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2-PLECTIC GEOMETRY, COURANT ALGEBROIDS, AND CATEGORIFIED PREQUANTIZATION

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A 2-plectic manifold is a manifold equipped with a closed nondegenerate 3-form, just as a symplectic manifold is equipped with a closed nondegenerate 2-form. In 2-plectic geometry one finds the higher analogues of many structures familiar from symplectic geometry. For example, any 2-plectic manifold has a Lie 2-algebra consisting of smooth functions and Hamiltonian 1-forms. This is equipped with a Poisson-like bracket which only satisfies the Jacobi identity up to "coherent chain homotopy". Over any 2-plectic manifold is a vector bundle equipped with extra structure called an exact Courant algebroid. This Courant algebroid is the 2-plectic analogue of a transitive Lie algebroid over a symplectic manifold. Its space of global sections also forms a Lie 2-algebra. We show that this Lie 2-algebra contains an important sub-Lie 2-algebra which is isomorphic to the Lie 2-algebra of Hamiltonian 1-forms. Furthermore, we prove that it is quasi-isomorphic to a central extension of the (trivial) Lie 2-algebra of Hamiltonian vector fields, and therefore is the higher analogue of the well-known Kostant–Souriau central extension in symplectic geometry. We interpret all of these results within the context of a categorified prequantization procedure for 2-plectic manifolds. In doing so, we describe how $U(1)$-gerbes, equipped with a connection and curving, and Courant algebroids are the 2-plectic analogues of principal $U(1)$ bundles equipped with a connection and their associated Atiyah Lie algebroids.

1. Introduction

A multisymplectic manifold is a smooth manifold equipped with a closed, nondegenerate form of degree ≥ 2 [**10**]. In this paper, we call a manifold "n-plectic" if the form has degree $(n+1)$. These manifolds naturally arise in certain covariant Hamiltonian formalisms for classical field theory [**16, 18, 28**]. In these formalisms, one describes a $(n+1)$-dimensional field theory by using a finite-dimensional n-plectic manifold as a "multi-phase space" instead of an infinite-dimensional phase space. The n-plectic form can be used to define a system of partial differential equations which are the analogue of Hamilton's equations in classical mechanics. The solutions to these equations correspond to particular submanifolds of the multi-phase space that encode the value of the field at each point in space-time as well as the values of its time and spatial derivatives.

Other formalisms, such as higher gauge theory [**4, 5, 33**], suggest that structures found in classical mechanics can be generalized by using higher category and homotopy theory and then applied to the study of field, string, and brane theories. Motivated by these ideas, we hypothesized in our previous work with Baez et al. [**2**] that the higher analogues of well-known algebraic and geometric structures on symplectic manifolds should naturally arise on n-plectic manifolds. Algebraically, this is indeed true. Just as a symplectic structure makes the ring of smooth functions a Poisson algebra, an n-plectic structure gives a Lie n-algebra on a n-term chain complex consisting of differential p-forms for $0 \leq p \leq n-2$ and certain $(n-1)$-forms, which we call Hamiltonian [**27**]. A Lie n-algebra (or n-term L_∞-algebra) is a higher analogue of a differential graded Lie algebra. It consists of a graded vector space concentrated in degrees $0, \ldots, n-1$ equipped with a collection of skew-symmetric k-ary brackets, for $1 \leq k \leq n+1$, that satisfy a generalized Jacobi identity [**20, 21**]. In particular, the $k = 2$ bilinear bracket behaves like a Lie bracket that only satisfies the ordinary Jacobi identity up to higher coherent chain homotopy. When $n = 1$, the relevant Lie 1-algebra is just the underlying Lie algebra of the usual Poisson algebra. When $n = 2$, we obtain a Lie 2-algebra whose underlying 2-term chain complex consists of smooth functions and Hamiltonian 1-forms.

Now let us consider the geometric picture. Interesting geometric structures appear, in particular, on prequantizable symplectic manifolds, i.e., those manifolds (M, ω) with the property that the integral of the symplectic form ω over any closed oriented 2-surface is an integer multiple of $2\pi\sqrt{-1}$. In this case, there exists a principal $U(1)$-bundle $P \xrightarrow{\pi} M$ over the manifold equipped with a connection whose curvature is π-related to the symplectic form. Equivalently, in terms of cohomology, the symplectic structure gives a representative of a degree 2 class in integer-valued cohomology, while the data encoding the principal bundle with connection give a representative of a degree 1 class in Deligne cohomology. Deligne cohomology can be interpreted as a refinement of the more familiar $U(1)$-valued Čech cohomology. In degree 1, it classifies not just principal $U(1)$-bundles, but principal $U(1)$-bundles equipped with connection.

Another geometric structure, called the Atiyah algebroid, is also present on a prequantized symplectic manifold. The Atiyah algebroid is an example of a Lie algebroid: roughly, a vector bundle $A \to M$ equipped with a bundle map to the tangent bundle of M, and a Lie algebra structure on its space of global sections. The total space of the Atiyah algebroid is the quotient $A = TP/U(1)$, where $P \to M$ is the aforementioned principal $U(1)$-bundle. Sections of A are $U(1)$-invariant vector fields on P. A connection on P is equivalent to a splitting of the short exact sequence

$$0 \to \mathbb{R} \times M \to A \xrightarrow{\pi_*} TM \to 0,$$

where the map $\mathbb{R} \times M \to A$ corresponds to identifying the vertical subspace of T_pP with the Lie algebra $\mathfrak{u}(1) \cong \mathbb{R}$. Those sections of A which preserve the connection (or splitting) form a Lie subalgebra that is isomorphic to the Poisson algebra. This implies that there is a well-defined action of the Poisson algebra on the $\mathbb{C}$-valued functions on P. Compactly supported global sections of the line bundle associated to P form a pre-Hilbert space and can be identified with $U(1)$-homogeneous $\mathbb{C}$-valued functions on P of degree -1. In this way, one obtains a faithful representation, or a quantization, of the Poisson algebra by linear operators on a Hilbert space. Moreover, if the symplectic manifold is connected, then the Poisson algebra gives what is known as the Kostant–Souriau central extension of the Lie algebra of Hamiltonian vector fields [**19**]. The symplectic form, evaluated at a point, gives a representative of the degree 2 class in the Lie algebra cohomology of the Hamiltonian vector fields (with values in the trivial representation) corresponding to this extension. The fact that this central extension is quantized, rather than the Hamiltonian vector fields themselves, is the reason why the concept of "phase" is introduced in quantum mechanics.

The process described above is known as prequantization [**19**]. It is the first step towards geometrically quantizing a symplectic manifold [**19, 36**]. We are interested in the higher analogues of the geometric structures described above. Indeed, the geometric quantization of what we call an n-plectic manifold remains a long-standing open problem. In this paper, we focus particularly on the prequantization of 2-plectic manifolds, since this is the first really new case of n-plectic geometry. Hence, we study prequantized 2-plectic manifolds and the 2-plectic analogues of principal $U(1)$-bundles, Atiyah algebroids, and the Kostant–Souriau central extension.

In analogy with the symplectic case, a 2-plectic manifold (M, ω) is prequantizable if the integral of the 2-plectic form ω over any closed oriented 3-surface is an integer multiple of $2\pi\sqrt{-1}$. Hence, the 2-plectic structure gives a representative of a degree 3 class in integer-valued cohomology. This degree 3 class corresponds to a (not necessarily unique) degree 2 class in Deligne cohomology. It is well known that a geometric object that realizes this degree 2 class is a $U(1)$-gerbe over M equipped with a connection and curving whose 3-curvature is ω [**8**]. Roughly, a $U(1)$-gerbe is a stack (or sheaf of groupoids) over M that is locally isomorphic to the stack of $U(1)$-bundles over M. Just as a connection on a principal $U(1)$-bundle is equivalent to specifying local 1-forms on M satisfying a cocycle condition, the connection and curving on a $U(1)$-gerbe correspond to specifying local 1-forms and 2-forms on M satisfying a pair of cocycle conditions. So, by going from symplectic geometry to 2-plectic geometry, we are replacing sets of local sections of a principal bundle (i.e., sheaves) by categories of principal bundles defined over open sets (i.e., stacks). Therefore the prequantization of a 2-plectic manifold is in some sense "categorified prequantization".

What is the 2-plectic analogue of the Atiyah algebroid? We answer this question by considering a more general problem: understanding the relationship between 2-plectic geometry and the theory of Courant algebroids. Roughly, a Courant algebroid is a vector bundle that generalizes the structure of a Lie algebroid equipped with a symmetric nondegenerate bilinear form on the fibers. They were first used by Courant [**12**] to study generalizations of pre-symplectic and Poisson structures in the theory of constrained mechanical systems. Curiously, many of the ingredients found in 2-plectic geometry are also found in the theory of "exact" Courant algebroids. An exact Courant algebroid is a Courant algebroid whose underlying vector bundle $C \to M$ is an extension of the tangent bundle by the cotangent bundle:

$$0 \to T^*M \to C \to TM \to 0.$$

In a letter to Weinstein, Ševera [**34**] described how exact Courant algebroids arise in 2-dimensional variational problems (e.g., bosonic string theory) and showed that they are classified up to isomorphism by the degree 3 de Rham cohomology of M. From any closed 3-form on M, one can explicitly construct an exact Courant algebroid equipped with an "isotropic" splitting of the above short exact sequence using local 1-forms and 2-forms that satisfy cocycle conditions [**6, 15, 17**].

Obviously, Ševera's classification implies that every 2-plectic manifold (M, ω) gives a unique exact Courant algebroid (up to isomorphism) C whose class is represented by the 2-plectic structure. However, there are more interesting similarities between 2-plectic structures and exact Courant algebroids. Roytenberg and Weinstein [**29**] showed that the bracket on the space of global sections of a Courant algebroid induces an L_∞ structure. If we are considering an exact Courant algebroid, then the global sections can be identified with vector fields and 1-forms on the base space. Roytenberg and Weinstein's results imply that these sections, when combined with the smooth functions on the base space, form a Lie 2-algebra [**32**]. Moreover, the "higher brackets" of the Lie 2-algebra encode a closed 3-form representing the Ševera class [**35**].

The first result we present in this paper is that there exists a Lie 2-algebra morphism which embeds the Lie 2-algebra of Hamiltonian 1-forms on a 2-plectic manifold (M, ω) into the Lie 2-algebra of global sections of the corresponding exact Courant algebroid C equipped with an isotropic splitting. Moreover, this morphism gives an isomorphism between the Lie 2-algebra of Hamiltonian 1-forms and the sub Lie 2-algebra consisting of sections of C, which preserve the splitting via a particular kind of adjoint action. This result holds without any integrality condition on the 2-plectic structure. However, its meaning becomes clear in the context of prequantization: it is the higher analogue of the isomorphism between the underlying Lie algebra of the Poisson algebra on a prequantized symplectic manifold and the Lie sub-algebra of sections of the Atiyah algebroid that preserve the connection on the associated principal bundle. Hence, we see that the 2-plectic analogue of the Atiyah algebroid associated to a principal $U(1)$-bundle is an exact Courant algebroid associated to a $U(1)$-gerbe. This idea that exact Courant algebroids are "higher Atiyah algebroids" has been discussed previously in the literature [**6, 15**]. However, this is the first time the analogy has been understood using Lie n-algebras within the context of multisymplectic geometry.

The second result presented here involves identifying the 2-plectic analogue of the Kostant–Souriau central extension and therefore the source of "phase" in categorified prequantization. On a 2-plectic manifold, associated to every Hamiltonian 1-form is a Hamiltonian vector field. These vector fields form a Lie algebra, which we can view as a trivial Lie 2-algebra whose underlying chain complex is concentrated in degree 0, and whose bracket satisfies the Jacobi identity on the nose. For any 1-connected (i.e., connected and simply connected) 2-plectic manifold, we show that the Lie 2-algebra of Hamiltonian 1-forms is quasi-isomorphic to a "strict central extension" of the trivial Lie 2-algebra of Hamiltonian vector fields by the abelian Lie 2-algebra $\mathbb{R} \to 0$. This abelian Lie 2-algebra is known as $b\mathfrak{u}(1)$. Furthermore, we show this extension corresponds to a degree 3 class in the Lie algebra cohomology of the Hamiltonian vector fields with values in the trivial representation. In analogy with the symplectic case, a 3-cocycle representing this class can be constructed by using the 2-plectic form. It follows from the aforementioned results relating a 2-plectic manifold (M, ω) to the Courant algebroid C that the sub Lie 2-algebra of sections of C that preserve the splitting is also quasi-isomorphic to this central extension, and can be interpreted as the quantization of the Lie 2-algebra of Hamiltonian 1-forms. Phases originate from the presence of $b\mathfrak{u}(1)$, which integrates to an important Lie 2-group called $BU(1)$.

In the next section, we briefly review the construction of transitive Lie algebroids on symplectic manifolds and describe an embedding of the Poisson algebra into the Lie algebra of sections of the algebroid. We recall some basic facts concerning Deligne cohomology and then consider prequantized symplectic manifolds. We emphasize the role played by the Atiyah algebroid in prequantization and the construction of the Kostant–Souriau central extension. The remainder of the paper is devoted to the 2-plectic analogue.

In Sections 3 and 4, we introduce 2-plectic manifolds and Courant algebroids as well as review Ševera's classification theorem for exact Courant algebroids. Section 5 contains a description of the geometric relationship between 2-plectic manifolds and exact Courant algebroids. After reviewing Lie 2-algebras in Section 6, we present the algebraic relationship between 2-plectic and Courant in Section 7. In Section 8, we introduce prequantized 2-plectic manifolds and describe how the exact Courant algebroid plays the role of a higher Atiyah algebroid. We then present in Section 9 the 2-plectic analogue of the Kostant–Souriau central extension. We assume the reader is comfortable with basic results in symplectic geometry and geometric quantization, but not necessarily familiar with Deligne cohomology, gerbes, Courant algebroids, or Lie 2-algebras. Therefore, our presentation of these topics is mostly self-contained.

