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A SYMPLECTICALLY NON-SQUEEZABLE SMALL SET AND THE REGULAR COISOTROPIC CAPACITY

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We prove that for $n \geq 2$ there exists a compact subset X of the closed ball in $\mathbb{R}^{2n}$ of radius $\sqrt{2}$, such that X has Hausdorff dimension n and does not symplectically embed into the standard open symplectic cylinder. The second main result is a lower bound on the dth regular coisotropic capacity, which is sharp up to a factor of 3. For an open subset of a geometrically bounded, aspherical symplectic manifold, this capacity is a lower bound on its displacement energy. The proofs of the results involve a certain Lagrangian submanifold of linear space, which was considered by Audin and Polterovich.

1. Motivation and results

Continuing our previous work [**SZ1, SZ2**], the present paper is motivated by the following question.

Question (A). *How much symplectic geometry can a small subset of a symplectic manifold carry?*

More concretely, we are concerned with the problem of finding a small subset of $\mathbb{R}^{2n}$ that cannot be squeezed symplectically. To be specific, we interpret "smallness" in two ways: in the sense of Hausdorff dimension and in terms of the size of a ball containing the subset. The first main result is the following. Let (M, ω) and (M', ω') be symplectic manifolds, and $X \subseteq M$ a subset. We say that X *(symplectically) embeds into* M' iff there exists an open neighborhood $U \subseteq M$ of X and a symplectic embedding $\varphi\colon U \to M'$. For $n \in \mathbb{N}$ and $a > 0$ we denote by $B^{2n}(a)$ and $\overline{B}^{2n}(a)$ the open and closed balls in $\mathbb{R}^{2n}$, of radius $\sqrt{a/\pi}$, around 0. (These balls have Gromov-width a.)

We denote

$$B^{2n} := B^{2n}(\pi), \quad \overline{B}^{2n} := \overline{B}^{2n}(\pi), \quad \mathbb{D} := \overline{B}^{2},$$
$$Z^{2n}(a) := B^2(a) \times \mathbb{R}^{2n-2}, \quad Z^{2n} := Z^{2n}(\pi),$$
$$\overline{P}_n := \begin{cases} \mathbb{D}^n, & \text{if } n \text{ is even,} \\ \mathbb{D}^{n-1} \times \mathbb{R}^2, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1 (Non-squeezable small set). *For every $n \geq 2$ there exists a compact subset*

$$X \subseteq \overline{P}_n \cap \overline{B}^{2n}(2\pi)$$

of Hausdorff dimension n, which does not symplectically embed into the open cylinder Z^{2n}. In fact, we may choose this set to be the union of a closed[1] *Lagrangian submanifold and the image of a smooth map from S^2 to $\mathbb{R}^{2n}$.*[2]

The set X in this result is "almost minimal": If n is even then the statement of Theorem 1 is wrong, if $\overline{P}_n$ is replaced by $(\mathbb{D} \setminus \{z\}) \times \mathbb{D}^{n-1}$, where z is an arbitrary point in $S^1 = \partial\mathbb{D}$. This follows from an elementary argument, using compactness of X and Moser isotopy in two dimensions. (A similar assertion holds in the case in which n is odd.) Furthermore, the condition $X \subseteq \overline{B}^{2n}(2\pi)$ is "sharp up to a factor of 2". In fact, based on a two-dimensional Moser type argument, we will show the following:

Proposition 2. *For $n \in \mathbb{N}$ every compact subset of $\overline{B}^{2n}$ with vanishing $(2n-1)$-dimensional Hausdorff measure symplectically embeds into Z^{2n}.*

In the proof of Theorem 1 we will consider a rotated and rescaled version $\widetilde{L}$ of a closed Lagrangian submanifold studied by L. Polterovich in [**Po**]. We will choose a map from S^2 to $\mathbb{R}^{2n}$ with image equal to the union of the cones over some loops in $\widetilde{L}$ that generate the fundamental group of $\widetilde{L}$. The union X of $\widetilde{L}$ and these cones cannot be squeezed into Z^{2n}. This will be a consequence of a result by Chekanov about the displacement energy of a Lagrangian submanifold.

We may ask whether the condition in Theorem 1 on the Hausdorff dimension of X is optimal:

Question (B). *Does every compact set $X \subseteq \mathbb{R}^{2n}$ of Hausdorff dimension $< n$ symplectically embed into an arbitrarily small symplectic cylinder or ball? Is this even true for any compact set X with vanishing n-dimensional Hausdorff measure?*

To our knowledge these questions are open.

[1] This means "compact and without boundary".

[2] It follows from the hypothesis $n \geq 2$ and standard arguments (cf. [**Fe,** p. 176]) that such a union has Hausdorff dimension equal to n.

