 must vanish since $0 \leq k_1 \leq k_i$ by assumption. But this too implies that

$H(\mathcal{P})^2 = 0$. So, it must be the case $t \neq 0$ and $k_i > 0$ for all i. We may assume (rescaling as necessary) that $t = -k_r$. Then

(5.18) $$s_i = -k_r(1/k_r + 1/k_{r-1} + \cdots + 1/k_{i+1}).$$

We define s_0 using this formula as well; note that $s_0 = mN/k_1$.

The triply periodic domain $\mathcal{P}'_{\text{sum}}$, given by the relation in Equation (5.17), is composed of square regions, two triangular regions, a pentagonal region and a region F that has genus g, one boundary component, and $4(r-1)$ corners (see figure 15). It is easy to check that the contributions of the triangular regions and the pentagonal region to $\widehat{\chi}(\mathcal{P}_{\text{sum}'})$ cancel. Since the region F has multiplicity k_r in $\mathcal{P}'_{\text{sum}}$,

$$\widehat{\chi}(\mathcal{P}_{\text{sum}}) = \widehat{\chi}(\mathcal{P}'_{\text{sum}}) = k_r(2 - 2g - r),$$

and, hence,

(5.19) $$\widehat{\chi}(\mathcal{P}) = k_r(2 - 2g - r)/m.$$

Let $u : \Delta \to \text{Sym}^n(\Sigma)$ be a representative of the homotopy class ψ_0. The local contribution of Δ_i to the dual spider number $\sigma(u, \mathcal{P})$ is $-|s_i|$. On the other hand, the number of boundary components of $\mathcal{P}$ is $|N| + 2|s_1| + \cdots + 2|s_n|$. So, the quantity

$$\langle c_1(\mathfrak{s}_z(\psi_0)), H(\mathcal{P}) \rangle = \widehat{\chi}(\mathcal{P}) + \#(\partial\mathcal{P}) + 2\sigma(u, \mathcal{P})$$

is simply

$$\begin{aligned} k_r(2 - 2g - r)/m + |N| \\ = k_r(2 - 2g - r)/m - N \\ = k_r(2 - 2g - r)/m - k_1 s_0/m. \end{aligned}$$

And we saw in the previous subsection that $H(\mathcal{P})^2 = -Ns_1 = -k_1 s_0 s_1/m$. As a result,

(5.20)
$$c_1(\mathfrak{s}_0)^2 = \frac{(k_r(2 - 2g - r)/m - k_1 s_0/m)^2}{-k_1 s_0 s_1/m} = \frac{(k_r(2 - 2g - r) - k_1 s_0)^2}{-mk_1 s_0 s_1},$$

by equation (5.16).

We now restrict our focus to genus one open books. Suppose that ξ is a tight contact structure supported by an open book $(S_{1,1}, \phi)$ with periodic ϕ. The table below lists the grading of $c^+(\xi)$ as a function of the FDTCs of ϕ. This follows from the grading calculations in [4, Section 6] (for non-integral FDTC's) and [21, Proposition 9] (for integral FDTC's).

Let f be the function, specified by this table, which takes an FDTC c and outputs $f(c) = gr(c^+(\xi))$. The theorem below then follows from Proposition 5.1, the grading shift formula in equation (5.9), and the expression for $c_1(\mathfrak{s}_0)^2$ in equation (5.20).
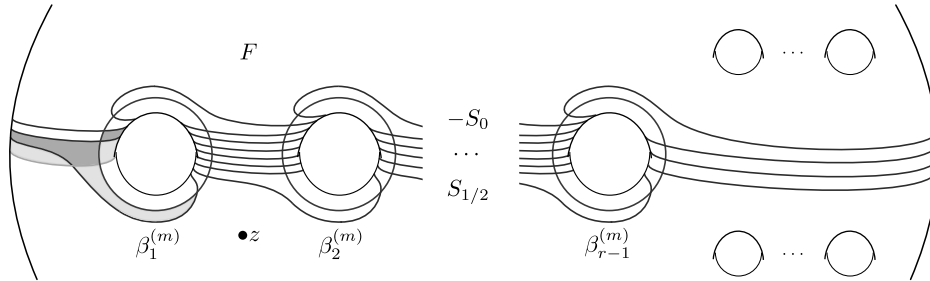
Figure 15. A portion of the surface Σ. The region bounded by the α, $\beta^{(m)}$ and γ curves is the periodic domain $\mathcal{P}'_{\text{sum}}$. It is composed of squares, two triangular regions (gray), a pentagonal region (dark gray) and the region F.