2. Lie algebroids, symplectic manifolds, and prequantization

2.1. Lie algebroids from closed 2-forms. We begin by reviewing the construction of a Lie algebroid which ultimately will describe how phases arise in the prequantization of symplectic manifolds. A section of this Lie algebroid is a vector field on the base manifold together with a "phase", or more precisely, a real-valued function.

Recall that a **Lie algebroid** [**24**] over a manifold M is a real vector bundle $A \to M$ equipped with a bundle map (called the anchor) $\rho\colon A \to TM$, and a Lie algebra bracket $[\cdot,\cdot]_A\colon \Gamma(A) \otimes \Gamma(A) \to \Gamma(A)$ such that the induced map

$$\Gamma(\rho)\colon \Gamma(A) \to \mathfrak{X}(M)$$

is a morphism of Lie algebras, and for all $f \in C^\infty(M)$ and $e_1, e_2 \in \Gamma(A)$ we have the Leibniz rule

$$[e_1, fe_2]_A = f\,[e_1, e_2]_A + \rho(e_1)(f)e_2.$$

A Lie algebroid with surjective anchor map is called a **transitive Lie algebroid**.

The main ideas of the following construction are presented in Section 17 of Cannas da Silva and Weinstein [**9**]. We provide the details here in order to compare to the 2-plectic case in Section 5. Let (M,ω) be a manifold equipped with a closed 2-form, e.g., a pre-symplectic manifold. By a **trivialization** of ω, we mean a cover $\{U_i\}$ of M, equipped with 1-forms $\theta_i \in \Omega^1(U_i)$, and smooth functions $g_{ij} \in C^\infty(U_i \cap U_j)$ such that

$$\omega|_{U_i} = d\theta_i, \tag{2.1}$$

$$(\theta_j - \theta_i)|_{U_{ij}} = dg_{ij}, \tag{2.2}$$

where $U_{ij} = U_i \cap U_j$. Every manifold admits a good cover (i.e., a cover where all nonempty finite intersections $U_{i_i \dots i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$ are contractible), hence every closed 2-form admits a trivialization. Given such a trivialization of ω, we can construct a transitive Lie algebroid over M. Over each U_i we consider the Lie algebroid

$$A_i = TU_i \oplus \mathbb{R} \to U_i,$$

with bracket

$$[v_1 + f_1, v_2 + f_2]_i = [v_1, v_2] + v_1(f_2) - v_2(f_1)$$

for all $v_i + f_i \in \mathfrak{X}(U_i) \oplus C^\infty(U_i)$, and anchor ρ given by the projection onto TU_i. From the 1-forms $dg_{ij} \in \Omega^1(U_{ij})$, we can construct transition functions

$$G_{ij} \colon U_{ij} \to GL(n+1),$$
$$G_{ij}(x) = \begin{pmatrix} 1 & 0 \\ dg_{ij}|_x & 1 \end{pmatrix},$$

which act on a point $v_x + r \in A_i|_{U_{ij}}$ by

$$G_{ij}(x)(v_x + r) = v_x + r + dg_{ij}(v_x).$$

Clearly, each G_{ij} satisfies the cocycle conditions on U_{ijk} by virtue of equation (2.2). Therefore, we have over M the vector bundle

$$A = \coprod_{x \in M} T_x U_i \oplus \mathbb{R}/\sim,$$

where the equivalence is defined via the functions G_{ij} in the usual way. For any sections $v_i + f_i$ of $A_i|_{U_{ij}}$, a direct calculation shows that

$$[G_{ij}(v_1 + f_1), G_{ij}(v_2 + f_2)]_i = G_{ij}([v_1, v_2] + v_1(f_2) - v_2(f_1)).$$

Hence the local bracket descends to a well-defined bracket $[\cdot,\cdot]_A$ on the quotient. Henceforth, $(A, [\cdot,\cdot]_A, \rho)$ will denote this transitive Lie algebroid associated to the closed 2-form ω.

It is easy to see that the above Lie algebroid is an extension of the tangent bundle

$$0 \to M \times \mathbb{R} \to A \xrightarrow{\rho} TM \to 0.$$

Moreover, the 1-forms $\theta_i \in \Omega^1(U_i)$ induce a splitting

$$s \colon TM \to A$$

of the above sequence defined as

$$s(v_x) = v_x - \theta_i(v_x), \quad \forall\ v_x \in TU_i. \tag{2.3}$$

By a slight abuse of notation, we denote the horizontal lift $\Gamma(s) \colon \mathfrak{X}(M) \to \Gamma(A)$ also by s. Hence, every section $e \in \Gamma(A)$ is of the form $e = s(v) + f$, for some $v \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Using the local definition of the splitting and the fact that $\omega|_{U_i} = d\theta_i$, a direct calculation shows that

$$[s(v_1) + f_1, s(v_2) + f_2]_A = s\big([v_1, v_2]\big) + v_1(f_2) - v_2(f_1) - \iota_{v_2}\iota_{v_1}\omega, \tag{2.4}$$

for all sections $s(v_i) + f_i$. The failure of the splitting $s \colon TM \to A$ to preserve the Lie bracket on sections is measured by the 2-form ω:

$$[s(v_1), s(v_2)]_A = s([v_1, v_2]) - \omega(v_1, v_2), \quad \forall v_1, v_2 \in \mathfrak{X}(M).$$

It is a simple exercise to show that a different choice of trivialization gives a Lie algebroid equipped with a splitting that is isomorphic to A equipped with the splitting given in equation (2.3).

2.2. The Poisson algebra. Let (M, ω) be a symplectic manifold. Here $\{f, g\} = \omega(v_f, v_g)$ denotes the Poisson bracket on smooth functions. The vector field v_f, satisfying the equality $df = -\iota_{v_f}\omega$, is the unique Hamiltonian vector field corresponding to the function f. We denote the Lie algebra of Hamiltonian vector fields by $\mathfrak{X}_{\mathrm{Ham}}(M)$. Let $(A, [\cdot,\cdot]_A, \rho)$ be the Lie algebroid associated to ω and $s \colon TM \to A$ be the splitting defined in equation (2.3). We are interested in a particular Lie sub-algebra of $\Gamma(A)$ acting on the subspace $s(\mathfrak{X}(M)) \subseteq \Gamma(A)$ via the adjoint action.

Definition 2.1. A section $a = s(v) + f \in \Gamma(A)$ **preserves the splitting** $s \colon TM \to A$ iff $\forall v' \in \mathfrak{X}(M)$

$$\big[a, s(v')\big]_A = s([v, v']).$$

The subspace of sections that preserve the splitting is denoted as $\Gamma(A)^s$.

Proposition 2.2. *$\Gamma(A)^s$ is a Lie subalgebra of $\Gamma(A)$.*

Proof. Follows directly from the fact that the bracket on $\Gamma(A)$ and the bracket on $\mathfrak{X}(M)$ both satisfy the Jacobi identity. □

It is easy to show that a section $s(v) + f$ preserves the splitting if and only if $v = v_f$. In fact:

Proposition 2.3. *The underlying Lie algebra of the Poisson algebra $\big(C^\infty(M), \{\cdot,\cdot\}\big)$ is isomorphic to the Lie algebra $\big(\Gamma(A)^s, [\cdot,\cdot]_A\big)$.*

Proof. For any vector field $v' \in \mathfrak{X}(M)$, it follows from equation (2.4) that we have $[s(v) + f, s(v')]_A = s([v, v'])$ if and only if

$$v'(f) + \omega(v, v') = 0,$$

and hence $df = -\iota_v\omega$. Therefore the injective map

$$\phi \colon C^\infty(M) \to \Gamma(A)^s, \quad \phi(f) = s(v_f) + f$$

is also surjective. If v_f and v_g are Hamiltonian vector fields corresponding to the functions f and g, respectively, then

$$\begin{aligned}
[\phi(f), \phi(g)]_A &= [s(v_f) + f, s(v_g) + g]_A \\
&= s([v_f, v_g]) + \big(v_f(g) - v_g(f)\big) - \iota_{v_g}\iota_{v_f}\omega \\
&= s([v_f, v_g]) + \omega(v_f, v_g) \\
&= \phi(\{f, g\}).
\end{aligned}$$

□

2.3. Deligne cohomology. We now briefly review some basic facts concerning smooth Deligne cohomology. We will mainly use this as a convenient language for dealing with geometric objects, such as principal $U(1)$-bundles or $U(1)$-gerbes, equipped with extra structure. Our presentation follows Section 3 of Carey et al. [**11**]. For more details, we refer the reader to the book by Brylinski [**8**].

Let $\underline{U(1)}$ and Ω^k denote the sheaves of smooth $U(1)$-valued functions and differential k-forms, respectively, on a manifold M. Consider the exact sequence of sheaves $D_p^\bullet$:

$$\underline{U(1)} \xrightarrow{d\log} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p, \quad p \geq 1.$$

Define the Deligne cohomology $H^\bullet(M, D_p^\bullet)$ to be the Čech hyper-cohomology of $D_p^\bullet$. This is the total cohomology of the double complex:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & & \vdots \\
\underline{U(1)}(\mathcal{U}^{[2]}) \arrow[r, "d\log"] \arrow[u, "\delta"] & \Omega^1(\mathcal{U}^{[2]}) \arrow[r, "d"] \arrow[u, "\delta"] & \Omega^2(\mathcal{U}^{[2]}) \arrow[r, "d"] \arrow[u, "\delta"] & \cdots \arrow[r, "d"] & \Omega^p(\mathcal{U}^{[2]}) \arrow[u, "\delta"] \\
\underline{U(1)}(\mathcal{U}^{[1]}) \arrow[r, "d\log"] \arrow[u, "\delta"] & \Omega^1(\mathcal{U}^{[1]}) \arrow[r, "d"] \arrow[u, "\delta"] & \Omega^2(\mathcal{U}^{[1]}) \arrow[r, "d"] \arrow[u, "\delta"] & \cdots \arrow[r, "d"] & \Omega^p(\mathcal{U}^{[1]}) \arrow[u, "\delta"] \\
\underline{U(1)}(\mathcal{U}^{[0]}) \arrow[r, "d\log"] \arrow[u, "\delta"] & \Omega^1(\mathcal{U}^{[0]}) \arrow[r, "d"] \arrow[u, "\delta"] & \Omega^2(\mathcal{U}^{[0]}) \arrow[r, "d"] \arrow[u, "\delta"] & \cdots \arrow[r, "d"] & \Omega^p(\mathcal{U}^{[0]}) \arrow[u, "\delta"]
\end{tikzcd}$$

where $\mathcal{U} = \{U_i\}$ is a good cover of M, δ is the usual Čech co-boundary operator, and $\underline{U(1)}(\mathcal{U}^{[n]})$ and $\Omega^k(\mathcal{U}^{[n]})$ denote the abelian groups

$$\underline{U(1)}(\mathcal{U}^{[n]}) = \prod_{i_0 \neq i_1 \neq \cdots \neq i_n} \underline{U(1)}\left(U_{i_0} \cap U_{i_2} \cdots \cap U_{i_n}\right),$$

$$\Omega^k(\mathcal{U}^{[n]}) = \prod_{i_0 \neq i_1 \neq \cdots \neq i_n} \Omega^k\left(U_{i_0} \cap U_{i_2} \cdots \cap U_{i_n}\right).$$

We will focus on the groups $H^p(M, D_p^\bullet)$. They can be thought of as a refinement of the usual Čech cohomology groups $H^\bullet(M, \underline{U(1)})$. In particular, there is a surjection

$$H^p(M, D_p^\bullet) \twoheadrightarrow H^p(M, \underline{U(1)}).$$

Hence, via the usual isomorphism $H^p(M, \underline{U(1)}) \cong H^{p+1}(M, \mathbb{Z})$, we have a surjection

$$c\colon H^p(M, D_p^\bullet) \twoheadrightarrow H^{p+1}(M, \mathbb{Z}). \tag{2.5}$$

If $[\xi] \in H^p(M, D_p^\bullet)$, then $c([\xi])$ is called the **Chern class** of $[\xi]$.

There is also a map of complexes

$$\begin{tikzcd}
\underline{U(1)} \arrow[r, "d\log"] \arrow[d] & \Omega^1 \arrow[r, "d"] \arrow[d] & \Omega^2 \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & \Omega^p \arrow[d, "d"] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & \Omega^{p+1}
\end{tikzcd}$$

where d is the de Rham differential. This induces a map

$$\kappa\colon H^p(M, D_p^\bullet) \to \Omega_{\mathrm{cl}}^{p+1}(M), \tag{2.6}$$

where $\Omega_{\mathrm{cl}}^{p+1}(M)$ are the closed $(p+1)$-forms on M. If $[\xi] \in H^p(M, D_p^\bullet)$, then $\kappa([\xi])$ is called the **$(\mathbf{p+1})$-curvature** of $[\xi]$. If $\jmath\colon H^k(M, \mathbb{Z}) \to H^k(M, \mathbb{R})$ is the map induced from the inclusion of the constant sheaves $\mathbb{Z} \hookrightarrow \mathbb{R}$, then one can prove that $\jmath(c([\xi])) \in H^{p+1}(M, \mathbb{R})$ corresponds to the class $(-1)^{p-1}[\kappa([\xi])] \in H_{\mathrm{DR}}^{p+1}(M)$ via the isomorphism between Čech and de Rham cohomology.

2.4. Prequantization. A symplectic manifold (M, ω) admits a **prequantization** iff the cohomology class $[\omega]$ lies in the image of the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \cong H_{\mathrm{DR}}^2(M)$. By virtue of equation (2.5), there exists a Deligne class in $H^1(M, D_1^\bullet)$ whose 2-curvature is ω. By definition, a representative of this class defined on a good cover $\{U_i\}$ is a collection of 1-forms $\{\theta_i \in \Omega^1(U_i)\}$, and $U(1)$-valued functions $\{g_{ij}\colon U_{ij} \to U(1)\}$ such that

$$\begin{aligned}
\omega &= d\theta_i \quad \text{on } U_i,\\
\theta_j - \theta_i &= g_{ij}^{-1} dg_{ij} \quad \text{on } U_{ij},\\
g_{jk} g_{ik}^{-1} g_{ij} &= 1 \quad \text{on } U_{ijk}.
\end{aligned}$$

Hence a 1-cocycle is a principal $U(1)$-bundle $P \xrightarrow{\pi} M$ equipped with a connection $\theta \in \Omega^1(P)$ with curvature $d\theta = \pi^*\omega$. A symplectic manifold equipped with such a 1-cocycle is said to be **prequantized**.

