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AFFINE YANG–MILLS–HIGGS METRICS

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Let (E, φ) be a flat Higgs bundle on a compact special affine manifold M equipped with an affine Gauduchon metric. We prove that (E, φ) is polystable if and only if it admits an affine Yang–Mills–Higgs metric.

1. Introduction

An affine manifold of dimension n is a smooth real manifold M of dimension n equipped with a flat torsion–free connection D on its tangent bundle. Equivalently, an affine structure on M is provided by an atlas of M whose transition functions are affine maps of the form $x \longmapsto Ax + b$, where $A \in \mathrm{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. The total space of the tangent bundle TM of an affine manifold M admits a natural complex structure; for the above transition function on $U \subset \mathbb{R}^n$, the corresponding transition map on $TU \subset T\mathbb{R}^n$ is $z \longmapsto Az + b$, where $z = x + \sqrt{-1}\, y$ with y being the fiber coordinate for the natural trivialization of the tangent bundle of U. There is a dictionary between the locally constant sheaves on M and the holomorphic sheaves on TM which are invariant in the fiber directions (cf. [**Lo09**]). In particular, a flat complex vector bundle over M naturally extends to a holomorphic vector bundle over TM.

An affine manifold M is called *special* if it admits a volume form which is covariant constant with respect to the flat connection D on M. In [**Lo09**], a Donaldson–Uhlenbeck–Yau-type correspondence was established for flat vector bundles over a compact special affine manifold equipped with an affine Gauduchon metric. This correspondence states that such a bundle admits an affine Yang–Mills metric if and only if it is polystable. The proof of it is an adaptation to the affine situation of the methods of Uhlenbeck–Yau [**UY86, UY89**] for the compact Kähler manifolds and their modification by Li–Yau [**LY87**] for the complex Gauduchon case.

Hitchin and Donaldson extended the correspondence between polystable bundles and Yang–Mills connections to Higgs bundles on Riemann surfaces [**Hi87, Do87b**]. Simpson extended it to Higgs bundles on compact Kähler manifolds (also to non–compact cases under some assumption) using Donaldson's heat flow technique (see [**Si88, Do85, Do87a**]). Recently, this has been adapted for the compact Gauduchon case by Jacob [**Ja11**].

Our aim here is to introduce Higgs fields on flat vector bundles over a compact special affine manifold equipped with an affine Gauduchon metric, and to establish a correspondence between polystable Higgs bundles and Yang–Mills–Higgs connections.

We prove the following theorem (see Theorem 2.10, Propositions 2.6 and 2.9):

Theorem 1.1. *Let M be a compact special affine manifold equipped with an affine Gauduchon metric. If (E, φ) is a stable flat Higgs vector bundle over M, then E admits an affine Yang–Mills–Higgs metric, which is unique up to a positive constant scalar.*

The analog of Theorem 1.1 holds for flat real Higgs bundles (see Corollary 4.3). We also note that Theorem 1.1 extends to the flat principal Higgs G-bundles, where G is any reductive affine algebraic group over $\mathbb{C}$ or of split type over $\mathbb{R}$; see Section 4.1.

We recall that a tt^* bundle on a complex manifold (M, J) is a triple (E, ∇, S), where E is a C^∞ real vector bundle over M, ∇ is a connection on E and S is a smooth section of $T^*M \otimes \mathrm{End}(E)$, such that the connection

$$\nabla^\theta_v := \nabla_v + \cos(\theta) \cdot S(v) + \sin(\theta) \cdot S(J(v)), \quad v \in TM$$

is flat for all $\theta \in \mathbb{R}$; see [**Sc05, Sc07**]. It would be interesting to develop tt^* bundles on affine manifolds.

2. Preparations and statement of the theorem

Let M be an affine manifold of dimension n. As mentioned before, TM has a natural complex structure. This complex manifold will be denoted by $M^{\mathbb{C}}$. The zero section of $TM = M^{\mathbb{C}}$ makes M a real submanifold of $M^{\mathbb{C}}$. Given an atlas on M compatible with the affine structure (so the transition functions are affine maps) the corresponding coordinates $\{x^i\}$ are called *local affine coordinates*. If $\{x^i\}$ is defined on $U \subset M$, then on $TU \subset TM$, we have the holomorphic coordinate function $z^i := x^i + \sqrt{-1}\, y^i$, where y^i is the fiber coordinate corresponding to the local trivialization of the tangent bundle given by $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$.

Define the bundle of (p,q) forms on M by

$$\mathcal{A}^{p,q} := \bigwedge^p T^*M \otimes \bigwedge^q T^*M.$$

Given local affine coordinates $\{x^i\}_{i=1}^n$ on M, we will denote the induced frame on $\mathcal{A}^{p,q}$ as

$$\big\{dz^{i_1} \wedge \cdots \wedge dz^{i_p} \otimes d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}\big\},$$

where $z^i = x^i + \sqrt{-1}\, y^i$ are the complex coordinates on $M^{\mathbb{C}}$ defined above; note that $dz^i = d\bar{z}^i = dx^i$ on M. There is a natural restriction map from $(p,\, q)$-forms on the complex manifold $M^{\mathbb{C}}$ to $(p,\, q)$-forms on M given in local affine coordinates on an open subset $U \subset M$ by

$$\begin{aligned}
&\sum \phi_{i_1,\ldots,i_p,j_1,\ldots,j_q} (dz^{i_1} \wedge \cdots \wedge dz^{i_p}) \wedge (d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}) \\
\longmapsto &\sum \phi_{i_1,\ldots,i_p,j_1,\ldots,j_q}|_U (dz^{i_1} \wedge \cdots \wedge dz^{i_p}) \otimes (d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}),
\end{aligned} \tag{2.1}$$

where $\phi_{i_1,\ldots,i_p,j_1,\ldots,j_q}$ are smooth functions on $TU \subset TM = M^{\mathbb{C}}$, U is considered as the zero section of $TU \longrightarrow U$, and the sums are taken over all $1 \leqslant i_1 < \cdots < i_p \leqslant n$ and $1 \leqslant j_1 < \cdots < j_q \leqslant n$.

One can define natural operators

$$\begin{aligned}
\partial &: \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p+1,q} \quad \text{and} \\
\overline{\partial} &: \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p,q+1}
\end{aligned}$$

given in local affine coordinates by

$$\partial\big(\phi \otimes (d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q})\big) := \frac{1}{2}\,(d\phi) \otimes (d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q})$$

if ϕ is a p–form, respectively by

$$\overline{\partial}\big((dz^{i_1} \wedge \cdots \wedge dz^{i_p}) \otimes \psi\big) := (-1)^p \frac{1}{2}\,(dz^{i_1} \wedge \cdots \wedge dz^{i_p}) \otimes (d\psi)$$

if ψ is a q-form. These operators are the restrictions of the corresponding operators on $M^{\mathbb{C}}$ with respect to the restriction map given in (2.1).

Similarly, there is a wedge product defined by

$$(\phi_1 \otimes \psi_1) \wedge (\phi_2 \otimes \psi_2) := (-1)^{q_1 p_2} (\phi_1 \wedge \phi_2) \otimes (\psi_1 \wedge \psi_2)$$

if $\phi_i \otimes \psi_i$ are forms of type $(p_i,\, q_i)$, $i = 1, 2$; as above, it is the restriction of the wedge product on $M^{\mathbb{C}}$.

The tangent bundle TM is equipped with a flat connection, which we will denote by D. The flat connection on T^*M induced by D will be denoted by D^*.

The affine manifold M is called *special* if it admits a volume (= top-degree) form ν which is covariant constant with respect to the flat connection D on TM.

In the case of special affine structures, ν induces natural maps

$$\mathcal{A}^{n,q} \longrightarrow \bigwedge^q T^*M, \quad \nu\otimes\chi \longmapsto (-1)^{\frac{n(n-1)}{2}}\chi,$$
$$\mathcal{A}^{p,n} \longrightarrow \bigwedge^p T^*M, \quad \chi\otimes\nu \longmapsto (-1)^{\frac{n(n-1)}{2}}\chi,$$

which are called *division* by ν. In particular, any (n,n) form χ can be integrated by considering the integral of $\frac{\chi}{\nu}$. (See [**Lo09**].)

A smooth Riemannian metric g on M gives rise to a $(1,1)$ form expressed in local affine coordinates as

$$\omega_g = \sum_{i,j=1}^n g_{ij}\, dz^i \otimes d\overline{z}^j; \tag{2.2}$$

it is the restriction of the corresponding $(1,1)$ form on $M^{\mathbb{C}}$ given by the extension of g to $M^{\mathbb{C}}$. The metric g is called an *affine Gauduchon metric* if

$$\partial\overline{\partial}(\omega_g^{n-1}) = 0$$

(recall that $n = \dim M$). By [**Lo09,** Theorem 5], on a compact affine manifold, every conformal class of Riemannian metrics contains an affine Gauduchon metric, which is unique up to a positive scalar.

Take a pair (E,∇), where E is a complex vector bundle on M, and ∇ is a flat connection on E. (In the following, we will always be concerned with complex vector bundles until we give analogs to our results for real vector bundles in Corollary 4.3.) The pullback of E to $TM = M^{\mathbb{C}}$ by the natural projection $TM \longrightarrow M$ will be denoted by $E^{\mathbb{C}}$. The flat connection ∇ pulls back to a flat connection on $E^{\mathbb{C}}$. This flat vector bundle on $M^{\mathbb{C}}$ can be considered as an extension of the flat vector bundle (E,∇) on the zero section of TM.

Let h be a Hermitian metric on E; it defines a Hermitian metric on the pulled back vector bundle $E^{\mathbb{C}}$. Let d^h be the Chern connection associated to this Hermitian metric on $E^{\mathbb{C}}$. Then d^h corresponds to a pair

$$(\partial^h, \overline{\partial}) = (\partial^{h,\nabla}, \overline{\partial}^{\nabla})$$

of operators on $\mathcal{A}^{p,q}(E) := \mathcal{A}^{p,q}\otimes E$. This pair is called the *extended Hermitian connection* of (E,h) (see [**Lo09**]). Similarly, we have an *extended connection form*

$$\theta \in \mathcal{A}^{1,0}(\operatorname{End} E),$$

an *extended curvature form* $\Omega = \overline{\partial}\theta \in \mathcal{A}^{1,1}(\operatorname{End} E)$, an *extended mean curvature*

$$K = \operatorname{tr}_g \Omega \in \mathcal{A}^{0,0}(\operatorname{End} E)$$

and an *extended first Chern form* $c_1(E,h) = \operatorname{tr}\Omega \in \mathcal{A}^{1,1}$, which are the restrictions of the corresponding objects on $E^{\mathbb{C}}$. Here tr_g denotes contraction of differential forms using the Riemannian metric g, and tr denotes the trace map on the fibers of $\operatorname{End} E$.

