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NONCOMMUTATIVE INTEGRABILITY AND ACTION–ANGLE VARIABLES IN CONTACT GEOMETRY

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We introduce a notion of the noncommutative integrability within a framework of contact geometry.

1. Introduction

In Hamiltonian mechanics solving by quadratures is closely related to the regularity of dynamics that is described in the Arnold–Liouville theorem. A Hamiltonian system on $2n$-dimensional symplectic manifold M is called *integrable* if it has n-smooth Poisson-commuting, almost everywhere independent integrals $f_1, f_2, \dots, f_n$. Regular compact connected invariant manifolds of the system are Lagrangian tori. Moreover, in a neighborhood of any torus, there exist canonical action–angle coordinates $(\varphi, I) = (\varphi_1, \dots, \varphi_n, I_1, \dots, I_n)$, integrals f_i depend only on actions I and the flow is translation in φ coordinates [**1**].

Therefore, an integrable Hamiltonian system can be considered as a toric Lagrangian fibration $\pi : M \to W$ (see Duistermaat [**11**]). This approach is reformulated to contact manifolds $(M, \mathcal{H})$ by Banyaga and Molino [**2**]. Instead of a toric Lagrangian fibration, one consider an invariant toric fibration transversal to the contact distribution $\mathcal{H}$, such that intersection of tori and $\mathcal{H}$ is a Lagrangian distribution with respect to the conformal class of the symplectic structure on $\mathcal{H}$ (see Section 5).

Slightly different notion of a contact integrability is given recently by Khesin and Tabachnikov [**19**]. They defined integrability in terms of the existence of an invariant foliation $\mathcal{F}$, called a co-Legendrian foliation (here we refer to $\mathcal{F}$ as a pre-Legendrian foliation). $\mathcal{F}$ is transversal to $\mathcal{H}$, $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ is a Legendrian foliation of M with an additional property that on every

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leaf F of $\mathcal{F}$, the foliation $\mathcal{G}|_F$ has a holonomy invariant transverse smooth measure.[1] It turns out that this condition implies the existence of a global contact form α (see [**19**]) and $\mathcal{G}$ is a α-complete Legendrian foliation studied by Libermann [**27**] and Pang [**32**]. Recall that a foliation $\mathcal{F}$ is *α-complete* if for any pair f_1, f_2 of first integrals of $\mathcal{F}$ (where f_i may be a constant), the Jacobi bracket $[f_1, f_2]$ is also a first integral of $\mathcal{F}$ (eventually a constant).

Due to the presence of symmetries, many Hamiltonian systems have more than n noncommuting integrals. Illustrative examples are G-invariant geodesic flows on homogeneous spaces [**4, 16**]. An appropriate framework for the study of these systems is noncommutative integrability introduced by Nehoroshev [**31**] and Mishchenko and Fomenko [**30**] (see also [**4,16,21,36**]). Here, we recall the Nehoroshev formulation: a Hamiltonian system on $2n$-dimensional symplectic manifold M is *noncommutatively integrable* if it has $2n-r$ almost everywhere independent integrals $f_1, f_2, \dots, f_{2n-r}$ and $f_1, \dots, f_r$ commute with all integrals

$$\{f_i, f_j\} = 0, \quad i = 1, \dots, 2n-r, \quad j = 1, \dots, r.$$

Regular compact connected invariant manifolds of the system are isotropic tori. In a neighborhood of a regular torus, there exist canonical *generalized action–angle coordinates* such that integrals f_i, $i = 1, \dots, r$ depend only on actions and the flow is translation in angle coordinates.

One of the basic examples of contact manifolds are unit co-sphere bundles $SQ \subset T^*Q$ of Riemannian manifolds (Q, g). The restriction of a geodesic flow to SQ is a contact flow of the Reeb vector field of the associated contact form. It is clear that noncommutatively integrable geodesic flows, considered as Reeb vector flows, have a geometrical structure that need to be described by a noncommutative variant of integrability.

We introduce an appropriate concept of a contact noncommutative integrability.