The Deligne 1-cocycle also gives, of course, a trivialization of the 2-form ω, and therefore the transitive Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ over M equipped with the splitting $s\colon TM \to A$. However in this case, the functions $\{g_{ij}\colon U_{ij} \to U(1)\}$ are the transition functions of the bundle P. Therefore, by identifying $\mathfrak{u}(1)$ with $\mathbb{R}$, we see that A is isomorphic to the **Atiyah algebroid** $TP/U(1)$. A point in A corresponds to a vector field along the fiber $\pi^{-1}(x)$ that is invariant under the right $U(1)$ action. Hence, a global section of A corresponds to a $U(1)$-invariant vector field on P.

In general, splittings of $0 \to M \times \mathbb{R} \to A \to TM \to 0$ correspond to connection 1-forms on P. The connection 1-form $\theta \in \Omega^1(P)$ induces a "left-splitting" $\hat{\theta}\colon A \to M \times \mathbb{R}$ such that $\hat{\theta} \circ s = 0$. It is straightforward to show that $a \in \Gamma(A)^s$ if and only if

$$\mathcal{L}_a \theta = 0.$$

That is, a section of the Atiyah algebroid preserves the splitting if and only if it preserves the corresponding connection on P. For a prequantized symplectic manifold, the Lie algebra $\Gamma(A)^s$ is a Lie sub-algebra of derivations on $C^\infty(P)_\mathbb{C}$ and therefore on the global sections of the associated line bundle of P. Proposition 2.3 then implies that we have a faithful representation, or quantization, of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$.

2.5. The Kostant–Souriau central extension. If (M,ω) is a connected symplectic manifold, then we have a short exact sequence of Lie algebras

$$0 \to \mathfrak{u}(1) \to C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M) \to 0. \tag{2.7}$$

The underlying Lie algebra of the Poisson algebra is known as the Kostant–Souriau central extension of the Lie algebra of Hamiltonian vector fields [**19**]. If $\sigma\colon \mathfrak{X}_{\mathrm{Ham}}(M) \to C^\infty(M)$ is a splitting of the underlying sequence of vector spaces, then the failure of σ to be a strict (i.e., bracket-preserving) Lie algebra morphism is measured by the difference

$$\{\sigma(v_1), \sigma(v_2)\} - \sigma([v_1, v_2]),$$

which represents a degree 2 class in the Chevalley–Eilenberg cohomology $H^2_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$. This class can be represented by using the symplectic form. More specifically, pick a point $x \in M$ and let $c \in \mathrm{Hom}(\Lambda^2 \mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$ be the cochain given by:

$$c(v, v') = -\omega(v, v')|_x, \quad \forall v, v' \in \mathfrak{X}_{\mathrm{Ham}}(M).$$

The fact that c is a cocycle follows from the bracket $\{\cdot,\cdot\}$ satisfying the Jacobi identity. One can show that the class $[c]$ does not depend on the choice of $x \in M$.

If (M,ω) is a prequantized connected symplectic manifold, then Prop. 2.3 implies that the "quantized Poisson algebra" gives an isomorphic central extension

$$0 \to \mathfrak{u}(1) \to \Gamma(A)^s \to \mathfrak{X}_{\mathrm{Ham}}(M) \to 0.$$

This central extension is responsible for introducing phases into the quantized system. Two functions f and f' differing by a constant $r \in \mathfrak{u}(1)$ will have the same Hamiltonian vector fields and therefore give the same flows on M. However, their quantizations will give unitary transformations, which differ by a phase $\exp(2\pi\sqrt{-1}r)$.

3. 2-plectic geometry

In this section, we give an overview of 2-plectic geometry. Motivation for the definitions presented here, as well as examples and applications can be found in previous work [**2,3**]. All of the following definitions and propositions generalize to arbitrary n-plectic manifolds, so we refer the reader to our recent work [**27**] for proofs and more details.

Definition 3.1. A 3-form ω on a smooth manifold M is **2-plectic**, or more specifically a **2-plectic structure**, if it is both closed:

$$d\omega = 0,$$

and nondegenerate:

$$\forall v \in T_x M, \quad \iota_v \omega = 0 \Rightarrow v = 0$$

If ω is a 2-plectic form on M we call the pair (M, ω) a **2-plectic manifold**.

The 2-plectic structure induces an injective map from the space of vector fields on M to the space of 2-forms on M. This leads us to the following definition:

Definition 3.2. Let (M, ω) be a 2-plectic manifold. A 1-form α on M is **Hamiltonian** if there exists a vector field v_α on M such that

$$d\alpha = -\iota_{v_\alpha} \omega.$$

We say v_α is the **Hamiltonian vector field** corresponding to α. The set of Hamiltonian 1-forms and the set of Hamiltonian vector fields on a 2-plectic manifold are both vector spaces and are denoted as $\Omega^1_{\mathrm{Ham}}(M)$ and $\mathfrak{X}_{\mathrm{Ham}}(M)$, respectively.

The Hamiltonian vector field v_α is unique if it exists, but there may be 1-forms α having no Hamiltonian vector field. Furthermore, two distinct Hamiltonian 1-forms may differ by a closed 1-form and therefore share the same Hamiltonian vector field.

We can generalize the Poisson bracket on functions in symplectic geometry by defining a bracket on Hamiltonian 1-forms.

Definition 3.3. Given $\alpha, \beta \in \Omega^1_{\mathrm{Ham}}(M)$, the **bracket** $\{\alpha, \beta\}$ is the 1-form given by

$$\{\alpha, \beta\} = \iota_{v_\beta}\iota_{v_\alpha}\omega.$$

Proposition 3.4. *Let $\alpha, \beta, \gamma \in \Omega^1_{\mathrm{Ham}}(M)$ and let $v_\alpha, v_\beta, v_\gamma$ be the respective Hamiltonian vector fields. The bracket $\{\cdot, \cdot\}$ has the following properties:*

(1) *The bracket of Hamiltonian forms is Hamiltonian:*

$$d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]}\omega, \tag{3.1}$$

so in particular we have

$$v_{\{\alpha,\beta\}} = [v_\alpha, v_\beta].$$

(2) *The bracket is skew-symmetric:*

$$\{\alpha, \beta\} = -\{\beta, \alpha\} \tag{3.2}$$

(3) *The bracket satisfies the Jacobi identity up to an exact 1-form:*

$$\{\alpha, \{\beta, \gamma\}\} - \{\{\alpha, \beta\}, \gamma\} - \{\beta, \{\alpha, \gamma\}\} = d\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega. \tag{3.3}$$

Proof. See Propositions 3.5 and 3.6 in [**27**]. ☐

Note that equation (3.1) in the above proposition implies that $\mathfrak{X}_{\mathrm{Ham}}(M)$ is a Lie algebra.

4. Courant algebroids

Here, we recall some basic facts and examples of Courant algebroids and then we proceed to describe Ševera's classification of exact Courant algebroids. There are several equivalent definitions of a Courant algebroid found in the literature. The following definition, due to Roytenberg [**30**], is equivalent to the original definition given by Liu et al. [**23**].

Definition 4.1. A **Courant algebroid** is a vector bundle $C \to M$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a skew-symmetric bracket $[\cdot, \cdot]_C$ on $\Gamma(C)$, and a bundle map (called the **anchor**) $\rho \colon C \to TM$ such that for all $e_1, e_2, e_3 \in \Gamma(C)$ and for all $f, g \in C^\infty(M)$ the following properties hold:

(1) $[e_1, [e_2, e_3]_C]_C - [[e_1, e_2]_C, e_3]_C - [e_2, [e_1, e_3]_C]_C = -DT(e_1, e_2, e_3)$,
(2) $\rho([e_1, e_2]_C) = [\rho(e_1), \rho(e_2)]$,
(3) $[e_1, fe_2]_C = f[e_1, e_2]_C + \rho(e_1)(f)e_2 - \frac{1}{2}\langle e_1, e_2\rangle Df$,
(4) $\langle Df, Dg\rangle = 0$,
(5) $\rho(e_1)(\langle e_2, e_3\rangle) = \langle [e_1, e_2]_C + \frac{1}{2}D\langle e_1, e_2\rangle, e_3\rangle + \langle e_2, [e_1, e_3]_C + \frac{1}{2}D\langle e_1, e_3\rangle\rangle$,

where $[\cdot, \cdot]$ is the Lie bracket of vector fields, $D \colon C^\infty(M) \to \Gamma(C)$ is the map defined by $\langle Df, e\rangle = \rho(e)f$, and

$$T(e_1, e_2, e_3) = \frac{1}{6}\left(\langle [e_1, e_2]_C, e_3\rangle + \langle [e_3, e_1]_C, e_2\rangle + \langle [e_2, e_3]_C, e_1\rangle\right).$$

The bracket in Definition 4.1 is skew-symmetric, but the first property implies that it needs only to satisfy the Jacobi identity "up to DT". Note that the vector bundle $C \to M$ may be identified with $C^* \to M$ via the bilinear form $\langle \cdot, \cdot \rangle$ and therefore we have the dual map

$$\rho^* \colon T^*M \to C.$$

Hence the map D is simply the pullback of the de Rham differential by ρ^*.

There is an alternate definition given by Ševera [**34**] for Courant algebroids which uses a bracket operation on sections that satisfies a Jacobi identity but is not skew-symmetric.

Definition 4.2. A **Courant algebroid** is a vector bundle $C \to M$ together with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a bilinear operation $[\![\cdot, \cdot]\!]_C$ on $\Gamma(C)$, and a bundle map $\rho \colon C \to TM$ such that for all $e_1, e_2, e_3 \in \Gamma(C)$ and for all $f \in C^\infty(M)$ the following properties hold:

(1) $[\![e_1, [\![e_2, e_3]\!]_C]\!]_C = [\![[\![e_1, e_2]\!]_C, e_3]\!]_C + [\![e_2, [\![e_1, e_3]\!]_C]\!]_C$,
(2) $\rho([\![e_1, e_2]\!]_C) = [\rho(e_1), \rho(e_2)]$,
(3) $[\![e_1, fe_2]\!]_C = f\,[\![e_1, e_2]\!]_C + \rho(e_1)(f)e_2$,
(4) $[\![e_1, e_1]\!]_C = \frac{1}{2} D\langle e_1, e_1 \rangle$,
(5) $\rho(e_1)\left(\langle e_2, e_3 \rangle\right) = \langle [\![e_1, e_2]\!]_C, e_3 \rangle + \langle e_2, [\![e_1, e_3]\!]_C \rangle$,

where $[\cdot, \cdot]$ is the Lie bracket of vector fields, and $D \colon C^\infty(M) \to \Gamma(C)$ is the map defined by $\langle Df, e \rangle = \rho(e)f$.

Roytenberg [**30**] showed that $C \to M$ is a Courant algebroid in the sense of Definition 4.1 with bracket $[\cdot, \cdot]_C$, bilinear form $\langle \cdot, \cdot \rangle$ and anchor ρ if and only if $C \to M$ is a Courant algebroid in the sense of Definition 4.2 with the same anchor and bilinear form but with bracket $[\![\cdot, \cdot]\!]_C$ given by

$$[\![e_1, e_2]\!]_C = [e_1, e_2]_C + \frac{1}{2} D\langle e_1, e_2 \rangle. \tag{4.1}$$

All Courant algebroids mentioned in this paper are Courant algebroids in the sense of Definition 4.1. We introduced Definition 4.2 mainly to connect our discussion here with previous results in the literature.

Example 1. An important example of a Courant algebroid is the **standard Courant algebroid** $C = TM \oplus T^*M$ over any manifold M equipped with the **standard Courant bracket**:

$$[v_1 + \alpha_1, v_2 + \alpha_2]_C = [v_1, v_2] + \mathcal{L}_{v_1}\alpha_2 - \mathcal{L}_{v_2}\alpha_1 - \frac{1}{2} d\big\langle v_1 + \alpha_1, v_2 + \alpha_2 \big\rangle^-, \tag{4.2}$$

where

$$\big\langle v_1 + \alpha_1, v_2 + \alpha_2 \big\rangle^- = \iota_{v_1}\alpha_2 - \iota_{v_2}\alpha_1 \tag{4.3}$$

is the **standard skew-symmetric pairing**. The bilinear form is given by the **standard symmetric pairing**:

$$\big\langle v_1 + \alpha_1, v_2 + \alpha_2 \big\rangle^+ = \iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1. \tag{4.4}$$

The anchor $\rho\colon C \to TM$ is the projection map, and $D = d$ is the de Rham differential. The bracket $[\cdot,\cdot]_C$ is the skew-symmetrization of the **standard Dorfman bracket** [**13, 14**]:

$$\llbracket v_1 + \alpha_1, v_2 + \alpha_2 \rrbracket_C = [v_1, v_2] + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2} d\alpha_1, \tag{4.5}$$

which plays the role of the bracket given in Definition 4.2.

The standard Courant algebroid is the prototypical example of an **exact Courant algebroid** [**6**].

Definition 4.3. A Courant algebroid $C \to M$ with anchor map $\rho\colon C \to TM$ is **exact** iff

$$0 \to T^*M \xrightarrow{\rho^*} C \xrightarrow{\rho} TM \to 0$$

is an exact sequence of vector bundles.

4.1. The Ševera class of an exact Courant algebroid.

Ševera's classification [**34**] originates in the idea that a particular kind of splitting of the above short exact sequence corresponds to defining a connection.