The extended first Chern form $c_1(E,h)$ and the extended mean curvature are related by

$$(\operatorname{tr} K)\,\omega_g^n = n\,c_1(E,h)\wedge\omega_g^{n-1}. \tag{2.3}$$

The *degree* of a flat vector bundle E over a compact special affine manifold M equipped with an affine Gauduchon metric g is defined to be

$$\deg_g E := \int_M \frac{c_1(E,h)\wedge\omega_g^{n-1}}{\nu}, \tag{2.4}$$

where h is any Hermitian metric on E. This is well–defined by [**Lo09,** p. 109]. Even though E admits a flat connection ∇, there is no reason in general for the degree to be zero in the Gauduchon case. In particular, we can extend ∇ to a flat extended connection on E and then define an extended first Chern form $c_1(E,\,\nabla)$. But

$$c_1(E,\,\nabla) - c_1(E,\,h) = \operatorname{tr}\overline{\partial}\theta_\nabla - \overline{\partial}\partial\log\det h_{\alpha\bar\beta}$$

is $\overline{\partial}$-exact but not necessarily $\partial\overline{\partial}$–exact. Thus, under integration by parts, the Gauduchon condition is insufficient to force the degree to be zero.

If $\operatorname{rank} E \neq 0$, the *slope* of E is defined as

$$\mu_g(E) := \frac{\deg_g E}{\operatorname{rank} E}.$$

Now, we introduce Higgs fields on flat vector bundles.

Definition 2.1. Let (E,∇) be a smooth vector bundle on M equipped with a flat connection. A *flat Higgs field* on (E,∇) is defined to be a smooth section φ of $T^*M\otimes\operatorname{End} E$ such that

(i) φ is covariant constant, meaning the connection operator

$$\widetilde{\nabla} : T^*M\otimes\operatorname{End} E \longrightarrow T^*M\otimes T^*M\otimes\operatorname{End} E \tag{2.5}$$

defined by the connections ∇ and D^* on E and T^*M respectively, annihilates φ, and

(ii) $\varphi\wedge\varphi = 0$.

If φ is a flat Higgs field on (E,∇), then (E,∇,φ) (or (E,φ) if ∇ is understood from the context) is called a *flat Higgs bundle.*

Note that (i) means that the homomorphism

$$\varphi : TM \longrightarrow \operatorname{End} E$$

is a homomorphism of flat vector bundles, where TM (respectively, $\operatorname{End} E$) is equipped with the flat connection D (respectively, the flat connection induced by the flat connection ∇ on E). The homomorphism φ induces a homomorphism

$$\varphi' : TM \otimes E \longrightarrow E.$$

The connections D and ∇ together define a connection on $TM \otimes E$. The condition (i) means that φ' takes locally defined flat sections of $TM \otimes E$ to locally defined flat sections of E.

Let

$$d^{\nabla} : T^*M \otimes \operatorname{End} E \longrightarrow \left(\wedge^2 T^*M\right) \otimes \operatorname{End} E \tag{2.6}$$

be the composition

$$T^*M \otimes \operatorname{End} E \xrightarrow{\widetilde{\nabla}} T^*M \otimes T^*M \otimes \operatorname{End} E \xrightarrow{\mathrm{pr} \times \mathrm{id}_{\operatorname{End} E}} \left(\wedge^2 T^*M\right) \otimes \operatorname{End} E,$$

where $\mathrm{pr} : T^*M \otimes T^*M \longrightarrow \wedge^2 T^*M$ is the natural projection, and $\widetilde{\nabla}$ is defined in (2.5). So if φ is a flat Higgs field on (E, ∇), then $d^{\nabla}(\varphi) = 0$.

The space of all connections on E is an affine space for the vector space of smooth sections of $T^*M \otimes \operatorname{End} E$; a family of connections $\{\nabla_t\}_{t \in \mathbb{R}}$ is called *affine* if there is a smooth section α of $T^*M \otimes \operatorname{End} E$ such that $\nabla_t = \nabla_0 + t \cdot \alpha$.

Lemma 2.2. *Giving a flat Higgs bundle (E, ∇, φ) is equivalent to giving a smooth vector bundle E together with a one-dimensional affine family $\{\nabla_t := \nabla_0 + t \cdot \alpha\}_{t \in \mathbb{R}}$ of flat connections on E such that the* $\operatorname{End} E$*-valued* 1*-form α is flat with respect to the connection on $T^*M \otimes \operatorname{End} E$ defined by ∇_0 and D^*.*

Proof. Given a flat Higgs bundle (E, ∇, φ), we define a family of connections on E by $\nabla_t := \nabla + t\varphi$. In a locally constant frame of E with respect to ∇, we have $d^{\nabla}(\varphi) = 0$ (see (2.6) for d^{∇}) and the curvature of ∇_t is as follows:

$$d^{\nabla}(t\varphi) + (t\varphi) \wedge (t\varphi) = t d^{\nabla}(\varphi) + t^2 \varphi \wedge \varphi = 0, \tag{2.7}$$

so $\{\nabla_t\}_{t \in \mathbb{R}}$ is a 1–dimensional affine family of flat connections on E. From the definition of a flat Higgs field given in Definition 2.1 it follows that this one-dimensional affine family of connections satisfies the condition in the lemma.

For the converse direction, assume that we are given a one-dimensional affine family of flat connections $\{\nabla_0 + t \cdot \alpha\}_{t \in \mathbb{R}}$ on E, satisfying the condition that α is flat with respect to the connection on $T^*M \otimes \operatorname{End} E$ defined by ∇_0 and D^*. Since

$$0 = d^{\nabla_0}(t\alpha) + (t\alpha) \wedge (t\alpha) = t d^{\nabla_0}(\alpha) + t^2 \alpha \wedge \alpha,$$

we conclude that $\alpha \wedge \alpha = 0$.

Since α is flat with respect to the connection on $T^*M \otimes \operatorname{End} E$ defined by ∇_0 and D^*, and $\alpha \wedge \alpha = 0$, it follows that (E, ∇_0, α) is a flat Higgs bundle. $\square$

A Higgs field will always be understood as a section of $\mathcal{A}^{1,0}(\operatorname{End} E)$, meaning it is expressed in local affine coordinates as

$$\varphi = \sum_{i=1}^{n} \varphi_i \otimes dz^i,$$

where φ_i are locally defined flat sections of $\operatorname{End} E$; note that $dz^i = dx^i$ on M. Given a Hermitian metric h on E, the adjoint φ^* of φ with respect to h will be regarded as an element of $\mathcal{A}^{0,1}(\operatorname{End} E)$. In local affine coordinates, this means that

$$\varphi^* = \sum_{j=1}^{n} (\varphi_j)^* \otimes d\overline{z}^j.$$

In particular, the Lie bracket $[\varphi, \varphi^*]$ is an element of $\mathcal{A}^{1,1}(\operatorname{End} E)$. Locally,

$$[\varphi, \varphi^*] = \varphi \wedge \varphi^* + \varphi^* \wedge \varphi = \sum_{i,j=1}^{n} \big(\varphi_i \circ (\varphi_j)^* - (\varphi_j)^* \circ \varphi_i\big) \otimes dz^i \otimes d\overline{z}^j.$$

Let E be a flat vector bundle on M equipped with a flat Higgs field φ as well as a Hermitian metric h. The *extended connection form* θ^φ of the Hermitian flat Higgs bundle (E, φ, h) is defined to be

$$\theta^\varphi := (\theta + \varphi, \varphi^*) \in \mathcal{A}^{1,0}(\operatorname{End} E) \oplus \mathcal{A}^{0,1}(\operatorname{End} E),$$

where φ^* denotes the adjoint of φ with respect to h, and θ as before is the extended connection for the flat Hermitian bundle (E, h). This extended connection form corresponds to the connection form of $d^h + \varphi + \varphi^*$ on $E^{\mathbb{C}} \longrightarrow M^{\mathbb{C}}$. Analogously, the *extended curvature form* Ω^φ of (E, φ, h) is defined to be

$$\begin{aligned}\Omega^\varphi := \big(\partial^h \varphi, \overline{\partial}\theta + [\varphi, \varphi^*], \overline{\partial}(\varphi^*)\big) &\in \mathcal{A}^{2,0}(\operatorname{End} E) \oplus \mathcal{A}^{1,1}(\operatorname{End} E)\\ &\oplus \mathcal{A}^{0,2}(\operatorname{End} E).\end{aligned}$$

It corresponds to the curvature form of the connection $d^h + \varphi + \varphi^*$ on $E^{\mathbb{C}}$. As in the usual case, the *extended mean curvature* K^φ of (E, φ, h) is obtained by contracting the $(1,1)$ part of the extended curvature Ω^φ using the Riemannian metric g, so

$$K^\varphi := \operatorname{tr}_g \left(\overline{\partial}\theta + [\varphi, \varphi^*]\right) \in \mathcal{A}^{0,0}(\operatorname{End} E). \tag{2.8}$$

Since $\operatorname{tr}[\varphi, \varphi^*] = 0$, we have $\operatorname{tr} K^\varphi = \operatorname{tr} K$, and so by (2.3), the extended mean curvature K^φ of (E, φ, h) also is related to the first Chern form $c_1(E, h)$ by

$$(\operatorname{tr} K^\varphi)\omega_g^n = n c_1(E, h) \wedge \omega_g^{n-1}. \tag{2.9}$$

Definition 2.3. An *affine Yang–Mills–Higgs metric* on a flat Higgs bundle (E, φ) is a Hermitian metric h on E such that the extended mean curvature K^φ of (E, φ, h) satisfies the equation

$$K^\varphi = \gamma \cdot \mathrm{id}_E \tag{2.10}$$

for some constant scalar γ, which is called the *Einstein factor*.

We show the uniqueness of affine Yang–Mills–Higgs metrics for simple flat Higgs bundles.

Definition 2.4. A flat Higgs bundle (E, φ) is called *simple* if every locally constant section f of $\operatorname{End} E$ satisfying $[\varphi, f] = 0$ is a constant scalar multiple of the identity automorphism of E.

