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TOPOLOGICAL RECURSION RELATIONS IN NON-EQUIVARIANT CYLINDRICAL CONTACT HOMOLOGY

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It was pointed out by Eliashberg in his ICM 2006 plenary talk that the integrable systems of rational Gromov–Witten theory very naturally appear in the rich algebraic formalism of symplectic field theory (SFT). Carefully generalizing the definition of gravitational descendants from Gromov–Witten theory to SFT, one can assign to every contact manifold a Hamiltonian system with symmetries on SFT homology and the question of its integrability arises. While we have shown how the well-known string, dilaton and divisor equations translate from Gromov–Witten theory to SFT, the next step is to show how genus-zero topological recursion translates to SFT. Compatible with the example of SFT of closed geodesics, it turns out that the corresponding localization theorem requires a non-equivariant version of SFT, which is generated by parameterized instead of unparameterized closed Reeb orbits. Since this non-equivariant version is so far only defined for cylindrical contact homology, we restrict ourselves to this special case. As an important result we show that, as in rational Gromov–Witten theory, all descendant invariants can be computed from primary invariants, i.e., without descendants.

Contents

1. Introduction

Symplectic field theory (SFT), introduced by Hofer et al. in 2000 [**EGH**], is a very large project and can be viewed as a topological quantum field theory approach to Gromov–Witten theory. Besides providing a unified view on established pseudoholomorphic curve theories such as symplectic Floer homology, contact homology and Gromov–Witten theory, it leads to numerous new applications and opens new routes yet to be explored.

Although SFT leads to algebraic invariants with very rich algebraic structures, it was pointed out by Eliashberg in his ICM 2006 plenary talk [**E**] that the integrable systems of rational Gromov–Witten theory very naturally appear in rational SFT by using the link between the rational SFT of prequantization spaces in the Morse–Bott version and the rational Gromov–Witten potential of the underlying symplectic manifold; see the recent papers [**R1, R2**] by the second author. Indeed, after introducing gravitational descendants as in Gromov–Witten theory, it is precisely the rich algebraic formalism of SFT with its Weyl and Poisson structures that provides a natural link between SFT and (quantum) integrable systems.

On the other hand, carefully defining a generalization of gravitational descendants and adding them to the picture, the first author has shown in [**F2**] that one can assign to every contact manifold an infinite sequence of commuting Hamiltonian systems on SFT homology and the question of their integrability arises. For this, it is important to fully understand the algebraic structure of gravitational descendants in SFT.

While it is well known that in Gromov–Witten theory the topological meaning of gravitational descendants leads to new differential equations for the Gromov–Witten potential, in this paper we want to proceed with our project of understanding how these rich algebraic structures carry over from Gromov–Witten theory to SFT. While we have already shown in [**FR**] how the well-known string, dilaton and divisor equations translate from Gromov–Witten theory to SFT, as a next step we want to show how classical genus-zero topological recursion generalizes to SFT.

Although this is a first concrete step in the study of integrability of the Hamiltonian systems of SFT, notice that topological recursion relations in the forms we study here might not be enough to answer the question of integrability: the Hamiltonian systems arising from SFT are, a priori, much more general than those associated with Gromov–Witten invariants, involving in particular more than just local functionals (see [**R2**]), and topological recursion relations, even together with string, dilaton and divisor equations, might not yet restrictive enough to grant complete control over the algebra of commuting Hamiltonians. They seem however to give an affirmative answer to the fundamental question of the reconstructability of the gravitational descendants from the primary invariants (i.e., without descendants) in genus 0.

From the computation of the SFT of a Reeb orbit with descendants in [**F2**] it can be seen that the genus-zero topological recursion requires a non-equivariant version of SFT, which is generated by parameterized instead of unparameterized Reeb orbits. The definition of this non-equivariant version of SFT is currently a very active field of research and related to the work of Bourgeois and Oancea in [**BO**], where a Gysin-type spectral sequence relating linearized contact homology (a slight generalization of cylindrical contact homology depending on a symplectic filling) and symplectic homology of this filling is established by viewing the one as the (non-)equivariant version of the other.

Since the topological recursion relation is already interesting in the case of cylindrical contact homology and the non-equivariant version of it is already understood, in this first paper on topological recursion we restrict ourselves to cylindrical contact homology, i.e., study the algebraic structure of gravitational descendants only for this special case.

This paper is organized as follows: While in section two we review the most important definitions and results about SFT with gravitational descendants and its relation with integrable systems in [**F2, FR**], in section three we first show, as a motivation for our main result, how the topological recursion relations in Gromov–Witten theory carry over to symplectic Floer theory. Since this example suggests that the localization theorem for gravitational descendants needs a non-equivariant version of cylindrical contact homology which, similar to symplectic Floer homology, is generated by parameterized instead of unparameterized closed Reeb orbits, we then recall the definition of non-equivariant cylindrical homology from [**BO**] and prove the topological recursion relations in the non-equivariant situation. Finally, in section four we discuss two important applications of our main result. First we show how the topological recursion formulas carry over from the non-equivariant to the equivariant situation and use this result to show that, as in rational Gromov–Witten theory, all descendant invariants can be computed from primary invariants, that is, without descendants. After this we show that our results can be further used to define an action of the (quantum) cohomology on non-equivariant cylindrical homology similar to the corresponding action on symplectic Floer homology defined in [**PSS**]. At the end we show that in the Floer case of SFT we just get back the topological recursion relations in Floer homology and that the action of quantum cohomology on non-equivariant homology splits and agrees with the action on Floer homology as defined in [**PSS**].

2. SFT with gravitational descendants

2.1. SFT. SFT is a very large project, initiated by Eliashberg et al. in their paper [**EGH**], designed to describe in a unified way the theory of pseudoholomorphic curves in symplectic and contact topology. Besides providing a unified view on well-known theories like symplectic Floer homology and Gromov–Witten theory, it shows how to assign algebraic invariants to closed manifolds with a stable Hamiltonian structure.

Following [**BEHWZ**] a Hamiltonian structure on a closed $(2m-1)$-dimensional manifold V is a closed two-form ω on V, which is maximally non-degenerate in the sense that $\ker\omega = \{v \in TV : \omega(v,\cdot) = 0\}$ is a one-dimensional distribution. The Hamiltonian structure is required to be stable in the sense that there exists a one-form λ on V such that $\ker\omega \subset \ker d\lambda$ and $\lambda(v) \neq 0$ for all $v \in \ker\omega - \{0\}$. Any stable Hamiltonian structure (ω,λ) defines a symplectic hyperplane distribution $(\xi = \ker\lambda, \omega_\xi)$, where ω_ξ is the restriction of ω, and a vector field R on V by requiring $R \in \ker\omega$ and $\lambda(R) = 1$, which is called the Reeb vector field of the stable Hamiltonian structure. Examples for closed manifolds V with a stable Hamiltonian structure (ω,λ) are contact manifolds, symplectic mapping tori and principal circle bundles over symplectic manifolds [**BEHWZ**]:

First observe that when λ is a contact form on V, it is easy to check that $(\omega := d\lambda, \lambda)$ is a stable Hamiltonian structure and the symplectic hyperplane distribution agrees with the contact structure. For the other two cases, let (M, ω_M) be a symplectic manifold. Then every principal circle bundle $S^1 \to V \to M$ and every symplectic mapping torus $M \to V \to S^1$, i.e., $V = M_\phi = \mathbb{R} \times M/\{(t,p) \sim (t+1, \phi(p))\}$ for $\phi \in \mathrm{Symp}(M,\omega)$ also carries a stable Hamiltonian structure. For the circle bundle the Hamiltonian structure is given by the pullback $\pi^*\omega$ under the bundle projection and we can choose as one-form λ any S^1-connection form. On the other hand, the stable Hamiltonian structure on the mapping torus $V = M_\phi$ is given by lifting the symplectic form to $\omega \in \Omega^2(M_\phi)$ via the natural flat connection $TV = TS^1 \oplus TM$ and setting $\lambda = dt$ for the natural S^1-coordinate t on M_ϕ.

While in the mapping torus case ξ is always integrable, in the circle bundle case the hyperplane distribution ξ may be integrable or non-integrable, even contact.

SFT assigns algebraic invariants to closed manifolds V with a stable Hamiltonian structure. The invariants are defined by counting $\underline{J}$-holomorphic curves in $\mathbb{R}\times V$ with finite energy, where the underlying closed Riemann surfaces are explicitly allowed to have punctures, i.e., single points are removed. The almost complex structure $\underline{J}$ on the cylindrical manifold $\mathbb{R}\times V$ is required to be cylindrical in the sense that it is $\mathbb{R}$-independent, links the two natural vector fields on $\mathbb{R}\times V$, namely the Reeb vector field R and the $\mathbb{R}$-direction ∂_s, by $\underline{J}\partial_s = R$, and turns the symplectic hyperplane distribution on V into a complex subbundle of TV, $\xi = TV \cap \underline{J}TV$. It follows that a cylindrical almost complex structure $\underline{J}$ on $\mathbb{R}\times V$ is determined by its restriction $\underline{J}_\xi$ to $\xi \subset TV$, which is required to be ω_ξ-compatible in the sense that $\omega_\xi(\cdot, \underline{J}_\xi\cdot)$ defines a metric on ξ. Note that such almost complex structures $\underline{J}$ are called compatible with the stable Hamiltonian structure and that the set of these almost complex structures is non-empty and contractible.

Let us recall the definition of moduli spaces of holomorphic curves studied in rational SFT in the general setup. Let Γ^+, Γ^- be two ordered sets of closed orbits γ of the Reeb vector field R on V, i.e., $\gamma : \mathbb{R} \to V$, $\gamma(t+T) = \gamma(t)$, $\dot{\gamma} = R$, where $T > 0$ denotes the period of γ. Here we assume that the stable Hamiltonian structure is Morse in the sense that all closed orbits of the Reeb vector field are non-degenerate in the sense of [**BEHWZ**]; in particular, the set of closed Reeb orbits is discrete. Then the (parameterized) moduli space $\mathcal{M}^0_{r,A}(\Gamma^+,\Gamma^-)$ consists of tuples $(u, (z_k^\pm), (z_i))$, where $(z_k^\pm), (z_i)$ are three disjoint ordered sets of points on $\mathbb{CP}^1$, which are called positive and negative punctures, and additional marked points, respectively. The map $u : \dot{S} \to \mathbb{R}\times V$ starting from the punctured Riemann surface $\dot{S} = \mathbb{CP}^1 - \{(z_k^\pm)\}$ is required to satisfy the Cauchy–Riemann equation

$$\bar{\partial}_{\underline{J}} u = du + \underline{J}(u)\cdot du \cdot i = 0$$

with the complex structure i on $\mathbb{CP}^1$. Assuming we have chosen cylindrical coordinates $\psi_k^\pm : \mathbb{R}^\pm \times S^1 \to \dot{S}$ around each puncture $z_k^\pm$ in the sense that $\psi_k^\pm(\pm\infty, t) = z_k^\pm$, the map u is additionally required to show for all $k = 1, \dots, n^\pm$ the asymptotic behaviour

$$\lim_{s\to\pm\infty} (u \circ \psi_k^\pm)(s, t+t_0) = (\pm\infty, \gamma_k^\pm(T_k^\pm t))$$

with some $t_0 \in S^1$ and the orbits $\gamma_k^\pm \in \Gamma^\pm$, where $T_k^\pm > 0$ denotes period of $\gamma_k^\pm$. In order to assign an absolute homology class A to a holomorphic curve $u : \dot{S} \to \mathbb{R}\times V$ we have to employ spanning surfaces u_γ connecting a given closed Reeb orbit γ in V to a linear combination of circles c_s representing a basis of $H_1(V)$,

$$\partial u_\gamma = \gamma - \sum_s n_s \cdot c_s$$

in order to define

$$A = [u_{\Gamma^+}] + [u(\dot{S})] - [u_{\Gamma^-}],$$

where $[u_{\Gamma^\pm}] = \sum_{n=1}^{s^\pm} [u_{\gamma_n^\pm}]$ viewed as singular chains.

Observe that the group $\mathrm{Aut}(\mathbb{CP}^1)$ of Moebius transformations acts on elements in $\mathcal{M}^0 = \mathcal{M}^0_{r,A}(\Gamma^+, \Gamma^-)$ in an obvious way,

$$\varphi.(u, (z_k^\pm), (z_i)) = (u \circ \varphi^{-1}, (\varphi(z_k^\pm)), (\varphi(z_i))), \quad \varphi \in \mathrm{Aut}(\mathbb{CP}^1),$$

and we obtain the moduli space $\mathcal{M} = \mathcal{M}_{r,A}(\Gamma^+, \Gamma^-)$ studied in SFT by dividing out this action and the natural $\mathbb{R}$-action on the target manifold $(\mathbb{R} \times V, \underline{J})$. Furthermore it was shown in [**BEHWZ**] that this moduli space can be compactified to a moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{r,A}(\Gamma^+, \Gamma^-)$ by adding moduli space of multi-floor curves with nodes. In particular, the moduli space has codimension-one boundary given by (fibre) products $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2 = \overline{\mathcal{M}}_{r_1,A_1}(\Gamma_1^+, \Gamma_1^-) \times \overline{\mathcal{M}}_{r_2,A_2}(\Gamma_2^+, \Gamma_2^-)$ of lower-dimensional moduli spaces.

Let us now briefly introduce the algebraic formalism of rational SFT as described in [**EGH**]:

Let us fix a trivialization of the symplectic bundle $(\xi, \omega|_\xi)$ over each curve C_i. This induces a trivialization a hoomotopically unique trivialization of the same bundle over each periodic Reeb orbit γ via the spanning surface u_γ. Let us use this trivialization to define the Conley–Zehnder index of the Reeb orbit (the Maslov index of the path in $Sp(2m-2, \mathbb{R})$ given by the linearized Reeb flow along γ). Recall that a multiply covered Reeb orbit γ^k is called bad if $\mathrm{CZ}(\gamma^k) \neq \mathrm{CZ}(\gamma) \mod 2$, where $\mathrm{CZ}(\gamma)$ denotes the Conley–Zehnder index of γ. Calling a Reeb orbit γ *good* if it is not bad, we assign to every good Reeb orbit γ two formal graded variables p_γ, q_γ with grading

$$|p_\gamma| = m - 3 - \mathrm{CZ}(\gamma), |q_\gamma| = m - 3 + \mathrm{CZ}(\gamma)$$

when $\dim V = 2m - 1$.

Assuming we have chosen a basis $A_0, \ldots, A_M$ of $H_2(V)$, we assign to every A_i a formal variable z_i with grading $|z_i| = -2c_1(A_i)$. In order to include higher-dimensional moduli spaces we further assume that a string of closed (homogeneous) differential forms $\Theta = (\theta_1, \ldots, \theta_N)$ on V is chosen and assign to every $\theta_\alpha \in \Omega^*(V)$ a formal variables t^α with grading

$$|t^\alpha| = 2 - \deg \theta_\alpha.$$

With this let $\mathfrak{P}$ be the Poisson algebra of formal power series in the variables p_γ and t_i with coefficients which are polynomials in the variables q_γ and

Laurent series in z_n with Poisson bracket given by

$$\{f,g\} = \sum_{\gamma} \kappa_\gamma \left(\frac{\partial f}{\partial p_\gamma}\frac{\partial g}{\partial q_\gamma} - (-1)^{|f||g|}\frac{\partial g}{\partial p_\gamma}\frac{\partial f}{\partial q_\gamma} \right).$$

As in Gromov–Witten theory we want to organize all moduli spaces $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ into a generating function $\mathbf{h} \in \mathfrak{P}$, called *Hamiltonian.* In order to include also higher-dimensional moduli spaces, in [**EGH**] the authors follow the approach in Gromov–Witten theory to integrate the chosen differential forms θ_α over the moduli spaces after pulling them back under the evaluation map from target manifold V. The Hamiltonian $\mathbf{h}$ is then defined by

$$\mathbf{h} = \sum_{\Gamma^+,\Gamma^-,I} \frac{1}{r!n^+!n^-!\kappa_{\Gamma^+}\kappa_{\Gamma^-}} \int_{\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)/\,\mathbb{R}} \mathrm{ev}_1^*\,\theta_{\alpha_1}\wedge\ldots\wedge\mathrm{ev}_r^*\,\theta_{\alpha_r}\; t^I p^{\Gamma^+} q^{\Gamma^-} z^A$$

with $t^I = t^{\alpha_1}\ldots t^{\alpha_r}$, $p^{\Gamma^+} = p_{\gamma_1^+}\ldots p_{\gamma_{n^+}^+}$, $q^{\Gamma^-} = q_{\gamma_1^-}\ldots q_{\gamma_{n^-}^-}$, $z^A = z_0^{d_0}\cdot\ldots\cdot z_M^{d_M}$ for $A = d_0A_0 + \cdots + d_MA_M$ and $\kappa_{\Gamma^\pm} = \kappa_{\gamma_1^\pm}\cdot\ldots\cdot\kappa_{\gamma_{s^\pm}^\pm}$.