Definition 4.4. A **splitting** of an exact Courant algebroid C over a manifold M is a map of vector bundles $s\colon TM \to C$ such that

(1) $\rho \circ s = \mathrm{id}_{TM}$,
(2) $\langle s(v_1), s(v_2)\rangle = 0$ for all $v_1, v_2 \in TM$,

where $\rho\colon C \to TM$ and $\langle \cdot,\cdot\rangle$ are the anchor and bilinear form, respectively.

In other words, a splitting of an exact Courant algebroid is an isotropic splitting of the sequence of vector bundles. Bressler and Chervov call splittings "connections" [**6**]. If s is a splitting and $B \in \Omega^2(M)$ is a 2-form then one can construct a new splitting:

$$(s + B)(v) = s(v) + \rho^* B(v,\cdot). \tag{4.6}$$

Furthermore, one can show that any two splittings on an exact Courant algebroid must differ by a 2-form on M in this way. Hence the space of splittings on an exact Courant algebroid is an affine space modeled on the vector space of 2-forms $\Omega^2(M)$ [**6**].

The failure of a splitting to preserve the bracket gives a suitable notion of curvature:

Definition 4.5 ([6]). If C is an exact Courant algebroid over M with bracket $[\cdot,\cdot]_C$ and $s\colon TM \to C$ is a splitting then the **curvature** is a map $F\colon TM \times TM \to C$ defined by

$$F(v_1, v_2) = [s(v_1), s(v_2)]_C - s([v_1, v_2]).$$

If F is the curvature of a splitting s, then given $v_1, v_2 \in TM$, it follows from exactness and axiom 2 in Definition 4.1 that there exists a 1-form $\alpha_{v_1,v_2} \in \Omega^1(M)$ such that $F(v_1, v_2) = \rho^*(\alpha_{v_1,v_2})$. Since s is a splitting, its image is isotropic in C. Therefore for any $v_3 \in TM$ we have:

$$\langle F(v_1, v_2), s(v_3)\rangle = \langle [s\,(v_1)\,, s\,(v_2)]_C\,, s(v_3)\rangle.$$

The above formula allows one to associate the curvature F to a 3-form on M:

Proposition 4.6. *Let C be an exact Courant algebroid over a manifold M with bracket $[\cdot,\cdot]_C$ and bilinear form $\langle\cdot,\cdot\rangle$. Let $s\colon TM \to C$ be a splitting on C. Then given vector fields v_1, v_2, v_3 on M:*

(1) *The function*

$$\omega(v_1, v_2, v_3) = \langle [s\,(v_1)\,, s\,(v_2)]_C\,, s(v_3)\rangle$$

defines a closed 3-form on M.

(2) *If $\theta \in \Omega^2(M)$ is a 2-form and $\tilde{s} = s + \theta$ then*

$$\begin{aligned}\tilde{\omega}(v_1, v_2, v_3) &= \langle [\tilde{s}\,(v_1)\,, \tilde{s}\,(v_2)]_C\,, \tilde{s}(v_3)\rangle \\ &= \omega(v_1, v_2, v_3) + d\theta(v_1, v_2, v_3).\end{aligned}$$

Proof. The statements in the proposition are proved in Lemmas 4.2.6, 4.2.7, and 4.3.4 in the paper by Bressler and Chervov [**6**]. In their work, they define a Courant algebroid using Definition 4.2, and therefore their bracket satisfies the Jacobi identity, but is not skew-symmetric. In our notation, their definition of the curvature 3-form is:

$$\nu(v_1, v_2, v_3) = \langle [\![s\,(v_1)\,, s\,(v_2)]\!]_C\,, s(v_3)\rangle.$$

In particular, they show that $[\![\cdot,\cdot]\!]_C$ satisfying the Jacobi identity implies ν is closed. The bracket $[\cdot,\cdot]_C$ does not satisfy the Jacobi identity in general. However the isotropicity of the splitting and equation (4.1) imply

$$[\![s(v_1), s(v_2)]\!]_C = [s(v_1), s(v_2)]_C \quad \forall v_1, v_2 \in TM.$$

Hence $\nu = \omega$, so all the needed results in [**6**] apply here. $\square$

Thus, the above proposition implies that the curvature 3-form of an exact Courant algebroid over M gives a well-defined cohomology class in $H^3_{\mathrm{DR}}(M)$, independent of the choice of splitting.

Definition 4.7 ([15]). The **Ševera class** of an exact Courant algebroid with bracket $[\cdot,\cdot]_C$ and bilinear form $\langle\cdot,\cdot\rangle$ is the cohomology class $[-\omega] \in H^3_{\mathrm{DR}}(M)$, where

$$\omega(v_1, v_2, v_3) = \langle [s\,(v_1)\,, s\,(v_2)]_C\,, s(v_3)\rangle.$$

5. The Courant algebroid associated to a 2-plectic manifold

In this section, we recall how to explicitly construct an exact Courant algebroid with Ševera class $[\omega]$. This is the 3-form analogue of the construction that gives a transitive Lie algebroid over a pre-symplectic manifold, which was previously discussed in Section 2.1. The approach given here is essentially identical to those given by Gualtieri [**15**], Hitchin [**17**], and Ševera [**34**].

Let (M, ω) be a manifold equipped with a closed 3-form. A trivialization of ω is an open cover$\{U_i\}$ of M equipped with 2-forms $B_i \in \Omega^2(U_i)$, and 1-forms $A_{ij} \in \Omega^1(U_{ij})$ on intersections such that

$$\begin{aligned} \omega|_{U_i} &= dB_i, \\ (B_j - B_i)|_{U_{ij}} &= dA_{ij}. \end{aligned} \tag{5.1}$$

Given such a trivialization, over each open set U_i consider the bundle $C_i = TU_i \oplus T^*U_i \to U_i$ equipped with the standard pairing

$$\left\langle v_1 + \alpha_1, v_2 + \alpha_2 \right\rangle_i^+ = \iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1, \tag{5.2}$$

$v_1, v_2 \in \mathfrak{X}(U_i)$, $\alpha_1, \alpha_2 \in \Omega^1(U_i)$, which has signature (n, n). On double intersections, it is easy to see that

$$\left\langle v_1 + \iota_{v_1} dA_{ij} + \alpha_1, v_2 + \iota_{v_2} dA_{ij} + \alpha_2 \right\rangle_i^+ = \left\langle v_1 + \alpha_1, v_2 + \alpha_2 \right\rangle_i^+.$$

Hence, the 2-forms $\{dA_{ij}\}$ generate transition functions

$$\begin{aligned} G_{ij} \colon U_{ij} &\to SO(n, n), \\ G_{ij}(x) &= \begin{pmatrix} 1 & 0 \\ dA_{ij}|_x & 1 \end{pmatrix}, \end{aligned}$$

which satisfy the cocycle conditions on U_{ijk} by virtue of equation (5.1). Therefore, we have over M the vector bundle

$$C = \coprod_{x \in M} T_x U_i \oplus T_x^* U_i / \sim,$$

equipped with a bilinear form denoted as $\left\langle \cdot, \cdot \right\rangle^+$. C sits in the exact sequence

$$0 \to T^*M \xrightarrow{\jmath} C \xrightarrow{\rho} TM \to 0,$$

where the anchor ρ is induced by the projection $T^*U_i \oplus TU_i \to TU_i$, and $\jmath$ is the inclusion.

The 2-forms $\{B_i\}$ induce a bundle map $s \colon TM \to C$

$$s(v_x) = v_x - \iota_{v_x} B_i \quad \text{if } x \in U_i. \tag{5.3}$$

It follows from equation (5.1) that s is well defined when $x \in U_{ij}$. It is easy to see that this map is an isotropic splitting (Definition 4.4). Hence every section $e \in \Gamma(C)$ can be uniquely expressed as

$$e = s(v) + \alpha,$$

for some $v \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$. As before, we use s to also denote the map $\Gamma(s)\colon \mathfrak{X}(M) \to \Gamma(C)$. The anchor map is just

$$\rho\bigl(s(v) + \alpha\bigr) = v. \tag{5.4}$$

Given sections $s(v_1) + \alpha_1,\ s(v_2) + \alpha_2 \in \Gamma(C)$, a local calculation using equation (5.3) gives

$$\begin{aligned} \bigl\langle s(v_1) + \alpha_1, s(v_2) + \alpha_2 \bigr\rangle^+ &= \iota_{v_1}\alpha_2 - \iota_{v_1}\iota_{v_2}B_i + \iota_{v_2}\alpha_1 - \iota_{v_2}\iota_{v_1}B_i \\ &= \bigl\langle v_1 + \alpha_1, v_2 + \alpha_2 \bigr\rangle^+. \end{aligned} \tag{5.5}$$

The above equality holds, in fact, for any splitting $s'\colon TM \to C$, since $s - s'$ is a 2-form on M and therefore skew-symmetric. The bracket on $\Gamma(C)$ is defined over the open set U_i by:

$$[s(v_1) + \alpha_1, s(v_2) + \alpha_2]_C \,|_{U_i} = [s(v_1) + \alpha_1, s(v_2) + \alpha_2]_i$$

where $[\cdot,\cdot]_i$ is the standard Courant bracket (4.2) on C_i. Since the 2-forms $\{dA_{ij}\}$ are closed, it follows by direct computation that on double intersections U_{ij}:

$$[G_{ij}(v_1 + \alpha_1), G_{ij}(v_2 + \alpha_2)]_i = G_{ij}\bigl([v_1 + \alpha_1, v_2 + \alpha_2]_i\bigr).$$

Hence the bracket $[\cdot,\cdot]_C$ is indeed globally well-defined. Using the local definition of the bracket and the splitting, as well as the fact that $dB_i = \omega$, it is easy to show that

$$\begin{aligned} [s(v_1) + \alpha_1, s(v_2) + \alpha_2]_C = s\bigl([v_1, v_2]\bigr) &+ \mathcal{L}_{v_1}\alpha_2 - \mathcal{L}_{v_2}\alpha_1 \\ &- \frac{1}{2} d\bigl\langle v_1 + \alpha_1, v_2 + \alpha_2 \bigr\rangle^- - \iota_{v_2}\iota_{v_1}\omega. \end{aligned} \tag{5.6}$$

The bracket $[\cdot,\cdot]_C$ is called the **twisted Courant bracket**. A analogous construction using the standard Dorfman bracket (4.5) on C_i gives the **twisted Dorfman bracket**:

$$[\![s(v_1) + \alpha_1, s(v_2) + \alpha_2]\!]_C = s\bigl([v_1, v_2]\bigr) + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2} d\alpha_1 - \iota_{v_2}\iota_{v_1}\omega. \tag{5.7}$$

These brackets were studied in detail by Ševera and Weinstein [**34, 35**].

It is straightforward to check that $C \to M$ equipped with the aforementioned bilinear form, anchor, and bracket $[\cdot,\cdot]_C$ is an exact Courant algebroid (Definition 4.1). Just as in Lie algebroid case (Section 2.1), the construction of C is independent of the choice of trivialization up to a splitting-preserving isomorphism.

A direct calculation shows that

$$-\omega(v_1, v_2, v_3) = \bigl\langle [s\,(v_1)\,, s\,(v_2)]_C\,, s(v_3) \bigr\rangle^+.$$

Hence by Prop. 4.6, the Courant algebroid C has Ševera class $[\omega]$. Of course, we are interested in the special case when ω is a 2-plectic structure. We summarize the above discussion with the following proposition:

Proposition 5.1. *Let (M,ω) be a 2-plectic manifold. Up to isomorphism, there exists a unique exact Courant algebroid C over M, with bilinear form $\langle\cdot,\cdot\rangle^{+}$, anchor map ρ, and bracket $[\cdot,\cdot]_C$ given in equations* (5.2)*,* (5.4)*, and* (5.6)*, respectively, and equipped with a splitting whose curvature is $-\omega$.*

6. Lie 2-algebras

Both the Courant bracket and the bracket on Hamiltonian 1-forms are, roughly, Lie brackets which satisfy the Jacobi identity up to an exact 1-form. This leads us to the notion of a Lie 2-algebra. In general, a Lie n-algebra is a n-term L_∞-algebra. It consists of a graded vector space concentrated in degrees $0,\ldots,n-1$ and is equipped with a collection of skew-symmetric k-ary brackets, for $1 \leq k \leq n+1$, that satisfy a generalized Jacobi identity [**20, 21**]. In particular, the $k = 2$ bilinear bracket behaves like a Lie bracket that only satisfies the ordinary Jacobi identity up to higher coherent chain homotopy. Baez and Crans showed that Lie 2-algebras are equivalent to categories internal to the category of vector spaces over $\mathbb{R}$ equipped with structures analogous to those of a Lie algebra, for which the usual law involving the Jacobi identity holds only up to natural isomorphism [**1**]. (Note that what we call a Lie 2-algebra is called a “semistrict Lie 2-algebra” in [**1, 2, 32**].)

As L_∞-algebras, Lie 2-algebras are relatively easy to work with and one can write out the axioms explicitly. Therefore we use the following definition which is equivalent the usual definition for an L_∞-algebra [**21**] when the underlying complex is concentrated in degrees 0 and 1.