Lemma 2.5. *Let (E, φ) be a flat Higgs bundle over a compact affine manifold M equipped with a Riemannian metric g. Assume that E admits an affine Yang–Mills–Higgs metric h with Einstein factor γ. Let s be a locally constant section of E with $\varphi(s) = 0$.*

- *If $\gamma < 0$, then $s = 0$.*
- *If $\gamma = 0$, then $\partial^h s = 0$ and $\varphi^*(s) = 0$, where φ^* is the adjoint of φ with respect to h.*

Proof. For any locally constant section s of E with $\varphi(s) = 0$, compute

$$\operatorname{tr}_g \partial\overline{\partial}|s|^2 = -\gamma|s|^2 + |\partial^h s|^2 + |\varphi^*(s)|^2$$

and apply the maximum principle. □

Proposition 2.6. *Let (E, φ) be a flat Higgs bundle on a compact affine manifold M equipped with a Riemannian metric g. If (E, φ) is simple, then an affine Yang–Mills–Higgs metric on E is unique up to a positive scalar.*

Proof. Let h_1 and h_2 be two affine Yang–Mills–Higgs metrics on E with Einstein factors γ_1 and γ_2, respectively. Then there is an endomorphism f of E which is positive definite and self–adjoint with respect to h_1 (and h_2) such that

$$h_2(s, t) = h_1(f(s), t)$$

for all sections s and t of E.

Let ∇ be the flat connection on E. Define

$$\nabla' := f^{\frac{1}{2}} \circ \nabla \circ f^{-\frac{1}{2}} \quad \text{and} \quad \varphi' := f^{\frac{1}{2}} \circ \varphi \circ f^{-\frac{1}{2}}.$$

Then ∇' is another flat connection on E. Denote by E' the new flat structure on the underlying vector bundle of E induced by ∇'. Since φ' is locally constant with respect to ∇', we obtain a new flat Higgs bundle (E', φ'). The endomorphism $f^{\frac{1}{2}}$ is a locally constant section of the flat vector bundle $\mathrm{Hom}(E, E')$ and satisfies the equation

$$\varphi_{\mathrm{Hom}}(f^{\frac{1}{2}}) = 0,$$

where φ_{Hom} is the flat Higgs field on $\mathrm{Hom}(E, E')$ induced by φ and φ'. We observe that h_1 is an affine Yang–Mills–Higgs metric on (E', φ') with Einstein factor γ_2, and so the metric h on $\mathrm{Hom}(E, E')$ induced by h_1 on both E and E' is an affine Yang–Mills–Higgs metric with Einstein factor $\gamma_2 - \gamma_1$.

As $f^{\frac{1}{2}} \neq 0$, Lemma 2.5 implies that $\gamma_2 - \gamma_1 \geqslant 0$. By reversing the roles of h_1 and h_2, we obtain $\gamma_2 - \gamma_1 = 0$, and so from Lemma 2.5 we conclude that $\partial^h f^{\frac{1}{2}} = 0$ and $\varphi^*_{\mathrm{Hom}}(f^{\frac{1}{2}}) = 0$.

We write $(\partial_1, \overline{\partial}) = (\partial^{h_1,\nabla}, \overline{\partial}^{\nabla})$ and $(\partial', \overline{\partial}') = (\partial^{h_1,\nabla'}, \overline{\partial}^{\nabla'})$ for the extended Hermitian connections of (E, h_1) and (E', h_1), respectively, and calculate

$$0 = \partial^h f^{\frac{1}{2}} = \partial' \circ f^{\frac{1}{2}} - f^{\frac{1}{2}} \circ \partial_1 \; = \; f^{-\frac{1}{2}} \circ \partial_1 \circ f - f^{\frac{1}{2}} \circ \partial_1 = f^{-\frac{1}{2}} \circ \partial_1 f,$$

which implies that $\partial_1 f = 0$. Since f is self–adjoint with respect to h_1, it follows that $\overline{\partial} f = 0$.

In an analogous way, we compute

$$0 = \varphi^*_{\mathrm{Hom}}(f^{\frac{1}{2}}) = (\varphi')^* \circ f^{\frac{1}{2}} - f^{\frac{1}{2}} \circ \varphi^* = f^{-\frac{1}{2}} \circ \varphi^* \circ f - f^{\frac{1}{2}} \circ \varphi^* = f^{-\frac{1}{2}} \circ [\varphi^*, f],$$

which implies that $[\varphi^*, f] = 0$. Again, since f is self-adjoint with respect to h_1, it follows that $[\varphi, f] = 0$. As (E, φ) is simple, f must be a constant scalar multiple of the identity automorphism of E. □

Definition 2.7. Let (E, φ) be a flat Higgs bundle on a compact special affine manifold M equipped with an affine Gauduchon metric g.

(i) (E, φ) is called *stable* (respectively, *semistable*) if for every flat subbundle F of E with $0 < \operatorname{rank} F < \operatorname{rank} E$ which is preserved by φ, meaning $\varphi(F) \subset T^*M \otimes F$, we have

$$\mu_g(F) < \mu_g(E) \quad \text{(respectively, } \mu_g(F) \leqslant \mu_g(E)). \tag{2.11}$$

(ii) (E, φ) is called *polystable* if

$$(E, \varphi) = \bigoplus_{i=1}^{N} (E_i, \varphi_i)$$

with stable flat Higgs bundles (E_i, φ_i) of the same slope $\mu_g(E_i) = \mu_g(E)$.

Remark 2.8. If $\{\nabla_t\}_{t\in\mathbb{R}}$ is the family of flat connections on E satisfying the condition in Lemma 2.2 and corresponding to the flat Higgs bundle (E,φ), then Definition 2.7 (i) is equivalent to the condition that (2.11) holds for every smooth subbundle F of E with $0 < \operatorname{rank} F < \operatorname{rank} E$ which is preserved by ∇_t for all t.

Proposition 2.9. *Every stable flat Higgs bundle over a compact special affine manifold is simple.*

Proof. Apply the proof of [**Lo09,** Proposition 30], and note that the condition $[\varphi, f] = 0$ implies that the subbundle $H := (f - a\,\mathrm{id}_E)(E)$ of E is preserved by φ. □

We can now state our main theorem.

Theorem 2.10. *Let M be a compact special affine manifold equipped with an affine Gauduchon metric g. Let (E,φ) be a stable flat Higgs vector bundle over M. Then E admits an affine Yang–Mills–Higgs metric.*

Consider the special case where $\operatorname{rank} E = 1$, meaning (E,∇) is a flat line bundle over M. In this case, the statement of Theorem 2.10 turns out to be independent of the Higgs field φ. More precisely, a flat Higgs field on (E,∇) is nothing but a smooth 1–form on M which is flat with respect to the flat connection D^* on T^*M. Given a Hermitian metric h on E, the extended mean curvature K^φ of (E,φ,h) coincides with the usual mean curvature K of (E, h), and thus the Yang–Mills–Higgs equation (2.10) for (E,φ,h) reduces to the usual Yang–Mills equation for (E, h). Since, as a line bundle, E is automatically stable, this equation has a solution by [**Lo09,** Theorem 1].

3. Existence of Yang–Mills–Higgs metrics

This section is dedicated to the proof of Theorem 2.10.

Let M be a compact special affine manifold equipped with a covariant constant volume form ν and an affine Gauduchon metric g. Let (E,φ) be a flat Higgs bundle over M. For any Hermitian metric h on E, (2.4) and (2.9) together imply that

$$\int_M (\operatorname{tr} K^\varphi)\frac{\omega_g^n}{\nu} = n\deg_g E,$$

where K^φ denotes the extended mean curvature of (E,φ,h). Therefore, the Einstein factor γ of any affine Yang–Mills–Higgs metric on (E,φ) must satisfy the equation

$$\gamma\int_M \frac{\omega_g^n}{\nu} = n\,\mu_g(E). \tag{3.1}$$

Choose a background Hermitian metric h_0 on E. Any Hermitian metric h on E is represented by an endomorphism f of E such that

$$h(s,t) = h_0(f(s),t)$$

for all sections s and t of E. This endomorphism f is positive definite and self–adjoint with respect to h_0. As we pass from h_0 to h, the extended connection form, curvature form and mean curvature change as follows:

$$\theta^\varphi = \theta_0^\varphi + \big(f^{-1}\partial_0 f, f^{-1}[\varphi^*, f]\big), \tag{3.2}$$

$$\Omega^\varphi = \Omega_0^\varphi + \big([f^{-1}\partial_0 f, \varphi], \overline{\partial}(f^{-1}\partial_0 f) + [\varphi, f^{-1}[\varphi^*, f]], \overline{\partial}(f^{-1}[\varphi^*, f])\big), \tag{3.3}$$

$$K^\varphi = K_0^\varphi + \operatorname{tr}_g \overline{\partial}(f^{-1}\partial_0 f) + \operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]], \tag{3.4}$$

$$\operatorname{tr} K^\varphi = \operatorname{tr} K_0^\varphi - \operatorname{tr}_g \partial\overline{\partial}\log(\det f). \tag{3.5}$$

Here, θ^φ, Ω^φ and K^φ are defined with respect to h, and θ_0^φ, Ω_0^φ and K_0^φ are defined with respect to h_0. Moreover, $(\partial_0, \overline{\partial}) = (\partial^{h_0}, \overline{\partial})$ denotes the extended Hermitian connection on (E, h_0), and φ^* is the adjoint of φ with respect to h_0.

According to (3.4), we need to solve the equation

$$K_0^\varphi - \gamma \operatorname{id}_E + \operatorname{tr}_g \overline{\partial}(f^{-1}\partial_0 f) + \operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]] = 0,$$

where γ is determined by (3.1).

As done in the usual case, we will solve this equation by the continuity method. For $\varepsilon \in [0,1]$, consider the equation

$$L_\varepsilon(f) := K_0^\varphi - \gamma \operatorname{id}_E + \operatorname{tr}_g \overline{\partial}(f^{-1}\partial_0 f) + \operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]] + \varepsilon \log f = 0, \tag{3.6}$$

and let

$$J := \big\{\varepsilon \in (0,1] \big| \text{there is a smooth solution } f \text{ to } L_\varepsilon(f) = 0\big\}.$$

We will use the continuity method to show that $J = (0,1]$ for any simple flat Higgs bundle (E, φ), and then show that we may take $\varepsilon \longrightarrow 0$ to get a limit of solutions if (E, φ) is stable. Note that by Proposition 2.9, if (E, φ) is stable, then it is automatically simple.

The first step in the continuity method is to show that $1 \in J$ and so J is non–empty. The following proposition also yields, apart from the above mentioned inclusion, an appropriately normalized background metric h_0 on E.

Proposition 3.1. *There is a smooth Hermitian metric h_0 on E such that the equation $L_1(f) = 0$ has a smooth solution f_1. The metric h_0 satisfies the normalization $\operatorname{tr} K_0^\varphi = r\gamma$, where r is the rank of E, and γ is given by* (3.1).