Theorem 5.5. *Suppose* (M, ξ) *is compatible with a genus one open book* $(S_{1,r}, \phi)$, *where* ϕ *is periodic with FDTCs* $0 \leq k_1/m \leq \cdots \leq k_r/m$. *Let I be the smallest integer such that* $k_I \neq 0$. *For* $i = I - 1, \ldots, r - 1$, *define*

$$s_i = -k_r(1/k_r + 1/k_{r-1} + \cdots + 1/k_{i+1}).$$

Then

$$d_3(\xi) = -f\left(\frac{k_r}{m}\right) + \frac{3I - r - 4}{4} + \frac{1}{4}\sum_{j=I-1}^{r-2} \frac{(k_r(j-r) - k_{j+1}s_j)^2}{mk_{j+1}s_j s_{j+1}}.$$

6. Capping off and pseudo-Anosov open books

If ϕ is a boundary-fixing diffeomorphism of S which is neither periodic nor reducible, then ϕ is said to be pseudo-Anosov. In this case (and only in this case), ϕ is freely isotopic to a homeomorphism ϕ_0 for which there exists a transverse pair of singular measured foliations, $(\mathcal{F}_s, \mu_s)$ and $(\mathcal{F}_u, \mu_u)$, of S such that $\phi_0(\mathcal{F}_s, \mu_s) = (\mathcal{F}_s, \lambda\mu_s)$ and $\phi_0(\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda^{-1}\mu_u)$ for some $\lambda > 1$ [**36**]. The singularities of $\mathcal{F}_s$ and $\mathcal{F}_u$ which lie in the interior of S are required to be "p-pronged saddles" with $p \geq 3$, as shown in Figure 16. Each foliation must have at least one singularity on every boundary component, and each boundary singularity must have a neighborhood of the form shown in figure 17.

The *FDTC* of ϕ around a boundary component of S measures the amount of twisting around this component that takes place in the free isotopy from ϕ to ϕ_0. More precisely, let $x_1, \ldots, x_n$ be the singularities of $\mathcal{F}_s$ which lie on some boundary component B, labeled in order as one traverses B in the direction specified by its orientation. The map ϕ_0 permutes these singularities; in fact, we may assume that there exists an integer k for which ϕ_0 sends x_i to x_{i+k} for all i (where the subscripts are taken modulo n). If $H : S \times [0, 1] \rightarrow S$ is the free isotopy from ϕ to ϕ_0, and
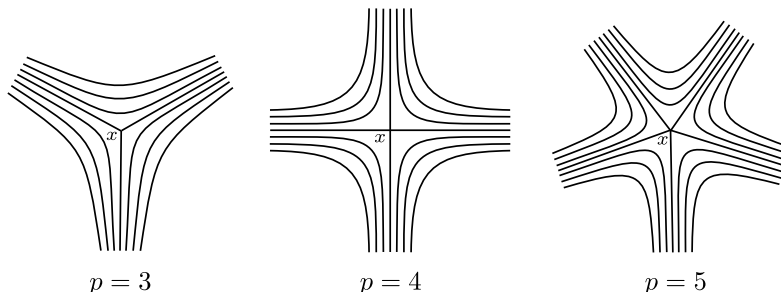
Figure 16. Neighborhoods of interior singularities. The singular leaves in each neighborhood are called "prongs". From left to right, x is a p-pronged singularity with $p = 3, 4, 5$.
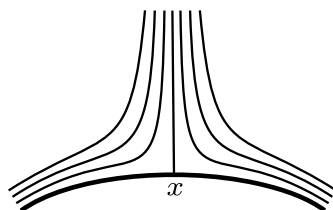

Figure 17. A neighborhood of a boundary singularity x. The thickened segment represents a portion of the boundary.