Returning to Question (A), consider the class of "small" subsets of a given symplectic manifold consisting of coisotropic submanifolds. Based on these submanifolds, in [**SZ1**] for a fixed dimension $2n$ we defined a collection of capacities, one for each $d \in \{n, \dots, 2n-1\}$, as follows. Recall that a symplectic manifold (M, ω) is called *(symplectically) aspherical* iff for every $u \in C^\infty(S^2, M)$ we have $\int_{S^2} u^*\omega = 0$. For a coisotropic submanifold $N \subseteq M$ we denote by $A(N) = A(M, \omega, N)$ its minimal (symplectic) area (or action). (See (3.2) below.) We define the *dth regular coisotropic capacity* to be the map

$$(1.1)\ A^d_{\mathrm{coiso}}\colon \big\{\text{aspherical symplectic manifold, } \dim M = 2n\big\} \to [0,\infty],$$
$$A^d_{\mathrm{coiso}}(M,\omega) \coloneqq \sup A(N),$$

where $N \subseteq M$ runs over all non-empty closed regular (i.e., "fibering") coisotropic submanifolds of dimension d, satisfying the following condition:

$$\forall \text{ isotropic leaf } F \text{ of } N, \forall x \in C(S^1, F)\colon\ x \text{ is contractible in } M. \tag{1.2}$$

(For explanations see Subsection 3.1.) By [**SZ1,** Theorem 4] the map A^d_{coiso} is a (not necessarily normalized) symplectic capacity. For $d = n$ we abbreviate

$$A_{\mathrm{Lag}} \coloneqq A^n_{\mathrm{coiso}}.$$

Since every Lagrangian submanifold is regular, $A_{\mathrm{Lag}}(M,\omega)$ equals the supremum of all minimal areas $A(L)$, where L runs over all those non-empty closed Lagrangian submanifolds of M, for which every continuous loop in L is contractible in M. (Here $A(L) = \inf\big(S(L) \cap (0,\infty)\big)$, where the symplectic area spectrum $S(L)$ is given by (3.3) below.)

Our second main result provides a lower bound on A^d_{coiso} for the unit ball B^{2n}, equipped with the standard symplectic form ω_0:

Theorem 3 (Regular coisotropic capacity). *For every $n \geq 2$ we have*

$$A_{\mathrm{Lag}}(B^{2n}) \coloneqq A_{\mathrm{Lag}}(B^{2n}, \omega_0) \geq \tfrac{\pi}{2}, \tag{1.3}$$

$$A^d_{\mathrm{coiso}}(B^{2n}) \geq \tfrac{\pi}{3}, \quad \forall d \in \{n+1, \dots, 2n-3\}. \tag{1.4}$$

The proof of this result uses again the closed Lagrangian submanifold of $\mathbb{R}^{2n}$ studied by Polterovich. To put Theorem 3 into context, note that in [**SZ1,** Theorem 4] we proved the (in-)equalities

$$A^d_{\mathrm{coiso}}(Z^{2n}) \leq \pi, \forall d \in \{n, \dots, 2n-1\},$$
$$A^{2n-1}_{\mathrm{coiso}}(B^{2n}) = \pi,$$
$$A^{2n-2}_{\mathrm{coiso}}(B^{2n}) \geq \tfrac{\pi}{2}.$$

Combining these with Theorem 3, it follows that the capacity A^d_{coiso} is normalized for $d = 2n-1$, normalized up to a factor of 2 for $d = n$ and $2n-2$, and up to a factor of 3, otherwise.

2. Remarks and related work

About Theorem 1. Note that we may not just take a closed Lagrangian submanifold L of $\mathbb{R}^{2n}$ for X, since every such submanifold "symplectically embeds" (in the above sense) into an arbitrarily small ball. To see this, let $B \subseteq \mathbb{R}^{2n}$ be an open ball. We choose a number $c > 0$ such that the rescaled Lagrangian cL is contained in B. It follows from Weinstein's neighborhood theorem that there exist open neighborhoods U and U' of L and cL, respectively, and a symplectomorphism $\varphi : U \to U'$ that maps L to cL. The restriction of φ to $U \cap \varphi^{-1}(B)$ is a symplectic embedding of a neighborhood of L into B.

Theorem 1 has the following application. For $n \in \mathbb{N}$ and $d \in [0, 2n]$ consider the quantity

$$a(n,d) := \inf a \in [0, \infty],$$

where the infimum runs over all numbers $a > 0$, for which there exists a compact subset X of $B^{2n}(a)$ of Hausdorff dimension at most d, such that X does not symplectically embed into Z^{2n}. (Our convention is that $\inf \emptyset = \infty$.) Note that we always have $a(n,d) \geq \pi$, and $a(n,d)$ is decreasing in d. Theorem 1 implies that

$$a(n,d) \leq 2\pi, \quad \forall d \geq n,$$

and hence we know these numbers up to a factor of 2. This improves our previous result [**SZ1,** Theorem 6]. That result implies that $a(n,d)$ is bounded above by π times some integer, depending on n and d in a combinatorial way. For $n = d$ this integer behaves asymptotically like $\sqrt{n}$, as $n \to \infty$.

Gromov's non-squeezing result (cf. [**Gr**]) implies that $a(n, 2n) = \pi$. This can be strengthened to the equality $a(n, 2n-1) = \pi$, which follows from [**SZ1,** Theorem 6]. In the case $d < 2$ we have $a(n,d) = \infty$. This is a consequence of the following result.