Proof. As we have $\operatorname{tr} K^\varphi = \operatorname{tr} K$ for the extended mean curvature of any Hermitian metric on E, the proof of [**Lo09,** Proposition 7] also works for Higgs bundles. □

So we choose h_0 according to Proposition 3.1 and obtain the following:

Corollary 3.2. *The inclusion* $1 \in J$ *holds.*

3.1. Openness of *J*. Let $\mathrm{Herm}(E, h_0)$ be the real vector bundle of endomorphisms of E which are self-adjoint with respect to h_0. For any Hermitian metric h on E, we know that $[\varphi, \varphi^*]$ is anti-self-adjoint. Therefore, as in [**LT95,** Lemma 3.2.3], for any $f \in \mathrm{Herm}(E, h_0)$, we have

$$\widehat{L}(\varepsilon, f) := f L_\varepsilon(f) = f K^\varphi - \gamma f + \varepsilon f \log f \in \mathrm{Herm}(E, h_0). \tag{3.7}$$

Let $1 < p < \infty$, and let k be a sufficiently large integer.

Assume that $\varepsilon \in J$, meaning there is a smooth solution f_ε to $L_\varepsilon(f) = 0$, or equivalently $\widehat{L}(\varepsilon, f) = 0$. We will use the implicit function theorem to show that there is some $\delta > 0$, such that for every $\varepsilon' \in (\varepsilon - \delta, \varepsilon + \delta)$, there is a solution $f_{\varepsilon'}$ to $\widehat{L}(\varepsilon', f) = 0$ lying in $L^p_k \mathrm{Herm}(E, h_0)$. By choosing k large enough, it then follows that each $f_{\varepsilon'}$ is smooth. Thus $(\varepsilon - \delta, \varepsilon + \delta) \cap (0, 1] \subset J$, implying that J is open.

In order to be able to apply the implicit function theorem, we have to show that

$$\Xi := \frac{\delta}{\delta f} \widehat{L}(\varepsilon, f) : L^p_k \mathrm{Herm}(E, h_0) \longrightarrow L^p_{k-2} \mathrm{Herm}(E, h_0) \tag{3.8}$$

is an isomorphism of Banach spaces. For $\phi \in \mathrm{Herm}(E, h_0)$, the Higgs field φ does not contribute any derivatives of ϕ to $\Xi(\phi)$. So the following lemma from [**Lo09**] is still valid for Higgs bundles (see [**Lo09,** Lemma 9]):

Lemma 3.3. *The linear operator* Ξ *in* (3.8) *is elliptic Fredholm of index* 0.

Consequently, in order to be able to apply the implicit function theorem, it is enough to show that Ξ is injective.

As in [**Lo09,** p. 116], for an endomorphism f of E, which is positive definite and self-adjoint with respect to h_0, define

$$\partial_0^f := \mathrm{Ad}\, f^{-\frac{1}{2}} \circ \partial_0 \circ \mathrm{Ad}\, f^{\frac{1}{2}} \quad \text{and} \quad \overline{\partial}^f := \mathrm{Ad}\, f^{\frac{1}{2}} \circ \overline{\partial} \circ \mathrm{Ad}\, f^{-\frac{1}{2}}$$

and also

$$\varphi^f := (\mathrm{Ad}\, f^{\frac{1}{2}})(\varphi), \tag{3.9}$$

where

$$(\mathrm{Ad}\, s)(\psi) := s \circ \psi \circ s^{-1}$$

for an automorphism s and an endomorphism ψ of E.

Proposition 3.4. *Let* $\alpha \in \mathbb{R}$ *and* $\varepsilon \in (0, 1]$. *Let* f *be an endomorphism of* E*, which is positive definite and self-adjoint with respect to* h_0*, and let* $\phi \in \mathrm{Herm}(E, h_0)$. *Assume that* $\widehat{L}(\varepsilon, f) = 0$ *(see* (3.7)*) and*

$$\frac{\delta}{\delta f} \widehat{L}(\varepsilon, f)(\phi) + \alpha f \log f = \Xi(\phi) + \alpha f \log f = 0, \tag{3.10}$$

where Ξ is defined in (3.8)*. Then for* $\eta := f^{-\frac{1}{2}} \circ \phi \circ f^{-\frac{1}{2}}$*, we have*

$$-\operatorname{tr}_g \partial\overline{\partial}|\eta|^2 + 2\varepsilon|\eta|^2 + |\partial_0^f \eta|^2 + |\overline{\partial}^f \eta|^2 + 2\big|[\varphi^f, \eta]\big|^2 \leqslant -2\alpha h_0(\log f, \eta).$$

Proof. By definition of $\widehat{L}$, we have

$$\Xi(\phi) = \phi \circ L_\varepsilon(f) + f \circ \frac{\delta}{\delta f} L_\varepsilon(f)(\phi).$$

The first term vanishes because $\widehat{L}(\varepsilon, f) = 0$. From (3.10) it follows that

$$\frac{\delta}{\delta f} L_\varepsilon(f)(\phi) = -\alpha \log f.$$

The left-hand side can be computed as in [**LT95,** proof of Proposition 3.2.5]. The additional contribution due to the Higgs field is as follows:

$$\begin{aligned}\frac{d}{dt} \operatorname{tr}_g[\varphi, (f + t\phi)^{-1}[\varphi^*, f + t\phi]]\Big|_{t=0} &= \operatorname{tr}_g[\varphi, [f^{-1}\varphi^* f, f^{-1}\phi]] \\ &= f^{-\frac{1}{2}} \circ \operatorname{tr}_g[\varphi^f, [(\varphi^f)^*, \eta]] \circ f^{\frac{1}{2}}.\end{aligned}$$

Following [**LT95**], we write

$$P^f := \operatorname{tr}_g \overline{\partial}^f \partial_0^f \quad \text{and} \quad \Phi := f^{\frac{1}{2}} \circ \frac{\delta}{\delta f} (\log f)(\phi) \circ f^{-\frac{1}{2}},$$

and obtain

$$P^f(\eta) + \operatorname{tr}_g[\varphi^f, [(\varphi^f)^*, \eta]] + \varepsilon\Phi = -\alpha \log f.$$

We compute

$$\operatorname{tr}_g \left(h_0\big([\varphi^f, [(\varphi^f)^*, \eta]], \eta\big) + h_0\big(\eta, [\varphi^f, [(\varphi^f)^*, \eta]]^*\big)\right) = 2\big|[\varphi^f, \eta]\big|^2,$$

and together with the estimates in [**LT95**], the proposition follows. □

Proposition 3.5. *The subset J is open.*

Proof. We show that Ξ is injective. Take any ϕ such that $\Xi(\phi) = 0$. Setting $\alpha = 0$ in Proposition 3.4 we see that

$$-\operatorname{tr}_g \partial\overline{\partial}|\eta|^2 + 2\varepsilon|\eta|^2 \leqslant 0.$$

Therefore, the maximum principle gives that $|\eta|^2 = 0$. So $\phi = 0$, proving that Ξ is injective. As explained before, this completes the proof of Proposition 3.5. □

3.2. Closedness of J. As in [**Lo09,** Lemma 12], we have the following:

Lemma 3.6. *Let f be an endomorphism of E, which is positive definite and self-adjoint with respect to h_0. If $L_\varepsilon(f) = 0$ (defined in (3.6)) for some $\varepsilon > 0$, then $\det f = 1$.*

Let

$$f = f_\varepsilon \tag{3.11}$$

be the family of solutions constructed for $\varepsilon \in (\varepsilon_0, 1]$ in Corollary 3.2 and Proposition 3.5. Define

(3.12)
$$m := m_\varepsilon := \max_M |\log f_\varepsilon|, \quad \phi := \phi_\varepsilon := \frac{df_\varepsilon}{d\varepsilon}, \quad \eta := \eta_\varepsilon := f_\varepsilon^{-\frac{1}{2}} \circ \phi_\varepsilon \circ f_\varepsilon^{-\frac{1}{2}}.$$

As in [**Lo09,** Lemma 13], Lemma 3.6 implies the following:

Lemma 3.7. *For η_ε in (3.12),*

$$\operatorname{tr} \eta_\varepsilon = 0.$$

On M, consider the L^2 inner products on $\mathcal{A}^{p,q}(\operatorname{End} E)$ given by h_0, g and the volume form $\frac{\omega_g^n}{\nu}$. Then, we have the following proposition.

Proposition 3.8. *Let (E, φ) be a simple flat Higgs bundle over M. Let f be as in (3.11). Then there is a constant $C(m)$ depending only on M, g, ν, φ, h_0, and $m = m_\varepsilon$ such that for $\eta = \eta_\varepsilon$, we have*

$$||\overline{\partial}^f \eta||_{L^2}^2 + \big\|[\varphi^f, \eta]\big\|_{L^2}^2 \geqslant C(m) ||\eta||_{L^2}^2,$$

where φ^f is defined in (3.9).

Remark 3.9. Following [**Lo09**], henceforth $C(m)$ will always denote a constant depending on M, g, ν, φ, h_0 and m. However, the particular constant may change with the context. Similarly, C will denote a constant depending only on the initial data M, g, ν, φ and h_0, but not on ε or m.

Proof. Let $\psi := f^{-\frac{1}{2}} \circ \eta \circ f^{\frac{1}{2}}$. Then pointwise, we have

$$\begin{aligned} |\overline{\partial}^f \eta|^2 + \big|[\varphi^f, \eta]\big|^2 &= |f^{\frac{1}{2}} \circ \overline{\partial}\psi \circ f^{-\frac{1}{2}}|^2 + \big|f^{\frac{1}{2}} \circ [\varphi, \psi] \circ f^{-\frac{1}{2}}\big|^2 \\ &\geqslant C(m) \cdot \big(|\overline{\partial}\psi|^2 + \big|[\varphi, \psi]\big|^2\big). \end{aligned}$$

Integrating both sides of it over M with respect to the volume form $\frac{\omega_g^n}{\nu}$, we obtain

$$||\overline{\partial}^f \eta||_{L^2}^2 + \big\|[\varphi^f, \eta]\big\|_{L^2}^2 \geqslant C(m)\big(||\overline{\partial}\psi||_{L^2}^2 + \big\|[\varphi, \psi]\big\|_{L^2}^2\big). \tag{3.13}$$

The space $\mathcal{A}^{1,0}(\operatorname{End} E) \oplus \mathcal{A}^{0,1}(\operatorname{End} E)$ on M corresponds to the space of 1-forms on $M^{\mathbb{C}}$ with values in $\operatorname{End} E^{\mathbb{C}}$. It has a natural L^2 inner product induced by the L^2 inner products on $\mathcal{A}^{1,0}(\operatorname{End} E)$ and $\mathcal{A}^{0,1}(\operatorname{End} E)$. Consider the operator

$$L : \mathcal{A}^{0,0}(\operatorname{End} E) \longrightarrow \mathcal{A}^{1,0}(\operatorname{End} E) \oplus \mathcal{A}^{0,1}(\operatorname{End} E), \quad \chi \longmapsto \big([\varphi, \chi], \overline{\partial}\chi\big).$$