$\beta : B \times [0, 1] \to B \times [0, 1]$ is the map that sends (x, t) to $(H(x, t), t)$, then $\beta(x_i \times [0, 1])$ is an arc from $(x_i, 0)$ to $(x_{i+k}, 1)$. The FDTC of ϕ around B is defined to be the fraction $c \in \mathbb{Q}$, where $c \equiv k/n$ modulo 1 is the number of times that $\beta(x_i \times [0, 1])$ wraps around $B \times [0, 1]$ (see [**18**] for more details).

Suppose that $(S_{g,r}, \phi)$ is an open book with pseudo-Anosov ϕ, and let $B_1, \ldots, B_r$ denote the boundary components of $S_{g,r}$. Let c_i be the fractional Dehn twists coefficient of ϕ around B_i, and suppose that $\mathcal{F}_s$ and $\mathcal{F}_u$ are the singular foliations associated to ϕ. If each of these foliations has $p > 1$ singularities on B_r ($\mathcal{F}_s$ and $\mathcal{F}_u$ will have the same number), then they can be extended to transverse singular measured foliations, $\mathcal{F}_s'$ and $\mathcal{F}_u'$, of the surface $S_{g,r-1}$ obtained by capping off the boundary component B_r. To see this, remove the leaf corresponding to B_r in each of these foliations, and extend them across the capping disc, creating a p-pronged singularity at the center of the disc (if $p = 2$, then the foliations extend without singularity over the disc). The induced diffeomorphism ϕ' of the capped off surface $S_{g,r-1}$ is then pseudo-Anosov with associated foliations $\mathcal{F}_s'$ and $\mathcal{F}_u'$. Moreover, c_i is the FDTC of ϕ' around the boundary component B_i for $i = 1, \ldots, r-1$ since this modification took place locally. The requirement that $\mathcal{F}_s$ and $\mathcal{F}_u$ have $p > 1$ singularities on B_r is critical in order for this to work; otherwise, there is no obvious way of extending these foliations across the capping disc so that

the new interior singularities have $p \geq 3$ prongs. In fact, there *are* pseudo-Anosov diffeomorphisms of $S_{g,r}$ for which the induced diffeomorphism on the capped off surface $S_{g,r-1}$ is *not* pseudo-Anosov.

In [2], we show that if ϕ is a pseudo-Anosov diffeomorphism of $S_{1,1}$ with FDTC less than 1 then $M_{S_{1,1},\phi}$ is an L-space, which implies that $c^+(S_{1,1}, \phi)$ is in the image of U^d for all $d \in \mathbb{N}$.[2] The corollary below follows immediately from this fact.

Corollary 6.1. *Suppose that (M, ξ) is supported by a genus one open book $(S_{1,r}, \phi)$ with pseudo-Anosov ϕ such that the associated foliations have exactly two singularities on every boundary component of $S_{1,r}$. If any of the fractional Dehn twists coefficients of ϕ are less than 1, then $c^+(\xi)$ is in the image of U^d for all $d \in \mathbb{N}$.*

Remark 6.2. The assumption in Corollary 6.1 that there are *exactly* two singularities on every boundary component is equivalent to the condition that there are *at least* two singularities on each boundary component (and no interior singularities), and is also equivalent to the condition that the foliations associated to ϕ are orientable. Finally, note that any open book $(S_{1,r}, \phi)$ of the sort considered in the corollary above arises from an Anosov map ϕ_0 of $S_{1,0}$ by puncturing the torus (creating boundary components) at r fixed points of ϕ_0.