Proposition 4 (Two-dimensional squeezing). *For all $n \in \mathbb{N}$ and $a > 0$, every subset X of $\mathbb{R}^{2n}$ with vanishing 2-dimensional Hausdorff measure symplectically embeds into $Z^{2n}(a)$.*

The proof of this result is based on Moser isotopy. In contrast with this proposition, a straight-forward argument shows that $a(1,2) = \pi$. Hence in the case $n = 1$, the values $a(1,d)$ are all known.

Theorem 1 is related to the following results by Sikorav and Schlenk. In [**Si**] Sikorav proved that there does not exists a symplectomorphism of $\mathbb{R}^{2n}$, which maps the standard Lagrangian torus $\mathbb{T}^n$ into Z^{2n}. Schlenk noted in [**Sch,** p. 8] that combining this result with the Extension after Restriction Principle implies the "Symplectic Hedgehog Theorem": for every $n \geq 2$, no star-shaped domain in $\mathbb{R}^{2n}$ containing $\mathbb{T}^n$ symplectically embeds into the cylinder Z^{2n}. It follows that no neighborhood of the set

$$[0,1]\cdot\mathbb{T}^n := \big\{cx \,\big|\, c\in[0,1],\, x\in\mathbb{T}^n\big\}$$

can be squeezed into Z^{2n}. This set has Hausdorff dimension $n+1$ and is contained in the ball $\overline{B}^{2n}(n\pi)$. Theorem 1 improves this statement in two ways: The set X in that result has Hausdorff dimension only n and is contained in the ball $\overline{B}^{2n}(2\pi)$, whose size does not depend on n.

About Proposition 2. In the case $n\geq 2$ the condition on the Hausdorff measure in this result is necessary, since then no neighborhood of the unit sphere symplectically embeds into Z^{2n}. See [**SZ1,** Corollary 5].

About the regular coisotropic capacity and Theorem 3. A motivation for the definition of A^d_{coiso} as in (1.1) is that for an open subset U of an aspherical symplectic manifold (M,ω) the number $A^d_{\mathrm{coiso}}(U)$ is a lower bound on the displacement energy of U, if (M,ω) is geometrically bounded. This follows from [**Zi,** Theorem 1.1].

For $d=n$ the capacity $A_{\mathrm{Lag}}=A^n_{\mathrm{coiso}}$ is closely related to the Lagrangian capacity introduced by Cieliebak and Mohnke: We denote

$$\mathcal{M} := \big\{(M,\omega) \text{ symplectic manifold } \big| \\ \dim M = 2n,\ \pi_i(M) \text{ trivial},\ i=1,2\big\}.$$

In [**CM**][3] Cieliebak and Mohnke defined the *Lagrangian capacity* to be the map $c_L\colon\mathcal{M}\to[0,\infty)$, given by

$$c_L(M,\omega) := \sup\big\{A(L)\,\big|\, L\subseteq M \text{ embedded Lagrangian torus}\big\},$$

where $A(L)=\inf\big(S(L)\cap(0,\infty)\big)$ denotes the minimal symplectic area of L (see (3.3) below). The authors proved that

$$c_L(B^{2n},\omega_0)=\frac{\pi}{n}. \tag{2.1}$$

The capacity c_L is bounded above by A_{Lag}. For $n\geq 3$, it is strictly smaller than A_{Lag}, when applied to (B^{2n},ω_0). This follows from inequalities (1.3) and (2.1).

For $d=2n-1$ the capacity $A^{2n-1}_{\mathrm{coiso}}$ is related to a definition recently introduced by Geiges and Zehmisch: In [**GZ1, GZ2**] these authors defined, for any symplectic manifold (V,ω),

$$c(V,\omega) := \sup_{(M,\alpha)}\big\{\inf(\alpha)\,\big|\,\exists \text{ contact type embedding } (M,\alpha)\hookrightarrow(V,\omega)\big\},$$

where the supremum is taken over all closed contact manifolds (M,α), and $\inf(\alpha)$ denotes the infimum of all positive periods of closed orbits of the Reeb

[3] See also [**CHLS**], Section 2.4, p. 11.

vector field R_α. They showed that c is a normalized symplectic capacity. (See [**GZ2,** Theorem 4.5].)

As a consequence of Theorem 3 and [**SZ1,** Theorem 4], the value of the capacity $A_{\mathrm{Lag}} = A^n_{\mathrm{coiso}}$ on the ball B^{2n} lies between $\frac{\pi}{2}$ and π. In the case $n = 2$ this value can be exactly calculated, if we modify the definition of A_{Lag} by restricting to *orientable* Lagrangian submanifolds. Namely, the so obtained capacity A^+_{Lag} satisfies

$$A^+_{\mathrm{Lag}}(B^4) = \frac{\pi}{2}.$$

To see this, we denote by $\mathbb{T}^2 = (S^1)^2$ the standard torus in $\mathbb{R}^4$. For every $r < \frac{1}{\sqrt{2}}$ the rescaled torus $r\mathbb{T}^2$ is a Lagrangian submanifold of B^4, with minimal area πr^2. It follows that $A^+_{\mathrm{Lag}}(B^4) \geq \frac{\pi}{2}$. To see the opposite inequality, note that every orientable closed connected Lagrangian submanifold $L \subseteq B^4$ is diffeomorphic to the torus $\mathbb{T}^2$, since its Euler characteristic vanishes. For such an L, Cieliebak and Mohnke proved [**CM**] that $A(L) < \frac{\pi}{2}$. The statement follows.