Definition 6.1. A **Lie 2-algebra** is a 2-term chain complex of vector spaces $L_\bullet = (L_1 \xrightarrow{d} L_0)$ equipped with:

- a skew-symmetric chain map $[\cdot,\cdot]\colon L_\bullet \otimes L_\bullet \to L_\bullet$ called the **bracket**;
- a skew-symmetric chain homotopy $J\colon L_\bullet \otimes L_\bullet \otimes L_\bullet \to L_\bullet$ from the chain map

$$\begin{aligned} L_\bullet \otimes L_\bullet \otimes L_\bullet &\to L_\bullet \\ x \otimes y \otimes z &\longmapsto [x,[y,z]], \end{aligned}$$

to the chain map

$$\begin{aligned} L_\bullet \otimes L_\bullet \otimes L_\bullet &\to L_\bullet \\ x \otimes y \otimes z &\longmapsto [[x,y],z] + [y,[x,z]] \end{aligned}$$

called the **Jacobiator**,

such that the following equation holds:

$$
\begin{aligned}
&[x, J(y,z,w)] + J(x,[y,z],w) + J(x,z,[y,w]) \\
&\quad + [J(x,y,z),w] + [z, J(x,y,w)] \\
&= J(x,y,[z,w]) + J([x,y],z,w) + [y, J(x,z,w)] \\
&\quad + J(y,[x,z],w) + J(y,z,[x,w]).
\end{aligned} \tag{6.1}
$$

We will also need a suitable notion of morphism:

Definition 6.2. Given semistrict Lie 2-algebras $L = (L_\bullet, [\cdot,\cdot], J)$ and $L' = (L'_\bullet, [\cdot,\cdot]', J')$ a **morphism** from L to L' consists of:

- a chain map $\phi_\bullet \colon L_\bullet \to L'_\bullet$, and
- a chain homotopy $\Phi \colon L_\bullet \otimes L_\bullet \to L'_\bullet$ from the chain map
$$
\begin{aligned}
L_\bullet \otimes L_\bullet &\to L'_\bullet \\
x \otimes y &\longmapsto \phi_\bullet([x,y])
\end{aligned}
$$
to the chain map
$$
\begin{aligned}
L_\bullet \otimes L_\bullet &\to L'_\bullet \\
x \otimes y &\longmapsto [\phi_\bullet(x), \phi_\bullet(y)]',
\end{aligned}
$$

such that the following equation holds:

$$
\begin{aligned}
&\phi_1(J(x,y,z)) - J'(\phi_0(x),\phi_0(y),\phi_0(z)) \\
&= \Phi(x,[y,z]) - \Phi([x,y],z) - \Phi(y,[x,z]) - [\Phi(x,y),\phi_0(z)]' \\
&\quad + [\phi_0(x), \Phi(y,z)]' - [\phi_0(y), \Phi(x,z)]'.
\end{aligned} \tag{6.2}
$$

We say a morphism is **strict** iff $\Phi = 0$.

This definition is equivalent to the definition of a morphism between 2-term L_∞-algebras [**20**].

Definition 6.3. A Lie 2-algebra morphism $(\phi_\bullet, \Phi) \colon L \to L'$ is a **quasi-isomorphism** iff the chain map $\phi_\bullet$ induces an isomorphism on the homology of the underlying chain complexes of L and L'.

6.1. The Lie 2-algebra from a 2-plectic manifold. Any n-plectic manifold gives a Lie n-algebra which can be understood as the n-plectic analogue of the Poisson algebra [**27**]. We now review this construction for the 2-plectic case. The underlying 2-term chain complex of our Lie 2-algebra is:

$$L_\bullet = C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{Ham}}(M)$$

where d is the de Rham differential. This chain complex is well-defined, since any exact form is Hamiltonian, with 0 as its Hamiltonian vector field. We can construct a chain map

$$[\cdot,\cdot] \colon L_\bullet \otimes L_\bullet \to L_\bullet,$$

by extending the bracket $\{\cdot,\cdot\}$ on $\Omega^1_{\mathrm{Ham}}(M)$ trivially to $L_\bullet$. In other words, in degree 0, the chain map is given as in Definition 3.3:

$$[\alpha,\beta] = \{\alpha,\beta\} = \iota_{v_\beta}\iota_{v_\alpha}\omega,$$

and in degrees 1 and 2, we set it equal to zero:

$$[\alpha, f] = 0, \quad [f,\alpha] = 0, \quad [f,g] = 0.$$

The precise construction of this Lie 2-algebra is given in the following theorem:

Theorem 6.4. *If (M,ω) is a 2-plectic manifold, there is a Lie 2-algebra $L_\infty(M,\omega) = (L_\bullet, [\cdot,\cdot], J)$ where:*

- $L_0 = \Omega^1_{\mathrm{Ham}}(M)$,
- $L_1 = C^\infty(M)$,
- *the differential $L_1 \xrightarrow{d} L_0$ is the de Rham differential,*
- *the bracket $[\cdot,\cdot]$ is $\{\cdot,\cdot\}$ in degree 0 and trivial otherwise,*
- *the Jacobiator is given by the linear map $J\colon \Omega^1_{\mathrm{Ham}}(M)\otimes\Omega^1_{\mathrm{Ham}}(M)\otimes\Omega^1_{\mathrm{Ham}}(M)\to C^\infty$, where $J(\alpha,\beta,\gamma) = \iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega$.*

Proof. See Theorem 5.2 in [**27**]. $\square$

6.2. The Lie 2-algebra from a Courant algebroid. Similarly, given any Courant algebroid $C\to M$ with bilinear form $\langle\cdot,\cdot\rangle$, bracket $[\cdot,\cdot]_C$, and anchor $\rho\colon C\to TM$, one can construct a 2-term chain complex

$$L_\bullet \quad = \quad C^\infty(M) \xrightarrow{D} \Gamma(C),$$

with differential $D=\rho^* d$ where d is the de Rham differential. The bracket $[\cdot,\cdot]_C$ on global sections can be extended to a chain map $[\cdot,\cdot] : L_\bullet\otimes L_\bullet\to L_\bullet$. If e_1, e_2 are degree 0 chains then $[e_1,e_2]$ is the original bracket. If e is a degree 0 chain and f,g are degree 1 chains, then we define:

$$[e,f] = -[f,e] = \frac{1}{2}\langle e, Df\rangle$$
$$[f,g] = 0.$$

It was shown by Roytenberg and Weinstein [**29**] that this extended bracket gives a L_∞-algebra. Roytenberg's later work [**31, 32**] implies that a brutal truncation of this L_∞-algebra is a Lie 2-algebra whose underlying complex is $L_\bullet$. For the Courant algebroid C constructed in Section 5, their result implies:

Theorem 6.5. *If C is the exact Courant algebroid given in Proposition 5.1 then there is a Lie 2-algebra $L_\infty(C) = (L_\bullet, [\cdot,\cdot], J)$ where:*

- $L_0 = \Gamma(C)$,
- $L_1 = C^\infty(M)$,
- *the differential $L_1 \xrightarrow{D} L_0$ is $D=\rho^* d$,*

- *the bracket* $[\cdot,\cdot]$ *is*
$$[e_1, e_2] = [e_1, e_2]_C \quad \textit{in degree 0}$$
and
$$[e, f] = -[f, e] = \frac{1}{2}\langle e, df\rangle^+ \quad \textit{in degree 1},$$
- *the Jacobiator is the linear map* $J\colon \Gamma(C) \otimes \Gamma(C) \otimes \Gamma(C) \to C^\infty(M)$ *defined by*
$$\begin{aligned} J(e_1, e_2, e_3) &= -T(e_1, e_2, e_3) \\ &= -\frac{1}{6}\Big(\big\langle [e_1, e_2]_C, e_3\big\rangle^+ + \big\langle [e_3, e_1]_C, e_2\big\rangle^+ \\ &\quad + \big\langle [e_2, e_3]_C, e_1\big\rangle^+\Big). \end{aligned}$$

More precisely, the theorem follows from Example 5.4 of [**32**] and Section 4 of [**31**]. On the other hand, the original construction of Roytenberg and Weinstein gives a L_∞-algebra on the complex:
$$0 \to \ker D \xrightarrow{\iota} C^\infty(M) \xrightarrow{D} \Gamma(C),$$
with trivial structure maps l_n for $n \geq 3$. Moreover, the map l_2 (corresponding to the bracket $[\cdot,\cdot]$ given above) is trivial in degree > 1 and the map l_3 (corresponding to the Jacobiator J) is trivial in degree > 0. Hence, these maps induce the above Lie 2-algebra structure on $C^\infty(M) \xrightarrow{D} \Gamma(C)$.

7. The algebraic relationship between 2-plectic and Courant

In Section 5, we described how one can construct over a 2-plectic manifold (M, ω), an exact Courant algebroid $(C, [\cdot,\cdot]_C, \langle\cdot,\cdot\rangle^+, \rho)$ equipped with a splitting $s\colon TM \to C$ whose curvature is $-\omega$. In this section, we show there is a complementary algebraic relationship. We can interpret these results as the 2-plectic analogues of those given in Section 2.2.

Theorem 7.1. *Let (M, ω) be a 2-plectic manifold and let C be its corresponding Courant algebroid. Let $L_\infty(M, \omega)$ and $L_\infty(C)$ be the Lie 2-algebras corresponding to (M, ω) and C, respectively. There exists a morphism of Lie 2-algebras embedding $L_\infty(M, \omega)$ into $L_\infty(C)$.*

Before we prove the theorem, we introduce some technical lemmas to ease the calculations. Recall from equation (4.3) that the formula for the standard skew-symmetric pairing on $\mathfrak{X}(M) \oplus \Omega^1(M)$:
$$\langle v_1 + \alpha_1, v_2 + \alpha_2\rangle^- = \iota_{v_1}\alpha_2 - \iota_{v_2}\alpha_1.$$
In what follows, by the symbol "c.p" we mean cyclic permutations of the symbols α, β, γ.

Lemma 7.2. *If $\alpha, \beta \in \Omega^1_{\mathrm{Ham}}(M)$ with corresponding Hamiltonian vector fields v_α, v_β, then $\mathcal{L}_{v_\alpha}\beta = \{\alpha, \beta\} + d\iota_{v_\alpha}\beta$.*

Proof. Since $\mathcal{L}_v = \iota_v d + d\iota_v$,

$$\mathcal{L}_{v_\alpha}\beta = \iota_{v_\alpha} d\beta + d\iota_{v_\alpha}\beta = -\iota_{v_\alpha}\iota_{v_\beta}\omega + d\iota_{v_\alpha}\beta = \{\alpha, \beta\} + d\iota_{v_\alpha}\beta.$$

□

Lemma 7.3. *If $\alpha, \beta, \gamma \in \Omega^1_{\mathrm{Ham}}(M)$ with corresponding Hamiltonian vector fields $v_\alpha, v_\beta, v_\gamma$, then*

$$\begin{aligned}\iota_{[v_\alpha, v_\beta]}\gamma + \text{c.p} = -3\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega + \iota_{v_\alpha} d\langle v_\beta + \beta, v_\gamma + \gamma\rangle^- \\ + \iota_{v_\gamma} d\langle v_\alpha + \alpha, v_\beta + \beta\rangle^- + \iota_{v_\beta} d\langle v_\gamma + \gamma, v_\alpha + \alpha\rangle^-.\end{aligned}$$

Proof. The identity $\iota_{[v_\alpha, v_\beta]} = \mathcal{L}_{v_\alpha}\iota_{v_\beta} - \iota_{v_\beta}\mathcal{L}_{v_\alpha}$ and Lemma 7.2 imply:

$$\begin{aligned}\iota_{[v_\alpha, v_\beta]}\gamma &= \mathcal{L}_{\,v_\alpha}\,\iota_{v_\beta}\gamma - \iota_{v_\beta}\mathcal{L}_{\,v_\alpha}\,\gamma \\ &= \mathcal{L}_{\,v_\alpha}\,\iota_{v_\beta}\gamma - \iota_{v_\beta}\left(\{\alpha, \gamma\} + d\iota_{v_\alpha}\gamma\right) \\ &= \iota_{v_\alpha} d\iota_{v_\beta}\gamma - \iota_{v_\beta}\iota_{v_\gamma}\iota_{v_\alpha}\omega - \iota_{v_\beta} d\iota_{v_\alpha}\gamma,\end{aligned}$$

where the last equality follows from the definition of the bracket.

Therefore we have:

$$\begin{aligned}\iota_{[v_\gamma, v_\alpha]}\beta &= \iota_{v_\gamma} d\iota_{v_\alpha}\beta - \iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega - \iota_{v_\alpha} d\iota_{v_\gamma}\beta, \\ \iota_{[v_\beta, v_\gamma]}\alpha &= \iota_{v_\beta} d\iota_{v_\gamma}\alpha - \iota_{v_\gamma}\iota_{v_\alpha}\iota_{v_\beta}\omega - \iota_{v_\gamma} d\iota_{v_\beta}\alpha,\end{aligned}$$

and equation (4.3) implies

$$\iota_{v_\alpha} d\iota_{v_\beta}\gamma - \iota_{v_\alpha} d\iota_{v_\gamma}\beta = \iota_{v_\alpha} d\langle v_\beta + \beta, v_\gamma + \gamma\rangle^-.$$

The statement then follows. □

Lemma 7.4. *If $\alpha, \beta \in \Omega^1_{\mathrm{Ham}}(M)$ with corresponding Hamiltonian vector fields v_α, v_β, then*

$$\mathcal{L}_{\,v_\alpha}\,\beta - \mathcal{L}_{\,v_\beta}\,\alpha = 2\,\{\alpha, \beta\} + d\langle v_\alpha + \alpha, v_\beta + \beta\rangle^-.$$

Proof. Follows immediately from Lemma 7.2 and equation (4.3). □

Proof of Theorem 7.1. We will construct a morphism from $L_\infty(M, \omega)$ to $L_\infty(C)$. Let

$$\begin{gathered} L_\bullet = C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{Ham}}(M), \\ [\cdot,\cdot]_L : L_\bullet \otimes L_\bullet \to L_\bullet, \\ J_L \colon L_\bullet \otimes L_\bullet \otimes L_\bullet \to L_\bullet \end{gathered}$$

denote the underlying chain complex, bracket, and Jacobiator of the Lie 2-algebra $L_\infty(M,\omega)$. Similarly,

$$L'_\bullet = C^\infty(M) \xrightarrow{d} \Gamma(C),$$
$$[\cdot,\cdot]_{L'} : L'_\bullet \otimes L'_\bullet \to L'_\bullet,$$
$$J_{L'} \colon L'_\bullet \otimes L'_\bullet \otimes L'_\bullet \to L'_\bullet$$

denotes the underlying chain complex, bracket, and Jacobiator of the Lie 2-algebra $L_\infty(C)$.