Proof of Corollary 6.1. Suppose the FDTC of ϕ around some boundary component is less than 1. After capping off every other boundary component, we obtain an open book $(S_{1,1}, \phi')$ where ϕ' is pseudo-Anosov with FDTC less than 1 (we may do this since the foliations associated to ϕ have more than one singularity on every boundary component). Then, $c^+(S_{1,1}, \phi')$ is in the image of U^d for all $d \in \mathbb{N}$. Combined with Theorem 1.2, this proves the corollary. □

In [19, Theorem 1.1], Honda et al. show (using the taut foliations constructed by Roberts in [35] along with a result of Eliashberg and Thurston [11]) that if a pseudo-Anosov diffeomorphism ϕ of $S_{1,1}$ has FDTC at least 1, then the contact structure compatible with the open book $(S_{1,1}, \phi)$ is weakly symplectically fillable by a filling W with $b_2^+(W) > 0$. This prompts the following question.

Question 6.3. *Suppose that ϕ is a pseudo-Anosov diffeomorphism of $S_{1,r}$ whose FDTCs are all at least 1. Is the contact structure compatible with the open book $(S_{1,r}, \phi)$ necessarily weakly symplectically fillable by a filling with $b_2^+(W) > 0$?*

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[2]The results in [2] are not stated in terms of FDTC; the translation is provided by the fact that the FDTC of $\phi_{\mathbf{n},d}$, to borrow the notation of Section 7, is $d/2$.

This question has recently been answered in the negative by the author and Etnyre in [**6**]. There, we provide an infinite family of genus one, two boundary component open books whose monodromies have arbitrarily large FDTCs such that the genus one, one boundary component open books resulting from capping off either boundary component support overtwisted contact structures. In our genus one, two boundary component examples, the associated stable and unstable foliations each have exactly one singularity on every boundary component, so capping off does not preserve the foliation structure. We therefore propose the following modification of Question 6.3:

Question 6.4. *Suppose that ϕ is a pseudo-Anosov diffeomorphism of $S_{1,r}$ whose FDTCs are all at least 1 and whose associated foliations have two singularities on every boundary components of $S_{1,r}$. Is the contact structure compatible with the open book $(S_{1,r}, \phi)$ necessarily weakly symplectically fillable by a filling with $b_2^+(W) > 0$?*

In [**27**], Ozsváth and Szabó show that if (M, ξ) is weakly symplectically fillable by such a filling, and $b_1(M) = 0$ (in which case this weak filling may be perturbed to a strong filling [**26**]), then there exists some $d \in \mathbb{N}$ for which $c^+(\xi)$ is *not* in the image of U^d. The conjecture below follows from this fact, together with Corollary 6.1 and a positive answer to Question 6.4.

Conjecture 6.5. *Suppose that (M, ξ) is supported by a genus one open book $(S_{1,r}, \phi)$ with pseudo-Anosov ϕ such that the associated foliations have two singularities on every boundary component of $S_{1,r}$. If $b_1(M) = 0$, then ξ is strongly symplectically fillable by a filling with $b_2^+(W) > 0$ if and only if there exist some $d \in \mathbb{N}$ for which $c^+(\xi)$ is* not *in the image of U^d.*

One may view Conjecture 6.5 as a potential obstruction, via Heegaard Floer homology, to a contact structure being supported by a certain type of genus one open book. Even if true, however, this obstruction appears rather cumbersome. One wonders whether there is a more geometric interpretation of the condition that the foliations associated to ϕ have two singularities on every boundary component. Such a condition, combined with a result of the sort proposed in Conjecture 6.5 could be helpful in formulating a usable obstruction to $sg(\xi) = 1$.