In the first part of the paper (Sections 3 and 4), foliations on contact manifolds $(M, \mathcal{H})$ are considered. We refer to a foliation $\mathcal{F}$ as *pre-isotropic* if it is transversal to $\mathcal{H}$ and $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ is an isotropic subbundle of $\mathcal{H}$.

Let $\mathcal{F}$ be a pre-isotropic foliation containing the Reeb vector field Z on a co-oriented contact manifold (M, α). The foliation $\mathcal{F}$ is α-complete if and only if $\mathcal{E}$ is completely integrable, where $\mathcal{E} = \mathcal{F}^\perp$ is the *pseudo-orthogonal distribution* of $\mathcal{F}$ and we have a flag of foliations $\mathcal{G} \subset \mathcal{F} \subset \mathcal{E}$. Furthermore, each leaf of $\mathcal{G}$ and $\mathcal{F}$ has an affine structure (Theorem 3.2).

Thus, if $\mathcal{F}$ has compact leaves, they are tori. Locally, in an invariant neighborhood of any leaf, the foliation $\mathcal{F}$ can be seen as a fibration over some base manifold. Also, affine translations provide an Abelian Lie algebra of contact transformations with orbits that coincide with $\mathcal{F}$.

[1] Throughout the paper we use the same notation for foliations and their integrable distributions of tangent spaces.

Next, we consider a pre-isotropic foliation $\mathcal{F}$ on a contact manifold $(M, \mathcal{H})$ with the mentioned properties of α-complete pre-isotropic foliations: $\mathcal{F}$ is defined via submersion $\pi : M \to W$ and it is given an Abelian Lie algebra of contact symmetries $\mathcal{X}$ with orbits equal to $\mathcal{F}$. We refer to a triple $(M, \mathcal{H}, \mathcal{X})$ as a *complete pre-isotropic contact structure*.

For a given complete pre-isotropic structure $(M, \mathcal{H}, \mathcal{X})$, locally, there always exists an invariant contact form α such that $\mathcal{F}$ is α-complete (Theorem 4.1). Note, if $\mathcal{F}$ has the maximal dimension (i.e., it is pre-Legendrian) and fibers of π are connected, $(M, \mathcal{H}, \mathcal{X})$ is a *regular completely integrable contact structure* studied by Banyaga and Molino [**2**]. The analysis above lead us to the following definition (Section 5).

Let X be a contact vector field. We shall say that a contact equation

$$\dot{x} = X \tag{1.1}$$

is *contact noncommutatively integrable* if there is an Abelian Lie algebra of contact symmetries $\mathcal{X}$, an open dense set $M_{\text{reg}} \subset M$, and a submersion $\pi : M_{\text{reg}} \to W$ such that

(i) The contact vector field X is tangent to the fibers of π;
(ii) $(M_{\text{reg}}, \mathcal{H}, \mathcal{X})$ is a complete pre-isotropic contact structure.

Analogs to the Mishchenko–Fomenko–Nehoroshev theorem, we prove that in a neighborhood of any invariant torus, there exist canonical generalized contact action–angle coordinates and (1.1) is a translation in angle variables, where frequencies depend only on actions and in which the contact distribution $\mathcal{H}$ is presented by the canonical 1-form α_0 (Theorem 5.1).

For the co-oriented case, we also formulate the statement involving only integrals of a motion (Theorem 5.2): a contact equation (1.1) is noncommutatively integrable if it possesses a collection of first integrals $f_1, f_2, \dots, f_{2n-r}$, that are all in involution with the constant functions and with the first r integrals:

$$[1, f_i] = 0, \quad [f_i, f_j] = 0, \quad i = 1, \dots, 2n - r, \quad j = 1, \dots, r.$$

Note that, besides integrable geodesic flows on homogeneous spaces restricted to the unit co-sphere bundles [**4, 16**], a natural class of examples of contact flows integrable in a noncommutative sense are the Reeb flows on K-contact manifolds (M^{2n+1}, α) where the rank of the manifold is less then $n + 1$ (see Yamazaki [**35**] and Lerman [**23**]).