2.2. Gravitational descendants. After introducing SFT in the sense of [**EGH**], we now recall the definition of gravitational descendants in SFT in [**F2**], which we will use to enrich the SFT Hamiltonian. In the same way as the above Hamiltonian $\mathbf{h}$ explicitly depends on the chosen contact form, the cylindrical almost complex structure, the differential forms and abstract polyfold perturbations making all moduli spaces regular, it will turn out that the enriched Hamiltonian further depends on the additional auxiliary choices we have to make to define gravitational descendants. In the next subsection we will show how to construct algebraic invariants, which just depend on the contact structure and the cohomology classes of the differential forms and are independent of the other auxiliary choices.

In complete analogy to Gromov–Witten theory we can introduce r tautological line bundles $\mathcal{L}_1,\ldots,\mathcal{L}_r$ over each moduli space $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$. The fibre of $\mathcal{L}_i$ over a punctured curve is again given by the cotangent line to the underlying, possibly unstable nodal Riemann surface (without ghost components) at the i.th marked point. Note that it can still be formally defined as the pull-back of the vertical cotangent line bundle of $\pi_i : \overline{\mathcal{M}}_{r+1,A}(\Gamma^+,\Gamma^-) \to \overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ under the canonical section $\sigma_i : \overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-) \to \overline{\mathcal{M}}_{r+1,A}(\Gamma^+,\Gamma^-)$ mapping to the ith marked point in the fibre.

Recall that in Gromov–Witten theory the gravitational descendants were defined by integrating powers of the first Chern class of the tautological line bundle over the moduli space, which by Poincare duality corresponds to counting common zeroes of sections in this bundle. On the other hand, in SFT, more generally every holomorphic curves theory where curves with punctures and/or boundary are considered, we are faced with the problem that the moduli spaces generically have codimension-one boundary, so that the count of zeroes of sections in general depends on the chosen sections in the boundary. It follows that the integration of the first Chern class of the tautological line bundle over a single moduli space has to be replaced by a construction involving all moduli space at once, which we will recall now from [**F2**].

Following the compactness statement in [**BEHWZ**], the codimension-one boundary of a moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ of SFT holomorphic curves consists of curves with two levels (in the sense of [**BEHWZ**]). More precisely, each component of the boundary has the form of a (fibred) product $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2 = \overline{\mathcal{M}}_{r_1,A_1}(\Gamma_1^+,\Gamma_1^-) \times \overline{\mathcal{M}}_{r_2,A_2}(\Gamma_2^+,\Gamma_2^-)$ of moduli spaces of strictly lower dimension, where the marked points distribute on the two levels. Consider a boundary component where the ith marked point sits, say, on the first level $\overline{\mathcal{M}}_1$: it directly follows from the definition of the tautological line bundle $\mathcal{L}_i$ at the ith marked point over $\overline{\mathcal{M}}$ that, over such boundary component,

$$\mathcal{L}_i\,|_{\overline{\mathcal{M}}_1\times\overline{\mathcal{M}}_2} = \pi_1^*\,\mathcal{L}_{i,1}$$

where $\mathcal{L}_{i,1}$ denotes the tautological line bundle over the moduli space $\overline{\mathcal{M}}_1$ and $\pi_1 : \overline{\mathcal{M}}_1\times\overline{\mathcal{M}}_2 \to \overline{\mathcal{M}}_1$ is the projection onto the first factor. With this we can now give the definition of coherent collections of sections in tautological line bundles from [**F2**].

Definition 2.1. Assume that we have chosen sections s_i in the tautological line bundles $\mathcal{L}_i$ over all moduli spaces $\overline{\mathcal{M}}$ of $\underline{J}$-holomorphic curves of SFT. Then these collections of sections (s_i) are called *coherent* if for every section s_i of $\mathcal{L}_i$ over a moduli space $\overline{\mathcal{M}}$ the following holds: over each codimension-one boundary component $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2$ of $\overline{\mathcal{M}}$, the section s_i agrees with the pull-back $\pi_1^* s_{i,1}$ $(\pi_2^* s_{i,2})$ of the chosen section $s_{i,1}$ $(s_{i,2})$ of the tautological line bundle $\mathcal{L}_{i,1}$ over $\overline{\mathcal{M}}_1$ $(\mathcal{L}_{i,2}$ over $\overline{\mathcal{M}}_2)$, assuming that the ith marked point sits on the first (second) level.

Since in the end we will again be interested in the zero sets of these sections, we will assume that all occurring sections are sufficiently generic, in particular, transversal to the zero section. Furthermore, we want to assume that all the chosen sections are indeed invariant under the obvious symmetries like reordering of punctures and marked points. In order to meet both requirements, it follows that actually need to employ multi-sections (in the sense of branched manifolds). On the other hand, it is clear that one can always find coherent collections of (transversal) sections (s) by using induction on the dimension of the underlying moduli space.

For every tuple $(j_1,\ldots,j_r)$ of natural numbers we choose j_i coherent collections of sections $(s_{i,k})$ of $\mathcal{L}_i$. Then we define for every moduli space

$\overline{\mathcal{M}} = \overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$,

$$\overline{\mathcal{M}}^{(j_1,\dots,j_r)} = s_{1,1}^{-1}(0) \cap \dots \cap s_{1,j_1}^{-1}(0) \cap \dots \cap s_{r,1}^{-1}(0) \cap \dots \cap s_{r,j_r}^{-1}(0) \subset \overline{\mathcal{M}}.$$

Note that by choosing all sections sufficiently generic, we can assume $\overline{\mathcal{M}}^{(j_1,\dots,j_r)} = \overline{\mathcal{M}}_{r,A}^{(j_1,\dots,j_r)}(\Gamma^+,\Gamma^-)$ is a branched-labelled submanifold of the moduli space $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$. Note that by definition

$$\overline{\mathcal{M}}^{(j_1,\dots,j_r)} = \overline{\mathcal{M}}^{(j_1,0,\dots,0)} \cap \dots \cap \overline{\mathcal{M}}^{(0,\dots,0,j_r)},$$

and it follows from the coherency condition that the codimension-one boundary of $\overline{\mathcal{M}}^{(0,\dots,0,j,0,\dots,0)}$ is given by the products $\overline{\mathcal{M}}_1^{(0,\dots,0,j,0,\dots,0)} \times \overline{\mathcal{M}}_2$ or $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2^{(0,\dots,0,j,0,\dots,0)}$ (depending on the ith marked points sitting on the first or second level).

With this we can define the descendant Hamiltonian of SFT, which we will continue denoting by $\mathbf{h}$, while the Hamiltonian defined in [**EGH**] will from now on be called *primary*. In order to keep track of the descendants we will assign to every chosen differential form θ_α now a sequence of formal variables $t^{\alpha,j}$ with grading

$$|t^{\alpha,j}| = 2(1-j) - \deg\theta_\alpha.$$

Then the descendant Hamiltonian $\mathbf{h} \in \mathfrak{P}$ of (rational) SFT is defined by

$$\mathbf{h} = \sum_{\Gamma^+,\Gamma^-,I} \frac{1}{r!n^+!n^-!\kappa_{\Gamma^+}\kappa_{\Gamma^-}} \times \int_{\overline{\mathcal{M}}_{r,A}^{(j_1,\dots,j_r)}(\Gamma^+,\Gamma^-)/\mathbb{R}} \mathrm{ev}_1^*\theta_{\alpha_1} \wedge \dots \wedge \mathrm{ev}_r^*\theta_{\alpha_r}\; t^I p^{\Gamma^+} q^{\Gamma^-} z^A,$$

where $p^{\Gamma^+} = p_{\gamma_1^+}\cdots p_{\gamma_{n^+}^+}$, $q^{\Gamma^-} = q_{\gamma_1^-}\cdots q_{\gamma_{n^-}^-}$, $t^I = t^{\alpha_1,j_1}\cdots t^{\alpha_r,j_r}$, $z^A = z_0^{d_0}\cdots z_M^{d_M}$ for $A = d_0A_0 + \dots + d_MA_M$ and $\kappa_{\Gamma^\pm} = \kappa_{\gamma_1^\pm}\cdots\kappa_{\gamma_{s^\pm}^\pm}$.

2.3. Quantum Hamiltonian systems with symmetries. In [**F2**] it is shown that, after introducing gravitational descendants, SFT assigns to every contact manifold not only a Poisson algebra, the well-known rational SFT homology, but also a Hamiltonian system in it with an infinite number of symmetries.

Theorem 2.2. *Differentiating the rational Hamiltonian $\mathbf{h} \in \mathfrak{P}$ with respect to the formal variables $t^{\alpha,p}$ defines a sequence of classical Hamiltonians*

$$\mathbf{h}_{\alpha,p} = \frac{\partial\,\mathbf{h}}{\partial t^{\alpha,p}} \in H_*(\mathfrak{P},\{\mathbf{h},\cdot\})$$

in the rational SFT homology algebra with differential $d = \{\mathbf{h},\cdot\} : \mathfrak{P} \to \mathfrak{P}$, which commute with respect to the bracket on $H_*(\mathfrak{P},\{\mathbf{h},\cdot\})$,

$$\{\mathbf{h}_{\alpha,p},\mathbf{h}_{\beta,q}\} = 0,\ (\alpha,p),(\beta,q) \in \{1,\dots,N\} \times \mathbb{N}.$$

Note that everything is an immediate consequence of the master equation $\{\mathbf{h},\mathbf{h}\}=0$, which can be proven in the same way as in the case without descendants using the results in [**F2**]. While the boundary equation $d\circ d=0$, $d=\{\mathbf{h},\cdot\}$ is well-known to follow directly from the identity $\{\mathbf{h},\mathbf{h}\}=0$, the fact that every $\mathbf{h}_{\alpha,p}$, $(\alpha,p)\in\{1,\dots,N\}\times\mathbb{N}$ defines an element in the homology $H_*(\mathfrak{P},\{\mathbf{h},\cdot\})$ follows from the identity

$$\{\mathbf{h},\mathbf{h}_{\alpha,p}\}=0,$$

which can be shown by differentiating the master equation with respect to the $t^{\alpha,p}$-variable and using the graded Leibniz rule,

$$\frac{\partial}{\partial t^{\alpha,p}}\{f,g\}=\left\{\frac{\partial f}{\partial t^{\alpha,p}},g\right\}+(-1)^{|t^{\alpha,p}||f|}\left\{f,\frac{\partial g}{\partial t^{\alpha,p}}\right\}.$$

On the other hand, in order to see that any two $\mathbf{h}_{\alpha,p}$, $\mathbf{h}_{\beta,q}$ commute *after passing to homology* it suffices to see that by differentiating twice (and using that all summands in $\mathbf{h}$ have odd degree) we get the identity

$$\{\mathbf{h}_{\alpha,p},\mathbf{h}_{\beta,q}\}+(-1)^{|t^{\alpha,p}|}\left\{\mathbf{h},\frac{\partial^2\mathbf{h}}{\partial t^{\alpha,p}\partial t^{\beta,q}}\right\}=0.$$

We now turn to the question of independence of these nice algebraic structures from the choices like contact form, cylindrical almost complex structure, abstract polyfold perturbations and, of course, the choice of the coherent collection of sections. This is the content of the following theorem proven in [**F2**].

Theorem 2.3. *For different choices of contact form $\lambda^\pm$, cylindrical almost complex structure $\underline{J}^\pm$, abstract polyfold perturbations and sequences of coherent collections of sections $(s_j^\pm)$ the resulting systems of commuting functions $\mathbf{h}^-_{\alpha,p}$ on $H_*(\mathfrak{P}^-,d^-)$ and $\mathbf{h}^+_{\alpha,p}$ on $H_*(\mathfrak{P}^+,d^+)$ are isomorphic, i.e., there exists an isomorphism of the Poisson algebras $H_*(\mathfrak{P}^-,d^-)$ and $H_*(\mathfrak{P}^+,d^+)$ which maps $\mathbf{h}^-_{\alpha,p}\in H_*(\mathfrak{P}^-,d^-)$ to $\mathbf{h}^+_{\alpha,p}\in H_*(\mathfrak{P}^+,d^+)$.*

This theorem is an extension of the theorem in [**EGH**], which states that for different choices of auxiliary data the Poisson algebras $H_*(\mathfrak{P}^-,d^-)$ and $H_*(\mathfrak{P}^+,d^+)$ with $d^\pm=\{\mathbf{h}^\pm,\cdot\}$ are isomorphic. For the extension in [**F2**] the first author introduced the notion of a collection of sections (s_j) in the tautological line bundles over all moduli spaces of holomorphic curves in the cylindrical cobordism interpolating between the auxiliary structures which are *coherently connecting* the two coherent collections of sections $(s_j^\pm)$. On the other hand, assuming that the contact form, the cylindrical almost complex structure and also the abstract polyfold sections are fixed to have well-defined moduli spaces, the isomorphism of the homology algebras is the identity and hence the theorem states the sequence of commuting Hamiltonians is indeed independent of the chosen sequences of coherent collections of sections *after passing to homology*,

$$\mathbf{h}^{1,-}_{i,j} = \mathbf{h}^{1,+}_{i,j} \in H_*(\mathfrak{P}, \{\mathbf{h}, \cdot\}).$$

We want to point out the fact that the Poisson SFT homology algebra can be thought of as the space of functions on some abstract infinite-dimensional Poisson space. Indeed the kernel $\ker(\{\mathbf{h}, \cdot\})$ can be seen as the algebra of functions on the space $\mathcal{O}$ of orbits of the Hamiltonian $\mathbb{R}$-action given by $\mathbf{h}$, that is, the flow lines of the Hamiltonian vector field $X_{\mathbf{h}}$ associated to $\mathbf{h}$. Even in a finite dimensional setting the space $\mathcal{O}$ can be very wild. Anyhow the image $\mathrm{im}(\{\mathbf{h}, \cdot\})$ is an ideal of such algebra and hence identifies a sub-space of $\mathcal{O}$ given by all of those orbits $o \in \mathcal{O}$ at which, for any $f \in \mathfrak{P}$, $\{\mathbf{h}, f\}|_o = 0$. But such orbits are simply the constant ones, where $X_{\mathbf{h}}$ vanishes. Hence the Poisson SFT-homology algebra $H_*(\mathfrak{P}, \{\mathbf{h}, \cdot\})$ can regarded as the algebra of functions on $X_{\mathbf{h}}^{-1}(0)$, seen as a subspace of the space $\mathcal{O}$ of orbits of $\mathbf{h}$, endowed with a Poisson structure by singular, stationary reduction.

While it is well-known that in Gromov–Witten theory the topological meaning of gravitational descendants leads to new differential equations for the Gromov–Witten potential, it is natural to ask how these rich algebraic structures carry over from Gromov–Witten theory to SFT. As a first step, the authors have shown in the paper [**FR**] how the well-known string, dilaton and divisor equations generalize from Gromov–Witten theory to SFT. Here the main problem is to deal with the fact that the SFT Hamiltonian indeed depends on auxiliary data like the chosen differential forms θ_i and coherent collections of sections (s_j) used to define gravitational descendants. As customary in Gromov–Witten theory we will assume that the chosen string of differential forms on V contains a two-form θ_2. It turns out that we obtain the same equations as in Gromov–Witten theory (up to contributions of constant curves), but these however only hold after passing to SFT homology.

Theorem 2.4. *For any choice of differential forms and coherent sections the following* string, dilaton and divisor equations *hold* after *passing to SFT-homology*

$$\frac{\partial}{\partial t^{0,0}}\,\mathbf{h} = \int_V t \wedge t + \sum_k t^{\alpha,k+1}\frac{\partial}{\partial t^{\alpha,k}}\,\mathbf{h} \in H_*(\mathfrak{P}, \{\mathbf{h}, \cdot\}),$$

$$\frac{\partial}{\partial t^{0,1}}\,\mathbf{h} = \mathbf{D}_{\mathrm{Euler}}\,\mathbf{h} \in H_*(\mathfrak{P}, \{\mathbf{h}, \cdot\}),$$

$$\left(\frac{\partial}{\partial t^{2,0}} - z_0\frac{\partial}{\partial z_0}\right)\mathbf{h} = \int_V t \wedge t \wedge \theta_2 + \sum_k t^{\alpha,k+1} c^{\beta}_{2\alpha}\frac{\partial\,\mathbf{h}}{\partial t^{\beta,k}} \in H_*(\mathfrak{P}, \{\mathbf{h}, \cdot\}),$$

with the first-order differential operator

$$\mathbf{D}_{\text{Euler}} := 2 - \sum_{\gamma} p_{\gamma} \frac{\partial}{\partial p_{\gamma}} - \sum_{\gamma} q_{\gamma} \frac{\partial}{\partial q_{\gamma}} - \sum_{\alpha,p} t^{\alpha,p} \frac{\partial}{\partial t^{\alpha,p}}.$$

As computed example we review the SFT of a closed Reeb orbit discussed in [**F2**]. For this recall that in [**F2**] the first author has shown that every algebraic invariant of SFT has a natural analog defined by counting only orbit curves. In particular, in the same way as we define sequences of descendant Hamiltonians $\mathbf{h}^1_{i,j}$ by counting general curves in the symplectization of a contact manifold, we can define sequences of descendant Hamiltonians $\mathbf{h}^1_{\gamma,i,j}$ by just counting branched covers of the orbit cylinder over γ with signs (and weights), where the preservation of the contact area under splitting and gluing of curves proves that for every theorem from above we have a version for γ.