Its adjoint with respect to the L^2 inner products is

$$\begin{aligned} L^* &: \mathcal{A}^{1,0}(\operatorname{End} E) \oplus \mathcal{A}^{0,1}(\operatorname{End} E) \longrightarrow \mathcal{A}^{0,0}(\operatorname{End} E), \\ &(u, v) \longmapsto \operatorname{tr}_g[\varphi^*, u] + \overline{\partial}^* v. \end{aligned}$$

Using this, the right-hand side in (3.13) can be written as

$$C(m)||L\psi||^2_{L^2} = C(m)\,\langle L^*L\psi, \psi\rangle_{L^2}\,.$$

The operator L^*L is self-adjoint, and it is elliptic because $L^*L\chi$ is equivalent to $\overline{\partial}^*\overline{\partial}\chi$ up to zeroth-order derivatives of χ. For any χ in the kernel of L^*L, we have

$$0 = \langle L^*L\chi, \chi\rangle_{L^2} = ||L\chi||^2_{L^2} = ||\overline{\partial}\chi||^2_{L^2} + \big\|[\varphi\,, \chi]\big\|^2_{L^2},$$

and so χ is a locally constant section of $\operatorname{End} E$ satisfying $[\varphi, \chi] = 0$. Since (E, φ) is simple, it follows that the kernel of L^*L consists only of the constant multiples of the identity automorphism. As in [**Lo09,** proof of Proposition 14], Lemma 3.7 implies that ψ is L^2-orthogonal to the kernel of L^*L, and hence there is a constant $\lambda_1 > 0$ (the smallest positive eigenvalue of L^*L) such that

$$\langle L^*L\psi, \psi\rangle_{L^2} \geqslant \lambda_1||\psi||^2_{L^2}.$$

Combining this with the inequality in (3.13), it now follows that

$$||\overline{\partial}^f\eta||^2_{L^2} + \big\|[\varphi^f, \eta]\big\|^2_{L^2} \geqslant C(m)\,\langle L^*L\psi, \psi\rangle_{L^2} \geqslant C(m)||\psi||^2_{L^2} \geqslant C(m)||\eta||^2_{L^2}.$$

□

Proposition 3.10. *Let (E, φ) be a simple flat Higgs bundle over M. Then*

$$\max_M |\phi_\varepsilon| \leqslant C(m),$$

where ϕ_ε is defined in (3.12)*.*

Proof. This follows as in [**Lo09,** Proposition 16] from Propositions 3.4 and 3.8. □

Lemma 3.11. *Let f be as in Proposition* 3.8*. Then*

$$-\frac{1}{2}\operatorname{tr}_g \partial\overline{\partial}|\log f|^2 + \varepsilon|\log f|^2 \leqslant |K_0^\varphi - \gamma\operatorname{id}_E| \cdot |\log f|,$$

where K_0^φ is defined as in (2.8) *for h_0 (see* (3.5)*).*

Proof. Since $L_\varepsilon(f) = 0$ (see equation (3.6)), we have

$$K_0^\varphi - \gamma \operatorname{id}_E = -\operatorname{tr}_g \overline{\partial}(f^{-1}\partial_0 f) - \operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]] - \varepsilon \log f. \tag{3.14}$$

This implies that

$$\begin{aligned} |K_0^\varphi - \gamma \operatorname{id}_E| \cdot |\log f| &\geqslant \big|h_0(-(K_0^\varphi - \gamma \operatorname{id}_E), \log f)\big| \\ &\geqslant h_0(\operatorname{tr}_g \overline{\partial}(f^{-1}\partial_0 f) + \varepsilon \log f, \log f) \\ &\quad + h_0(\operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]], \log f) \end{aligned}$$

if both summands on the right-hand side are real. From [**LT95,** proof of Lemma 3.3.4 (i)], we know that the first summand is real and satisfies the condition

$$h_0(\operatorname{tr}_g \overline{\partial}(f^{-1}\partial_0 f) + \varepsilon \log f, \log f) \geqslant -\frac{1}{2} \operatorname{tr}_g \partial\overline{\partial}|\log f|^2 + \varepsilon|\log f|^2.$$

So to complete the proof of the proposition, it suffices to show that

$$h_0(\operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]], \log f) \in \mathbb{R}^{\geqslant 0}. \tag{3.15}$$

The argument for it is similar to the one in [**LT95**]. Over each point of M, we can write

$$f = \sum_{\alpha=1}^{r} \exp(\lambda_\alpha)\, e_\alpha \otimes e^\alpha$$

in a h_0-unitary frame $\{e_\alpha\}$ of E, where r is the rank of E, and $\{e^\alpha\}$ is the dual frame of E^*; the eigenvalues λ_α are real. Then we have

$$\log f = \sum_{\alpha=1}^{r} \lambda_\alpha\, e_\alpha \otimes e^\alpha,$$

and writing

$$\varphi = \sum_{\alpha,\beta=1}^{r} \varphi_\beta^\alpha\, e_\alpha \otimes e^\beta,$$

we compute

$$\begin{aligned} h_0(\operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]], \log f) &= -\operatorname{tr}_g \operatorname{tr}(f^{-1}[\varphi^*, f] \wedge [\varphi^*, \log f]^*) \\ &= -\operatorname{tr}_g \sum_{\alpha,\beta=1}^{r} (\exp(\lambda_\alpha - \lambda_\beta) - 1)\, \overline{\varphi_\beta^\alpha} \wedge (\lambda_\alpha - \lambda_\beta)\varphi_\beta^\alpha \\ &= \sum_{\alpha,\beta=1}^{r} (\exp(\lambda_\alpha - \lambda_\beta) - 1)(\lambda_\alpha - \lambda_\beta)|\varphi_\beta^\alpha|^2. \end{aligned}$$

Therefore, (3.15) holds because $x(\exp(x) - 1) \in \mathbb{R}^{\geqslant 0}$ for all $x \in \mathbb{R}$. We already noted that (3.15) completes the proof of the proposition. □

The following corollary can be derived from Lemma 3.11 as in [**Lo09,** Corollaries 18 and 19].

Corollary 3.12.

(i) $m \leqslant \varepsilon^{-1} C$, *where* m *is defined in* (3.12), *and* C *is as in Remark* 3.9, *and*
(ii) $m \leqslant C\big(||\log f||_{L^2} + 1\big)^2$.

Proposition 3.13. *Let* (E, φ) *be a simple flat Higgs bundle over* M. *Suppose there is an* $m \in \mathbb{R}$ *such that* $m_\varepsilon \leqslant m$ *for all* $\varepsilon \in (\varepsilon_0, 1]$. *Let* ϕ_ε *and* f_ε *be as in* (3.12). *Then for all* $p > 1$ *and* $\varepsilon \in (\varepsilon_0, 1]$,

$$||\phi_\varepsilon||_{L^p_2} \leqslant C(m)(1 + ||f_\varepsilon||_{L^p_2}),$$

where $C(m)$ *may depend on* p *as well as* m *along with the initial data.*

Proof. We proceed as in the proof of [**Lo09,** Proposition 21]. Similar to [**Lo09,** equation (19)], for the operator

$$\Lambda := n\, \partial_0^* \partial_0 + \mathrm{id}_E,$$

we obtain

$$\begin{aligned}\Lambda\phi = &- \phi\big(K_0^\varphi - (\gamma+1)\,\mathrm{id}_E + \varepsilon \log f + \mathrm{tr}_g[\varphi, f^{-1}[\varphi^*, f]]\big) \\ &- \mathrm{tr}_g(\overline{\partial} f \wedge f^{-1}\phi f^{-1}\partial_0 f) + \mathrm{tr}_g(\overline{\partial} f \wedge f^{-1}\partial_0\phi) + \mathrm{tr}_g(\overline{\partial}\phi \wedge f^{-1}\partial_0 f) \\ &- f\log f - \varepsilon f\left(\frac{\delta}{\delta f}\,\log f\right)(\phi) - n\,\frac{\partial_0\phi \wedge \overline{\partial}\omega_g^{n-1}}{\omega_g^n} \\ &- f\,\mathrm{tr}_g[\varphi, [f^{-1}\varphi^* f, f^{-1}\phi]].\end{aligned} \tag{3.16}$$

Compared to [**Lo09,** equation (19)], the right-hand side of equation (3.16) contains the two additional terms

$$-\phi\,\mathrm{tr}_g[\varphi, f^{-1}[\varphi^*, f]] \quad \text{and} \quad -f\,\mathrm{tr}_g[\varphi, [f^{-1}\varphi^* f, f^{-1}\phi]].$$

By Proposition 3.10, these are both bounded in L^p norm by $C(m)$. Consequently, the proposition follows as in [**Lo09**]. □

As in [**Lo09,** Corollary 22], we obtain the following corollary.

Corollary 3.14. *Suppose there is an* $m \in \mathbb{R}$ *such that* $m_\varepsilon \leqslant m$ *for all* $\varepsilon \in (\varepsilon_0, 1]$. *Then for all* $\varepsilon \in (\varepsilon_0, 1]$, *we have* $||f_\varepsilon||_{L^p_2} \leqslant C(m)$, *where* $C(m)$ *is independent of* ε.

Proposition 3.15. *Let (E,φ) be a simple flat Higgs bundle over M. Then*

(i) $J=(0,1]$, *and*
(ii) *if $||f_\varepsilon||_{L^2}$ (see (3.11)) is bounded independently of $\varepsilon\in(0,1]$, then there exists a smooth solution f_0 to the Yang–Mills–Higgs equation $L_0(f)=0$.*

Proof. For (i), it is enough to show that if $J=(\varepsilon_0,1]$ for $\varepsilon_0\in(0,1)$, then there is a smooth solution f_{ε_0} to $L_{\varepsilon_0}(f)=0$. Indeed, this implies that J is closed and so (i) follows from Corollary 3.2 and Proposition 3.5.

For (ii), we need to show the same for $\varepsilon_0=0$. In both cases, we know that there is a constant $C>0$ such that $||f_\varepsilon||_{L^p_2}\leqslant C$ for all $\varepsilon\in(\varepsilon_0,1]$. Indeed, in case of (i), this follows from Corollaries 3.12 (i) and 3.14, and in case of (ii), it follows from Corollaries 3.12 (ii), 3.14 and the hypothesis of (ii).

So assume that $J=(\varepsilon_0,1]$ for $\varepsilon_0\in[0,1)$ and that there is a constant $C>0$, such that $||f_\varepsilon||_{L^p_2}\leqslant C$ for all $\varepsilon\in(\varepsilon_0,1]$. We will find a sequence $\varepsilon_i\longrightarrow\varepsilon_0$ such that the limit $f_{\varepsilon_0}=\lim_{i\to\infty}f_{\varepsilon_i}$ is the required solution.