3. Background and proofs of the results of section 1

3.1. Background. Let (M, ω) be a symplectic manifold and $N \subseteq M$ a submanifold. Then N is called *coisotropic* iff for every $x \in N$ the subspace

$$T_xN^\omega = \big\{v \in T_xM \,\big|\, \omega(v, w) = 0, \quad \forall w \in T_xN\big\}$$

of T_xM is contained in T_xN. Examples include $N = M$, hypersurfaces, and Lagrangian submanifolds of M. Let $N \subseteq M$ be a coisotropic submanifold. Then ω gives rise to the isotropic (or characteristic) foliation on N. We define the *isotropy relation on* N to be the subset

$$R^{N,\omega} := \big\{(x(0), x(1)) \,\big|\, x \in C^\infty([0, 1], N) \colon \dot{x}(t) \in (T_{x(t)}N)^\omega, \quad \forall t\big\}$$

of $N \times N$. This is an equivalence relation on N. For a point $x_0 \in N$ we call the $R^{N,\omega}$-equivalence class of x_0 the *isotropic leaf* through x_0. (This is the leaf of the isotropic foliation that contains x_0.) We call N *regular* iff $R^{N,\omega}$ is a closed subset and a submanifold of $N \times N$. This holds if and only if there exists a manifold structure on the set of isotropic leaves of N, such that the canonical projection π_N from N to this set is a submersion, cf. [**Zi,** Lemma 15]. If N is closed then by C. Ehresmann's theorem this implies that π_N is a smooth (locally trivial) fiber bundle. (See the proposition on p. 31 in [**Eh**].)

We define the *(symplectic) area (or action) spectrum* and the *minimal (symplectic) area* of N as

$$S(N) := S(M,\omega,N) := \left\{ \int_{\mathbb{D}} u^*\omega \,\middle|\, u \in C^\infty(\mathbb{D},M) \colon \exists \text{ isotropic leaf } F \text{ of } N \colon u(S^1) \subseteq F \right\}, \tag{3.1}$$

$$A(N) := A(M,\omega,N) := \inf\big(S(M,\omega,N) \cap (0,\infty)\big) \in [0,\infty]. \tag{3.2}$$

(Our convention is that $\inf \emptyset = \infty$.) Note that if $L = N$ is a Lagrangian submanifold of M then the isotropic leaf of a point $x \in L$ is the connected component of L containing x, and therefore the area spectrum of L is given by

$$S(L) = \left\{ \int_{\mathbb{D}} u^*\omega \,\middle|\, u \in C^\infty(\mathbb{D},M) \colon u(S^1) \in L \right\}. \tag{3.3}$$

3.2. Proof of Theorem 1 (Non-squeezable small set). The proof of Theorem 1 is based on the following result.

Proposition 5. *Let $n \geq 2$ and $L \subseteq \mathbb{R}^{2n}$ be a non-empty closed Lagrangian submanifold. Then there exists a smooth map*

$$u : S^2 \to [0,1]\cdot L := \big\{ cx \,\big|\, c \in [0,1], x \in L \big\} \subseteq \mathbb{R}^{2n},$$

such that the union $L \cup u(S^2)$ does not symplectically embed into the cylinder $Z^{2n}(A(\mathbb{R}^{2n},\omega_0,L))$.

The proof of Proposition 5 follows the lines of the proof of [**SZ1,** Proposition 21]. It is based on the following result, which is due to Chekanov. Let (M,ω) be a symplectic manifold. We denote by $\mathcal{H}(M,\omega)$ the set of all functions $H \in C^\infty\big([0,1]\times M,\mathbb{R}\big)$ whose Hamiltonian time t flow $\varphi_H^t \colon M \to M$ exists and is surjective, for every $t \in [0,1]$.[4]

We define the *Hofer norm*

$$\|\cdot\| \colon \mathcal{H}(M,\omega) \to [0,\infty], \quad \|H\| := \int_0^1 \big(\sup_M H^t - \inf_M H^t\big)\,dt,$$

and the *displacement energy* of a subset $X \subseteq M$ to be

$$e(X,M,\omega) := \inf\big\{ \|H\| \,\big|\, H \in \mathcal{H}(M,\omega) \colon \varphi_H^1(X) \cap X = \emptyset \big\}.$$[5]

[4]The time t flow of a time-dependent vector field on a manifold M is always an injective smooth immersion on its domain of definition. (This follows for example from [**Le,** Theorem 17.15, p. 451, and Problem 17–15, p. 463].) Hence if it is everywhere well-defined and surjective then it is a diffeomorphism of M. The second condition is not a consequence of the first one. As an example, consider $M := (0,\infty)\times\mathbb{R}$, $\omega := \omega_0$, $H(q,p) := p$, and $t > 0$. The Hamiltonian time t flow of H is everywhere well-defined and given by $\varphi_H^t(q,p) = (q+t,p)$. However, the map $\varphi_H^t : M \to M$ is not surjective.