Let $\mathfrak{P}^0_{\gamma}$ be the graded Poisson subalgebra of the Poisson algebra $\mathfrak{P}^0$ obtained from the Poisson algebra $\mathfrak{P}$ by setting all t-variables to zero, which is generated only by those p- and q-variables $p_n = p_{\gamma^n}$, $q_n = q_{\gamma^n}$ corresponding to Reeb orbits which are multiple covers of the fixed orbit γ. In his paper [**F2**] the first author computed the corresponding Poisson-commuting sequence in the special case where the contact manifold is the unit cotangent bundle S^*Q of a (m-dimensional) Riemannian manifold Q, so that every closed Reeb orbit γ on $V = S^*Q$ corresponds to a closed geodesic $\bar{\gamma}$ on Q (and the string of differential forms just contains a one-form which integrates to one over the closed Reeb orbit).

Theorem 2.5. *The system of Poisson-commuting functions $\mathbf{h}^1_{\gamma,j}$, $j \in \mathbb{N}$ on $\mathfrak{P}^0_{\gamma}$ is isomorphic to a system of Poisson-commuting functions $\mathbf{g}^1_{\bar{\gamma},j}$, $j \in \mathbb{N}$ on $\mathfrak{P}^0_{\bar{\gamma}} = \mathfrak{P}^0_{\gamma}$, where for every $j \in \mathbb{N}$ the descendant Hamiltonian $\mathbf{g}^1_{\bar{\gamma},j}$ given by*

$$\mathbf{g}^1_{\bar{\gamma},j} = \sum \epsilon(\vec{n}) \frac{q_{n_1} \cdot \ldots \cdot q_{n_{j+2}}}{(j+2)!}$$

where the sum runs over all ordered monomials $q_{n_1} \cdot \ldots \cdot q_{n_{j+2}}$ with $n_1 + \cdots + n_{j+2} = 0$ ***and which are of degree*** *$2(m+j-3)$. Further $\epsilon(\vec{n}) \in \{-1, 0, +1\}$ is fixed by a choice of coherent orientations in SFT and is zero if and only if one of the orbits $\gamma^{n_1}, \ldots, \gamma^{n_{j+2}}$ is bad in the sense of [**BEHWZ**].*

2.4. Cylindrical contact homology. While the punctured curves in SFT may have arbitrary genus and arbitrary numbers of positive and negative punctures, it is shown in [**EGH**] that there exist algebraic invariants counting only special types of curves. While in rational SFT one counts punctured curves with genus zero, contact homology is defined by further restricting to punctured spheres with only one positive puncture. Further restricting to spheres with both just one negative and one positive puncture, i.e., cylinders, the resulting algebraic invariant is called cylindrical contact homology.

Note however that contact homology and cylindrical contact homology are not always defined. In order to prove the well-definedness of (cylindrical) contact homology it however suffices to show that there are no punctured holomorphic curves where all punctures are negative (or all punctures are positive). To be more precise, for the well-definedness of cylindrical contact homology it actually suffices to assume that there are no holomorphic planes and that there are either no holomorphic cylinders with two positive or no holomorphic cylinders with two negative ends.

While the existence of holomorphic curves without positive punctures can be excluded for all contact manifolds using the maximum principle, which shows that contact homology is well-defined for all contact manifolds, it can be seen from homological reasons that for mapping tori M_ϕ there cannot exist holomorphic curves in $\mathbb{R}\times M_\phi$ carrying just one type of punctures, which shows that in this case both contact homology and cylindrical contact homology are defined.

Similarly to what happens in Floer homology, the chain space for cylindrical homology C is defined be the vector space generated by the formal variables q_γ with coefficients which are formal power series in the $t^{\alpha,j}$-variables and Laurent series in the z_n-variables. Counting only holomorphic cylinders defines a differential $\partial : C_* \to C_*$ by

$$\partial q_{\gamma^+} = \kappa_{\gamma^+} \sum_{\gamma^-} \frac{\partial^2 \mathbf{h}}{\partial p_{\gamma^+}\partial q_{\gamma^-}}|_{p=q=0} \cdot q_{\gamma^-}$$

with

$$\frac{\partial^2 \mathbf{h}}{\partial p_{\gamma^+}\partial q_{\gamma^-}}|_{p=q=0} = \sum \frac{1}{r!\kappa_{\gamma^+}\kappa_{\gamma^-}} \int_{\overline{\mathcal{M}}_{r,A}^{(j_1,\ldots,j_r)}(\gamma^+,\gamma^-)} \mathrm{ev}_1^*\,\theta_{\alpha_1}\wedge\ldots\wedge\mathrm{ev}_r^*\,\theta_{\alpha_r}\; t^I z^A.$$

It follows from the master equation $\{\mathbf{h},\mathbf{h}\} = 0$ of rational SFT that $\partial\circ\partial = 0$ when there do not exist any holomorphic planes, so that one can define the cylindrical homology of the closed stable Hamiltonian manifold as the homology of the chain complex (C,∂). The sequence of commuting Hamiltonians $\mathbf{h}_{\alpha,p}$ in rational SFT gets now replaced by linear maps

$$\partial_{\alpha,p} = \frac{\partial}{\partial t^{\alpha,p}}\circ\partial : C_*\to C_*,\ \partial_{\alpha,p}q_{\gamma^+} = \kappa_{\gamma^+}\sum_{\gamma^-}\frac{\partial^3 \mathbf{h}}{\partial t^{\alpha,p}\partial p_{\gamma^+}\partial q_{\gamma^-}}|_{p=q=0}\cdot q_{\gamma^-},$$

which by the same arguments descend to maps on homology, $\partial_{\alpha,p} : H_*(C,\partial) \to H_*(C,\partial)$, and commute on homology, $[\partial_{\alpha,p},\partial_{\beta,q}]_- = 0$, with respect to the graded commutator $[f,g]_- = f\circ g - (-1)^{\deg(f)\deg(g)} g\circ f$.

While we have already shown how the well-known string, dilaton and divisor equations translate from Gromov–Witten theory to SFT, in this paper we want to proceed with our project of understanding how the rich algebraic structures from Gromov–Witten theory carry over to SFT. As the next step we want to show how classical genus-zero topological recursion generalizes to SFT. As we will outline in a forthcoming paper, it follows from the computation of the SFT of a Reeb orbit with descendants outlined above that the genus-zero topological recursion requires a non-equivariant version of SFT, which is generated by parameterized instead of unparameterized Reeb orbits.

The definition of this non-equivariant version of SFT is currently a very active field of research and related to the work of Bourgeois and Oancea in [**BO**], where a Gysin-type spectral sequence relating linearized contact homology (a slight generalization of cylindrical contact homology depending on a symplectic filling) and symplectic homology of this filling is established by viewing the one as the (non-)equivariant version of the other. On the other hand, since the topological recursion relations are already interesting in the case of cylindrical contact homology and the non-equivariant version of it is already understood, in this paper we will study the algebraic structure of gravitational descendants just for this special case first.

Remarks on transversality We end this section with a short discussion of the analytical foundations. As in [**BO**] and other papers on SFT, the algebraic results we prove rely on the fact that all appearing moduli spaces of holomorphic curves are (weighted branched) manifolds with corners of dimension equal to the Fredholm index of the Cauchy–Riemann operator. On the other hand, it is well-known that the required transversality result for the Cauchy–Riemann operator does not hold even for generic choices of almost complex structures due to the appearance of multiply covered curves.

While this problem is already present in Gromov–Witten theory and Floer homology, the polyfold approach of Hofer et al. [**HWZ**] seems to be the approach that solves all the challenges in the most satisfactory way, see also the first author's survey [**F3**]. Since, in contrast to relative SFT and other papers about SFT using Morse–Bott techniques, we only need to add the classical transversality result for sections in the finite-dimensional tautological line bundles, our results are indeed rigorous when Hofer and his collaborators have completed their work.

Apart from the fact that we follow other papers in the field and state our results as theorems for the general case, we remark that there are special but interesting cases where transversality is already established. For this observe that in the case when the stable Hamiltonian manifold is a symplectic mapping torus, the cylindrical contact homology just splits into the direct sum of the Floer homologies of powers of the underlying symplectomorphisms; see [**F1**]. It follows that one can directly apply the transversality result for symplectic Floer homology, where we again just will assume that the closed symplectic manifold is monotone in order to deal with bubbling of holomorphic spheres. Since we must allow the almost complex structure to depend on the circle coordinate on the holomorphic cylinder, note that the translation back to the SFT-picture leads to multi-valued cylindrical almost complex structures when the period of the asymptotic orbits is greater than one.

3. Topological recursion in non-equivariant cylindrical homology

3.1. Motivation from Floer homology. It is well known that there is a very close relation between Gromov–Witten theory and symplectic Floer theory. As a motivation for the topological recursion relations in (non-equivariant) cylindrical homology, which we will prove in the next section, we sketch in this subsection how the topological recursion relations in Gromov–Witten theory carry over to symplectic Floer homology using a Morse–Bott correspondence.

Assuming we have chosen a basis $A_0,\dots,A_N$ of $H_2(X)$ and a string of closed (homogeneous) differential forms $\Theta=(\theta_1,\dots,\theta_N)$ on X, to which we assign graded formal variable z_i and $t^{\alpha,j}$ with grading $|z_i|=-2c_1(A_i)$, $|t^{\alpha,j}|=2-2j-\deg\theta_\alpha$, recall that the rational descendant potential of Gromov–Witten theory $\mathbf{f}$ is defined as

$$\mathbf{f}=\sum_I\frac{1}{r!}\int_{\overline{\mathcal{M}}^{(j_1,\dots,j_r)}_{r,A}(X)}\mathrm{ev}_1^*\,\theta_{\alpha_1}\wedge\cdots\wedge\mathrm{ev}_r^*\,\theta_{\alpha_r}\;t^Iz^A,$$

where $t^I=t^{\alpha_1,j_1}\cdots t^{\alpha_r,j_r}$ and $z^A=z_0^{d_0}\cdot z_M^{d_M}$ for $A=d_0A_0+\cdots+d_MA_M$. Here $\overline{\mathcal{M}}^{(j_1,\dots,j_r)}_{r,A}(X)\subset\overline{\mathcal{M}}_{r,A}(X)$ denotes the corresponding zero divisor inside the moduli space of closed J-holomorphic curves $u:(S^2,i)\to(X,J)$, which is Poincaré dual to the product of psi-classes $\psi_1^{j_1}\wedge\cdots\wedge\psi_r^{j_r}$.

Topological recursion relations are differential equations for the descendant potential $\mathbf{f}$ which are proven using the geometric meaning of gravitational descendants, as for the string, dilaton and divisor equations. Fix $1\le i\le r$ and choose $1\le j,k\le r$ such that i,j,k are pairwise different. While the string, dilaton and divisor equations are proven (see also [**FR**]) by studying the behaviour of the tautological line bundle under the natural map $\overline{\mathcal{M}}_r(X)\to\overline{\mathcal{M}}_{r-1}(X)$, the topological recursion relations follow by studying the behaviour of the tautological line bundle under the natural map $\overline{\mathcal{M}}_r(X)\to\overline{\mathcal{M}}_3=\{\text{point}\}$, where the map and all marked points except i,j,k are forgotten. Now it is a standard result in Gromov–Witten theory that one can construct a special non-generic section in the tautological line bundle $\mathcal{L}_{i,r}$ over $\overline{\mathcal{M}}_r(X)$ such that the zero divisor $\overline{\mathcal{M}}_r^{(0,\dots,0,1,0,\dots,0)}(X)$ agrees with the divisor $\overline{\mathcal{M}}_r^{i,(j,k)}(X)$ of holomorphic spheres with one node, where the i.th marked point lies on one component and the jth and the kth fixed marked points lie on the other component.

Translating the above localization result for descendants into a differential equation for the descendant potential $\mathbf{f}$ we get that he descendant potential $\mathbf{f}$ of rational Gromov–Witten theory satisfies the topological recursion relations. Note that this above localization result for descendants is not symmetric with respect to permutation of marked points. Instead of writing down these relations in their usual form, we want to emphasize that the above localization result also leads to the following averaged version of the usual topological recursion relations,

$$N(N-1)\frac{\partial \mathbf{f}}{\partial t^{\alpha,i}} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu,0}}\,\eta^{\mu\nu}N(N-1)\frac{\partial \mathbf{f}}{\partial t^{\nu,0}},$$

where $N := \sum_{\beta,j} t^{\beta,j}\frac{\partial}{\partial t^{\beta,j}}$ is the differential operator which counts the number of marked points.

Recall that the goal of this section is to translate the above topological recursion relations from Gromov–Witten theory to symplectic Floer theory. In order to do so, we recall in this subsection the well-known relation between Gromov–Witten theory and symplectic Floer theory.

First recall that the Floer cohomology $HF^* = HF^*(H)$ of a time-dependent Hamiltonian $H : S^1 \times X \to \mathbb{R}$ is defined as the homology of the chain complex (CF^*, ∂), where CF^* is the vector space freely generated by the formal variables q_γ assigned to all one-periodic orbits of H with coefficients which are Laurent series in the variables z_n. On the other hand, the differential $\partial : CF^* \to CF^*$ is given by counting elements in the moduli spaces $\overline{\mathcal{M}}(\gamma^+, \gamma^-)$ of cylinders $u : \mathbb{R} \times S^1 \to X$ satisfying the perturbed Cauchy–Riemann equation $\bar{\partial}_{J,H}(u) = \partial_s u + J_t(u)(\partial_t u - X^H_t(u)) = 0$ with a one-periodic family of almost complex structures J_t and where X^H_t denotes the symplectic gradient of H_t, and which converge to the one-periodic orbits $\gamma^\pm$ as $s \to \pm\infty$, $u(s,\cdot) \to \gamma^\pm$. In the same way as the group of Moebius transforms acts on the solution space of Gromov–Witten theory and the moduli space is defined only after dividing out this obvious symmetries, $\mathbb{R}$ acts on the above space of Floer cylinders by translations in the domain, so that the moduli space is again defined after dividing out this natural action. On the other hand, since the Hamiltonian and the almost complex structure depends on the S^1-coordinate, it will become important that we do not divide out the action of the circle.

In order to prove the Arnold conjecture about the number of one-periodic orbits of H one shows that the Floer cohomology groups are isomorphic to the quantum cohomology groups $QH^*(X)$ of the underlying symplectic manifold. One way to prove the above isomorphism is by studying the behaviour of the moduli spaces of Floer cylinders as the Hamiltonian H converges to zero. In the limit $H = 0$ the removable singularity theorem for (unperturbed) holomorphic curves that in the limit the moduli spaces of Floer trajectories $\overline{\mathcal{M}}(\gamma^+,\gamma^-)$ are replaced by the moduli spaces of holomorphic spheres $\overline{\mathcal{M}}_{0,2}(X)$ with two marked points from Gromov–Witten theory. On the other hand, note that there is a natural product structure on quantum cohomology, the so-called quantum cup product, given by counting holomorphic spheres with three marked points. In [**PSS**] it was already shown that one can define the corresponding action of $QH^*(X)$ on the Floer cohomology groups $HF^*(H)$ by counting Floer cylinders $u:\mathbb{R}\times S^1\to X$, $\bar{\partial}_{J,H}(u)=0$ with an additional marked point with fixed position $(0,0)\in\mathbb{R}\times S^1$, as in the description of the moduli spaces of the quantum product.

Note that on the quantum cohomology side we can either assume that also the third marked point is fixed or it varies and we divide out the symmetry group $\mathbb{R}\times S^1$ of the two-punctured sphere afterwards. On the other hand, since in the Floer case we now only divide by $\mathbb{R}$ and not by $\mathbb{R}\times S^1$ on the domain, it follows that in the second case with varying position the third marked point must still be constrained to lie on some ray $\mathbb{R}\times\{t_0\}\subset\mathbb{R}\times S^1$, where without loss of generality we can assume that $t_0=0$. Keeping the picture of points with varying positions of marked points as in Gromov–Witten theory and SFT we hence have the following

Proposition 3.1. *With respect to the above Morse–Bott correspondence, counting holomorphic spheres with three or more marked points in Gromov–Witten theory corresponds on the Floer side to counting Floer cylinders with additional marked points, where only the first marked point is constrained to* $\mathbb{R}\times\{0\}\subset\mathbb{R}\times S^1$.