Choose $p>n$. Then L^p_1 maps compactly into C^0. The uniform L^p_2 norm bound implies that there is a sequence $\varepsilon_i\longrightarrow\varepsilon_0$ such that $f_{\varepsilon_i}\longrightarrow f_{\varepsilon_0}$ converges weakly in L^p_2 norm and strongly in L^p_1 norm as well as in C^0 norm.

For a smooth section α of $\operatorname{End}E$, we compute in the sense of distributions:

$$\begin{aligned}\langle L_{\varepsilon_0}(f_{\varepsilon_0}),\alpha\rangle_{L^2}&=\langle L_{\varepsilon_0}(f_{\varepsilon_0})-L_{\varepsilon_i}(f_{\varepsilon_i}),\alpha\rangle_{L^2}\\&=\int_M h_0\big(\operatorname{tr}_g\overline{\partial}(f_{\varepsilon_0}^{-1}\partial_0 f_{\varepsilon_0}-f_{\varepsilon_i}^{-1}\partial_0 f_{\varepsilon_i}),\alpha\big)\,\frac{\omega_g^n}{\nu}\\&\quad+\int_M h_0(\varepsilon_0\log f_{\varepsilon_0}-\varepsilon_i\log f_{\varepsilon_i},\alpha)\,\frac{\omega_g^n}{\nu}\\&\quad+\int_M h_0\big(\operatorname{tr}_g[\varphi,f_{\varepsilon_0}^{-1}[\varphi^*,f_{\varepsilon_0}]-f_{\varepsilon_i}^{-1}[\varphi^*,f_{\varepsilon_i}]],\alpha\big)\,\frac{\omega_g^n}{\nu}.\end{aligned}$$

The first two integrals go to zero as $i\longrightarrow\infty$ by [**Lo09,** proof of Proposition 23]. For the third integral, we can assume that $f_{\varepsilon_i}^{-1}\longrightarrow f_{\varepsilon_0}^{-1}$ strongly in L^p_1 norm and thus in C^0 norm (after going to a subsequence) since $f^{-1}=\exp(-\log f)$, and both exp and log maps on functions are continuous in L^p_1 norm. As $f_{\varepsilon_i}\longrightarrow f_{\varepsilon_0}$ in C^0 norm, the third integral also goes to zero as $i\longrightarrow\infty$. Therefore, $L_{\varepsilon_0}(f_{\varepsilon_0})=0$ in the sense of distributions.

In the same way, it can be shown that for $f_{\varepsilon_0}\in L^p_2$, we have $\operatorname{tr}_g\overline{\partial}\partial_0 f_{\varepsilon_0}\in L^p_1$. As in [**Lo09**], it then follows that f_{ε_0} is smooth and satisfies the equation $L_{\varepsilon_0}(f)=0$. □

3.3. Construction of a destabilizing subbundle. We will construct a destabilizing flat subbundle of (E,φ) if $\limsup_\varepsilon||f_\varepsilon||_{L^2}=\infty$. For a sequence $\varepsilon_i\longrightarrow 0$, we will re-scale by the reciprocal ρ_i of the largest eigenvalue of f_{ε_i}.

Then, we will show that the limit

$$\lim_{\sigma\to 0}\lim_{i\to\infty}(\rho_i f_{\varepsilon_i})^\sigma$$

exists, and each of its eigenvalues is 0 or 1. A projection to the destabilizing subbundle will be given by id_E minus this limit.

Proposition 3.16. *Let $\varepsilon > 0$ and $0 < \sigma \leqslant 1$. If $L_\varepsilon(f) = 0$, then*

$$-\frac{1}{\sigma}\operatorname{tr}_g \partial\overline{\partial}(\operatorname{tr} f^\sigma) + \varepsilon h_0(\log f, f^\sigma) + \left|f^{-\frac{\sigma}{2}}\partial_0(f^\sigma)\right|^2 + \left|f^{-\frac{\sigma}{2}}[\varphi^*, f^\sigma]\right|^2 \\ \leqslant -h_0(K_0^\varphi - \gamma\,\mathrm{id}_E, f^\sigma),$$

where f is as in (3.11) *and K_0^φ is defined as in* (2.8) *for h_0.*

Proof. Using (3.14), we have

$$-h_0\big(K_0^\varphi - \gamma\,\mathrm{id}_E, f^\sigma\big) = h_0\big(\operatorname{tr}_g \overline{\partial}(f^{-1}\partial_0 f) + \varepsilon\log f, f^\sigma\big) \\ + h_0\big(\operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]], f^\sigma\big).$$

By [**LT95,** proof of Lemma 3.4.4 (ii)], the first summand satisfies

$$h_0\big(\operatorname{tr}_g \overline{\partial}(f^{-1}\partial_0 f) + \varepsilon\log f, f^\sigma\big) \geqslant -\frac{1}{\sigma}\operatorname{tr}_g \partial\overline{\partial}(\operatorname{tr} f^\sigma) + \varepsilon h_0(\log f, f^\sigma) \\ + \left|f^{-\frac{\sigma}{2}}\partial_0(f^\sigma)\right|^2.$$

It remains to show that

$$h_0\big(\operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]], f^\sigma\big) \geqslant \left|f^{-\frac{\sigma}{2}}[\varphi^*, f^\sigma]\right|^2. \tag{3.17}$$

In the notation of Lemma 3.11, we have

$$\begin{aligned} &h_0\big(\operatorname{tr}_g[\varphi, f^{-1}[\varphi^*, f]], f^\sigma\big) \\ &\quad = -\operatorname{tr}_g \operatorname{tr}(f^{-1}[\varphi^*, f]\wedge[\varphi^*, f^\sigma]^*) \\ &\quad = -\operatorname{tr}_g \sum_{\alpha,\beta=1}^r (\exp(\lambda_\alpha - \lambda_\beta) - 1)\overline{\varphi^\alpha_\beta}\wedge(\exp(\sigma\lambda_\alpha) - \exp(\sigma\lambda_\beta))\varphi^\alpha_\beta \\ &\quad = \sum_{\alpha,\beta=1}^r (\exp(\lambda_\alpha - \lambda_\beta) - 1)(\exp(\sigma\lambda_\alpha) - \exp(\sigma\lambda_\beta))|\varphi^\alpha_\beta|^2 \\ &\quad \geqslant \sum_{\alpha,\beta=1}^r \exp(-\sigma\lambda_\beta)(\exp(\sigma\lambda_\alpha) - \exp(\sigma\lambda_\beta))^2|\varphi^\alpha_\beta|^2 \end{aligned}$$

because $(\exp(x) - 1)(\exp(\sigma x) - 1) \geqslant (\exp(\sigma x) - 1)^2$ for all $x \in \mathbb{R}$ and $0 \leqslant \sigma \leqslant 1$. The inequality in (3.17) now follows from

$$\begin{aligned}\left|f^{-\frac{\sigma}{2}}[\varphi^*, f^\sigma]\right|^2 &= -\operatorname{tr}_g \operatorname{tr}(f^{-\frac{\sigma}{2}}[\varphi^*, f^\sigma] \wedge (f^{-\frac{\sigma}{2}}[\varphi^*, f^\sigma])^*) \\ &= \sum_{\alpha,\beta=1}^{r} \exp(-\sigma\lambda_\beta)(\exp(\sigma\lambda_\alpha) - \exp(\sigma\lambda_\beta))^2 |\varphi^\alpha_\beta|^2.\end{aligned}$$

□

Now for $x \in M$, let $\lambda(\varepsilon, x)$ be the largest eigenvalue of $\log f_\varepsilon(x)$. Define

$$M_\varepsilon := \max_{x\in M} \lambda(\varepsilon, x) \quad \text{and} \quad \rho_\varepsilon := \exp(-M_\varepsilon). \tag{3.18}$$

Since $\det f_\varepsilon = 1$ by Lemma 3.6, it follows that $\rho_\varepsilon \leqslant 1$. As in [**Lo09,** Lemma 25], we have the following lemma.

Lemma 3.17. *Assume that* $\limsup_{\varepsilon\to 0} ||f_\varepsilon||_{L^2} = \infty$. *Then*

(i) $\rho_\varepsilon f_\varepsilon \leqslant \mathrm{id}_E$, *meaning that for every* $x \in M$, *and every eigenvalue* λ *of* $\rho_\varepsilon f_\varepsilon(x)$, *one has* $\lambda \leqslant 1$,
(ii) *for every* $x \in M$, *there is an eigenvalue* c_x *of* $\rho_\varepsilon f_\varepsilon(x)$ *with* $c_x \leqslant \rho_\varepsilon$, *where* ρ_ε *and* f_ε *are defined in* (3.18) *and* (3.11) *respectively,*
(iii) $\max_M \rho_\varepsilon |f_\varepsilon| \leqslant 1$, *and*
(iv) *there is a sequence* $\varepsilon_i \longrightarrow 0$ *such that* $\rho_{\varepsilon_i} \longrightarrow 0$.

Proposition 3.18. *There is a subsequence* $\varepsilon_i \longrightarrow 0$ *such that* $\rho_{\varepsilon_i} \longrightarrow 0$, *and for* $f_i := \rho_{\varepsilon_i} f_{\varepsilon_i}$ *(see Lemma* 3.17 *for notation),*

(i) f_i *converges weakly in* L^2_1 *norm to an* $f_\infty \neq 0$, *and*
(ii) *as* $\sigma \longrightarrow 0$, f^σ_∞ *converges weakly in* L^2_1 *norm to an* f^0_∞.

Proof. Apply the proof of [**Lo09,** Proposition 26] and use Corollary 3.12 (i), Proposition 3.16 and Lemma 3.17. □

Now let

$$\varpi := \mathrm{id}_E - f^0_\infty. \tag{3.19}$$

Proposition 3.19. *The endomorphism* ϖ *in* (3.19) *is an* h_0*-orthogonal projection onto a flat subbundle* $F := \varpi(E)$ *of* E, *which is preserved by the Higgs field* φ, *meaning it satisfies the identities*

$$\varpi^2 = \varpi, \quad \varpi^* = \varpi, \quad (\mathrm{id}_E - \varpi) \circ \overline{\partial}\varpi = 0 \quad \textit{and} \quad (\mathrm{id}_E - \varpi) \circ \varphi \circ \varpi = 0 \quad \textit{in } L^1.$$

Moreover, ϖ *is a smooth endomorphism of* E. *So the flat subbundle* F *is smooth.*

Proof. Following the proof of [**Lo09,** Proposition 27], using Proposition 3.16, Lemma 3.17 and Proposition 3.18 we conclude that

$$\varpi^2 = \varpi, \quad \varpi^* = \varpi \quad \text{and} \quad (\mathrm{id}_E - \varpi) \circ \overline{\partial}\varpi = 0 \quad \text{in } L^1$$

and that these imply that ϖ is a smooth endomorphism of E.