Let $s\colon TM \to C$ be the splitting. Let $\phi_0 \colon \Omega^1_{\mathrm{Ham}}(M) \to \Gamma(C)$ be given by

$$\phi_0(\alpha) = s(v_\alpha) + \alpha,$$

where v_α is the Hamiltonian vector field corresponding to α. Let $\phi_1 \colon C^\infty(M) \to C^\infty(M)$ be the identity. Then $\phi_\bullet \colon L_\bullet \to L'_\bullet$ is a chain map, since the Hamiltonian vector field of an exact 1-form is zero. Let $\Phi \colon \Omega^1_{\mathrm{Ham}}(M) \otimes \Omega^1_{\mathrm{Ham}}(M) \to C^\infty(M)$ be given by

$$\Phi(\alpha,\beta) = -\frac{1}{2}\big\langle v_\alpha + \alpha, v_\beta + \beta \big\rangle^-.$$

Now we show Φ is a well-defined chain homotopy in the sense of Definition 6.2. We have

$$\begin{aligned}
[\phi_0(\alpha),\phi_0(\beta)]_{L'} &= [s(v_\alpha)+\alpha, s(v_\beta)+\beta]_C \\
&= s([v_\alpha,v_\beta]) + \mathcal{L}_{v_\alpha}\beta - \mathcal{L}_{v_\beta}\alpha - \iota_{v_\beta}\iota_{v_\alpha}\omega \\
&\quad - \frac{1}{2} d\big\langle v_\alpha+\alpha, v_\beta+\beta\big\rangle^- \\
&= s([v_\alpha,v_\beta]) + \{\alpha,\beta\} + \frac{1}{2} d\big\langle v_\alpha+\alpha, v_\beta+\beta\big\rangle^- \\
&= s([v_\alpha,v_\beta]) + [\alpha,\beta]_L - d\Phi(\alpha,\beta).
\end{aligned} \tag{7.1}$$

The second line above is just the definition of the twisted Courant bracket (equation (5.6)), while the second to last line follows from Lemma 7.4 and Def. 3.3 of the bracket $\{\cdot,\cdot\}$. By Prop. 3.4, the Hamiltonian vector field of $\{\alpha,\beta\}$ is $[v_\alpha,v_\beta]$. Hence, we have:

$$\phi_0([\alpha,\beta]_L) - [\phi_0(\alpha),\phi_0(\beta)]_{L'} = d\Phi(\alpha,\beta).$$

In degree 1, the bracket $[\cdot,\cdot]_L$ is trivial. It follows from the definition of $[\cdot,\cdot]_{L'}$ that

$$\phi_1([\alpha,f]_L) - [\phi_0(\alpha),\phi_1(f)]_{L'} = -\frac{1}{2}\big\langle s(v_\alpha)+\alpha, df\big\rangle^+.$$

From equation (5.5), we have

$$\big\langle s(v_\alpha)+\alpha, df\big\rangle^+ = \big\langle s(v_\alpha)+\alpha, s(0)+df\big\rangle^+ = \iota_{v_\alpha} df.$$

Therefore

$$\phi_1([\alpha, f]_L) - [\phi_0(\alpha), \phi_1(f)]_{L'} = \Phi(\alpha, df),$$

and similarly

$$\phi_1([f, \alpha]_L) - [\phi_1(f), \phi_0(\alpha)]_{L'} = \Phi(df, \alpha).$$

Therefore, Φ is a chain homotopy.

It remains to show the coherence condition (equation (6.2) in Definition 6.2) is satisfied. First, we rewrite the Jacobiator $J_{L'}$ using the second to last line of (7.1):

$$\begin{aligned} J_{L'}(\phi_0(\alpha), \phi_0(\beta), \phi_0(\gamma)) &= -\frac{1}{6}\left\langle [\phi_0(\alpha), \phi_0(\beta)]_{L'}, \phi_0(\gamma)\right\rangle^+ + \text{c.p} \\ &= -\frac{1}{6}\left\langle s([v_\alpha, v_\beta]) + \{\alpha, \beta\} - d\Phi(\alpha,\beta), s(v_\gamma) + \gamma\right\rangle^+. \\ &\quad + \text{c.p}\,. \end{aligned}$$

From the definition of the bracket $\{\cdot,\cdot\}$ and the symmetric pairing, we have

(7.2)
$$J_{L'}(\phi_0(\alpha), \phi_0(\beta), \phi_0(\gamma)) = -\frac{1}{2}\iota_{v_\gamma}\iota_{v_\beta}\iota_{v_\alpha}\omega - \frac{1}{6}\big(\iota_{[v_\alpha, v_\beta]}\gamma - \iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\big).$$

Lemma 7.3 implies

$$\iota_{[v_\alpha, v_\beta]}\gamma + \text{c.p} = -3\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega - \big(2\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\big), \tag{7.3}$$

so equation (7.2) becomes

$$J_{L'}(\phi_0(\alpha), \phi_0(\beta), \phi_0(\gamma)) = \iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega + \left(\frac{1}{2}\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\right).$$

By definition, $J_L(\alpha,\beta,\gamma) = \iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega$. Therefore, in this case, the left-hand side of equation (6.2) is

$$\phi_1(J_L(\alpha,\beta,\gamma)) - J_{L'}(\phi_0(\alpha), \phi_0(\beta), \phi_0(\gamma)) = -\frac{1}{2}\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\,. \tag{7.4}$$

Since the brackets and homotopy Φ are skew-symmetric, the right-hand side of equation (6.2) can be rewritten as:

$$\big(\Phi(\alpha, [\beta,\gamma]_L) + \text{c.p}\big) - \big([\Phi(\alpha,\beta), \phi_0(\gamma)]_{L'} + \text{c.p}\big). \tag{7.5}$$

Consider the first term in equation (7.5). The Hamiltonian vector field corresponding to $[\beta,\gamma]_L = \{\beta,\gamma\}$ is $[v_\beta, v_\gamma]$. Therefore the definition of Φ implies

$$\Phi(\alpha, [\beta,\gamma]_L) + \text{c.p} = -\frac{3}{2}\iota_{v_\gamma}\iota_{v_\beta}\iota_{v_\alpha}\omega + \frac{1}{2}\big(\iota_{[v_\beta, v_\gamma]}\alpha + \text{c.p}\big).$$

It then follows from Lemma 7.3 (see equation (7.3)) that

$$\Phi(\alpha, [\beta,\gamma]_L) + \text{c.p} = -\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\,.$$

By definition of the bracket $[\cdot,\cdot]_{L'}$, the second term in equation (7.5) can be written as

$$[\Phi(\alpha,\beta),\phi_0(\gamma)]_{L'} + \text{c.p} = -\frac{1}{2}\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\,.$$

Hence, the coherence condition:

$$\begin{aligned}\phi_1(J_L(\alpha,\beta,\gamma)) &- J_{L'}(\phi_0(\alpha),\phi_0(\beta),\phi_0(\gamma))\\ &= \Phi(\alpha,[\beta,\gamma]_L) - [\Phi(\alpha,\beta),\phi_0(\gamma)]_{L'} + \text{c.p}\end{aligned}$$

is satisfied, and $(\phi_\bullet,\Phi)\colon L_\infty(M,\omega) \to L_\infty(C)$ is a morphism of Lie 2-algebras. □

We now focus on a particular sub-Lie 2-algebra of $L_\infty(C)$. The following definition is due to Ševera [**34**] and is a generalization of Def. 2.1:

Definition 7.5. Let C be the exact Courant algebroid given in Prop. 5.1 equipped with a splitting $s\colon TM \to C$. We say a section $e = s(v) + \alpha$ **preserves the splitting** iff $\forall v' \in \mathfrak{X}(M)$

$$[\![e, s(v')]\!]_C = s([v,v']).$$

The subspace of sections that preserve the splitting is denoted as $\Gamma(C)^s$.

Note that the twisted Dorfman bracket is used in the above definition rather than the twisted Courant bracket. Since it satisfies the Jacobi identity, it gives a "strict" adjoint action on sections of C. The 2-plectic analogue of Proposition 2.2 is:

Proposition 7.6. *If C is the exact Courant algebroid given in Proposition 5.1 equipped with the splitting $s\colon TM \to C$, then there is a Lie 2-algebra $L_\infty(C)^s = (L_\bullet,[\cdot,\cdot],J)$ where:*

- $L_0 = \Gamma(C)^s$,
- $L_1 = C^\infty(M)$,
- *the differential* $L_1 \xrightarrow{D} L_0$ *is* $D = \rho^* d$,
- *the bracket* $[\cdot,\cdot]$ *is*

  $$[e_1,e_2] = [e_1,e_2]_C \quad \text{in degree 0}$$

  and

  $$[e,f] = -[f,e] = \frac{1}{2}\langle e, df\rangle^+ \quad \text{in degree 1},$$

- *the Jacobiator is the linear map* $J\colon \Gamma(C)^s\otimes\Gamma(C)^s\otimes\Gamma(C)^s \to C^\infty(M)$ *defined by*

  $$\begin{aligned}J(e_1,e_2,e_3) &= -T(e_1,e_2,e_3)\\ &= -\frac{1}{6}\Big(\langle [e_1,e_2]_C, e_3\rangle^+ + \langle [e_3,e_1]_C, e_2\rangle^+ \\ &\quad + \langle [e_2,e_3]_C, e_1\rangle^+\Big).\end{aligned}$$

Proof. Let v' be a vector field on M. By the definition of the twisted Dorfman bracket (equation (5.7)), it follows that $[\![df, s(v')]\!]_C = 0\,\forall f \in C^\infty(M)$. Hence the complex $L_\bullet$ is well defined. We now show that $\Gamma^s(C)$ is closed under the twisted Courant bracket. Suppose e_1 and e_2 are sections preserving the splitting. Let $e_i = s(v_i) + \alpha_i$. Since the twisted Dorfman bracket and the Lie bracket of vector fields satisfy the Jacobi identity, we have:

$$\left[\!\left[[\![e_1, e_2]\!]_C\,, s(v') \right]\!\right]_C = s([[v_1, v_2], v']).$$

From equation (4.1), we have the identity:

$$[e_1, e_2]_C = [\![e_1, e_2]\!]_C - \frac{1}{2} d\big\langle e_1, e_2\big\rangle^+.$$

Therefore,

$$\begin{aligned}\left[\!\left[[e_1, e_2]_C\,, s(v') \right]\!\right]_C &= \left[\!\left[[\![e_1, e_2]\!]_C\,, s(v') \right]\!\right]_C - \frac{1}{2}\left[\!\left[d\big\langle e_1, e_2\big\rangle^+, s(v') \right]\!\right]_C \\ &= s([[v_1, v_2], v']).\end{aligned}$$

It follows from Theorem 6.5 that the Lie 2-algebra axioms are satisfied. □

This next result is essentially a corollary of Thm. 7.1. However, it is important since it is the 2-plectic analogue of Prop. 2.3.

Proposition 7.7. *$L_\infty(M, \omega)$ and $L_\infty(C)^s$ are isomorphic as Lie 2-algebras.*

Proof. Recall that in Theorem 7.1 we constructed a morphism of Lie 2-algebras given by a chain map $\phi_\bullet\colon L_\infty(M, \omega) \to L_\infty(C)$:

$$\phi_0(\alpha) = s(v_\alpha) + \alpha, \quad \phi_1 = \mathrm{id},$$

and a homotopy $\Phi\colon \Omega^1_{\mathrm{Ham}}(M) \otimes \Omega^1_{\mathrm{Ham}}(M) \to C^\infty(M)$:

$$\Phi(\alpha, \beta) = -\frac{1}{2}\big\langle v_\alpha + \alpha, v_\beta + \beta\big\rangle^-.$$

Let $v' \in \mathfrak{X}(M)$ and $e = s(v) + \alpha$. By definition of the twisted Dorfman bracket, $[\![e, s(v')]\!]_C = s[v, v']$ if and only if $\iota_{v'}\big(d\alpha + \iota_v\omega\big) = 0$. Hence, a section of C preserves the splitting if and only if it lies in the image of the chain map $\phi_\bullet$. Since this map is also injective, the statement follows. □

8. Categorified prequantization

In this section, we introduce a prequantization scheme for 2-plectic manifolds, and provide a brief exposition on the higher geometric structures which naturally appear. The relationship between the Courant algebroid C and the 2-plectic manifold (M, ω) has an interesting interpretation when we consider the special case of prequantized 2-plectic manifolds. In particular, we will see that C acts as the 2-plectic analogue of the Atiyah algebroid A described in Section 2.4.

Definition 8.1. A 2-plectic manifold (M, ω) admits a **prequantization** iff the cohomology class $[\omega]$ lies in the image of the map $H^3(M, \mathbb{Z}) \to H^3(M, \mathbb{R}) \cong H^3_{\mathrm{DR}}(M)$.

Let (M, ω) be prequantizable. By using the maps $c\colon H^2(M, D_2^\bullet) \twoheadrightarrow H^3(M, \mathbb{Z})$ and $\kappa\colon H^2(M, D_2^\bullet) \to \Omega^3_{\mathrm{cl}}(M)$ discussed in Section 2.3, we can find a Deligne class in $H^2(M, D_2^\bullet)$ whose 3-curvature is ω. By definition, a representative of this class defined on a good cover $\{U_i\}$ is a set of 2-forms $\{B_i \in \Omega^2(U_i)\}$, a set of 1-forms $\{A_{ij} \in \Omega^1(U_{ij})\}$ on double intersections, and a set of $U(1)$-valued functions $\{g_{ijk}\colon U_{ijk} \to U(1)\}$ on triple intersections such that

$$\begin{aligned} \omega &= dB_i \text{ on } U_i, \\ B_j - B_i &= dA_{ij} \text{ on } U_{ij}, \\ A_{jk} - A_{ik} + A_{ij} &= g_{ijk}^{-1} dg_{ijk} \text{ on } U_{ijk}, \\ g_{jkl} g_{ikl}^{-1} g_{ijl} g_{ijk}^{-1} &= 1 \text{ on } U_{ijkl}. \end{aligned} \tag{8.1}$$

A 2-plectic manifold equipped with such a Deligne 2-cocycle is said to be **prequantized**. We can use the 2-cocycle to construct the Courant algebroid C over M equipped with a splitting given locally by the 2-forms $\{B_i\}$. However, the fact that the cocycle data includes the Čech 2-cocycle $\{g_{ijk}\colon U_{ijk} \to U(1)\}$ implies that there is an additional geometric structure present on M. We would expect C to be related to this structure just as the Atiyah algebroid A described in Section 2.4 is related to its associated principal bundle.