Before we use the above translation scheme from Gromov–Witten theory to symplectic Floer theory to transfer the topological recursion relations from Gromov–Witten theory to symplectic Floer theory, we first enrich the Floer complex using descendants as we did for cylindrical contact homology,

$$\partial(q_{\gamma^+})=\sum\frac{1}{r!}\int_{\overline{\mathcal{M}}^{(j_1,\dots,j_r)}_{r,A}(\gamma^+,\gamma^-)}\mathrm{ev}_1^*\,\theta_{\alpha_1}\wedge\cdots\wedge\mathrm{ev}_r^*\,\theta_{\alpha_r}\;t^I q_{\gamma^-}z^A,$$

where $\overline{\mathcal{M}}^{(j_1,\dots,j_r)}_{r,A}(\gamma^+,\gamma^-)\subset\overline{\mathcal{M}}_r(\gamma^+,\gamma^-)$ now denotes the corresponding zero divisors inside the moduli space of Floer trajectories with r marked points. We again define $\partial_{(\alpha,i)}:=\frac{\partial}{\partial t^{\alpha,i}}\circ\partial$. As we have seen in the last subsection we further need to include cylinders with one constrained (to $\mathbb{R}\times\{0\}$) marked point. In order to distinguish these new linear maps from the linear maps $\partial_{\alpha,p}$ obtained by counting holomorphic cylinders with one *un*constrained marked point, we denote them by $\partial_{\check{\alpha},p}:CF^*\to CF^*$. In the same as for $\partial_{\alpha,p}$ it can be shown that $\partial_{\check{\alpha},p}$ descends to a linear map on homology, and commutes on homology, $[\partial_{\check{\alpha},p},\partial_{\check{\beta},q}]_-=0$, with respect to the graded commutator $[f,g]_-=f\circ g-(-1)^{\deg(f)\deg(g)}g\circ f$

With this we can formulate our proposition about topological recursion in symplectic Floer theory as follows. In contrast to Gromov–Witten theory, we now obtain three different equations, depending on whether we remember both punctures, one puncture and one additional marked point or two marked points. On the other hand, inside their class we want to treat the additional marked point in a symmetric way as in the above averaged version of topological recursion relations in Gromov–Witten theory. Note that this is needed for coherence as we will show in the proof of the corresponding relations for non-equivariant cylindrical contact homology.

Proposition 3.2. *With respect to the above Morse–Bott correspondence, the topological recursion relations from Gromov–Witten theory have the following translation to symplectic Floer theory: for three* different *non-generic special choices of coherent sections we have*

(2,0):

$$\partial_{(\check{\alpha},i)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}\partial_{\check{\nu}}$$

(1,1):

$$N\,\partial_{(\check{\alpha},i)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}N\,\partial_{\check{\nu}} + \frac{1}{2}[\partial_{(\check{\alpha},i-1)},\check{N}\partial]_+$$

(0,2):

$$(N-1)N\partial_{(\check{\alpha},i)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}(N-1)N\partial_{\check{\nu}} + [\partial_{(\check{\alpha},i-1)},(N-1)\check{N}\partial]_+$$

where $N := \sum_{\beta,j} t^{\beta,j}\frac{\partial}{\partial t^{\beta,j}}$, $\check{N}\partial := \sum_{\beta,j} t^{\beta,j}\partial_{\check{\beta},j}$ *and* $[f,g]_+ = f\circ g + (-1)^{\deg(f)deg(g)} g\circ f$ *denotes a graded anti-commutator with respect to the operator composition. Notice that, since the two entries in the bracket are having even degree, in the above formulas the anti-commutator always corresponds to a sum.*

Proof. In order to translate the localization result of Gromov–Witten theory to symplectic Floer theory, we first replace the holomorphic sphere with three or more marked points by a Floer cylinder with one marked point constrained to $\mathbb{R}\times\{0\}$ and possibly other unconstrained additional marked points, where we assume that the constrained marked point agrees with the ith marked point carrying the descendant. In order to obtain the three different equations we have to decide whether the jth and kth marked point agree with the positive or negative puncture or some other additional marked point and then use the localization theorem from the first subsection, which states that the zero divisor localizes on nodal spheres with two smooth components, where the ith marked point lies on one component and the jth and the kth marked point lie on the other component.

In order to obtain equation (2,0) we remember both punctures ($j = +, k = -$). While for each holomorphic curve in the corresponding divisor $\overline{\mathcal{M}}_r^{i,(+,-)}(\gamma^+,\gamma^-) \subset \overline{\mathcal{M}}_r(\gamma^+,\gamma^-)$ the component with the jth and the kth marked point is a Floer cylinder, the other component with the ith marked point is a sphere, since both components are still connected by a node (and not a puncture) by the compactness theorem in Floer theory.

On the other hand, in order to obtain equation (1,1), we remember one of the two punctures, $j = +$ (or $j = -$) and another marked point. While for each holomorphic curve in $\overline{\mathcal{M}}_r^{i,(+,k)}(\gamma^+,\gamma^-) \subset \overline{\mathcal{M}}_r(\gamma^+,\gamma^-)$ the component carrying the jth and the kth marked point still needs to be a Floer cylinder and not a holomorphic sphere as the jth marked point is a puncture, the other component carrying the ith marked point can either be a holomorphic sphere or Floer cylinder, depending on whether both components are connected by a node or a puncture. Note that in the second case this connecting puncture is necessarily the negative puncture for the Floer cylinder with the ith marked point and the positive puncture for the Floer cylinder with the kth marked point. On the other hand, both Floer cylinder carry a special marked point, namely the ith or the kth marked point, respectively, which by the above Morse–Bott correspondence are constrained to $\mathbb{R} \times \{0\}$.

Finally, in order to establish equation (0,2), we remember none of the two punctures. Since only Floer cylinders and holomorphic spheres appear in the compactification, it follows that for the above equation we must just sum over all choices for both components being either a cylinder or a sphere and again use the above Morse–Bott correspondence. □

Note that in order to make the above proof precise, one needs to rigorously establish an isomorphism between Gromov–Witten theory and symplectic Floer theory beyond the isomorphism between quantum cohomology and Floer cohomology groups together with the action of the quantum cohomology on them proven in [**PSS**] involving the full 'infinity structures'. On the other hand, while we expect that the Morse–Bott picture from above should lead to such an isomorphism in an obvious way, we are satisfied with the level of rigor, since it should just serve as a motivation for our topological recursion result in non-equivariant cylindrical contact homology. Note that our rigorous proof for that case will in turn directly lead to a rigorous proof of this proposition.

3.2. Non-equivariant cylindrical homology. Motivated by the topological recursion result in symplectic Floer homology discussed in the last subsection, we want to prove the corresponding topological recursion result for cylindrical contact homology. Since, in contrast to Floer homology, the closed orbits in cylindrical contact homology are not parameterized by S^1, it turns out that we need to work with a non-S^1-equivariant version of cylindrical contact homology, where we follow the ideas in Bourgeois–Oancea's paper [**BO**]. Note that in the contact case our results indeed immediately generalize from cylindrical contact homology to linearized contact homology which depends on a symplectic filling and is defined for any fillable contact manifold, e.g., since it is still isomorphic to the positive symplectic homology which only counts true cylinders.

Here the key observation is that a closed (unparameterized) Reeb orbit γ defines a S^1-family S_γ of parametrized Reeb orbits. In particular, the set of parameterized orbits is no longer discrete but is a disjoint union of circles. Following [**BO**] we introduce Morse–Bott moduli spaces $\mathcal{M}^0_{r,A}(S_{\gamma^+},S_{\gamma^-})$ of parameterized holomorphic cylinders with r additional marked points $(u,(z_i))$, where $u:\mathbb{R}\times S^1\to\mathbb{R}\times V$ satisfies $\bar\partial_J(u)=0$, $[u]=A\in H_2(V)$ and $u(s,\cdot)\to\gamma'\in S_{\gamma^\pm}$ as $s\to\pm\infty$. As in Floer homology and in contrast to the equivariant case, we now do *not* divide by the S^1-action on the domain $\mathbb{R}\times S^1$, but only divide out the action of $\mathbb{R}$ in the domain and in the target to obtain the moduli space $\mathcal{M}_{r,A}(S_{\gamma^+},S_{\gamma^-})$. Furthermore this moduli space can again be compactified as in Floer homology to obtain the compact moduli space $\overline{\mathcal{M}}_{r,A}(S_{\gamma^+},S_{\gamma^-})$ by adding multi-floor curves.

Since in the definition of the moduli space of parameterized cylinders $\mathcal{M}^0_{r,A}(\gamma^+,\gamma^-)$ in equivariant cylindrical homology we did not fix a parametrization of the asymptotic Reeb orbits, it follows immediately that $\mathcal{M}^0_{r,A}(\gamma^+,\gamma^-)$ agrees with $\mathcal{M}^0_{r,A}(S_{\gamma^+},S_{\gamma^-})$ (note that for the sake of simplicity we want to ignore the combinatorial factors for coherent orientations here). On the other hand, since in the definition of $\mathcal{M}_{r,A}(\gamma^+,\gamma^-)$ we divided out the S^1-action on the cylinder while for $\mathcal{M}_{r,A}(S_{\gamma^+},S_{\gamma^-})$ we did not, we get that $\mathcal{M}_{r,A}(S_{\gamma^+},S_{\gamma^-})$ is a trivial circle bundle over $\mathcal{M}_{r,A}(\gamma^+,\gamma^-)$. Since cylinders in $\mathcal{M}_{r,A}(S_{\gamma^+},S_{\gamma^-})$ approach Reeb orbits with a fixed parametrization, observe that we have additional evaluation maps

$$\mathrm{ev}_\pm:\mathcal{M}_{r,A}(S_{\gamma^+},S_{\gamma^-})\to S_{\gamma^\pm}\cong\gamma^\pm\cong S^1,$$

which also extend to the compactified moduli space.

Assigning to every closed Reeb orbit γ two new formal variables $\hat q_\gamma$, $\check q_\gamma$, we can write a general element in $\Omega^*(S_\gamma)$ as $q_\gamma=\hat q_{\gamma^+}\cdot d\phi+\check q_{\gamma^+}\cdot 1$. With this the chain space for non-equivariant cylindrical contact homology $HC^{\text{non-}S^1}_*(V)$ is the vector space generated by the formal variables $\hat q_\gamma$ and $\check q_\gamma$ with coefficients which are formal power series in the $t^{\alpha,j}$-variables and Laurent series in the z_n-variables. Note that the chain space naturally splits,

$$C^{\text{non-}S^1}_*=\hat C_*\oplus\check C_*,$$

where $\hat C_*$, $\check C_*$ are generated by the formal variables $\hat q_\gamma$, $\check q_\gamma$, respectively. Following [**BO**] the Morse–Bott differential $\partial:\hat C_*\oplus\check C_*\to\hat C_*\oplus\check C_*$ for non-equivariant cylindrical homology is then given by

$$\begin{aligned}\partial\hat{q}_{\gamma^+} &= \sum \frac{1}{r!}\int_{\overline{\mathcal{M}}^{(j_1,\dots,j_r)}_{r,A}(S_{\gamma^+},S_{\gamma^-})} \mathrm{ev}_1^*\,\theta_{\alpha_1}\wedge\cdots\wedge \mathrm{ev}_r^*\,\theta_{\alpha_r}\wedge \mathrm{ev}_+^*\,1\\ &\quad\wedge \mathrm{ev}_-^*\,q_{\gamma^-}\;t^I z^A,\\ \partial\check{q}_{\gamma^+} &= \sum \frac{1}{r!}\int_{\overline{\mathcal{M}}^{(j_1,\dots,j_r)}_{r,A}(S_{\gamma^+},S_{\gamma^-})} \mathrm{ev}_1^*\,\theta_{\alpha_1}\wedge\cdots\wedge \mathrm{ev}_r^*\,\theta_{\alpha_r}\wedge \mathrm{ev}_+^*\,d\phi\\ &\quad\wedge \mathrm{ev}_-^*\,q_{\gamma^-}\;t^I z^A,\end{aligned}$$

and we define as in Floer homology,

$$\partial_{(\alpha,i)} := \frac{\partial}{\partial t^{\alpha,i}}\circ\partial : \hat{C}_*\oplus\check{C}_*\to\hat{C}_*\oplus\check{C}_*.$$

Note that, as in Floer homology, the multiplicity of the orbit does not enter appear as a combinatorial factor, but enters after identifying the Morse–Bott moduli space $\overline{\mathcal{M}}(S_{\gamma^+},S_{\gamma^-})$ with the SFT moduli space $\overline{\mathcal{M}}(\gamma^+,\gamma^-)$ (where we ignored signs for simplicity).

Furthermore as for Floer homology we further need to include cylinders with one constrained (to $\mathbb{R}\times\{0\}$) marked point. For this observe that as in Floer homology we can also enrich the evaluation map for an additional marked points by not only remembering their image in the target manifold but also their S^1-coordinate on the cylinder,

$$(\mathrm{ev},\pi) : \overline{\mathcal{M}}_{r,A}(S_{\gamma^+},S_{\gamma^-})\to V^r\times(S^1)^r.$$

On the other hand, instead of integrating the pull-back form $\pi_{r+1}^*d\theta$, $d\theta\in\Omega^1(S^1)$ over the moduli space $\overline{\mathcal{M}}_{r+1,A}(S_{\gamma^+},S_{\gamma^-})$, we can equivalently consider the moduli space $\overline{\mathcal{M}}_{r,1,A}(S_{\gamma^+},S_{\gamma^-})\subset\overline{\mathcal{M}}_{r+1,A}(S_{\gamma^+},S_{\gamma^-})$ of holomorphic cylinders with $r+1$ additional marked points, where the $r+1$-st additional marked point is constrained to lie on the ray $\mathbb{R}\times\{0\}\subset\mathbb{R}\times S^1$. With this we again define new linear maps $\partial_{\check{\alpha},p}:\hat{C}_*\oplus\check{C}_*\to\hat{C}_*\oplus\check{C}_*$, where

$$\begin{aligned}&\partial_{\check{\alpha},p}q^*_{\gamma^+}\\ &=\frac{1}{r!}\int_{\overline{\mathcal{M}}^{(j_1,\dots,j_r,p)}_{r,1,A}(S_{\gamma^+},S_{\gamma^-})} \mathrm{ev}_1^*\,\theta_{\alpha_1}\wedge\cdots\wedge \mathrm{ev}_r^*\,\theta_{\alpha_r}\\ &\wedge \mathrm{ev}_{r+1}^*\,\theta_\alpha\wedge \mathrm{ev}_+^*\,\theta_+^*\wedge \mathrm{ev}_-^*\,q_{\gamma^-}\;t^I z^A.\end{aligned}$$

In the same way as for $\partial_{\alpha,p}$ it can be shown that $\partial_{\check{\alpha},p}$ descends to a linear map on non-equivariant cylindrical homology.

The significance of the moduli space $\mathcal{M}_{r,1,A}(S_{\gamma^+},S_{\gamma^-})$ and hence of the linear map $\partial_{\check{\alpha},p}$ for our desired localization result for descendants is as follows: While the moduli space $\mathcal{M}_{r+1,A}(S_{\gamma^+},S_{\gamma^-})$ is a trivial circle bundle over the usual SFT moduli space $\mathcal{M}_{r+1,A}(\gamma^+,\gamma^-)$, the submoduli space

$\mathcal{M}_{r,1,A}(S_{\gamma^+}, S_{\gamma^-}) \subset \mathcal{M}_{r+1,A}(S_{\gamma^+}, S_{\gamma^-})$ can be identified with the equivariant moduli space $\mathcal{M}_{r+1,A}(\gamma^+, \gamma^-)$ by determining the missing S^1-coordinate by the requirement that the $r+1$st marked point has to be constrained to $\mathbb{R}\times\{0\} \subset \mathbb{R}\times S^1$. On the other hand, as a subset of the non-equivariant moduli space $\mathcal{M}_{r+1,A}(S_{\gamma^+}, S_{\gamma^-})$, the moduli space $\mathcal{M}_{r,1,A}(S_{\gamma^+}, S_{\gamma^-})$ still has the two evaluation maps $\mathrm{ev}_\pm : \mathcal{M}_{r,1,A}(S_{\gamma^+}, S_{\gamma^-}) \to S_{\gamma^\pm} \cong S^1$, which we will use to determine a codimension-two locus in the codimension-one boundary of the moduli space $\mathcal{M}_{r,1,A}(S_{\gamma^+}, S_{\gamma^-}) \cong \mathcal{M}_{r+1,A}(\gamma^+, \gamma^-)$.