7. The support genera of contact structures compatible with $(S_{1,1}, \phi)$

The mapping class group of $S_{1,1}$ is generated by Dehn twists around the curves a and b shown on the right in Figure 1. It is well known that this group is isomorphic to the braid group B_3 by an isomorphism $\Phi : \mathrm{MCG}(S_{1,1}, \partial S_{1,1}) \to B_3$, which sends the Dehn twists t_a and t_b to the standard generators σ_1 and σ_2 of B_3. So, by a theorem of Murasugi on 3-braids [**25**], we have the following.

Lemma 7.1. *Let $h = (t_a t_b)^3$. Any diffeomorphism of $S_{1,1}$ that fixes the boundary pointwise and is freely isotopic to a pseudo-Anosov map is, up to conjugation, isotopic (rel. ∂) to a diffeomorphism*

$$\phi_{\mathbf{n},d} = h^d \cdot t_b t_a^{-n_1} \cdots t_b t_a^{-n_k}$$

for some k-tuple of non-negative integers $\mathbf{n} = (n_1, \ldots, n_k)$ for which some $n_i \neq 0$, and some $d \in \mathbb{Z}$.

The diffeomorphism h represents a "half-twist" around a curve δ parallel to the boundary of $S_{1,1}$; that is, $h^2 = t_\delta$. Let $\xi_{\mathbf{n},d}$ denote the contact structure compatible with the open book $(S_{1,1}, \phi_{\mathbf{n},d})$. In this short section, we prove the following.

Proposition 7.2. *The support genus of $\xi_{\mathbf{n},d}$ is zero for $d \leq 0$, and one for $d > 1$.*

Note that this proposition is inconclusive for $d = 1$.

Proof of Proposition 7.2. In [**2, 17**], it is shown that $\xi_{\mathbf{n},d}$ is tight if and only if $d > 0$. Recall from the introduction that $sg(\xi) = 0$, if ξ is overtwisted [**12**]. It therefore follows immediately that $sg(\xi_{\mathbf{n},d}) = 0$ for $d \leq 0$.

To simplify notation in this proof, we let $M_{\mathbf{n},d}$ denote the 3-manifold with open book decomposition $(S_{1,1}, \phi_{\mathbf{n},d})$. Observe that $M_{\mathbf{n},d}$ is the double cover of S^3 branched along the closed braid $B_{\mathbf{n},d}$ corresponding to the braid word $\Phi(\phi_{\mathbf{n},d}) \in B_3$ (see [**4**, Section 2], for example). Note that $B_{\mathbf{n},d}$ is obtained from the alternating braid $B_{\mathbf{n},0}$ by adding d full positive twists. It is clear that, as manifolds, links and contact structures, $M_{\mathbf{n},d}$, $B_{\mathbf{n},d}$ and $\xi_{\mathbf{n},d}$ are invariant under the action of cyclic permutation on the tuple $\mathbf{n}$ (the image $\Phi(h)$ is central in B_3).

The contact structure ξ' in Example 1.12 associated to the open book $(S_{1,1}, (t_a t_b)^5 t_b^2)$ is simply $\xi_{(1),2}$ in our notation. In that example, we showed that there is some $d \in \mathbb{N}$ for which $c^+(\xi_{(1),2}) \notin U^d \cdot HF^+(-M_{(1),2})$; let us call this condition on $c^+(\xi_{(1),2})$ *Condition* **R**. Recall that if $c^+(\xi)$ satisfies Condition **R**, then $sg(\xi) > 0$, by Proposition 1.11.