It remains to show that $(\mathrm{id}_E - \varpi) \circ \varphi \circ \varpi = 0$, so that the smooth flat subbundle $F = \varpi(E)$ is preserved by the Higgs field φ.

Applying the same argument as in [**Lo09**] and using Proposition 3.16, we compute for $0 < \sigma \leqslant 1$ and $0 < s \leqslant \frac{\sigma}{2}$:

$$\begin{aligned}\int_M \left|(\mathrm{id}_E - f_i^s)[\varphi^*, f_i^\sigma]\right|^2 \frac{\omega_g^n}{\nu} &\leqslant \left(\frac{s}{s+\frac{\sigma}{2}}\right)^2 \int_M \left|f_i^{-\frac{\sigma}{2}}[\varphi^*, f_i^\sigma]\right|^2 \frac{\omega_g^n}{\nu} \\ &\leqslant \left(\frac{s}{s+\frac{\sigma}{2}}\right)^2 \int_M |\varepsilon_i \log f_i + K_0^\varphi - \gamma \,\mathrm{id}_E| \cdot |f_i^\sigma| \cdot \frac{\omega_g^n}{\nu} \\ &\leqslant \left(\frac{s}{s+\frac{\sigma}{2}}\right)^2 C.\end{aligned}$$

A similar argument to the one in [**Lo09**] then gives that

$$\varpi \circ [\varphi^*, \mathrm{id}_E - \varpi] = 0.$$

Together with $\varpi^2 = \varpi$, this implies that

$$0 = -\varpi \circ [\varphi^*, \varpi] = \varpi \circ \varphi^* \circ (\mathrm{id}_E - \varpi),$$

and with $\varpi^* = \varpi$, it follows that

$$0 = \big(\varpi \circ \varphi^* \circ (\mathrm{id}_E - \varpi)\big)^* = (\mathrm{id}_E - \varpi) \circ \varphi \circ \varpi,$$

completing the proof of the proposition. □

Proposition 3.20. *The flat subbundle $F = \varpi(E) \subset E$ is a proper subbundle, meaning*

$$0 < \operatorname{rank} F < \operatorname{rank} E.$$

Proof. Apply the proof of [**Lo09,** Proposition 28], and use Lemma 3.17 and Proposition 3.18. □

Proposition 3.21. *The flat subbundle $F = \varpi(E)$ is a destabilizing subbundle, meaning*

$$\mu_g(F) \geqslant \mu_g(E).$$

Proof. As in [**Lo09,** proof of Proposition 29], by the Chern–Weil formula we have

$$\mu_g(F) = \mu_g(E) + \frac{1}{sn}\int_M \left(\operatorname{tr}((K_0 - \gamma \operatorname{id}_E)\varpi) - |\partial_0 \varpi|^2 \right) \frac{\omega_g^n}{\nu},$$

where s is the rank of F. Therefore, to complete the proof it suffices to show that

$$\int_M \operatorname{tr}((K_0 - \gamma \operatorname{id}_E)\varpi) \frac{\omega_g^n}{\nu} \geqslant \int_M |\partial_0 \varpi|^2 \frac{\omega_g^n}{\nu}. \tag{3.20}$$

Using the identity $K_0^\varphi = K_0 + \operatorname{tr}_g[\varphi, \varphi^*]$, we obtain that

$$\begin{aligned} \int_M \operatorname{tr}((K_0 - \gamma \operatorname{id}_E)\varpi) \frac{\omega_g^n}{\nu} &= \int_M \operatorname{tr}((K_0^\varphi - \gamma \operatorname{id}_E)\varpi) \frac{\omega_g^n}{\nu} \\ &\quad - \int_M \operatorname{tr}(\operatorname{tr}_g[\varphi, \varphi^*]\varpi) \frac{\omega_g^n}{\nu}. \end{aligned} \tag{3.21}$$

The first term in the right-hand side can be estimated as follows. Since

$$\varpi = \lim_{\sigma \to 0} \lim_{i \to \infty} (\operatorname{id}_E - f_i^\sigma)$$

strongly in L^2 norm, and $\operatorname{tr}(K_0^\varphi - \gamma \operatorname{id}_E) = 0$, we have

$$\int_M \operatorname{tr}((K_0^\varphi - \gamma \operatorname{id}_E)\varpi) \frac{\omega_g^n}{\nu} = - \lim_{\sigma \to 0} \lim_{i \to \infty} \int_M \operatorname{tr}((K_0^\varphi - \gamma \operatorname{id}_E) f_i^\sigma) \frac{\omega_g^n}{\nu},$$

and using equation (3.6), we see that

$$\begin{aligned} - \int_M \operatorname{tr}((K_0^\varphi - \gamma \operatorname{id}_E) f_i^\sigma) \frac{\omega_g^n}{\nu} &= \int_M \varepsilon_i \operatorname{tr}(\log(f_{\varepsilon_i}) f_i^\sigma) \frac{\omega_g^n}{\nu} \\ &\quad + \int_M \operatorname{tr}(\operatorname{tr}_g \overline{\partial}(f_i^{-1}\partial_0 f_i) f_i^\sigma) \frac{\omega_g^n}{\nu} \\ &\quad + \int_M \operatorname{tr}(\operatorname{tr}_g[\varphi, f_i^{-1}[\varphi^*, f_i]] f_i^\sigma) \frac{\omega_g^n}{\nu}. \end{aligned}$$

We estimate the first two integrals as in [**Lo09**] and the third integral as in the proof of Proposition 3.16. Together with $f_i \leqslant \operatorname{id}_E$, it then follows that

$$- \int_M \operatorname{tr}((K_0^\varphi - \gamma \operatorname{id}_E) f_i^\sigma) \frac{\omega_g^n}{\nu} \geqslant ||\partial_0(\operatorname{id}_E - f_i^\sigma)||_{L^2}^2 + \left\|[\varphi^*, \operatorname{id}_E - f_i^\sigma]\right\|_{L^2}^2.$$

Passing to the limit $i \longrightarrow \infty$ as in [**Lo09**], we obtain the following estimate of (3.21):

$$\int_M \operatorname{tr}((K_0 - \gamma \operatorname{id}_E)\varpi)\,\frac{\omega_g^n}{\nu} \geqslant ||\partial_0 \varpi||_{L^2}^2 + \big\|[\varphi^*, \varpi]\big\|_{L^2}^2 - \int_M \operatorname{tr}(\operatorname{tr}_g[\varphi, \varphi^*]\varpi)\,\frac{\omega_g^n}{\nu}. \tag{3.22}$$

Now, using $\varpi^2 = \varpi$, $\varpi^* = \varpi$ and $(\operatorname{id}_E - \varpi) \circ \varphi \circ \varpi = 0$, one shows that

$$\big\|[\varphi^*, \varpi]\big\|_{L^2}^2 = \int_M \operatorname{tr}(\operatorname{tr}_g[\varphi, \varphi^*]\varpi)\,\frac{\omega_g^n}{\nu}.$$

Therefore, the inequality in (3.20) follows from the one in (3.22). This completes the proof of the proposition. □

Proposition 3.21 completes the proof of Theorem 2.10.

4. Some consequences

Theorem 2.10 has the following corollary:

Corollary 4.1. *Let M be a compact special affine manifold equipped with an affine Gauduchon metric g, and let (E, φ) be a flat Higgs vector bundle over M. Then E admits an affine Yang–Mills–Higgs metric if and only if it is polystable. Moreover, a polystable flat Higgs vector bundle over M admits a unique Yang–Mills–Higgs connection.*

Proof. The "if" part follows immediately from Theorem 2.10.

For the "only if" part, assume that (E, φ) admits an affine Yang–Mills–Higgs metric h. Let F be a flat subbundle of E, which is preserved by the Higgs field φ.

The flat connection on E will be denoted by ∇. Let ∇_F be the flat connection on F induced by ∇, and let h_F be the Hermitian metric on F induced by h. Then for any section s of F, we have

$$\partial^{\nabla, h} s = \partial^{\nabla_F, h_F} s + A(s),$$

where $A \in \mathcal{A}^{1,0}(\operatorname{Hom}(F, F^\perp))$ is the second fundamental form, and $\partial^{\nabla,h}$ (respectively, ∂^{∇_F, h_F}) is the component of type $(1,0)$ of the extended Hermitian connection on E (respectively, F) with respect to h (respectively, h_F). Analogously, if φ_F is the flat Higgs field on F induced by φ, we write

$$\varphi^*(s) = \varphi_F^*(s) + \widetilde{\varphi}(s),$$

where φ^* and φ_F^* are the adjoints with respect to h and h_F, respectively, and $\widetilde{\varphi}$ is a $(0,1)$ form with values in $\operatorname{Hom}(F, F^\perp)$.

To complete the proof of the "only if" part, it suffices to show that $\mu_g(F) \leqslant \mu_g(E)$ with the equality holding if and only if A and $\widetilde{\varphi}$ vanish identically.

Denoting by s the rank of F, we compute

$$\mu_g(F) = \mu_g(E) - \frac{1}{sn}\int_M |A|^2 \, \frac{\omega_g^n}{\nu} - \frac{1}{sn}\int_M |\widetilde{\varphi}|^2 \, \frac{\omega_g^n}{\nu},$$

which implies that $\mu_g(F) \leqslant \mu_g(E)$ with the equality holding if and only if A and $\widetilde{\varphi}$ vanish identically.

To prove the uniqueness of the Yang–Mills–Higgs connection, first note that a stable flat Higgs bundle on M admits a unique Yang–Mills–Higgs connection, because any two Yang–Mills–Higgs metrics on it differ by a constant scalar (see Propositions 2.6 and 2.9). Write a polystable flat Higgs bundle (E, φ) as a direct sum of stable flat Higgs bundles. It was shown above that a Yang–Mills–Higgs connection on (E, φ) is the direct sum of Yang–Mills–Higgs connections on the stable direct summands. Therefore, (E, φ) admits a unique Yang–Mills–Higgs connection. □

Let us observe that the above results also hold for flat real Higgs bundles.

Definition 4.2. Let (E, φ) be a flat real Higgs bundle on a compact special affine manifold M equipped with an affine Gauduchon metric g.