[5]Alternatively, one can define a displacement energy, using only functions H with compact support. However, it seems more natural to allow for all functions in $\mathcal{H}(M,\omega)$. For some remarks on this issue see [**SZ2**].

Theorem 6. *Let $L \subseteq M$ be a closed Lagrangian submanifold. Assume that (M,ω) is geometrically bounded (see [**Ch**]). Then we have*

$$e(L,M,\omega) \geq A(M,\omega,L).$$

Proof of Theorem 6. This follows from the Main Theorem in [**Ch**] by an elementary argument. □

For the proof of Proposition 5, we also need the following.

Lemma 7. *Let (M,ω) and (M',ω') be symplectic manifolds of the same dimension, $N \subseteq M$ a coisotropic submanifold, and $\varphi\colon M \to M'$ a symplectic embedding. Assume that (M',ω') is aspherical, and every continuous loop in a leaf of N is contractible in M. Then we have*

$$A\big(M',\omega',\varphi(N)\big) = A(M,\omega,N).$$

Proof of Lemma 7. This follows from [**SZ1,** Remark 32 and Lemma 33]. □

Proof of Proposition 5. Without loss of generality we may assume that L is connected. We choose a point $x_0 \in L$. Since L is a closed manifold, there exists a finite set $\mathcal{L}$ of loops in L that generate the fundamental group $\pi_1(L,x_0)$. We choose these loops to be smooth, and define

$$f : \mathcal{L} \times [0,1] \times S^1 \to \mathbb{R}^{2n}, \quad f(x,t,z) := tx(z),$$
$$X := L \cup \mathrm{im}(f).$$

The statement of the proposition is a consequence of the following two claims.

Claim 1. *If $\mathcal{L} \neq \emptyset$*[6] *then there exists a smooth map from S^2 to $\mathbb{R}^{2n}$ with the same image as f.*

Proof of Claim 1. We denote $k := |\mathcal{L}|$, and choose a bijection

$$\{1,\dots,k\} \ni i \mapsto x_i \in \mathcal{L}$$

and a function $\rho \in C^\infty\big([0,1],[0,1]\big)$ that attains the value i in a neighborhood of $i = 0,1$. We define the map $u : [0,2k] \times S^1 \to \mathbb{R}^{2n}$ by

$$u(t,z) := \begin{cases} \rho(t-2i+2)x_i(z), & \text{for } t \in [2i-2,2i-1], \\ \rho(2i-t)x_i(z), & \text{for } t \in [2i-1,2i], \end{cases}$$

for $i = 1,\dots,k$. This map is smooth and has the same image as f. We identify $[0,2k] \times S^1$ with the two boundary circles collapsed with S^2. Since u is constant in neighborhoods of $\{0\} \times S^1$ and $\{2k\} \times S^1$, it descends to a map from S^2 to $\mathbb{R}^{2n}$. This map has the required properties. This proves Claim 1. □

[6] By a result of Gromov [**Gr**] this is always the case. However, we do not use this in the proof of Proposition 5.

Claim 2. *For every open neighborhood U of X, and every symplectic embedding $\varphi\colon U \to \mathbb{R}^{2n}$ we have $\varphi(U) \not\subseteq Z^{2n}\big(A(\mathbb{R}^{2n}, \omega_0, L)\big)$.*

Proof of Claim 2. In order to apply Lemma 7, we check that every continuous loop in L is contractible in U. Let x be such a loop. It follows from our choice of the set $\mathcal{L}$ that there exists a collection of loops $y_1, \dots, y_\ell \in \mathcal{L}$ and signs $\epsilon_1, \dots, \epsilon_\ell \in \{1, -1\}$, such that x is homotopic inside L to $y_1^{\epsilon_1} \# \cdots \# y_\ell^{\epsilon_\ell}$. Here $\#$ denotes concatenation of loops based at x_0, and y_i^{-1} denotes the time-reversed loop y_i. Since X contains the image of the map $[0,1] \times S^1 \ni (t, z) \mapsto t y_i(z) \in \mathbb{R}^{2n}$, for every $i = 1, \dots, \ell$, it follows that x is contractible in X, and hence in U. Therefore, the hypotheses of Lemma 7 are satisfied with $\big(M, \omega, M', \omega', N\big) := \big(U, \omega_0|U, \mathbb{R}^{2n}, \omega_0, L\big)$. (Here $\omega_0|U$ denotes the restriction of ω_0 to U.) Applying this result, it follows that

$$A\big(U, \omega_0|U, L\big) = A\big(\mathbb{R}^{2n}, \omega_0, \varphi(L)\big). \tag{3.4}$$

Similarly, applying Lemma 7 with φ replaced by the inclusion map of U into $\mathbb{R}^{2n}$, we have

$$A(\mathbb{R}^{2n}, \omega_0, L) = A\big(U, \omega_0|U, L\big). \tag{3.5}$$

By Theorem 6, we have

$$A\big(\mathbb{R}^{2n}, \omega_0, \varphi(L)\big) \le e\big(\varphi(L), \mathbb{R}^{2n}, \omega_0\big). \tag{3.6}$$