The geometric object we associate to the Čech 2-cocycle $\{g_{ijk}\colon U_{ijk} \to U(1)\}$ is a $U(1)$-gerbe. The precise definition of a gerbe is rather technical and can be found in Brylinski [**8**] or Moerdijk [**25**]. However, in what follows we hope to provide some intuitive geometric understanding of these structures and motivate their proposed role in the prequantization of 2-plectic manifolds. Additional details can be found in Sections 5.5 and 7.2 of [**26**].

8.1. *U*(1)-gerbes as stacks. For the purpose of comparison, it is helpful to momentarily return to the "1-plectic" case. Instead of associating a Čech 1-cocycle to a principal $U(1)$- bundle $P \xrightarrow{\pi} M$, we can just as well associate the cocycle to the bundle's sheaf of sections $\underline{P}$. The sheaf $\underline{P}$ is a $\underline{U(1)}$-torsor. This means that the sheaf of groups $\underline{U(1)}$ acts on $\underline{P}$ in such a way so that for all $x \in M$ there exists a neighborhood $x \in U$ and an equivariant isomorphism of sheaves $\underline{P}_U \xrightarrow{\sim} \underline{U(1)}_U$. In other words, $\underline{P}$ is locally isomorphic to the trivial torsor $\underline{U(1)}$. We recover the Čech 1-cocycle from $\underline{P}$ in the obvious way: Choose a good cover $\{U_i\}$ of M such that $\underline{P}_{U_i}$ is isomorphic to $\underline{U(1)}$ as a sheaf over U_i. Choose sections $\sigma_i \in \underline{P}_{U_i}$, and consider the restrictions $\sigma_i|_{U_{ij}}$, $\sigma_j|_{U_{ij}} \in \underline{P}(U_{ij})$. There exist sections $g_{ij} \in \underline{P}(U_{ij})$ such that $\sigma_j = \sigma_i \cdot g_{ij}$ on U_{ij}, which obey the usual cocycle condition on U_{ijk}.

Now let us consider the higher analogue. Just as the $\underline{U(1)}$ torsor $\underline{P}$ is a particular kind of sheaf, a $U(1)$-gerbe is a particular kind of stack. A stack $\mathcal{S}$ over M is, very roughly, a categorified sheaf over M. To every open set U of M, one assigns a groupoid $\mathcal{S}(U)$. To every inclusion of open sets $V \xrightarrow{\iota} U$, one assign a functor $\mathcal{S}(\iota)\colon \mathcal{S}(U) \to \mathcal{S}(V)$, which pulls back, or "restricts", objects and morphisms over U to those over V. However, given a composition of inclusions:

$$\begin{tikzcd} W \arrow[rr, "\iota_{UW}=\iota_{UV}\circ\iota_{VW}"] \arrow[dr, "\iota_{VW}"'] & & U \\ & V \arrow[ur, "\iota_{UV}"'] & \end{tikzcd}$$

one requires the corresponding functors $\mathcal{S}(\iota_{UW})$ and $\mathcal{S}(\iota_{VW}) \circ \mathcal{S}(\iota_{UV})$ to be equivalent via a coherent natural isomorphism instead of being equal. Just as the sheaf axioms involve gluing together local sections (i.e., elements of sets), the axioms for a stack involve gluing together objects and morphisms of groupoids.

Perhaps the most intuitive example of a stack is the classifying stack $\mathcal{B}U(1)$, which assigns to every open set $U \subseteq M$ the groupoid of principal $U(1)$-bundles over U. This stack has nice extra properties. For example, for any open set U and any two bundles $P_1, P_2 \in \mathcal{B}U(1)(U)$, there exists an open subset $V \subseteq U$ such that the pullback bundles $P_i|_V$ are isomorphic as objects in $\mathcal{B}U(1)(V)$. Moreover, V can be chosen so that the automorphism groups $\mathrm{Aut}(P_i|_V)$ are isomorphic to the group of $U(1)$-valued functions $\underline{U(1)}(V)$. Roughly, these are the defining properties of a $U(1)$-gerbe. We may think of a $U(1)$-gerbe $\mathcal{G}$ over M as a stack with the additional property that there exists an open cover $\{U_i\}$ of M such that for all open sets $V \subseteq U_i$, the groupoid $\mathcal{G}(V)$ is equivalent (as a category) to $\mathcal{B}U(1)(V)$.

One obtains a Čech 2-cocycle from a $U(1)$-gerbe $\mathcal{G}$ in the following way: Choose a good open cover $\{U_i\}$ of M such that there exists objects $P_i \in \mathcal{G}(U_i)$, isomorphisms $u_{ij}\colon P_i|_{U_{ij}} \to P_j|_{U_{ij}}$, and isomorphisms $\mathrm{Aut}(P_i|_V) \cong \underline{U(1)}(V)$ for all open subsets $V \subset U_i$. Such a cover exists since $\mathcal{G}$ is locally isomorphic to $\mathcal{B}U(1)$. By restricting these objects and isomorphisms to triple intersections U_{ijk}, we obtain an automorphism $u_{ik}^{-1} u_{ij} u_{jk}$ of $P_k|_{U_{ijk}}$. This gives a $U(1)$-valued function $g_{ijk} \in \underline{U(1)}(U_{ijk}) \cong \mathrm{Aut}(P_k|_{U_{ijk}})$, which satisfies the cocycle condition on quadruple intersections. One can show that the cohomology class given by the g_{ijk} is invariant with respect to all choices made in this construction. In particular, $\mathcal{B}U(1)$ gives the trivial class in $H^2(M, \underline{U(1)})$. We refer the reader to Brylinski [**8**] for the reverse construction which produces a gerbe from a 2-cocycle.

Since the "sections" of a $U(1)$-gerbe are locally principal $U(1)$-bundles, they can be equipped with connections (local 1-forms) which give their curvatures (local 2-forms). This fact leads to the notion of equipping the gerbe with a connection and curving. One can proceed further and show that gerbes equipped with such structures correspond to the aforementioned Deligne 2-cocycles (8.1). The precise definitions of connections and curvings and their relationships to Deligne cohomology are somewhat lengthy and technical, so we, again, refer the reader to [**8**] for the details.

8.2. Exact Courant algebroids as higher Atiyah algebroids. Recall that in Section 2.4, we discussed how the transitive Lie algebroid A on a prequantized symplectic manifold is isomorphic to the Atiyah algebroid associated to a principal $U(1)$-bundle $P \to M$ equipped with a connection. Sections of the Atiyah algebroid are $U(1)$-invariant vector fields on the total space of the bundle. Therefore they act as infinitesimal $U(1)$-equivariant diffeomorphisms on P. Prop. 7.7 implies that the quantized Poisson algebra is the subalgebra of infinitesimal diffeomorphisms that preserve the connection on P. Analogously, the above discussion and Prop. 5.1 imply that the Courant algebroid C on a prequantized 2-plectic manifold is associated to a $U(1)$-gerbe $\mathcal{G} \to M$ equipped with a connection and curving. Furthermore, Prop. 7.7 suggests that we interpret the Lie 2-algebra $L_\infty(C)^s$ as the quantization of the Lie 2-algebra of "observables" $L_\infty(M, \omega)$. Clearly, these results further support the idea that exact Courant algebroids play the role of higher Atiyah algebroids [**6, 15**]. However, interpreting $L_\infty(C)^s$ as "operators" or as infinitesimal symmetries of $\mathcal{G}$ is still a work in progress.

One possible strategy for addressing these issues is to work with principal $BU(1)$ 2-bundles and Lie groupoids rather than $U(1)$-gerbes and manifolds [**4, 5**]. $BU(1)$ is the one object Lie groupoid

$$U(1) \rightrightarrows *$$

It is also an example of a strict Lie 2-group, i.e., a Lie groupoid that is equipped with a strict (and smooth) monoidal structure such that all objects have inverses. The action of a Lie 2-group on a Lie groupoid is the higher analogue of the action of a Lie group on a manifold. The correct morphisms between Lie groupoids are not smooth functors, but rather "Morita maps", or "bibundles". (See Def. 3.25 in [**22**].) Since any manifold M is a trivial Lie groupoid $M \rightrightarrows M$, one can speak of a Lie groupoid morphism $M \to BU(1)$. By unfolding the definition of a bibundle, one can show that such a morphism corresponds to a principal $U(1)$-bundle over M. In other words, sections of the trivial principal $BU(1)$ 2-bundle over M correspond to principal $U(1)$-bundles over M, just as sections of the trivial principal $U(1)$-bundle over M corresponds to $U(1)$-valued functions. Bartels [**5**] showed that principal $BU(1)$ 2-bundles are classified by the usual Čech cohomology $H^2(M, \underline{U(1)})$.

Given a 2-cocycle, the corresponding $U(1)$-gerbe is the stack of sections of the corresponding 2-bundle. One can go further and equip a principal $BU(1)$ 2-bundle with a "2-connection". These correspond to Deligne 2-cocycles [**4**]. One could try to understand how sections of an exact Courant algebroid over a prequantized 2-plectic manifold correspond to $BU(1)$-invariant vector fields on a principal $BU(1)$ 2-bundle. This would be in complete analogy with the symplectic case. We will, in fact, see in the next section that there is a relationship between the Lie 2-algebra $L_\infty(C)^s$ and the Lie 2-algebra that integrates to $BU(1)$.

9. Central extensions of Lie 2-algebras

In this section, we push the analogy between prequantization and categorified prequantization further by constructing the 2-plectic version of the Kostant–Souriau central extension, which we discussed in Section 2.5. First some preliminary definitions:

Definition 9.1. A Lie 2-algebra $(L_\bullet, [\cdot,\cdot], J)$ is **trivial** iff $L_1 = 0$.

Any Lie algebra $\mathfrak{g}$ gives a trivial Lie 2-algebra whose underlying complex is

$$0 \to \mathfrak{g}.$$

In particular, the Lie algebra of Hamiltonian vector fields $\mathfrak{X}_{\mathrm{Ham}}(M)$ is a trivial Lie 2-algebra.

Definition 9.2. A Lie 2-algebra $(L_\bullet, [\cdot,\cdot], J)$ is **abelian** iff $[\cdot,\cdot] = 0$ and $J = 0$.

Hence an abelian Lie 2-algebra is just a 2-term chain complex.

Definition 9.3. If L, L', and L'' are Lie 2-algebras whose underlying chain complexes are $L_\bullet$, $L'_\bullet$, and $L''_\bullet$, respectively, then L' is a **strict extension** of L'' by L iff there exists Lie 2-algebra morphisms

$$(\phi_\bullet, \Phi)\colon L \to L', \quad (\phi'_\bullet, \Phi')\colon L' \to L''$$

such that

$$L_\bullet \overset{\phi_\bullet}{\to} L'_\bullet \overset{\phi'_\bullet}{\to} L''_\bullet$$

is a short exact sequence of complexes. We say L' is a **strict central extension** of L'' iff L' is a strict extension of L'' by L and

$$\left[\operatorname{im}\phi_\bullet, L'_\bullet\right]' = 0.$$

These definitions will be sufficient for our discussion here. However, they are, in general, too strict. For example, one can have homotopies between morphisms between Lie 2-algebras, and therefore we should consider sequences that are only exact up to homotopy as "exact". In what follows, by an extension we mean a strict extension in the sense of Def. 9.3.

We would like to understand how $L_\infty(M,\omega)$ is a central extension of $\mathfrak{X}_{\mathrm{Ham}}(M)$ as a Lie 2-algebra. Our first two results are quite general and hold for any 2-plectic manifold (M,ω).

Proposition 9.4. *If (M,ω) is a 2-plectic manifold, then the Lie 2-algebra $L_\infty(M,\omega)$ is a central extension of the trivial Lie 2-algebra $\mathfrak{X}_{\mathrm{Ham}}(M)$ by the abelian Lie 2-algebra*

$$C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{cl}}(M),$$

consisting of smooth functions and closed 1-forms.

Proof. Consider the following short exact sequence of complexes:

$$\begin{tikzcd} \Omega^1_{\mathrm{cl}}(M) \arrow[r, "\jmath"] & \Omega^1_{\mathrm{Ham}}(M) \arrow[r, "p"] & \mathfrak{X}_{\mathrm{Ham}}(M) \\ C^\infty(M) \arrow[u, "d"] \arrow[r, "\mathrm{id}"] & C^\infty(M) \arrow[u, "d"] \arrow[r] & 0 \arrow[u] \end{tikzcd} \tag{9.1}$$

The map $\jmath\colon \Omega^1_{\mathrm{cl}}(M) \to \Omega^1_{\mathrm{Ham}}(M)$ is the inclusion, and

$$p\colon \Omega^1_{\mathrm{Ham}}(M) \to \mathfrak{X}_{\mathrm{Ham}}(M), \quad p(\alpha) = v_\alpha$$

takes a Hamiltonian 1-form to its corresponding vector field. It follows from Prop. 3.4 that p preserves the bracket. In fact, all of the horizontal chain maps give strict Lie 2-algebra morphisms (i.e., all homotopies are trivial). The Hamiltonian vector field corresponding to a closed 1-form is zero. Thus, if α is closed, then for all $\beta \in \Omega^1_{\mathrm{Ham}}(M)$ we have $[\alpha,\beta]_{L_\infty(M,\omega)} = \{\alpha,\beta\} = 0$. Hence, $L_\infty(M,\omega)$ is a central extension of $\mathfrak{X}_{\mathrm{Ham}}(M)$. ☐

Proposition 9.5. *Let (M,ω) be a 2-plectic manifold. Given $x \in M$, there is a Lie 2-algebra $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x) = (L_\bullet,[\cdot,\cdot],J_x)$ where*

- *$L_0 = \mathfrak{X}_{\mathrm{Ham}}(M)$,*
- *$L_1 = \mathbb{R}$,*
- *the differential $L_1 \xrightarrow{d} L_0$ is trivial ($d = 0$),*
- *the bracket $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{X}_{\mathrm{Ham}}(M)$ in degree 0 and trivial in all other degrees*
- *the Jacobiator is the linear map*

$$J_x\colon \mathfrak{X}_{\mathrm{Ham}}(M) \otimes \mathfrak{X}_{\mathrm{Ham}}(M) \otimes \mathfrak{X}_{\mathrm{Ham}}(M) \to \mathbb{R}$$

defined by

$$J_x(v_1,v_2,v_3) = \iota_{v_1}\iota_{v_2}\iota_{v_3}\omega|_x.$$

Moreover, J_x is a 3-cocycle in the Chevalley–Eilenberg cochain complex $\mathrm{Hom}(\Lambda^\bullet\mathfrak{X}_{\mathrm{Ham}}(M),\mathbb{R})$.