For a k-tuple $\mathbf{n}$ as above, let $\mathbf{n}^-$ denote the k-tuple obtained from $\mathbf{n}$ by adding 1 to its kth entry, and let $\mathbf{n}^0$ denote the $(k+1)$-tuple obtained by appending a 0 to $\mathbf{n}$. Starting from the 1-tuple $\mathbf{n} = (1)$, we can obtain any k-tuple of the form described in Lemma 7.1 by repeated applications of the operations $\mathbf{n} \mapsto \mathbf{n}^-$, $\mathbf{n} \mapsto \mathbf{n}^0$, together with cyclic permutation. Moreover, for $d \geq 2$, the monodromy $\phi_{\mathbf{n},d}$ is obtained from $\phi_{\mathbf{n},2}$ by composition with $6d-12$ right handed Dehn twists around the curves a and b. Hence, by the naturality of the contact invariant under maps induced by Stein cobordisms [**17, 31**], $c^+(\xi_{\mathbf{n},d})$ satisfies Condition **R** as long as $c^+(\xi_{\mathbf{n},2})$ does.

Thus, in order to prove Proposition 7.2, it suffices to show that if $\xi_{\mathbf{n},2}$ satisfies Condition **R**, then so do $\xi_{\mathbf{n}^-,2}$ and $\xi_{\mathbf{n}^0,2}$. For the latter, observe

that $\phi_{\mathbf{n}^0,2} = \phi_{\mathbf{n},2} \cdot t_b$. So, the naturality of the contact invariant under maps induced by Stein cobordisms implies that $c^+(\xi_{\mathbf{n}^0,2})$ satisfies Condition **R** as long as $c^+(\xi_{\mathbf{n},2})$ does. Proving the other implication takes slightly more work.

Observe that $\phi_{\mathbf{n},2} = \phi_{\mathbf{n}^-,2} \cdot t_a$, and consider the map $\widehat{G} : \widehat{HF}(-M_{\mathbf{n},2}) \rightarrow \widehat{HF}(-M_{\mathbf{n}^-,2})$ induced by the corresponding Stein 2-handle cobordism (or, equivalently, by -1-surgery on a copy of the curve a in the open book for $M_{\mathbf{n}^-,2}$). To understand $\widehat{G}$, it helps to think of $M_{\mathbf{n},d}$ and $M_{\mathbf{n}^-,d}$ as the branched double covers $\Sigma(B_{\mathbf{n},d})$ and $\Sigma(B_{\mathbf{n}^-,d})$. Note that $B_{\mathbf{n},d}$ is obtained from $B_{\mathbf{n}^-,d}$ by taking the *oriented* resolution of $B_{\mathbf{n}^-,d}$ at a negative crossing. Let us denote the *unoriented* resolution at this crossing by $B_{\mathbf{n}^-,d}^u$. It is not hard to see that $B_{\mathbf{n}^-,d}^u$ is an alternating link and does not depend on d. Moreover, the double covers of S^3 branched along these braids fit into the surgery exact triangle below (for $d = 2$) [**33**].

$$\widehat{HF}(-\Sigma(B_{\mathbf{n},2})) \xrightarrow{\widehat{G}} \widehat{HF}(-\Sigma(B_{\mathbf{n}^-,2}))$$
$$\widehat{HF}(-\Sigma(B_{\mathbf{n}^-,2}^u)).$$

Let $\mathbb{F}$ denote the field with two elements, and let $\mathcal{T}^+$ denote the $\mathbb{F}[U]$-module $\mathbb{F}[U, U^{-1}]/\mathbb{F}[U]$. From the grading calculations in [**4**, Section 6], it follows that

$$(7.1) \qquad\qquad HF^+(-\Sigma(B_{\mathbf{n},2})) \cong (\mathcal{T}_0^+)^{\det(B_{\mathbf{n},0})} \oplus \mathbb{F}_1.$$

Here, the subscripts denote absolute $\mathbb{Z}_2$ gradings. The long exact sequence relating $\widehat{HF}$ and HF^+ therefore implies that

$$\widehat{HF}(-\Sigma(B_{\mathbf{n},2})) \cong (\mathbb{F}_0)^{\det(B_{\mathbf{n},0})+1} \oplus \mathbb{F}_1.$$