An elementary argument shows that

$$e\big(Z^{2n}(a), \mathbb{R}^{2n}, \omega_0\big) \le a, \quad \forall a > 0.$$

Combining this with (3.4, 3.5, 3.6), it follows that

$$A(\mathbb{R}^{2n}, \omega_0, L) \le a, \quad \forall a > 0 \text{ such that } \varphi(L) \subseteq Z^{2n}(a). \tag{3.7}$$

Assume by contradiction that $\varphi(U) \subseteq Z^{2n}\big(A(\mathbb{R}^{2n}, \omega_0, L)\big)$. Since L is compact and contained in U, it follows that $\varphi(L) \subseteq Z^{2n}(a)$ for some number $a < A(\mathbb{R}^{2n}, \omega_0, L)$. This contradicts (3.7). The statement of Claim 2 follows. This proves Proposition 5. $\square$

In the proof of Theorem 1 we will apply Proposition 5 with a rotated and rescaled version of the Lagrangian submanifold

$$L := \big\{ zq \,\big|\, z \in S^1 \subseteq \mathbb{C},\, q \in S^{n-1} \subseteq \mathbb{R}^n \big\} \subseteq \mathbb{C}^n. \tag{3.8}$$

This submanifold was used by Polterovich in [**Po,** Section 3] as an example of a monotone Lagrangian in $\mathbb{C}^n$ with minimal Maslov number n. Previously, it was considered by Weinstein in [**We,** Lecture 3] and Audin in [**Au,** p. 620].

Lemma 8. *For $n \ge 2$ the minimal symplectic area of the Lagrangian L in $\mathbb{R}^{2n}$ equals $\frac{\pi}{2}$.*

Proof of Lemma 8. Let $n \geq 2$. Recall the formula (3.3) for the area spectrum $S(L)$. We write a point in $\mathbb{R}^{2n}$ as (q,p), and denote by $\alpha := q \cdot dp$ the Liouville one-form. Since $d\alpha = \omega_0$, Stokes' theorem implies that

$$S(L) = \widetilde{S}(L) := \left\{ \int_{S^1} x^*\alpha \,\middle|\, x \in C^\infty(S^1, L) \right\}. \tag{3.9}$$

To calculate $\widetilde{S}(L)$, we need the following.

Claim. *If $x : S^1 \to L$, $\varphi : [0,1] \to \mathbb{R}$, and $q : [0,1] \to S^{n-1}$ are smooth maps, such that*

$$x(e^{2\pi it}) = \mathrm{e}^{\mathrm{i}\varphi(t)} q(t), \quad \forall t \in [0,1], \tag{3.10}$$

then we have

$$\int_{S^1} x^*\alpha = \frac{\varphi(1) - \varphi(0)}{2}. \tag{3.11}$$

Proof of the claim. We have $|q|^2 = 1$ and $q \cdot \dot{q} = 0$, and therefore,

$$\begin{aligned} \int_{S^1} x^*\alpha &= \int_0^1 \mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\varphi} q\big) \cdot \mathrm{Im}\left(\mathrm{e}^{\mathrm{i}\varphi}(i\dot{\varphi} q + \dot{q})\right) dt \\ &= \int_0^1 \cos(\varphi)^2 \dot{\varphi}\, dt \\ &= \left(\frac{1}{4}\sin(2\varphi(t)) + \frac{\varphi(t)}{2}\right)\Big|_{t=0}^1. \end{aligned} \tag{3.12}$$

On the other hand, equality (3.10) implies that $\varphi(1) - \varphi(0) \in \pi\mathbb{Z}$, and therefore, the first term in (3.12) vanishes. Equality (3.11) follows. This proves the claim. $\square$

We show that $\widetilde{S}(L) \subseteq \frac{\pi}{2}\mathbb{Z}$: Let $x \in C^\infty(S^1, L)$. The map $\mathbb{R} \times S^{n-1} \ni (\varphi, q) \mapsto \mathrm{e}^{\mathrm{i}\varphi} q \in L \subseteq \mathbb{C}^n$ is a smooth covering map. Therefore, there exist smooth paths $\varphi\colon [0,1] \to \mathbb{R}$ and $q\colon [0,1] \to S^{n-1}$ such that equality (3.10) holds. It follows that $\varphi(1) - \varphi(0) \in \pi\mathbb{Z}$. Combining this with the claim, we obtain $\int_{S^1} x^*\alpha \in \frac{\pi}{2}\mathbb{Z}$. This shows that $\widetilde{S}(L) \subseteq \frac{\pi}{2}\mathbb{Z}$.

To prove the opposite inclusion, we choose a path $q \in C^\infty([0,1], S^{n-1})$ that is constant near the ends and satisfies $q(1) = -q(0)$. (Here we use that $n \geq 2$, and therefore, S^{n-1} is connected.) We define $x : S^1 \to L$ by $x(\mathrm{e}^{2\pi it}) := \mathrm{e}^{\pi it} q(t)$, for $t \in [0,1)$. This is a smooth loop. By the above claim we have $\int_{S^1} x^*\alpha = \pi/2$. By considering multiple covers of x, it follows that $\widetilde{S}(L) \supseteq \frac{\pi}{2}\mathbb{Z}$.