(i) (E, φ) is called $\mathbb{R}$–*stable* (respectively, $\mathbb{R}$–*semistable*) if for every flat real subbundle F of E, with $0 < \operatorname{rank} F < \operatorname{rank} E$, which is preserved by φ, we have

$$\mu_g(F) < \mu_g(E) \quad \text{(respectively, } \mu_g(F) \leqslant \mu_g(E)\text{)}.$$

(ii) (E, φ) is called $\mathbb{R}$–*polystable* if

$$(E, \varphi) = \bigoplus_{i=1}^{N} (E_i, \varphi_i),$$

where each (E_i, φ_i) is an $\mathbb{R}$–stable flat real Higgs bundle with $\mu_g(E_i) = \mu_g(E)$.

Corollary 4.3. *Let M be a compact special affine manifold equipped with an affine Gauduchon metric, and let (E, φ) be a flat real Higgs vector bundle over M. Then (E, φ) admits an affine Yang–Mills–Higgs metric if and only if it is $\mathbb{R}$-polystable. Moreover, a polystable flat real Higgs vector bundle over M admits a unique Yang–Mills–Higgs connection.*

Proof. This follows from Corollary 4.1 as in [**Lo09,** Section 11]. □

4.1. Flat Higgs G-bundles. Any flat (real or complex) vector bundle over a compact affine manifold equipped with an affine Gauduchon metric has a unique Harder–Narasimhan filtration [**BL11**]. Using it and the above mentioned correspondence in [**Lo09**], the following can be proved:

Theorem 4.4 ([BL11]). *Let G be a reductive complex affine algebraic group. Let M be a compact special affine manifold equipped with an affine Gauduchon metric, and let E_G be a flat principal G-bundle over M. Then E admits an affine Yang–Mills connection if and only if E_G is polystable. Further, the Yang–Mills connection on a polystable flat bundle is unique.*

The above result remains valid if G is a reductive affine algebraic group over $\mathbb{R}$ of split type [**BL11**].

The proof of the existence and uniqueness of the Harder–Narasimhan filtration of a flat vector bundle goes through for a flat (real or complex) Higgs vector bundle. So a (real or complex) flat Higgs vector bundle over a compact affine manifold equipped with an affine Gauduchon metric has a unique Harder–Narasimhan filtration.

Let G be a reductive algebraic group. Let M be a compact affine manifold equipped with an affine Gauduchon metric g. Let E_G be a principal G-bundle over M equipped with a flat connection ∇^G. Let

$$\mathrm{ad}(E_G) := E_G \times^G \mathrm{Lie}(G)$$

be the adjoint vector bundle over M associated to E_G. Since the adjoint action of G on $\mathrm{Lie}(G)$ preserves the Lie algebra structure, each fiber of $\mathrm{ad}(E_G)$ is a Lie algebra isomorphic to $\mathrm{Lie}(G)$. If φ is a smooth section of $T^*M \otimes \mathrm{ad}(E_G)$, then using the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$, and the obvious projection $T^*M \otimes T^*M \longrightarrow \wedge^2 T^*M$, we get a smooth section of $(\wedge^2 T^*M) \otimes \mathrm{ad}(E_G)$, which we will denote by $[\varphi, \varphi]$.

The flat connection ∇^G on E_G induces a flat connection on $\mathrm{ad}(E_G)$; this flat connection on $\mathrm{ad}(E_G)$ will be denoted by ∇^{ad}. Let

$$\widetilde{\nabla}^{\mathrm{ad}} : T^*M \otimes \mathrm{ad}(E_G) \longrightarrow T^*M \otimes T^*M \otimes \mathrm{ad}(E_G)$$

be the flat connection on $T^*M \otimes \mathrm{ad}(E_G)$ defined by ∇^{ad} and the connection D^* on T^*M.

A *Higgs field* on the flat principal G-bundle (E_G, ∇^G) is a smooth section φ of $T^*M \otimes \mathrm{ad}(E_G)$ such that

(1) the section φ is flat with respect to the connection $\widetilde{\nabla}^{\mathrm{ad}}$ on $T^*M \otimes \mathrm{ad}(E_G)$, and
(2) $[\varphi, \varphi] = 0$.

A *Higgs G-bundle* is a flat principal G-bundle together with a Higgs field on it. (See [**Si92**] for Higgs G-bundles on complex manifolds.)

Let (E_G, ∇^G, φ) be a Higgs G-bundle on M. Fix a maximal compact subgroup $K \subset G$. Given a C^∞ reduction of structure group $E_K \subset E_G$, we have a natural connection ∇^{E_K} on the principal K-bundle E_K constructed using ∇^G; the connection on E_G induced by ∇^{E_K} will also be denoted by ∇^{E_K}. Given a C^∞ reduction of structure group $E_K \subset E_G$ to K, we may define as before the $(1,1)$-part of the extended curvature

$$\overline{\partial}\theta + [\varphi, \varphi^*],$$

which is a $(1,1)$-form with values in $\mathrm{ad}(E_G)$; as before, θ is a $(1,0)$-form with values in $\mathrm{ad}(E_G)$.

The reduction E_K is called a *Yang–Mills–Higgs* reduction of (E_G, ∇^G, φ) if there is an element γ of the center of $\mathrm{Lie}(G)$ such that the section

$$\mathrm{tr}_g(\overline{\partial}\theta + [\varphi, \varphi^*])$$

of $\mathrm{ad}(E_G)$ coincides with the one given by γ. If E_K is a Yang–Mills–Higgs reduction, then the connection ∇^{E_K} on E_G is called a *Yang–Mills–Higgs connection*.

The proof of Theorem 4.4 (see [**BL11**]) gives the following:

Corollary 4.5. *Let M be a compact special affine manifold equipped with an affine Gauduchon metric. Let G be either a reductive affine algebraic group over $\mathbb{C}$ or a reductive affine algebraic group over $\mathbb{R}$ of split type. Then a flat Higgs G-bundle (E_G, φ) over M admits a Yang–Mills–Higgs connection if and only if (E_G, φ) is polystable. Further, the Yang–Mills–Higgs connection on a polystable flat Higgs G-bundle is unique.*

4.2. A Bogomolov inequality. As before, M is a compact special affine manifold of dimension n equipped with a Gauduchon metric g. We assume that g is *astheno-Kähler*, meaning

$$\partial\overline{\partial}(\omega_g^{n-2}) = 0, \tag{4.1}$$

where ω_g is defined in (2.2) (see [**JY93,** p. 246]).

Proposition 4.6. *Let (E, φ) be a semistable flat Higgs vector bundle of rank r over M. Then*

$$\int_M \frac{c_2(\mathrm{End}(E)) \wedge \omega_g^{n-2}}{\nu} = \int_M \frac{(2r \cdot c_2(E) - (r-1)c_1(E)^2) \wedge \omega_g^{n-2}}{\nu} \geqslant 0.$$

Proof. First assume that (E, φ) is a polystable flat Higgs vector bundle. Consider an affine Yang–Mills–Higgs metric h on E given by Theorem 2.10. Then the integral of the n-form

$$\frac{(2r \cdot c_2(E, h) - (r-1)c_1(E, h)^2) \wedge \omega_g^{n-2}}{\nu}$$

on M coincides with the integral of a pointwise nonnegative n–form (see [**Si88,** p. 878–879, Proposition 3.4] and also [**LYZ,** p. 107] for the computation); here ν is the covariant constant volume form. Therefore,

$$\int_M \frac{(2r \cdot c_2(E,\, h) - (r-1)c_1(E\,,h)^2) \wedge \omega_g^{n-2}}{\nu} \geqslant 0.$$

Hence the inequality in the proposition is proved for polystable Higgs vector bundles.

If the flat Higgs bundle (E, φ) is semistable, then there is a filtration of flat subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$$

such that

- $\varphi(E_i) \subset T^*M \otimes E_i \subset T^*M \otimes E$ for all $i \in [0,\, \ell]$,
- the quotient E_i/E_{i-1} equipped with the Higgs field induced by φ is polystable for each $i \in [1\,, \ell]$, and
- $\mu_g(E_i/E_{i-1}) = \mu_g(E)$ for each $i \in [1\,, \ell]$.

We have shown that the inequality in the proposition holds for each E_i/E_{i-1}, $i \in [1\,, \ell]$. Therefore, the inequality holds for E. □

For a semistable flat Higgs G-bundle (E_G, φ) over M, the adjoint vector bundle $\mathrm{ad}(E_G)$ equipped with the Higgs field induced by φ is also semistable. Therefore, Proposition 4.6 gives a similar inequality for semistable flat Higgs G-bundles.

References

[BL11] I. Biswas and J. Loftin, *Hermitian–Einstein connections on principal bundles over flat affine manifolds*, Int. J. Math. **23**(4) (2012), `doi:10.1142/S0129167X12500395`.

[Do85] S.K. Donaldson, *Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985), 1–26.

[Do87a] S.K. Donaldson, *Infinite determinants, stable bundles and curvature*, Duke Math. J. **54** (1987), 231–247.

[Do87b] S.K. Donaldson, *Twisted harmonic maps and self-duality equations*, Proc. London Math. Soc. **55** (1987), 127–131.

[Hi87] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **55** (1987), 59–126.

[Ja11] A. Jacob, *Stable Higgs bundles and Hermitian–Einstein metrics on non-Kähler manifolds*, 2011, `arXiv:1110.3768v1` [math.DG].

[JY93] J. Jost and S.-T. Yau, *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*, Acta Math. **170** (1993), 221–254.

[LY87] J. Li and S.-T. Yau, *Hermitian–Yang–Mills connection on non-Kähler manifolds*, Mathematical aspects of string theory, Proc. Conf., San Diego/Calif. 1986, 560–573, Adv. Ser. Math. Phys. **1** (1987).

[LYZ] J. Li, S.-T. Yau and F. Zheng, *On projectively flat Hermitian manifolds*, Comm. Anal. Geom. **2** (1994), 103–109.

[Lo09] J. Loftin, *Affine Hermitian–Einstein metrics*, Asian J. Math. **13** (2009), 101–130.

[LT95] M. Lübke and A. Teleman, *The Kobayashi–Hitchin correspondence*, World Scientific, Singapore, 1995.

[Sc05] L. Schäfer, *tt^*-geometry and pluriharmonic maps*, Ann. Glob. Anal. Geom. **28** (2005), 285–300.

[Sc07] L. Schäfer, *A note on tt^*-bundles over compact nearly Kähler manifolds*, Geom. Dedicata **128** (2007), 107–112.

[Si88] C.T. Simpson, *Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), 867–918.

[Si92] C.T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 5–95.

[UY86] K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian–Yang–Mills connections in stable vector bundles*, Commun. Pure Appl. Math. **39** (1986), 257–293.

[UY89] K. Uhlenbeck and S.-T. Yau, *A note on our previous paper: on the existence of Hermitian Yang–Mills connections in stable vector bundles*, Commun. Pure Appl. Math. **42** (1989), 703–707.

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