In particular,

$$rk(\widehat{HF}(-\Sigma(B_{\mathbf{n},2}))) = 2 + \det(B_{\mathbf{n},0}).$$

Of course, the analogous formula holds for $rk(\widehat{HF}(-\Sigma(B_{\mathbf{n}^-,2})))$. Moreover, since $B_{\mathbf{n}^-,2}^u = B_{\mathbf{n}^-,0}^u$ is alternating, we know from [**33**] that

$$rk(\widehat{HF}(-\Sigma(B_{\mathbf{n}^-,2}^u))) = \det(B_{\mathbf{n}^-,0}^u).$$

And, since $B_{\mathbf{n}^-,0}$ is an alternating link, its determinant satisfies

$$(7.2) \qquad\qquad \det(B_{\mathbf{n}^-,0}) = \det(B_{\mathbf{n},0}) + \det(B_{\mathbf{n}^-,0}^u).$$

Combined with the rank formulae above, equation (7.2) implies that

$$rk(\widehat{HF}(-\Sigma(B_{\mathbf{n}^-,2}))) = rk(\widehat{HF}(-\Sigma(B_{\mathbf{n},2}))) + rk(\widehat{HF}(-\Sigma(B_{\mathbf{n}^-,2}^u))).$$

Therefore, our surgery exact triangle splits into a short exact sequence; in particular, the map $\widehat{G}$ is injective.

Now, let us assume that $c^+(\xi_{\mathbf{n},2})$ satisfies Condition **R**. Then, according to equation (7.1), $c^+(\xi_{\mathbf{n},2})$ must have absolute $\mathbb{Z}_2$ grading 1 (and, hence, so does $c(\xi_{\mathbf{n},2})$). Since $\widehat{G}$ sends $c(\xi_{\mathbf{n},2})$ to $c(\xi_{\mathbf{n}^-,2})$, and maps induced by cobordisms preserve relative $\mathbb{Z}_2$ gradings (and are homogeneous with respect to these gradings) [**28**], the injectivity of $\widehat{G}$ forces $c(\xi_{\mathbf{n}^-,2})$ (and, hence, $c^+(\xi_{\mathbf{n}^-,2})$) to have absolute $\mathbb{Z}_2$ grading 1 as well. But since $c^+(\xi_{\mathbf{n}^-,2})$ has absolute $\mathbb{Z}_2$ grading 1, it must satisfy Condition **R**, by the analogue of equation (7.1) for $HF^+(-\Sigma(B_{\mathbf{n}^-,2}))$. □

References

[1] K. Baker, J.B. Etnyre and J. Van Horn–Morris, *Cabling, contact structures and mapping class monoids*, 2010; `arXiv: math.SG/1005.1978`.

[2] J.A. Baldwin, *Tight contact structures and genus one fibered knots*, Algebr. Geom. Topol., **7** (2007), 701–735.

[3] J.A. Baldwin, *Comultiplicativity of the Ozsváth–Szabó contact invariant*, Math. Res. Lett., **15**(2) (2008), 273–287.

[4] J.A. Baldwin, *Heegaard Floer homology and genus one, one boundary component open books*, J. Topol., **1**(4) (2008), 963–992.

[5] J.A. Baldwin, *Contact monoids and Stein cobordisms*, Math. Res. Lett., **19**(1) (2012), 31–40.

[6] J.A. Baldwin and J.B. Etnyre, *Admissible transverse surgery does not preserve tightness*, 2012; `arXiv: math.SG/1203.2993`.

[7] V. Colin, K. Honda, *Reeb vector fields and open book decompositions*, 2008; `arXiv: math.GT/0809.5088`.

[8] F. Ding and H. Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, Math. Proc. Cambridge Philosophical Society, **136** (2004), 583–598.

[9] F. Ding, H. Geiges and A. Stipsicz, *Surgery diagrams for contact 3-manifolds*, Turkish J. Math., **28**(1) (2004), 41–74.