3.3. Topological recursion in non-equivariant cylindrical homology. With the reasonable assumption in mind that the topological recursion relations in Floer homology also hold true in (positive) symplectic homology and hence also carry over to non-equivariant cylindrical contact homology, we now formulate our main theorem. Since we assumed that there are no holomorphic planes in $\mathbb{R}\times V$ and hence the usual Gromov compactness result holds we define the Gromov–Witten potential $\mathbf{f}$ of a stable Hamiltonian manifold as the part of the rational SFT Hamiltonian $\mathbf{h}$ of V counting holomorphic spheres without punctures, $\mathbf{f} = \mathbf{h}|_{p=0=q}$. Note that in the contact case this agrees with the Gromov–Witten potential of a point due to the maximum principle and is determined by the Gromov–Witten potential of the symplectic fibre in the case when the stable Hamiltonian manifold is a symplectic mapping torus as every holomorphic map $\mathbb{CP}^1 \to \mathbb{R}\times M_\phi \to \mathbb{R}\times S^1 \cong \mathbb{C}^*$ is constant by Liouville's theorem.

Theorem 3.3. *For three* different *non-generic special choices of coherent sections the following three* topological recursion relations *hold in* non-equivariant *cylindrical contact homology*

(2,0):

$$\partial_{(\check{\alpha},i)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}\partial_{\check{\nu}}$$

(1,1):

$$N\,\partial_{(\check{\alpha},i)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}N\,\partial_{\check{\nu}} + \frac{1}{2}[\partial_{(\check{\alpha},i-1)}, \check{N}\partial]_+$$

(0,2):

$$(N-1)N\partial_{(\check{\alpha},i)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}(N-1)N\partial_{\check{\nu}} + [\partial_{(\check{\alpha},i-1)}, (N-1)\check{N}\partial]_+$$

where $N := \sum_{\beta,j} t^{\beta,j}\frac{\partial}{\partial t^{\beta,j}}$, $\check{N}\partial := \sum_{\beta,j} t^{\beta,j}\partial_{\check{\beta},j}$ *and* $[f,g]_+ = f\circ g + (-1)^{\deg(f)deg(g)}g\circ f$ *denotes a graded anti-commutator with respect to the operator composition. Note that, since the the two entries in the bracket are having even degree, in the above formulas the anti-commutator always corresponds to a sum.*

Indeed we claim that the three topological recursion relations only hold for three *special* coherent collections of sections and do not hold true for general choices. In particular, in general there does *not* exist a choice of coherent sections such that all three equations hold true *simultaneously*. Nevertheless we will show in Subsection 3.2 how we can generalize (and reprove) a well-known result from [**PSS**] by comparing two of the above topological recursion relations ((2,0) and (1,1)). Note that it is already known from [**PSS**] that the corresponding identity does not hold on the chain level, and we instead resolve the above invariance problem by proving an identity on homology. On the other hand, for the first equation we can solve the invariance problem directly.

Remark 3.4. Note that the first topological relation actually descends to homology, i.e., it holds for $\partial_{(\check{\alpha},i)}$ and $\partial_{\check{\nu}}$ *viewed as linear maps on non-equivariant cylindrical homology* $HC_*^{\text{non-}S^1}(V)$. In particular, while on the chain level all topological recursion relations only hold true for (three different) *special* choices of coherent sections, *after* passing to homology the first relation (2,0) holds for *all* coherent sections, i.e., is true for all auxiliary choices.

As in [**FR**], we furthermore need to make the following comment on the genericity of our choices.

While in the above theorem we make use of special choices of coherent collections of sections as in our proof in [**FR**] of the SFT analogues of the string, dilaton and divisor equations, recall that, for the definition of gravitational descendants in [**F2**], we need to choose sections in the tautological bundles over all moduli spaces which are generic in the sense that they are transversal to zero section, so that, in particular, all zero divisors are smooth. On the other hand, as we will see below, all our special choices of coherent collections of sections used in the proof are automatically non-generic, since, after the limit procedure described below, their zero sets localize on nodal curves and, in particular, are not smooth. In order to see that we can still use our special non-generic choices for computations, we have to use the fact that, by making small perturbations, the special non-generic choice of coherent collections of sections can be approximated arbitrarily closely (in the C^1-sense) by generic coherent collections of sections. While for two different coherent collections of sections the linear map in general depends on these choices, since for a given homotopy (coherent collection of sections coherently connecting the two different choices in the sense of [**F2**]) zeroes may run out of the codimension-one boundaries of the moduli spaces, we can further make use of the fact that the latter can be prevented from happening as long as the perturbation is small enough, as described in the following picture (we refer to [**FR**] for further details).

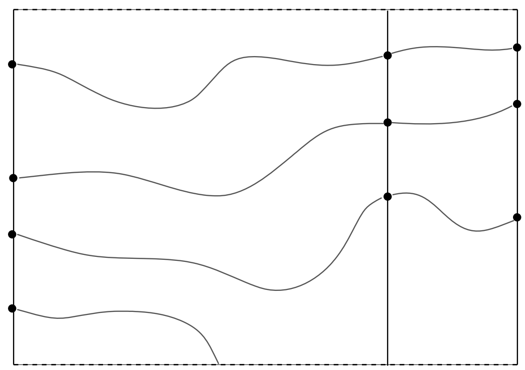


Figure 1. The picture represents the trivial cobordism between a moduli space and itself (vertical black lines) and the corresponding cobordism for the zeros of coherent sections (green lines). The number of zeroes (black dots) in each copy of the moduli space may change during a homotopy (from left to right) as zeroes may run out of the codimension-one-boundary (dashed lines above and below). This, however, can be excluded as long as the homotopy is chosen sufficiently small (like the one between the middle and the right vertical lines).

3.4. Proof of the main theorem. In this section we establish the psi-class localization result needed to prove Theorem 3.3, but also Proposition 3.2. In fact we will consider a more general situation, i.e., tautological bundles and coherent collections of sections in the general moduli space of SFT holomorphic curves. This will apply to non-equivariant cylindrical contact homology by the identification between the moduli space relevant for defining the differential $\partial_{(\check{\alpha},i)}$ and the moduli space of SFT curves with one positive and one negative puncture, and one special marked point constrained on a geodesic connecting the positive and negative puncture (besides carrying the ith descendant of $\theta_\alpha \in H^*(V)$). We will show how coherent sections of tautological bundles on the moduli space of SFT-curves can be chosen such that their zero locus localizes on nodal configurations and boundary strata (multi-level curves). In the case of curves with one positive and one negative punctures, the presence of a marked point constrained to the geodesic described above will be used to identify explicitly such zero locus inside the boundary.

The localization will be the analogue, in presence of coherence conditions, of the usual result in Gromov–Witten theory describing the divisor $\psi_{i,r} = c_1(\mathcal{L}_{i,r})$ on $\overline{\mathcal{M}}_{0,r,A}(X)$ as the locus of nodal curves where the ith marked point lies on a different component with respect to a pair of other reference marked points. We will divide the discussion in three parts, corresponding to the three kind of topological recursion relations $(2,0)$, $(1,1)$ and $(0,2)$.

For the moment we stay general and consider the full SFT moduli space of curves with any number of punctures and marked points in a general manifold with stable Hamiltonian structure. In particular we will describe a special class of coherent collections of sections $s_{i,r}$ for the tautological bundles $\mathcal{L}_{i,r}$ on the moduli spaces $\overline{\mathcal{M}}_{0,r,A}(\Gamma^+,\Gamma^-)$. We will then consider a sequence inside such class converging to a (no longer coherent) collection of sections whose zeros will completely be contained in the boundary strata (both nodal and multi-floor curves) of $\overline{\mathcal{M}}_{0,r,A}(\Gamma^+,\Gamma^-)$.

In [**FR**] we already explained how to choose a (non-generic) coherent collection of sections $s_{i,r}$ in such a way that, considering the projection $\pi_r : \overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-) \to \overline{\mathcal{M}}_{g,r-1,A}(\Gamma^+,\Gamma^-)$ consisting in forgetting the r-th marked point, the following *comparison formula* holds for their zero sets:

$$s_{i,r}^{-1}(0) = \pi_r^{-1}(s_{i,r-1}^{-1}(0)) + D_{i,r}^{\mathrm{const}}, \tag{3.1}$$

The sum in the right hand side means union with the submanifold $D_{i,r}^{\mathrm{const}}$ of nodal curves with a constant sphere bubble carrying the ith and rth marked points, transversally intersecting $\pi_r^{-1}(s_{i,r-1}^{-1}(0))$.

We wish to stress the fact that such choice is possible because any codimension 1 boundary of the moduli space $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ decomposes into a product of moduli spaces where the factor containing the ith marked point carries the same well defined projection map π_r. This is because codimension 1 boundary strata are always formed by non-constant maps, which remain stable after forgetting a marked point.

In fact coherence also requires that our choice of coherent collection of sections is symmetric with respect to permutations of the marked points (other than the ith, carrying the descendant). We can reiterate this procedure until we forget all the marked points but the i-th, getting easily

$$s_{i,r}^{-1}(0) = (\pi_1^* \circ \ldots \circ \hat{\pi}_i^* \circ \ldots \circ \pi_r^* \, s_{i,1})^{-1}(0) + \sum_{\substack{I \sqcup J = \{1,\ldots,r\} \\ \{i\} \subsetneq I \subseteq \{1,\ldots,r\}}} D_{(I|J)}^{\mathrm{const}} \tag{3.2}$$

where $D_{(I|J)}^{\mathrm{const}}$ is the submanifold of nodal curves with a constant sphere bubble carrying the marked points labelled by indices in I. Such choice of coherent sections is indeed symmetric with respect to permutation of the marked points.

However, forgetting all of the marked points is not what we want to do in general, so we may take another approach, that does not specify whether the points we are forgetting are marked points or punctures. Forgetting punctures only makes sense after forgetting the map too.

Indeed, consider the projection $\sigma : \overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-) \to \overline{\mathcal{M}}_{g,r+|\Gamma^+|+|\Gamma^-|}$ to the Deligne–Mumford moduli space of stable curves consisting in forgetting the holomorphic map and asymptotic markers, consequently stabilizing the curve, and considering punctures just as marked points. For simplicity, denote $n = r + |\Gamma^+| + |\Gamma^-|$. The tautological bundle $\mathcal{L}_{i,r}$ on $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ coincides, by definition, with the pull-back along σ of the tautological bundle $\mathcal{L}_{i,n}$ on $\overline{\mathcal{M}}_{g,n}$ away from the boundary stratum $D_i \subset \overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ of nodal curves with a (possibly non-constant) bubble carrying the i-th marked point alone and the boundary stratum $D_i' \subset \overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ of multi-level curves with a level consisting in a holomorphic disk bounded by a Reeb orbit and carrying the i-th marked point.

At this point we are going to make the following assumption, which will hold throughout the paper.

Assumption: In $V \times \mathbb{R}$ there is no holomorphic disk bounded by a Reeb orbit. This implies, in particular, $D_i' = \emptyset$.

We choose now a coherent collection of sections $\tilde{s}_{i,n}$ on the Deligne–Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. The definition of such coherent collection is the same as for the space of maps, but this time we impose coherence on each real-codimension 2 divisor of nodal curves (as opposed to the case of maps, where we only imposed coherence at codimension 1 boundary strata). Such a coherent collection pulls back to a coherent collection of sections on $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ away from the already considered boundary stratum D_i (the only one still present after the above assumption), where we use the bundle map induced by a local coordinate on the underlying curve to identify the bundles $\mathcal{L}_{i,r}$ and $\sigma^*\mathcal{L}_{i,n}$ on $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-) \setminus D_i$. Such map is a bundle isomorphism on $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-) \setminus D_i$ and becomes singular on D_i: the image of $\sigma^*\tilde{s}_{i,n}$ under this map extends to the whole $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ assuming the value zero on D_i. The zero appearing this way along D_i has degree 1 by construction. This way we get a coherent collection of sections on the full $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$.

Once more, such construction is possible because any codimension 1 boundary of the moduli space $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ decomposes into a product of moduli spaces where the factor containing the i-th marked point carries the same well defined projection map σ. This is because codimension 1 boundary strata are always formed by multi-level curves, each level carrying at least two punctures (by the above assumption) and the ith marked point, hence remaining stable after forgetting the map.

We then get, on $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$,

$$s_{i,r}^{-1}(0) = (\sigma^*\,\tilde{s}_{i,n})^{-1}(0) + D_i. \tag{3.3}$$

This construction of a coherent collection of sections for $\overline{\mathcal{M}}_{r,A}(\Gamma^+,\Gamma^-)$ moves the problem of explicitly describing their zero locus to the more tractable space of curves $\overline{\mathcal{M}}_{g,n}$. Note first of all that, since $\overline{\mathcal{M}}_{g,n}$ has no (codimension 1) boundary, any generic choice (coherent or not) of sections for the tautological bundles will give rise to a zero loci with the same homology class. However, when we pull-back such section via σ, we want to remember more than just the homology class of its zero (as required by coherence), so we need to make some specific choice.

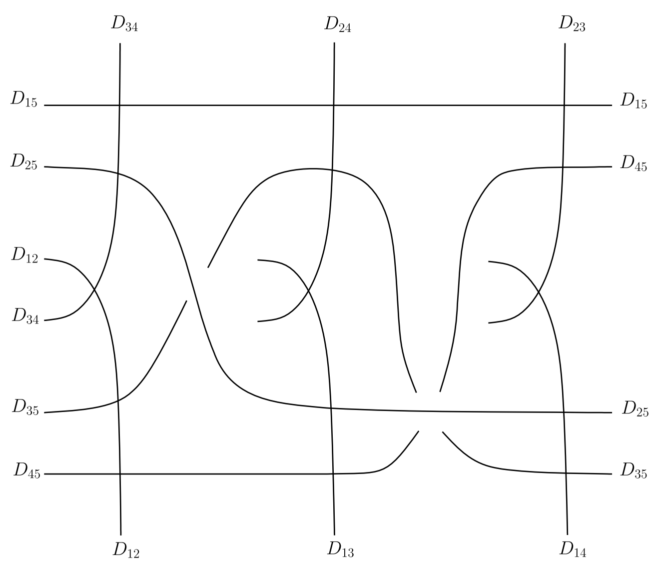
Let us now restrict ourselves to genus 0. In Gromov–Witten theory, where there is no need for coherence conditions, we are used to select two marked points besides the one carrying the psi-class and successively forget all the other ones until we drop on $\overline{\mathcal{M}}_{0,3} = \mathrm{pt}$, where the tautological bundle $\mathcal{L}_{i,3}$ is trivial. This approach is not possible in our SFT context where we require coherence on $\overline{\mathcal{M}}_{g,n}$. Indeed, if we select two punctures labelled by j and k, we automatically lose the required symmetry with respect to permutation of the marked points. To overcome this problem we need to use coherent collections of multi-sections (whose image is a branched manifold) in order to average over all the possible choices of a pair of punctures. In fact, we will only deal with multi-sections that are averages of ordinary sections of the tautological bundles, i.e., whose image is the union of images of ordinary sections each of them carrying a (rational) weight whose sum is one.

Let us choose an averaged (over all the possible ways of choosing two marked points out of n) multi-section for $\mathcal{L}_{i,n}$ on $\overline{\mathcal{M}}_{0,n}$ such that its zero locus has the (averaged) for

$$s_{i,n}^{-1}(0) = \frac{(n-3)!}{(n-1)!} \sum_{\substack{2\leq k\leq n-2 \\ I\sqcup J=\{1,\dots,n\} \\ i\in I, |J|=k}} \frac{k!}{(k-2)!} D_{(I|J)} \tag{3.4}$$

This formula is just the average (in the sense of branched manifolds) of the usual formula in Gromov–Witten theory expressing the psi-class on $\overline{\mathcal{M}}_{0,n}$ in terms of nodal divisors. The interest of such averaged (holomorphic, non-generic) multi-section is that it can be perturbed to a (smooth, generic) multi-section that is also coherent (this is in fact a statement about all of the sections $s_{i,j}$ together, for $3 \leq j \leq n$). The zero locus of such multi-section will form a (branched) codimension 2 locus in the tubular neighbourhood of the unperturbed zero locus $s_{i,j}^{-1}(0)$, transversally intersecting such locus. For notational and visualization simplicity we will analyse in detail such perturbation in the case of $\overline{\mathcal{M}}_{0,5}$ in the example below, the general case being just an easy extension of the same construction. Once such coherent collection of sections $\tilde{s}_{i,j}$ is constructed we consider a sequence of sections $\tilde{s}_{i,j}^{(k)}$ with $\tilde{s}_{i,j}^{0} = \tilde{s}_{i,j}$ and converging back to the old non-generic $s_{i,j}$ as $k \to \infty$. This limit construction determines a codimension 2 locus in the moduli space $\overline{\mathcal{M}}_{0,r}(\Gamma^+, \Gamma^-)$ completely contained in the boundary strata formed by nodal and multilevel curves, corresponding to the first summand in the right hand side of equation (3.3). The explicit form of the boundary components involved by such locus is described by formula (3.4), where the divisor $D_{(I|J)}$ refers to the source nodal Riemann surfaces for the multilevel curves in the target $V \times \mathbb{R}$.