Hence the equality $\widetilde{S}(L) = \frac{\pi}{2}\mathbb{Z}$ holds. Combining this with equality (3.9), it follows that $A(L) = \pi/2$. This proves Lemma 8. $\square$

Proof of Theorem 1. Let $n \geq 2$. We define L as in (3.8), and

$$\widetilde{L} := \{\sqrt{2}zw \,|\, z \in S^1 \subseteq \mathbb{C},\, w \in S^{2n-1} \subseteq \mathbb{C}^n : w_{n+1-j} = \overline{w}_j,\, \forall j = 1, \dots, n\}.$$

Claim. *There exists a unitary transformation U of $\mathbb{C}^n$, such that $\widetilde{L} = \sqrt{2}UL$.*

Proof of the claim. The set

$$W := \{w \in \mathbb{C}^n \,|\, w_{n+1-j} = \overline{w}_j,\, \forall j = 1, \dots, n\}$$

is a Lagrangian subspace of $\mathbb{C}^n$. Therefore, there exists a unitary transformation U of $\mathbb{C}^n$, such that $W = U\mathbb{R}^n$. The statement of the claim holds for every such U. □

We choose U as in the claim. Since U is a symplectic linear map, the set $\widetilde{L}$ is a Lagrangian submanifold of $\mathbb{C}^n$, and satisfies

$$A(\mathbb{C}^n, \omega_0, \widetilde{L}) = 2A(\mathbb{C}^n, \omega_0, L).$$

By Lemma 8 the right hand side equals π. Therefore, applying Proposition 5, it follows that there exists a smooth map $u : S^2 \to [0,1] \cdot \widetilde{L}$, such that the union $X := \widetilde{L} \cup u(S^2)$ does not symplectically embed into the cylinder Z^{2n}. The set X is contained in $\overline{B}^{2n}(2\pi)$, since $\widetilde{L} \subseteq \overline{B}^{2n}(2\pi)$.

Let $\widetilde{w} \in \widetilde{L}$. We choose $z \in S^1$ and $w \in S^{2n-1}$, such that $w_{n+1-j} = \overline{w}_j$, for all j, and $\widetilde{w} = \sqrt{2}zw$. If $j \in \{1, \dots, n\}$ is an index such that $j \neq \frac{n+1}{2}$, then we have

$$|\widetilde{w}_j|^2 = 2|w_j|^2 = |w_j|^2 + |w_{n+1-j}|^2 \leq |w|^2 = 1.$$

Therefore, if n is even then $\widetilde{L}$, and hence X is contained in $\mathbb{D}^n$. It follows that X has all the required properties in this case. Consider the case in which n is odd. We denote $n =: 2k+1$ and define

$$\Psi : \mathbb{C}^n \to \mathbb{C}^n, \quad \Psi(w) := (w_1, \dots, w_k, w_{k+2}, \dots, w_n, w_{k+1}).$$

It follows that $\Psi(\widetilde{L})$ is contained in $\mathbb{D}^{n-1} \times \mathbb{C}$, and hence the same holds for $\Psi(X)$. Therefore, $\Psi(X)$ has the required properties. This proves Theorem 1. □

3.3. Proof of Proposition 2. The proof of this result is based on the following. Let $n \in \mathbb{N}$ and $U \subseteq \mathbb{R}^n$ be an open set. We denote by $|U|$ the volume of U.

Lemma 9. *For every $c > |U|$ there exists an orientation and volume preserving embedding of U into the open ball (around 0) of volume c.*

The proof of this lemma is based on the following observation. For $r > 0$ we denote by $B^n_r \subseteq \mathbb{R}^n$ the open ball (around 0) of radius r.

Remark 10. Let $U \subseteq \mathbb{R}^n$ be a non-empty open set, and $r > r_0 > 0$ real numbers. Then there exists an orientation preserving embedding φ of U into the open ball in $\mathbb{R}^n$ of radius r, such that $B^n_{r_0} \subseteq \varphi(U)$. This follows from an elementary argument.

Proof of Lemma 9. By an elementary argument, we may assume without loss of generality that U is connected and non-empty. It follows from Remark 10 that there exists an orientation preserving embedding φ of U into the open ball of volume c, such that the ball of volume $|U|$ is contained in $\varphi(U)$. This condition ensures that $|\varphi(U)| > |U|$. Hence composing φ with a shrinking homothety of $\mathbb{R}^n$, we obtain an orientation preserving embedding ψ of U into the ball of volume c, such that $|\psi(U)| = |U|$. Denoting by Ω the standard volume form on $\mathbb{R}^n$, this means that $\int_U \Omega = \int_U \psi^*\Omega$. Therefore, a theorem by Greene and Shiohama ([**GS,** Theorem 1]) implies that there exists a diffeomorphism $\chi : U \to U$ such that $\chi^*\psi^*\Omega = \Omega$. (Here we use that $\int_U \Omega < \infty$. The result is based on Moser isotopy, see [**Mo**].) The map $\psi \circ \chi$ has the required properties. This proves Lemma 9. □