Proof. We have a bracket defined on a complex with trivial differential that satisfies the Jacobi identity "on the nose". Hence to show $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x)$ is a Lie 2-algebra, it sufficient to show that the Jacobiator $J_x(v_1,v_2,v_3)$

satisfies equation (6.1) in Def. 6.1 for $x \in M$. This follows immediately from Thm. 6.4. The classification theorem of Baez and Crans (Thm. 55 in [**1**]) implies that J_x satisfying equation (6.1) in the definition of a Lie 2-algebra is equivalent to J_x being a 3-cocycle with values in the trivial representation. □

Recall that in the symplectic case, if the manifold is connected, then the Poisson algebra is a central extension of the Hamiltonian vector fields by the Lie algebra $\mathfrak{u}(1) \cong \mathbb{R}$. The categorified analog of the Lie algebra $\mathfrak{u}(1)$ is the abelian Lie 2-algebra $b\mathfrak{u}(1)$ whose underlying chain complex is simply

$$\mathbb{R} \to 0.$$

This Lie 2-algebra integrates to the Lie 2-group $BU(1)$ discussed in Section 8.2. It is natural to suspect that, under suitable topological conditions, the abelian Lie algebra $C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{cl}}(M)$ introduced in Prop. 9.4 is related to $b\mathfrak{u}(1)$.

Let us first assume that the 2-plectic manifold is connected. Note that the Jacobiator J_x of the Lie 2-algebra $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$ introduced in Prop. 9.5 depends explicitly on the choice of $x \in M$. However, if M is connected, then the cohomology class J_x represents as a 3-cocycle does not depend on x. This fact has important implications for $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$:

Proposition 9.6. *If (M, ω) is a connected 2-plectic manifold and J_x is the 3-cocycle given in Prop.* 9.5*, then the cohomology class $[J_x] \in H^3_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$ is independent of the choice of $x \in M$. Moreover, given any other point $y \in M$, the Lie 2-algebras $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$ and $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), y)$ are quasi-isomorphic.*

Proof. To prove that $[J_x]$ is independent of x, we use a construction similar to the proof of Prop. 4.1 in [**7**]. The Chevalley–Eilenberg differential

$$\delta\colon \mathrm{Hom}(\Lambda^n \mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R}) \to \mathrm{Hom}(\Lambda^{n+1} \mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$$

is defined by

$$(\delta c)(v_1, \dots, v_{n+1}) = \sum_{1 \le i < j \le n} (-1)^{i+j} c([v_i, v_j], v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \dots, v_{n+1}).$$

Note that if c is an arbitrary 2-cochain then

$$(\delta c)(v_\alpha, v_\beta, v_\gamma) = -c([v_\alpha, v_\beta], v_\gamma) + c([v_\alpha, v_\gamma], v_\beta) - c([v_\beta, v_\gamma], v_\alpha).$$

Now let $y \in M$. Let $\Gamma\colon [0,1] \to M$ be a path from x to y. Given $v_\alpha, v_\beta \in \mathfrak{X}_{\mathrm{Ham}}(M)$, define

$$c(v_\alpha, v_\beta) = \int_\Gamma \omega(v_\alpha, v_\beta, \cdot).$$

Clearly, c is a 2-cochain. We claim

$$J_y(v_\alpha, v_\beta, v_\gamma) - J_x(v_\alpha, v_\beta, v_\gamma) = (\delta c)(v_\alpha, v_\beta, v_\gamma)$$

From part 3 of Prop. 3.4, we have:

$$d\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega = \{\alpha,\{\beta,\gamma\}\} - \{\{\alpha,\beta\},\gamma\} - \{\beta,\{\alpha,\gamma\}\}.$$

By definition of the bracket $\{\cdot,\cdot\}$, this implies

$$d\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega = -\omega([v_\alpha,v_\beta],v_\gamma,\cdot) + \omega([v_\alpha,v_\gamma],v_\beta,\cdot) - \omega([v_\beta,v_\gamma],v_\alpha,\cdot).$$

Integrating both sides of the above equation gives

$$\begin{aligned}\int_\Gamma d\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega &= J_y(v_\alpha,v_\beta,v_\gamma) - J_x(v_\alpha,v_\beta,v_\gamma)\\ &= -\int_\Gamma \omega([v_\alpha,v_\beta],v_\gamma,\cdot) + \int_\Gamma \omega([v_\alpha,v_\gamma],v_\beta,\cdot)\\ &\quad - \int_\Gamma \omega([v_\beta,v_\gamma],v_\alpha,\cdot)\\ &= (\delta c)(v_\alpha,v_\beta,v_\gamma).\end{aligned}$$

It follows from Thm. 57 in Baez and Crans [**1**] that $[J_x]=[J_y]$ implies $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x)$ and $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),y)$ are quasi-isomorphic (or "equivalent" in their terminology). ∎

Now we impose further conditions on our 2-plectic manifold. From here on, we assume (M,ω) is 1-connected (i.e., connected and simply connected). This is the 2-plectic analogue of the requirement that the symplectic manifold in Section 2.5 be connected. It will allow us to construct several elementary, yet interesting, quasi-isomorphisms of Lie 2-algebras.

Proposition 9.7. *If M is a 1-connected manifold, then the abelian Lie 2-algebra $C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{cl}}(M)$ is quasi-isomorphic to $b\mathfrak{u}(1)$.*

Proof. Let $x \in M$. The chain map

$$\begin{tikzcd} \Omega^1_{\mathrm{cl}}(M) \arrow[r] & 0 \\ C^\infty(M) \arrow[u, "d"] \arrow[r, "\mathrm{ev}_x"] & \mathbb{R} \arrow[u] \end{tikzcd}$$

is a quasi-isomorphism. ∎

Proposition 9.8. *If (M,ω) is a 1-connected 2-plectic manifold and $x \in M$, then the Lie 2-algebras $L_\infty(M,\omega)$ and $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x)$ are quasi-isomorphic.*

Proof. We construct a quasi-isomorphism from $L_\infty(M,\omega)$ to $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$. There is a chain map

$$\begin{tikzcd}
\Omega^1_{\mathrm{Ham}}(M) \arrow[r, "p"] & \mathfrak{X}_{\mathrm{Ham}}(M) \\
C^\infty(M) \arrow[u, "d"] \arrow[r, "\mathrm{ev}_x"] & \mathbb{R} \arrow[u, "0"']
\end{tikzcd}$$

with $\mathrm{ev}_x(f) = f(x)$ and $p(\alpha) = v_\alpha$. Since p preserves the bracket, we take Φ in Def. 6.2 to be the trivial homotopy. Equation (6.2) holds since:

$$\mathrm{ev}_x(\omega(v_\gamma, v_\beta, v_\alpha)) = J_x(v_\alpha, v_\beta, v_\gamma),$$

and therefore we have constructed a Lie 2-algebra morphism. Since M is connected, the homology of the complex $C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{Ham}}(M)$ is just $\mathbb{R}$ in degree 1 and $\Omega^1_{\mathrm{Ham}}(M)/dC^\infty(M)$ in degree 0. The kernel of the surjective map p is the space of closed 1-forms, which is $dC^\infty(M)$ since M is simply connected. $\square$

We can summarize the results given in Props. 9.4, 9.5, 9.7, and 9.8 with the following commutative diagram:

$$\begin{tikzcd}
\Omega^1_{\mathrm{cl}}(M) \arrow[rr, "\jmath"] \arrow[dr, squiggly] & & \Omega^1_{\mathrm{Ham}}(M) \arrow[rr, "p"] \arrow[dr, squiggly, "p"] & & \mathfrak{X}_{\mathrm{Ham}}(M) \arrow[dr, "\mathrm{id}"] & \\
& 0 \arrow[rr] & & \mathfrak{X}_{\mathrm{Ham}}(M) \arrow[rr] & & \mathfrak{X}_{\mathrm{Ham}}(M) \\
C^\infty(M) \arrow[uu, "d"] \arrow[rr] \arrow[dr, squiggly, "\mathrm{ev}_x"'] & & C^\infty(M) \arrow[uu] \arrow[rr] \arrow[dr, squiggly, "\mathrm{ev}_x"'] & & 0 \arrow[uu] \arrow[dr] & \\
& \mathbb{R} \arrow[uu] \arrow[rr] & & \mathbb{R} \arrow[uu] \arrow[rr] & & 0 \arrow[uu]
\end{tikzcd}$$

The back of the diagram shows $L_\infty(M,\omega)$ as the central extension of the trivial Lie 2-algebra $\mathfrak{X}_{\mathrm{Ham}}(M)$. The front shows $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$ as a central extension of $\mathfrak{X}_{\mathrm{Ham}}(M)$ by $b\mathfrak{u}(1)$. The morphisms going from back to front are all quasi-isomorphisms. Thus we have the 2-plectic analogue of the Kostant–Souriau central extension:

Corollary 9.9. *If (M,ω) is a 1-connected 2-plectic manifold, then $L_\infty(M,\omega)$ is quasi-isomorphic to a central extension of the trivial Lie 2-algebra $\mathfrak{X}_{\mathrm{Ham}}(M)$ by $b\mathfrak{u}(1)$.*

Also, from Prop. 7.7 we know that $L_\infty(M,\omega)$ is isomorphic to the Lie 2-algebra $L_\infty(C)^s$ consisting of sections of the Courant algebroid C which preserve a chosen splitting $s\colon TM \to C$. Therefore:

Corollary 9.10. *If (M,ω) is a 1-connected 2-plectic manifold, then $L_\infty(C)^s$ is quasi-isomorphic to a central extension of the trivial Lie 2-algebra $\mathfrak{X}_{\mathrm{Ham}}(M)$ by $b\mathfrak{u}(1)$.*

A comparison of the above corollary to the results discussed in Section 2.5 suggests that $L_\infty(C)^s$ be interpreted as the quantization of $L_\infty(M,\omega)$ with $b\mathfrak{u}(1)$ giving rise to the quantum phase.

Finally, note that a splitting of the short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{X}_{\mathrm{Ham}}(M) \arrow[r, "\mathrm{id}"] & \mathfrak{X}_{\mathrm{Ham}}(M) \\ \mathbb{R} \arrow[u] \arrow[r, "\mathrm{id}"] & \mathbb{R} \arrow[u, "0"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

is the identity map in degree 0 and the trivial map in degree 1. Obviously the splitting preserves the bracket but does not preserve the Jacobiator. Indeed, the failure of the splitting to be a strict Lie 2-algebra morphism between $\mathfrak{X}_{\mathrm{Ham}}(M)$ and $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x)$ is due to the presence of the 3-cocycle J_x.

10. Conclusion

Let us summarize the main points of the previous sections: If (M,ω) is a 0-connected, prequantized symplectic manifold, then there exists a principal $U(1)$-bundle over M equipped with a connection whose curvature is ω, and a corresponding Atiyah algebroid $A \to M$ equipped with a splitting such that the Lie algebra of sections of A which preserve the splitting is isomorphic to a central extension of the Lie algebra of Hamiltonian vector fields:

$$\mathfrak{u}(1) \to C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M).$$

This central extension gives a cohomology class in $H^2_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{Ham}}(M),\mathbb{R})$ which can be represented by the symplectic form evaluated at a point in M.

Analogously, if (M,ω) is a 1-connected, prequantized 2-plectic manifold, then there exists a $U(1)$-gerbe over M equipped with a connection and curving whose 3-curvature is ω, and a corresponding exact Courant algebroid $C \to M$ equipped with a splitting such that the Lie 2-algebra of sections of C which preserve the splitting is quasi-isomorphic to a central extension of the (trivial) Lie 2-algebra of Hamiltonian vector fields:

$$b\mathfrak{u}(1) \to L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M)) \to \mathfrak{X}_{\mathrm{Ham}}(M).$$

This central extension gives a cohomology class in $H^3_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{Ham}}(M),\mathbb{R})$ which can be represented by the 2-plectic form evaluated at a point in M.

In future work, we will develop this analogy further in order to obtain a categorified geometric quantization procedure for 2-plectic manifolds. In doing so, it is likely that we will make contact with related areas of interest including the representation theory of loop groups and extended topological quantum field theories (TQFTs). Such a procedure would also provided new insights into the theory of Courant algebroids.

However, there are several open problems in prequantization that we are currently addressing as we set our sights on full quantization. We have mentioned some of these throughout the text, and we summarize them here:

- For every principal $U(1)$ bundle with connection, there is an associated hermitian line bundle with connection, whose global sections give a Hilbert space. What is the corresponding geometric object for a $U(1)$-gerbe equipped with a connection and curving? (One possible answer is described in Section 5.5 of [**26**].)
- Sections of the Atiyah algebroid on a prequantized symplectic manifold are operators on this Hilbert space. How do sections of the Courant algebroid on a prequantized 2-plectic manifold act as operators on the higher analogue of this Hilbert space?
- Sections of the Atiyah algebroid are infinitesimal $U(1)$-equivariant symmetries of the corresponding principal $U(1)$-bundle. Integration gives elements of the gauge group, i.e., equivariant diffeomorphisms of the principal bundle. How can we understand sections of the Courant algebroid on a prequantized 2-plectic manifold as infinitesimal symmetries of the corresponding $U(1)$-gerbe?

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