Example 3.5. This example, and the understanding of the general phenomenon it describes, emerged in a discussion with Dimitri Zvonkine. Consider the moduli space $\overline{\mathcal{M}}_{0,5}$, whose boundary divisors, formed by nodal curves, we denote D_{ij}, $1 \leq i < j \leq 5$, where D_{ij} is the space of nodal curves with a bubble carrying the ith and jth points alone. The intersection structure of such divisors is represented by the following picture.



When two different nodal divisors intersect, they do it with intersection index $+1$. The self-intersection index of any of them is, on the other hand, -1. Each of the D_{ij} being a copy of $\mathbb{P}^1$ (representing a copy of the moduli space $\overline{\mathcal{M}}_{0,4}$ appearing at the boundary of $\overline{\mathcal{M}}_{0,5}$), this means that the normal bundle $N_{D_{ij}}$ of such D_{ij} has Chern class $c_1(N_{D_{ij}}) = -1$; hence the tubular neighbourhood of D_{ij} is a copy of $\tilde{\mathbb{C}}^2$, i.e., $\mathbb{C}^2$ blown up at 0, D_{ij} itself being the exceptional divisor. An intersecting D_{kl} can then be seen as a line through the origin of $\tilde{\mathbb{C}}^2$.

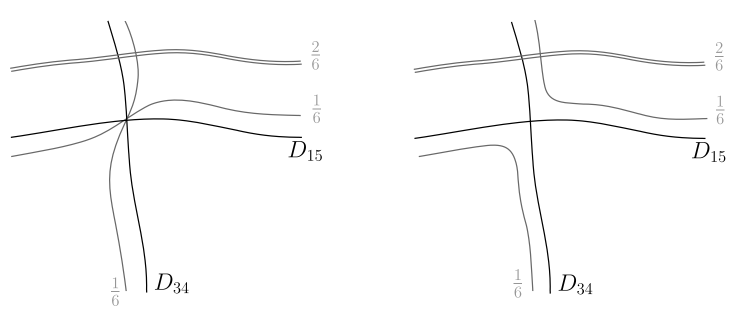
Consider now the tautological line bundle $\mathcal{L}_{1,5}$ on $\overline{\mathcal{M}}_{0,5}$. Using formula (3.4), the corresponding averaged psi-class, i.e., the (dual to the) zero locus of a averaged multi-section of $\mathcal{L}_{1,5}$, has the form

$$s_{1,5}^{-1}(0) = \frac{1}{2}(D_{12}+D_{13}+D_{14}+D_{15}) + \frac{1}{6}(D_{23}+D_{24}+D_{25}+D_{34}+D_{35}+D_{45})$$

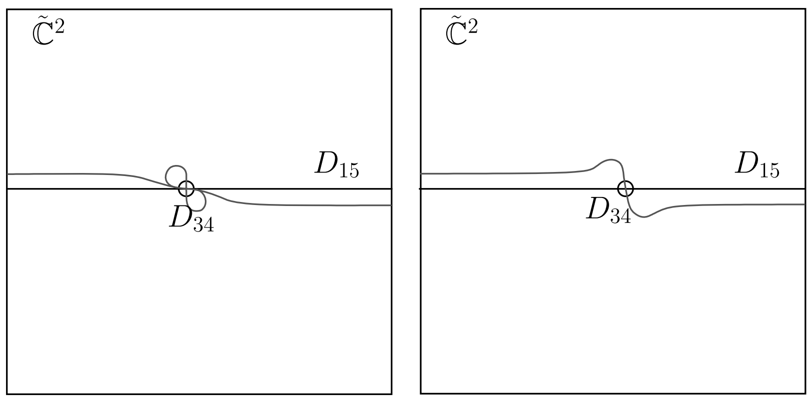
We now want to perturb such multi-section s to a coherent multi-section $\tilde{s}$ by a small perturbation int he neighbourhood of the nodal divisors. In fact it will be sufficient to describe how the zero locus is perturbed.

First notice that, once the line bundle $\mathcal{L}_{1,5}$ is chosen, the nodal divisors D_{ij} are split into two different sets, namely the ones for which $i = 1$ and the ones with $i \neq 1$ (appearing in the first and second summand in the above averaged formula). The perturbation will be symmetric with respect to permutations inside these two subsets separately. Looking at the picture above for visual help, let us start by perturbing $\frac{1}{6}D_{34}$ away from itself in such a way that it still intersects D_{34} at $D_{34} \cap D_{15}$ with index $-\frac{1}{6}$, at $D_{34} \cap D_{25}$ with index $+\frac{1}{6}$ and at $D_{34} \cap D_{12}$ with index $-\frac{1}{6}$ (notice that the total self-intersection index is $-\frac{1}{6}$, as it should for $\frac{1}{6}D_{34}$). Such perturbation is constructed starting from a section of the normal bundle to $\frac{1}{6}D_{34}$ with a zero at $D_{34} \cap D_{15}$ of index $-\frac{1}{6}$ (recall that, in general, the normal bundle to $D_{i,r}$ inside $\overline{\mathcal{M}}_{0,r}$ agrees with $\pi_r^* \mathcal{L}_{i,r-1}$ and, hence, the zeros of its sections are always localized at nodal divisors) and adding an extra zero of degree 0 (non-transversal) which spawns two zeros with opposite indices to be placed at $D_{34} \cap D_{25}$ and $D_{34} \cap D_{12}$ (using the fact that $D_{34} \cap D_{25}$ and $D_{34} \cap D_{12}$ are homologous inside D_{34} and more in general, for the case beyond $\overline{\mathcal{M}}_{0,5}$, the fact that two singular fibres of the forgetful map π_r are homologous inside $\overline{\mathcal{M}}_{0,r}$). The analogous choice is to be made for each of the divisors $D_{i,j}$ with $i \neq 1$.

Then we perturb $\frac{1}{2}D_{15}$ away from itself in such a way that it still intersects D_{15} at $D_{15} \cap D_{34}$, $D_{15} \cap D_{24}$ and $D_{15} \cap D_{23}$ always with intersection index $-\frac{1}{6}$ (summing to a total self-intersection index of $-\frac{1}{2}$, as it should be for $\frac{1}{2}D_{15}$). This is in fact a multi-section of the normal bundle to $\frac{1}{2}D_{15}$ formed by superimposing three sections of weight $\frac{1}{6}$ each, having a zero (of index $-\frac{1}{6}$) at $D_{15} \cap D_{34}$, $D_{15} \cap D_{24}$ and $D_{15} \cap D_{23}$ respectively (we are always using the fact that the normal bundle to $D_{i,r}$ inside $\overline{\mathcal{M}}_{0,r}$ agrees with $\pi_r^* \mathcal{L}_{i,r-1}$). Notice that such perturbation of $\frac{1}{2}D_{15}$ still intersects D_{34} in a punctured neighbourhood of $D_{34} \cap D_{15}$ with total intersection index $\frac{2}{6}$ and once more precisely at $D_{34} \cap D_{15}$ with intersection index $\frac{1}{6}$. The analogous choice is to be made for all of the divisors D_{1j}. See the left side of next picture for some intuition.



At the point $D_{34}\cap D_{15}$ we will combine the two perturbed divisors to annihilate their intersection there (their intersection indices being $-\frac{1}{6}$ and $+\frac{1}{6}$). This gives rise to a hyperbolic (and smooth, as we want to get generic sections) behaviour of the zero locus that will now avoid D_{15} completely (right side of the above picture). In a tubular neighbourhood of D_{34} the situation is described by the following picture, representing such neighbourhood as $\tilde{\mathbb{C}}^2$.



The circle at the origin is the exceptional divisor D_{34} and points on it are identified along diameters. Note that, in the second picture, we see the hyperbolic behaviour and the fact that the green zero locus does avoid D_{15}.

We are now ready to prove that such averaged perturbed section is coherent with the corresponding section on $\overline{\mathcal{M}}_{0,4}$, whose zero locus, using once more formula (3.4), will be

$$s_{1,4}^{-1}(0)=\frac{1}{3}(D_{12}+D_{13}+D_{14}).$$

Indeed, the perturbed zero locus does not intersect at all D_{1k}, coherently with the fact that $\mathcal{L}_{1,5}$ pulls back to the trivial bundle at such divisors, while at each of the D_{ij} with $i\neq 1$ the situation is the same as for D_{34}: two of the three branches of the multi-section of $N_{D_{15}}$, each with weight $\frac{1}{6}$, intersect it close to $D_{34}\cap D_{15}$ (total index $\frac{2}{6}$), and similarly for $N_{D_{12}}$, while, close to $D_{34}\cap D_{25}$, both the perturbation of $\frac{1}{6}D_{34}$ and the perturbation to $\frac{1}{6}D_{25}$ intersect D_{34} (total index $\frac{2}{6}$). This is coherent with the above averaged formula for $s_{1,4}^{-1}(0)$.

With the very same approach (only the combinatorics being more complicated) we can trat the general case of $\overline{\mathcal{M}}_{0,n}$, constructing the analogous perturbation to the averaged formula (3.4) for the psi-class on the Deligne–Mumford space of curves.

The same perturbation procedure of the above example can be applied to the general case of $\overline{\mathcal{M}}_{0,n}$. In particular one can proceed by induction, assuming that a coherent perturbation of the averaged multi-section corresponding to formula (3.4) exists for the moduli spaces $\overline{\mathcal{M}}_{0,k}$ with $k \leq n$ and proving that the perturbation can be coherently extended (as in the above example) to $\overline{\mathcal{M}}_{0,n+1}$. In order to do so, let us assume that the point carrying the descendant is the first marked point and consider the n possible forgetful maps $\pi_j : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$, $j = 2, \ldots, n+1$ which preserve the first marked point. Our perturbed averaged section $\tilde{s}_{1,n+1}$ on $\overline{\mathcal{M}}_{0,n+1}$ is constructed by starting with the superimposition of the n possible comparison lemmas (3.1), each with weight $\frac{1}{n}$, whose zero locus is

$$\bar{s}_{1,n+1}^{-1}(0) = \sum_{j=2}^{n+1} \frac{1}{n}(\pi_j^{-1}(\tilde{s}_{1,n}^{-1}(0)) + D_{1,j})$$

and then perturbing the divisors of zeros $D_{1,j}$ away from themselves (using as in the example above the fact that the normal bundle to $D_{1,j}$ inside $\overline{\mathcal{M}}_{0,n+1}$ agrees with $\pi_j^* \mathcal{L}_{1,n}$ to compensate any self-intersection of $D_{1,j}$ with an intersection of $D_{1,j}$ with $\pi_j^{-1}(\tilde{s}_{1,n}^{-1}(0))$) to obtain $\tilde{s}_{1,n+1}$.

Checking coherence on $\overline{\mathcal{M}}_{0,n+1}$ of such perturbed section $\tilde{s}_{1,n+1}$ is a somewhat involved combinatorial matter: one needs, in particular, to notice that the intersection of $\tilde{s}_{1,n+1}^{-1}(0)$ with each irreducible component of the boundary divisor $D_{(I|J)} \subset \overline{\mathcal{M}}_{0,n+1}$, $1 \in I$ (with $D_{(I,J)} \simeq \overline{\mathcal{M}}_{0,|I|} \times \overline{\mathcal{M}}_{0,|J|}$ and projections p_1 and p_2 on the two factors) gets a contribution of $\frac{|J|}{n} p_1^{-1}(s_{1,|I|}^{-1}(0))$ from the comparison lemma relative to forgetting each point in $j \in J$ and a contribution of $\frac{|I|-1}{n} p_1^{-1}(s_{1,|I|}^{-1}(0))$ from from the comparison lemma relative to forgetting each point in $j \in I$ with $j \neq 1$. This sums up to $\frac{|I|+|J|-1}{n} p_1^{-1}(s_{1,|I|}^{-1}(0)) = p_1^{-1}(s_{1,|I|}^{-1}(0))$ as desired.

In order to prove Theorem 3.3 we will make three different choices of special coherent collections of sections on the space of maps. Indeed, for equation $(2,0)$, the idea is not remembering any marked point, but only averaging with respect to the possible choices of two punctures. In this case we can choose an averaged coherent collection of multi-sections on the Deligne–Mumford moduli space of curves with $|\Gamma^+|+|\Gamma^-|+1$ marked points (using the perturbation technique of Example 3.5, where we are keeping all the punctures and the ith marked point, carrying the psi-class), and then use equations (3.3) and (3.2) to go to the space of maps $\overline{\mathcal{M}}_{0,r,A}(\Gamma^+, \Gamma^-)$. This coherent collection is evidently symmetric, with respect to permutations of marked points and punctures separately. Its zero locus, in the moduli space

$\overline{\mathcal{M}}_{0,r,A}(\Gamma^+,\Gamma^-)$, has the form

$$\frac{P(P-1)}{2} s^{-1}_{i,r,\Gamma^+,\Gamma^-}(0) = \sum_{\substack{i\in I,\ I\sqcup J=\{1,\dots,r\} \\ \Gamma_1^+\sqcup\Gamma_2^+=\Gamma^+ \\ \Gamma_1^-\sqcup\Gamma_2^-=\Gamma^- \\ P_2=|\Gamma_2^+|+|\Gamma_2^-|}} \frac{P_2(P_2-1)}{2}\ D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}$$

where $P=|\Gamma^+|+|\Gamma^-|$ and $D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}$ refers to a codimension 2 locus in the tubular neighbourhood of two-components curves joint at a node or puncture (hence nodal or two-level) where marked points and punctures split on each component as indicated by the subscript. The combinatorial factor on the left-hand side accounts for the possible ways of choosing two punctures to be remembered out of P, while the one on the right-hand side accounts for the number of ways a term of the form $D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}$ appears in the described averaging construction.

As a second possible choice we will start from an averaged coherent collection of multi-sections on the Deligne–Mumford moduli space of curves with $|\Gamma^+|+|\Gamma^-|+2$ marked points (like in Example 3.5 where, again, we keep all the punctures, the ith marked point carrying the psi-class, but also another marked point). There are exactly $r-1$ different forgetful projections from the space $\overline{\mathcal{M}}_{0,r,A}(\Gamma^+,\Gamma^-)$ to such $\overline{\mathcal{M}}_{0,|\Gamma^+|+|\Gamma^-|+2}$, corresponding to the numbering of the extra remembered marked point (excluding the ith, which is also always remembered). In order to obtain a coherent collection on $\overline{\mathcal{M}}_{0,r,A}(\Gamma^+,\Gamma^-)$ which is also symmetric with respect to permutations of the marked points we need to use the pull-back construction of equations (3.3) and (3.1) (the last one reiterated $r-2$ times), but also average among the $r-1$ different projections. After some easy combinatorics, its zero locus has the form

$$(r-1)\frac{P(P+1)}{2} s^{-1}_{i,r,\Gamma^+,\Gamma^-}(0) = \sum_{\substack{i\in I,\ I\sqcup J=\{1,\dots,r\} \\ |I|=r_1,\ |J|=r_2 \\ \Gamma_1^+\sqcup\Gamma_2^+=\Gamma^+ \\ \Gamma_1^-\sqcup\Gamma_2^-=\Gamma^- \\ P_2=|\Gamma_2^+|+|\Gamma_2^-|}} \left[r_2\frac{P_2(P_2+1)}{2} + (r_1-1)\frac{P_2(P_2-1)}{2}\right]\ D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}$$

Finally, as a last choice, we start from the moduli space of curves with $|\Gamma^+|+|\Gamma^-|+3$ marked points, so that we are keeping, after forgetting the map, two extra marked points (besides the ith). This time there will be exactly $\frac{(r-1)(r-2)}{2}$ forgetful maps from $\overline{\mathcal{M}}_{0,r,A}(\Gamma^+,\Gamma^-)$ to $\overline{\mathcal{M}}_{0,|\Gamma^+|+|\Gamma^-|+3}$, corresponding to the numbering of the two extra remembered marked points.