Proof of Proposition 2. Let $n \in \mathbb{N}$ and X be a compact subset of $\overline{B}^{2n}$ with vanishing $(2n-1)$-dimensional Hausdorff measure. Then X does not contain S^{2n-1}, and hence there exists an orthogonal linear symplectic map $\Psi \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, such that $(1, 0, \dots, 0) \notin \Psi(X)$. Since $\Psi(X)$ is compact and contained in $\overline{B}^{2n}$, an elementary argument shows that there exists $c < 1$, such that

$$\Psi(X) \subseteq Y := \big\{(q, p) \in \mathbb{D} \mid q < c\big\} \times \mathbb{R}^{2n-2}. \tag{3.13}$$

We choose an open neighborhood U of $\big\{(q, p) \in \mathbb{D} \mid q < c\big\}$ of area less than π. By Lemma 9 U symplectically embeds into the open unit ball in $\mathbb{R}^2$. Using (3.13), it follows that $\Psi(X)$ symplectically embeds into Z^{2n}. Hence the same holds for X. This proves Proposition 2. □

3.4. Proof of Theorem 3 (Regular coisotropic capacity). The idea is to consider the Lagrangian submanifold L defined in (3.8) (for inequality (1.3)) and a product of it with a sphere (for inequality (1.4)). We need the following result. Recall the definition of the area spectrum (3.1).

Lemma 11. *Let* (M, ω) *and* (M', ω') *be symplectic manifolds, and* $N \subseteq M$ *and* $N' \subseteq M'$ *coisotropic submanifolds. Then*

$$S(M \times M', \omega \oplus \omega', N \times N') = S(M, \omega, N) + S(M', \omega', N').$$

Proof. We refer to [**SZ1,** Remark 31]. □

Proof of Theorem 3. To prove **inequality** (1.3), we define L as in (3.8). Let $r < 1$. Then rL is a closed Lagrangian submanifold of B^{2n}. Furthermore, condition (1.2) is satisfied with $(M, \omega) := (B^{2n}, \omega_0)$, since B^{2n} is contractible. An elementary argument using Lemmas 8 and 7, shows that

$A(B^{2n},\omega_0,rL)=\frac{\pi}{2}r^2$. Therefore, for every $r<1$ we have $A_{\mathrm{Lag}}(B^{2n},\omega_0)\geq\frac{\pi}{2}r^2$. Inequality (1.3) follows.

We prove **inequality** (1.4). Let $d\in\{n+1,\ldots,2n-3\}$. We define L as in (3.8) with n replaced by $2n-d-1$. We denote by $S^{k-1}_r\subseteq\mathbb{R}^k$ the sphere of radius $r>0$, around 0. Let $r<1$. The set

$$N:=\sqrt{\frac{2}{3}}rL\times S^{2d-2n+1}_{\sqrt{1/3}r} \tag{3.14}$$

is a closed regular coisotropic submanifold of B^{2n}, of dimension d. Each factor has area spectrum in linear space given by $\frac{\pi r^2}{3}\mathbb{Z}$. (For the second factor this follows, e.g., from the proof of [**Zi,** Proposition 1.3].) Hence, Lemma 11 implies that $A(\mathbb{R}^{2n},\omega_0,N)=\frac{\pi r^2}{3}$. Lemma 7 implies that this number equals $A(B^{2n},\omega_0,N)$. It follows that $A^d_{\mathrm{coiso}}(B^{2n},\omega_0)\geq\frac{\pi r^2}{3}$, for every $r<1$. Inequality (1.4) follows. This proves Theorem 3. □

Remark. The ratio of the scaling factors used in the definition (3.14) above is optimal. Namely, for $r,r'>0$ consider the coisotropic submanifold $N:=rL\times S^{2d-2n+1}_{r'}$ of $\mathbb{R}^{2n}$. It follows from Lemma 11 that

$$A(\mathbb{R}^{2n},\omega_0,N)=\pi\gcd\left\{\frac{r^2}{2},{r'}^2\right\}. \tag{3.15}$$

Here, we define the greatest common divisor of two real numbers a,b to be

$$\gcd\{a,b\}:=\sup\left\{c\in(0,\infty)\,\middle|\,a,b\in c\mathbb{Z}\right\}.$$

(Here our convention is that the supremum over the empty set equals 0.) In order for N to be contained in B^{2n}, we need $r^2+{r'}^2<1$. For a given $c<1$, the expression (3.15) is largest (namely equal to $\frac{c\pi}{3}$) under the restriction $r^2+{r'}^2=c$, provided that $\frac{r^2}{2}={r'}^2$. This corresponds to the choice in (3.14).

3.5. Proof of Proposition 4 (Two-dimensional squeezing). We denote by $Y\subseteq\mathbb{R}^2$ the image of X under the canonical projection from $\mathbb{R}^{2n}=\mathbb{R}^2\times\mathbb{R}^{2n-2}$ onto the first component. The 2-dimensional Hausdorff measure of Y vanishes by a standard result. (See, e.g., [**Fe,** p. 176].) Therefore, there exists an open neighborhood $U\subseteq\mathbb{R}^2$ of Y of area less than a. By Lemma 9 there exists a symplectic embedding φ of U into the open ball in $\mathbb{R}^2$, of area a. The product $U\times\mathbb{R}^{2n-2}$ is an open neighborhood of X, and $\varphi\times\mathrm{id}$ is a symplectic embedding of this neighborhood into $Z^{2n}(a)$. This proves Proposition 4. □

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