[10] Y. Eliashberg, *A few remarks about symplectic fillings*, Geom. Topol., **8** (2004), 277–293.

[11] Y. Eliashberg and W. Thurston, *Confoliations*, volume 13 of *University Lecture Series*. Amer. Math. Soc., 1998.

[12] J.B. Etnyre, *Planar open book decompositions and contact structures*, Int. Math. Res. Not., **2004**(79) (2004), 4255–4267.

[13] J.B. Etnyre and K. Honda, *Knots and contact geometry I: torus knots and the figure eight knot*, J. Symp. Geom., **1**(1) (2001), 63–120.

[14] J.B. Etnyre and B. Ozbagci, *Invariants of contact structures from open books*, Trans. Amer. Math. Soc., **360**(6) (2008), 3133–3151.

[15] D.T. Gay and A.I. Stipsicz, *On symplectic caps*, In Perspectives in Analysis, Geometry, and Topology, Progress in Mathematics, **296** 199–212. Springer, 2012.

[16] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, In *Proc. Int. Cong. Math.*, vol. II, 405–414. Higher Ed. Press, 2002.

[17] K. Honda, W. Kazez and G. Matić, *On the contact class in Heegaard Floer homology*, 2006, `arXiv: math.GT/0609734`.

[18] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of a compact surface with boundary*, Inv. Math., **169**(2) (2007), 427–449.

[19] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of a compact surface with boundary II*, Geom. Topol., **12** (2008), 2057–2094.

[20] M. Hutchings and C. Taubes, *Proof of the Arnold chord conjecture in three dimensions II*, 2011; `arXiv: math.SG/1111.3324`.

[21] S. Jabuka and T. Mark, *Heegaard Floer homology of certain mapping tori II*, Geometry and Topology of Manifolds, **47**, 2005.

[22] Y. Lekili, *Planar open books with four binding components*, Algebr. Geom. Topol., **11** (2011), 909–928.

[23] P. Lisca and A. Stipsicz, *Contact surgery and transverse invariants*, J. Topology, **4**(4) (2011), 817–834.

[24] T. Mrowka and Y. Rollin, *Legendrian knots and monopoles*, Algebr. Geom. Topol., **6**, (2006), 1–69.

[25] K. Murasugi, *On closed 3-braids*, Number 151 in Memoirs of the American Mathematical Society. American Mathematical Society, 1974.

[26] H. Ohta and K. Ono, *Simple singularities and topology of symplectically filling 4-manifold*, Comment. Math. Helv., **74**, (1999), 575–590.

[27] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds*, Geom. Topol., **8** (2004), 311–334.

[28] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. Math., **159**(3) (2004), 1159–1245.

[29] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. Math., **159**(3) (2004), 1027–1158.

[30] P. Ozsváth and Z. Szabó, *Holomorphic triangle invariants and the topology of symplectic four-manifolds*, Duke Math. J., **121**(1) (2004), 1–34.

[31] P. Ozsváth and Z. Szabó, *Heegaard Floer homologies and contact structures*, Duke Math. J., **129**(1) (2005), 39–61.

[32] P. Ozsváth and Z. Szabó, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math., **202**(2) (2005), 326–400.

[33] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math., **194**(1) (2005), 1–33.

[34] P. Ozsváth, Z. Szabó and A.I. Stipsicz, *Planar open books and Floer homology*, Int. Math. Res. Not., **2005**(54) (2005), 3385–3401.

[35] R. Roberts, *Taut foliations in punctured surface bundles, I*, Proc. London Math. Soc., **3**(82) (2001), 747–768.

[36] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc, **19**(2) (1988), 417–431.

[37] C. Wendl, *A hierarchy of local symplectic filling obstructions for contact 3-manifolds*, 2010; `arXiv: math.SG/1009.2746`.

[38] C. Wendl, *Non-exact symplectic cobordisms between contact 3-manifolds*, 2010; `arXiv: math.SG/1008.2456`.

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