The pull-back construction of equations (3.3) and (3.1) (the last one reiterated $r-3$ times) needs then to be averaged among these possible choices in order to be symmetric with respect to permutations of marked points. This time the averaging combinatorics gives

$$\begin{aligned}
&\frac{(r-1)(r-2)}{2}\frac{(P+2)(P+1)}{2} s^{-1}_{i,r,\Gamma^+,\Gamma^-}(0)\\
&\quad=\sum_{\substack{i\in I,\ I\sqcup J=\{1,\dots,r\}\\ |I|=r_1,\ |J|=r_2\\ \Gamma_1^+\sqcup\Gamma_2^+=\Gamma^+\\ \Gamma_1^-\sqcup\Gamma_2^-=\Gamma^-\\ P_2=|\Gamma_2^+|+|\Gamma_2^-|}}\left[\frac{r_2(r_2-1)}{2}\frac{(P_2+2)(P_2+1)}{2}+r_2(r_1-1)\frac{P_2(P_2+1)}{2}\right.\\
&\qquad\left.+\frac{(r_1-1)(r_1-2)}{2}\frac{P_2(P_2-1)}{2}\right] D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}.
\end{aligned}$$

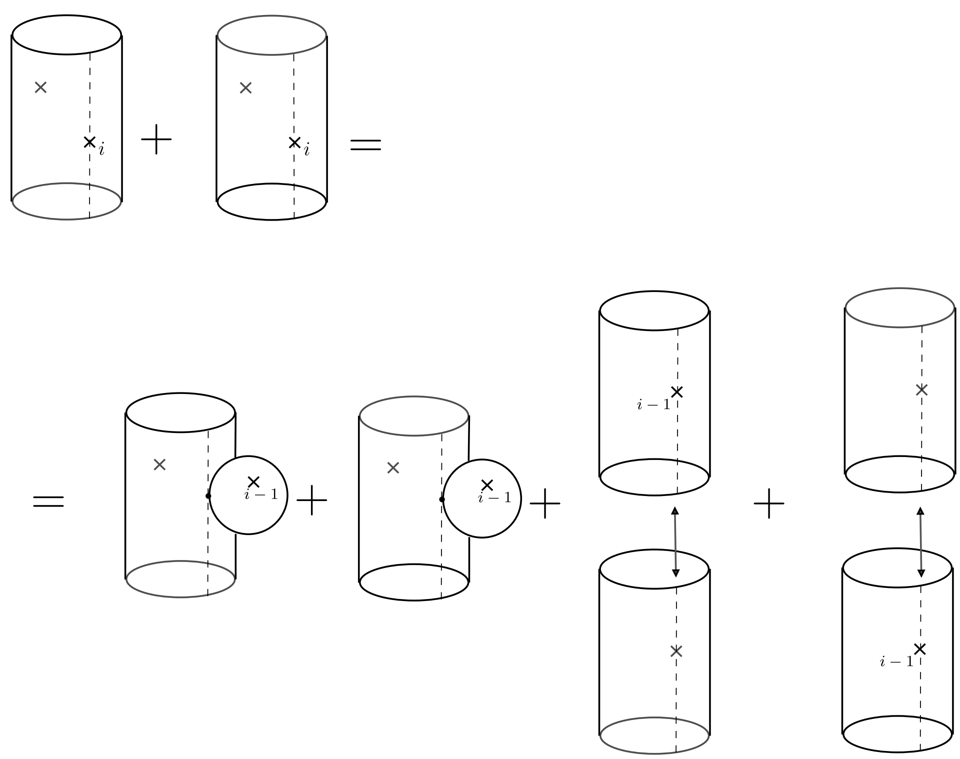
Such three different choices of multi-sections can also be superimposed (always in the sense of branched manifolds and multi-sections) to form further multi-sections which are, of course, still coherent. In particular, by taking respectively the first one, $\frac{3}{2}$ times the second one minus $\frac{1}{2}(r-1)$ the first one, and $\frac{3}{2}$ times the third one minus $\frac{3}{2}(r-2)$ times the second one plus the $(r-1)(r-2)$ times the first one, we get multi-sections whose zero loci have the form

$$\begin{aligned}
\frac{P(P-1)}{2}\ s^{-1}_{i,r,\Gamma^+,\Gamma^-}(0) &= \sum \frac{P_2(P_2-1)}{2}\ D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)},\\
(r-1)P\ s^{-1}_{i,r,\Gamma^+,\Gamma^-}(0) &= \sum r_2P_2\ D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)},\\
\frac{(r-1)(r-2)}{2}\ s^{-1}_{i,r,\Gamma^+,\Gamma^-}(0) &= \sum \frac{r_2(r_2-1)}{2}\ D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}.
\end{aligned}\tag{3.5}$$

To complete the proof of Theorem 3.3 we just need to notice that the limit procedure taking the perturbed sections $\tilde{s}_{i,n}$ of $\mathcal{L}_{i,n}$ back to their original non-generic limit $s_{i,n}$ corresponds, via equation (3.3) for the loci $D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}$ appearing above to select, in the space of maps relevant for cylindrical non-equivariant contact homology, either nodal configurations (and this is obvious), or two-level ones (i.e., where the two smooth components are connected by a puncture instead of a node). Since the two-level curves are of codimension one and not two in the space of maps (indeed this extra dimension remembers the information on the angular coordinate used for the gluing at the connecting puncture), we need to correct this error by fixing the decoration, i.e., the identification of the tangent planes at the connecting puncture, *a priori* as follows.

Since each of the two cylinders connected by the puncture carries one (or two in the case of (0,2)) of the remembered additional marked points, the position of these additional marked points can be used to fix unique

S^1-coordinates and hence asymptotic markers on each of the two cylinders, which in turn defines a natural decoration by simply requiring that the two asymptotic markers are identified. Note that this turns the additional markers used for fixing the S^1-coordinates automatically into additional marked points constrained to $\mathbb{R} \times \{0\}$. The following picture illustrate such phenomenon for the case relevant to equation $(1,1)$: the red puncture and marked point are those we are remembering, the black marked point is the one carrying the psi-class, whose power is specified by the index (i or $i-1$), the dashed line represents $\mathbb{R} \times \{0\}$ and the green arrow indicates the matching condition between the two dashed lines (notice that, for simplicity, we are not explicitly drawing any other marked point, which would correspond to having $N = 1$). This is a pictorial representation of each term in equation (1,1) once we divide each side by 2. The $(0,2)$ case is completely similar, only involving averaging between the two possible choices of marked point to be constrained to $\mathbb{R} \times \{0\}$.



We are only left with translating the three above equations (3.5) for the zero loci back into our generating functions language for the potential (recall the definition of the non equivariant cylindrical homology differential). There, considering that the number of punctures is always 2 in cylindrical contact homology, we see that, for each of the three formulae (3.5), the averaging combinatorial coefficients are absorbed in the right way to give rise to each of the three equations in the statement of Theorem 3.3. In particular, we put $P = 2$, $P_2 = 0, 2$ for nodal configurations (recall the absence of holomorphic planes), $P_2 = 1$ for two-level configurations (this in particular kills the two-level term in equation $(2, 0)$, since the corresponding first equation in (3.5) contains the factor $P_2 - 1$). Moreover we use the operators N to account for the number of marked points on the appropriate component when the limit of $D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}$, on the right hand side of formulae (3.5), corresponds to a nodal map, while we use $\check{N}$ to further constrain one of the marked points to $\mathbb{R} \times \{0\}$ when the limit of $D_{(I,\Gamma_1^+,\Gamma_1^-|J,\Gamma_2^+,\Gamma_2^-)}$ corresponds to a two-level map, as described above.

4. Applications

In this final section we want to apply the topological recursion result for non-equivariant cylindrical homology to (equivariant) cylindrical homology. As an important result we show that, as in rational Gromov–Witten theory, all descendant invariants can be computed from primary invariants, i.e., those without descendants. Furthermore we will prove that the topological recursion relations imply that one can define an action of the quantum cohomology ring $QH^*(V)$ of the target manifold (defined using the Gromov–Witten potential $\mathbf{f}$ of V introduced above) on the non-equivariant cylindrical homology $HC_*^{\text{non-}S^1}(V)$ by counting holomorphic cylinders with one constrained marked point.

4.1. Topological recursion in cylindrical homology. Since the chain space for non-equivariant cylindrical homology splits, $C_*^{\text{non-}S^1} = \hat{C}_* \oplus \check{C}_*$, it follows that the linear maps on the chain space, obtained by differentiating the differential of non-equivariant cylindrical homology with respect to $t^{\alpha,p}$- or $\check{t}^{\alpha,p}$-variables, can be restricted to linear maps between $\hat{C}_*$ and $\check{C}_*$, respectively. On the other hand, since each of the spaces $\hat{C}_*$ and $\check{C}_*$ is just a copy of the chain space for (equivariant) cylindrical homology, with degree shifted by one for the second space, $\hat{C}_* = C_*$, $\check{C}_* = C_*[1] = C_{*+1}$, we can translate the linear maps from non-equivariant cylindrical homology to (equivariant) cylindrical homology as follows.

While the restricted linear maps $\partial_{(\alpha,p)} : \hat{C}_* \to \hat{C}_*$ and $\partial_{(\alpha,p)} : \check{C}_* \to \check{C}_*$ indeed agree with the linear maps $\partial_{(\alpha,p)} : C_* \to C_*$ from cylindrical homology as defined in subsection 2.6, note that one can now introduce new linear maps $\partial_{(\check{\alpha},p)} : C_* \to C_*$ on cylindrical homology by requiring that they agree with the linear maps $\partial_{(\check{\alpha},p)} : \hat{C}_* \to \hat{C}_*$ (and hence $\partial_{(\check{\alpha},p)} : \check{C}_* \to \check{C}_*$) from non-equivariant cylindrical homology.

On the other hand, although the topological recursion relations we proved for the non-equivariant case are useful to compute the linear maps $\partial_{(\check{\alpha},p)}$ on $HC_*^{\text{non-}S^1}$, the goal of topological recursion in cylindrical contact homology (as in rational SFT) is to compute the linear maps $\partial_{(\alpha,p)} : HC_* \to HC_*$. In order to apply our results of the non-equivariant case to the equivariant case, we make use of the fact that (apart from the mentioned equivalence with $\partial_{(\alpha,p)} : \hat{C}_* \to \hat{C}_*$ and $\partial_{(\alpha,p)} : \check{C}_* \to \check{C}_*$) the linear map $\partial_{(\alpha,p)} : C_* \to C_*$ also agrees with the restricted linear map $\partial_{(\check{\alpha},p)} : \hat{C}_* \to \check{C}_*$.

In order to see this, observe that, while in the case of $\partial_{(\alpha,p)} : \hat{C}_* \to \hat{C}_*$ (or $\partial_{(\alpha,p)} : \check{C}_* \to \check{C}_*$) the free S^1-coordinate on the cylinder is fixed by the critical point on the negative (or positive) closed Reeb orbit, in the case of $\partial_{\check{\alpha},p} : \hat{C}_* \to \check{C}_*$ the free S^1-coordinate on the cylinder is fixed by the additional marked point (and thereby turning it into a constrained marked point).

With this we can prove the following corollary about topological recursion in (equivariant) cylindrical homology.

Corollary 4.1. *For three* different *non-generic special choices of coherent sections the following three* topological recursion relations *hold in (equivariant) cylindrical contact homology*

(2,0):

$$\partial_{(\alpha,i)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}\partial_{\nu}$$

(1,1):

$$N\,\partial_{(\alpha,i)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}N\,\partial_{\nu} + \frac{1}{2}[\partial_{(\alpha,i-1)}, \check{N}\,\partial]_+ + \frac{1}{2}[\partial_{(\check{\alpha},i-1)}, N\,\partial]_+$$

(0,2):

$$\begin{aligned}(N-1)N\partial_{(\alpha,i)} = {}& \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}(N-1)N\partial_{\nu} + [\partial_{(\alpha,i-1)}, (N-1)\check{N}\partial]_+ \\ &+ [\partial_{(\check{\alpha},i-1)}, (N-1)N\partial]_+.\end{aligned}$$

Proof. While the relation (2,0) is immediately follows by identifying the linear map $\partial_{(\alpha,p)} : C_* \to C_*$ with the restricted linear map $\partial_{(\check{\alpha},p)} : \hat{C}_* \to \check{C}_*$, for the relations (1,1) and (0,2) it suffices to observe that

$$\begin{aligned}(\partial_{(\check{\alpha},i-1)} \circ \check{N}\partial : \hat{C}_* \to \check{C}_*) &= (\partial_{(\check{\alpha},i-1)} : \hat{C}_* \to \check{C}_*) \circ (\check{N}\partial : \hat{C}_* \to \hat{C}_*) \\ &\quad + (\partial_{(\check{\alpha},i-1)} : \check{C}_* \to \check{C}_*) \circ (\check{N}\partial : \hat{C}_* \to \check{C}_*) \\ &= (\partial_{(\alpha,i-1)} : C_* \to C_*) \circ (\check{N}\partial : C_* \to C_*) \\ &\quad + (\partial_{(\check{\alpha},i-1)} : C_* \to C_*) \circ (N\partial : C_* \to C_*).\end{aligned}$$

□

While it follows that the second and the third topological recursion relation involve the linear maps $\partial_{\check{\alpha},p}: C_* \to C_*$ defined using non-equivariant contact homology and hence leave the frame of standard (equivariant) cylindrical homology, it is notable that the first topological recursion relation (2,0) indeed has the following important consequence.

Corollary 4.2. *All linear maps $\partial_{(\alpha,p)}: HC_*(V) \to HC_*(V)$ on cylindrical homology involving gravitational descendants can be computed from the linear maps $\partial_\alpha : HC_*(V) \to HC_*(V)$ with no gravitational descendants and the primary rational Gromov–Witten potential of the underlying stable Hamiltonian manifold, i.e., again involving no gravitational descendants.*

Proof. For the proof it suffices to observe that after applying the topological recursion relation (2,0) the marked point with the descendant sits on the attached sphere, so that the linear maps with descendants can indeed be computed from the linear maps without descendants and the rational Gromov–Witten potential of the target manifold with gravitational descendants. Together with the standard result of rational Gromov–Witten theory (generalized in the obvious way from symplectic manifolds to stable Hamiltonian manifolds without holomorphic planes) that the full descendant potential can be computed from the primary potential involving no descendants using the above mentioned topological recursion relations together with the divisor (to add more marked points on non-constant spheres), string and dilaton (for the case of constant spheres) equations, it follows the remarkable result that also in cylindrical homology the descendant invariants are determined by the primary invariants, that is, if we additionally include the primary Gromov–Witten potential. □

Remark 4.3. Note that the first topological relation actually descends to homology, i.e., it holds for $\partial_{(\alpha,i)}$ and ∂_ν *viewed as linear maps on cylindrical homology* $HC_*(V)$. In particular, while on the chain level all topological recursion relations only hold true for (three different) *special* choices of coherent sections, *after* passing to homology the first relation (2,0) holds for *all* coherent sections.

As we already remarked, it follows from the maximum principle that the Gromov–Witten potential of a contact manifold simply agrees with the Gromov–Witten potential of a point. Since in this case it follows from dimensional reasons that after setting all t-variables to zero we have

$$\frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^\mu}|_{t=0} = 0, \quad i > 0,$$

we have the following important vanishing result for contact manifolds.

For the rest of this subsection as well as the next one we will restrict ourselves to the case where all formal t-variables are set to zero.

Following the notation in [**F2,FR**] let us denote by $HC^0_*(V) = H(C^0_*, \partial^0)$ the cylindrical homology without additional marked points and hence without t-variables, which is obtained from the big cylindrical homology complex $HC_*(V) = H(C_*, \partial)$ by setting all t-variables to zero. In the same way, let us introduce the corresponding linear map $\partial^1_{(\alpha,p)} : HC^0_*(V) \to HC^0_*(V)$ obtained again by setting $t = 0$ and which now counts holomorphic cylinders with just one additional marked point (and descendants).

Corollary 4.4. *In the case when V is a contact manifold, after setting all t-variables to zero, the corresponding descendant linear maps $\partial^1_{(\alpha,p)}$: $HC^0_*(V) \to HC^0_*(V)$, $p > 0$ are zero.*

While this result shows that counting holomorphic cylinders with one additional marked point and gravitational descendants is not very interesting in the case of contact manifolds, it is clear from our ongoing work on topological recursion in full rational SFT that the arguments used above do *not* apply to the sequence of commuting Hamiltonians $\mathbf{h}^1_{(\alpha,p)}$ of rational SFT, which in the Floer case lead to the integrable hierarchies of Gromov–Witten theory. More precisely, we expect that the corresponding recursive procedure involves primary invariants belonging to a non-equivariant version of rational SFT.

4.2. Action of quantum cohomology on non-equivariant cylindrical homology. As we already mentioned in Subsection 2.1, in [**PSS**] Piunikhin–Salamon–Schwarz defined an action of the quantum cohomology ring of the underlying symplectic manifold on the Floer (co)homology groups by counting Floer cylinders with one additional marked point constrained to $\mathbb{R} \times \{0\} \subset \mathbb{R} \times S^1$. Note that for this the authors also needed to show that the concatenation of two maps on Floer cohomology corresponds to the ring multiplication in quantum cohomology.

While in [**PSS**] this result was proven by establishing appropriate compactness and gluing theorems for all appearing moduli spaces, in this final subsection we want to show how our topological recursion relation (1,1) together with the relation (2,0) can be used to define a corresponding action of the quantum cohomology (defined using the Gromov–Witten potential introduced above) on the non-equivariant cylindrical contact homology of a stable Hamiltonian manifold after setting all t-variables to zero.

In the same way as for closed symplectic manifolds we define the quantum cohomology $QH^*(V)$ of the stable Hamiltonian manifold V as the vector space freely generated by formal variables $t^\alpha = t^{\alpha,0}$, with coefficients which are Laurent series in the z_n-variables. Note that, as vector spaces, the only difference to the usual cohomology groups $H^*(V)$ again lies in the different choice of coefficients. On the other hand, while for general stable

Hamiltonian manifolds the quantum product defined using the Gromov–Witten three-point invariants is different from the usual product structure of $H^*(V)$, note that for contact manifolds we have $QH^*(V) = H^*(V)$ (with the appropriate choice of coefficients) as in this case the Gromov–Witten potential of V agrees with that of a point. Recalling that the linear maps $\partial^1_{\check{\alpha}} = \partial^1_{(\check{\alpha},0)}$ actually descend to maps on non-equivariant cylindrical homology $HC_*^{0,\text{non-}S^1}(V)$, we prove the following

Corollary 4.5. *The map*

$$QH^*(V) \otimes HC_*^{0,non\text{-}S^1}(V) \to HC_*^{0,non\text{-}S^1}(V), \quad (t^\alpha, \hat{q}_\gamma) \mapsto \partial^1_{\check{\alpha}}(\hat{q}_\gamma),$$
$$(t^\alpha, \check{q}_\gamma) \mapsto \partial^1_{\check{\alpha}}(\check{q}_\gamma),$$

defines an action of the quantum cohomology ring $QH^*(V)$ on the non-equivariant cylindrical homology $HC_*^{0,non\text{-}S^1}(V)$ (after setting all $t = 0$).

Proof. It follows from our topological recursion relations (2,0) and (1,1) for non-equivariant cylindrical contact homology that, after setting all t-variables to zero, we indeed have the following two non-averaged topological recursion relations,

$$\partial^2_{(\check{\alpha},i),(\beta,j)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^\mu}\eta^{\mu\nu}\partial^2_{\check{\nu},(\beta,j)} + \frac{\partial^3 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\beta,j}\partial t^\mu}\eta^{\mu\nu}\partial^1_{\check{\nu}},$$
$$\partial^2_{(\check{\alpha},i),(\beta,j)} = \frac{\partial^2 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^\mu}\eta^{\mu\nu}\partial^2_{\check{\nu},(\beta,j)} + \frac{1}{2}[\partial^1_{(\check{\alpha},i-1)}, \partial^1_{(\check{\beta},j)}]_+.$$

While the first equation follows from differentiating the recursion relation (2,0) with respect to the formal variable $t^{\beta,j}$, the second equation follows from the recursion relation (1,1) by first setting all t-variables except $t^{\beta,j}$ to zero.

Ignoring invariance problems for the moment, the desired result follows by comparing both equations. Since the left-hand side and the first summand on the right-hand side of both equations agree, it follows that

$$\frac{1}{2}[\partial^1_{(\check{\alpha},i-1)}, \partial^1_{(\check{\beta},j)}]_+ = \frac{\partial^3 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\beta,j}\partial t^\mu}\eta^{\mu\nu}\partial^1_{\check{\nu}}.$$

On the other hand, since $[\partial^1_{(\check{\alpha},i-1)}, \partial^1_{(\check{\beta},j)}]_- = 0$ on homology, and using the natural relation between the commutator and its corresponding anti-commutator, $\frac{1}{2}[\partial^1_{(\check{\alpha},i-1)}, \partial^1_{(\check{\beta},j)}]_+ + \frac{1}{2}[\partial^1_{(\check{\alpha},i-1)}, \partial^1_{(\check{\beta},j)}]_- = \partial^1_{(\check{\alpha},i-1)} \circ \partial^1_{(\check{\beta},j)}$, it follows that after passing to homology we have

$$\partial^1_{(\check{\alpha},i-1)} \circ \partial^1_{(\check{\beta},j)} = \frac{\partial^3 \mathbf{f}}{\partial t^{\alpha,i-1}\partial t^{\beta,j}\partial t^\mu}\eta^{\mu\nu}\partial^1_{\check{\nu}},$$

so that the desired result follows after setting $i = 1$ and $j = 0$.

We want emphasize that the above identity in general does not hold on the chain level, but only holds after passing to homology. Note that this is already known from [**PSS**]. While in their proof a chain homotopy adds an exact term to the above identity, in our proof this follows from the fact that the relations (2,0) and (1,1) only hold for two *different* (special) coherent collections of sections, i.e., they do not hold true simultaneously on the chain level. While the above identity should hold after passing to homology, note that the invariance problem cannot be resolved as for the topological recursion relation (2,0) by simply passing to homology, since the two equations with which we started involve linear maps counting holomorphic cylinders with more than one additional marked point.

In order to show that the desired composition rule still holds after passing to homology, we make use of the fact that we can choose nice coherent collections of sections interpolating between the two special coherent sections (in the sense of Subsection 1.3) as follows. Since it follows from the proof of the main theorem in Subsection 2.4 that our special coherent collections of sections are indeed pulled back from the moduli space of curves to the moduli space of maps (we can ignore the bubbles that we added afterwards here), we do not need to consider arbitrary interpolating coherent sections but only those which are again pull-backs of coherent sections on the underlying moduli space of curves. Since in the moduli space of curves (in contrast to the moduli space of maps) the strata of singular curves (maps) are of codimension at least two, we can further choose the homotopy such that it avoids all singular strata, so that all underlying curves in the homotopy are indeed smooth. Since we excluded holomorphic planes throughout the paper, it follows that the only singular maps that appear during the interpolation process are holomorphic maps where a cylinder without additional marked points splits off. But this implies that the difference between the two different special coherent collections of sections is indeed exact, so that the above equation indeed holds after passing to homology. □

While in the same way we can give an alternative proof of the result of Piunikhin–Salamon–Schwarz by using our topological recursion relations (1,1) and (2,0) in symplectic Floer theory of Subsection 2.1, in contrast note that in (equivariant) cylindrical homology, due to the differences in the topological recursion formulas in this case, neither ∂^1_α nor $\partial^1_{\check{\alpha}}$ defines an action of quantum cohomology on (equivariant) cylindrical homology.

Finally, using the isomorphism of Bourgeois–Oancea in [**BO**], we show how the latter result also establishes an action of the cohomology ring on the symplectic homology, and thereby generalizes the result of [**PSS**] in the obvious way from closed manifolds to compact manifolds X with contact boundary $\partial X = V$. For the proof we assume that there not only no holomorphic planes in $(\mathbb{R}\times)V$, but also no holomorphic planes in the filling X. Furthermore we assume that all t-variables are set to zero without explicitly mentioning it again.

Corollary 4.6. *Using the isomorphism between non-equivariant cylindrical homology and positive symplectic homology in [**BO**], our result defines an action of the cohomology ring $H^*(X)$ on (full) symplectic homology $SH^0_*(X)$ (at $t = 0$).*

Proof. Since we assume that there are no holomorphic planes in the filling X, it further follows from the computation of the differential in symplectic homology by Bourgeois and Oancea in [**BO**] that the symplectic homology is given by the direct sum, $SH^0_*(X) = SH^{0,+}_*(X) \oplus H^{\dim X - *}(X)$. While the action of $H^*(V)$ on non-equivariant cylindrical contact homology $HC^{0,\text{non-}S^1}_*(V)$ established above defines an action of $H^*(X)$ on $SH^{0,+}_*(X)$ using the isomorphism $HC^{0,\text{non-}S^1}_*(V) \cong SH^{0,+}_*(X)$ and the natural map $H^*(X) \to H^*(V)$ defined by the inclusion $V \hookrightarrow X$, together with the natural action of $H^*(X)$ on itself we get the desired result. □

Of course, we expect that this action agrees with the action of $H^*(X)$ on $SH^0_*(X)$ defined using the action on the Floer homology groups $FH^0_*(H)$ for admissible Hamiltonians and taking the direct limit, after generalizing the result in [**PSS**] from closed symplectic manifolds to compact symplectic manifolds with contact boundary in the obvious way.

Example 4.7. Using the natural map $H^*(Q) \to H^*(T^*Q)$ given by the projection, note that in the cotangent bundle case $X = T^*Q$ this defines an action of $H^*(Q)$ on $SH^0_*(T^*Q)$, which by [**AS, SW**] is isomorphic to $H_*(\Lambda Q)$, where ΛQ denotes the loop space of Q. Introducing additional marked points on the cylinders in the proofs of Abbondandolo–Schwarz and Salamon–Weber, we expect that it can be shown that this action agrees with the natural action of $H^*(Q)$ on $H_*(\Lambda Q)$ given by the cap product and the base point map $\Lambda Q \to Q$.

On the other hand, as mentioned above, in the equivariant setting we do *not* expect to find a natural action of $H^*(Q)$ on (equivariant) cylindrical homology $HC^0_*(S^*Q)$, which by [**CL**] is isomorphic to the S^1-equivariant singular homology $H^{S^1}_*(\Lambda Q, Q)$. But this fits with the well-known fact that there is *no* natural action of the cohomology $(H^*(Q) \to)H^*(\Lambda Q)$ on relative S^1-equivariant homology $H^{S^1}_*(\Lambda Q, Q)$.

4.3. Example: cylindrical homology in the Floer case. We end this paper by discussing briefly the important *Floer case* of SFT, which was worked out in the paper [**F1**] of the first author, including the necessary transversality proof. Here $V = S^1 \times M$ is equipped with a stable Hamiltonian structure ($\omega^H = \omega + dH \wedge dt, \lambda = dt$) for some time-dependent Hamiltonian $H : S^1 \times M \to \mathbb{R}$ on a closed symplectic manifold $M = (M, \omega)$. It follows that the Reeb vector field is given by $R^H = \partial_t + X^H_t$, so that in particular every one-periodic closed Reeb orbit is a periodic orbit of the time-dependent

Hamiltonian H. More precisely, it can be shown, see [**F1**], that the chain complex of equivariant cylindrical contact homology naturally splits into subcomplexes generated by Reeb orbits of a fixed integer period and that the equivariant cylindrical homology generated by Reeb orbits of period one agrees with the standard symplectic Floer homology for the time-dependent Hamiltonian $H : S^1 \times M \to \mathbb{R}$. In order to see that the differentials indeed agree, one uses that the holomorphic map from the cylinder to $\mathbb{R} \times V$ splits, $\tilde{u} = (h, u) : \mathbb{R} \times S^1 \to (\mathbb{R} \times S^1) \times M$, where the map $h : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ is just the identity (up to automorphisms of the domain). Note that in the Morse–Bott limit $H = 0$ one arrives in the trivial circle bundle case and just gets back the relation between SFT and Gromov–Witten theory from [**EGH**].We now show the following important

Proposition 4.8. *In the Floer case the topological recursion relations for equivariant cylindrical homology reproduce the topological recursion relations for symplectic Floer homology from section three. In particular, by passing to the Morse–Bott limit $H = 0$, they reproduce the standard topological recursion relations of Gromov–Witten theory. Furthermore, the action of the quantum cohomology on the non-equivariant cylindrical homology splits and agrees with the action of quantum cohomology on symplectic Floer homology defined in [**PSS**].*

Proof. In the same way as it follows from the fact that the map $h : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ is just the identity (up to automorphisms of the domain) that the differentials ∂ in equivariant cylindrical homology and symplectic Floer homology naturally agree, it follows that the linear maps $\partial_{\check{\alpha}}$ for $\alpha \in H^*(M)$ introduced in symplectic Floer homology in section three and in equivariant cylindrical homology (using the corresponding map in non-equivariant cylindrical homology) agree. Furthermore it follows from the same result that for $\alpha_1 = \alpha \wedge dt \in H^*(S^1 \times M)$ we have

$$\partial_{\alpha_1,p} = \partial_{\check{\alpha},p} : C_* \to C_*, \quad \partial_{\check{\alpha}_1,p} = 0.$$

Note that the last equation follows from the fact that the S^1-symmetry is divided out twice. Again only working out the second topological recursion relation, it then indeed follows that

$$\begin{aligned}(N\partial_{(\check{\alpha},i)} : CF_* \to CF_*) &= (N\partial_{(\alpha_1,i)} : C_* \to C_*) \\ &= \frac{\partial^2 \mathbf{f}_{S^1 \times M}}{\partial t^{\alpha_1,i-1} \partial t^{\mu}} \eta^{\mu\nu} (N\partial_\nu : C_* \to C_*) \\ &\quad + \left(\frac{1}{2}[\partial_{(\alpha_1,i-1)}, \check{N}\,\partial]_+ : C_* \to C_*\right) \\ &\quad + \left(\frac{1}{2}[\partial_{(\check{\alpha}_1,i-1)}, N\,\partial]_+ : C_* \to C_*\right)\end{aligned}$$

$$= \frac{\partial^2 \mathbf{f}_M}{\partial t^{\alpha,i-1}\partial t^{\mu}}\eta^{\mu\nu}(N\partial_{\check{\nu}} : CF_* \to CF_*) + \left(\frac{1}{2}[\partial_{(\check{\alpha},i-1)}, \check{N}\,\partial]_+ : CF_* \to CF_*\right),$$

where we further use that by similar arguments, namely that every holomorphic map $\mathbb{CP}^1 \to \mathbb{R}\times S^1$ is constant, the Gromov–Witten potential of the stable Hamiltonian manifold $S^1\times M$ is given by the Gromov–Witten potential of the symplectic manifold M, see also the discussion at the beginning of Subsection 2.3.

Apart from the fact that this proves the desired statement about the topological recursion relations, note that the same equation shows that in this special case the linear map $\partial_{\check{\alpha}} : C_* \to C_*$ indeed leads to an action of the quantum cohomology of $S^1\times M$ on the *equivariant* cylindrical homology, which agrees with the one defined by [**PSS**] on symplectic Floer homology. For the action of quantum cohomology $QH^*(S^1\times M)$ on the non-equivariant cylindrical homology $HC_*^{\text{non-}S^1}(S^1\times M)$, observe that the differential of non-equivariant cylindrical homology is indeed of diagonal form

$$\partial = \mathrm{diag}(\partial,\partial) : \hat{CF}_* \oplus \check{CF}_* \to \hat{CF}_* \oplus \check{CF}_*$$

with the Floer homology differential $\partial : CF_* \to CF_*$. This follows from the fact that in this case the only off-diagonal contribution $\delta : \hat{CF}_* \to \check{CF}_*$ is also zero, which as above again follows from the fact that the S^1-symmetry on the cylinder is divided out twice. Furthermore note that by the same argument the linear map $\partial_{\check{\alpha}}$ is also of diagonal form. It follows that the non-equivariant cylindrical homology is given as a direct sum, $HC_*^{\text{non-}S^1} = \hat{HF}_* \oplus \check{HF}_*$, that the quantum cohomology acts on both factors separately and agrees with the action defined in [**PSS**]. □

References

[AS] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), 254–316.

[BEHWZ] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Top. **7** (2003), 799–888.

[BO] F. Bourgeois and A. Oancea, *An exact sequence for contact- and symplectic homology*, Inv. Math. **175**(3) (2009), 611–680.

[CL] K. Cieliebak and J. Latschev, *The role of string topology in symplectic field theory*, 2007, arXiv:0706.3284.

[E] Y. Eliashberg, *Symplectic field theory and its applications*, Proc. ICM 2006.

[EGH] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory.* GAFA 2000 Visions in Mathematics special volume, Part II, 2000, 560–673.

[F1] O. Fabert, *Contact homology of Hamiltonian mapping tori*, Comm. Math. Helv. **85**, 2010.

[F2] O. Fabert, *Gravitational descendants in symplectic field theory*, 2009, arXiv: 0907.0789.

[F3] O. Fabert, *Transversality problems in symplectic field theory and a new Fredholm theory*, 2010, arXiv:1003.0651.

[FR] O. Fabert and P. Rossi, *String, dilaton and divisor equation in Symplectic Field Theory*, 2010, arXiv: 1001.3094.

[HWZ] H. Hofer, K. Wysocki and E. Zehnder, *A general Fredholm theory I: a splicing-based differential geometry*, 2006, arXiv: math.FA/0612604.

[PSS] S. Piunikhin, D. Salamon and M. Schwarz, *Symplectic Floer–Donaldson theory and quantum cohomology*, in 'Proc. Symp. Symplectic Geometry, held at the Isaac Newton Institute in Cambridge in 1994' (C.B. Thomas, ed.), LMS Lecture Note Series.

[R1] P. Rossi, *Gromov–Witten invariants of target curves via symplectic field theory*, 2007, arXiv: 0709.2860.

[R2] P. Rossi, *Integrable systems and holomorphic curves*, 2009, arXiv: 0912.0451.

[SW] D. Salamon and J. Weber, *Floer homology and the heat flow*, Geom. Funct. Anal. **16** (2006), 1050–1138.

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