Finally in Section 6, we consider a complete pre-isotropic contact structure $(M, \alpha, \mathcal{X})$ of the *Reeb type*, i.e., $\mathcal{H}$ is defined by a global $\mathcal{X}$-invariant form α and the Reeb vector field of α is π-vertical. Note that $(M, \mathcal{H}, \mathcal{X})$ can be a complete pre-isotropic structure with a global $\mathcal{X}$-invariant contact form α, which is not of the Reeb type (see Proposition 6.1). On the other hand, the invariant foliation $\mathcal{F}$ of a complete pre-isotropic contact structure $(M, \alpha, \mathcal{X})$ of the Reeb type is α-complete (Proposition 6.2). We describe the transition functions between the contact action–angle coordinates (Proposition 6.3) and prove the statement on the existence of global action–action variables in the case when $\pi : M \to W$ is a trivial principal $\mathbb{T}^{r+1}$-bundle (Theorem 6.1).

2. Contact manifolds and the Jacobi bracket

In the definitions and notations, we mostly follow Libermann and Marle [**25**].

A *contact form* α on a $(2n+1)$-dimensional manifold M is a Pfaffian form satisfying $\alpha \wedge (d\alpha)^n \neq 0$. By a *contact manifold* $(M, \mathcal{H})$ we mean a connected $(2n+1)$-dimensional manifold M equipped with a nonintegrable *contact* (or *horizontal*) *distribution* $\mathcal{H}$, locally defined by a contact form: $\mathcal{H}|_U = \ker \alpha|_U$, U is an open set in M.

Two contact forms α and α' define the same contact distribution $\mathcal{H}$ on U if and only if $\alpha' = a\alpha$ for some nowhere vanishing function a on U. The condition $\alpha \wedge (d\alpha)^n \neq 0$ implies that the form $d\alpha|_x$ is nondegenerate (symplectic) structure restricted to $\mathcal{H}_x$. The conformal class of $d\alpha|_x$ is invariant under the change $\alpha' = a\alpha$. If $\mathcal{V}$ is a linear subspace of $\mathcal{H}_x$, then we have well-defined orthogonal complement $\operatorname{orth}_{\mathcal{H}} \mathcal{V} \subset \mathcal{H}_x$ with respect to $d\alpha|_x$, as well as the notion of the *isotropic* ($\mathcal{V} \subset \operatorname{orth}_{\mathcal{H}} \mathcal{V}$), *coisotropic* ($\mathcal{V} \supset \operatorname{orth}_{\mathcal{H}} \mathcal{V}$) and the *Lagrange* subspaces ($\mathcal{V} = \operatorname{orth}_{\mathcal{H}} \mathcal{V}$) of $\mathcal{H}_x$.

A *contact diffeomorphism* between contact manifolds $(M, \mathcal{H})$ and $(M', \mathcal{H}')$ is a diffeomorphism $\phi : M \to M'$ such that $\phi_* \mathcal{H} = \mathcal{H}'$. If a local 1-parameter group of a vector field X is made of contact diffeomorphisms, X is called an *infinitesimal automorphism* of a contact structure $(M, \mathcal{H})$ or a *contact vector field*. Locally, if $\mathcal{H} = \ker \alpha$, then $\mathcal{L}_X \alpha = \lambda \alpha$, for some smooth function λ.

The existence of a global contact form α is equivalent to the coorientability of $\mathcal{H}$ [**15**]. From now on we consider a *co-oriented (or strictly) contact manifold* (M, α). The *Reeb vector field* Z is a vector field uniquely defined by

$$i_Z \alpha = 1, \quad i_Z d\alpha = 0.$$

The tangent bundle TM and the cotangent bundle T^*M are decomposed into

$$TM = \mathcal{Z} \oplus \mathcal{H}, \quad T^*M = \mathcal{Z}^0 \oplus \mathcal{H}^0, \tag{2.1}$$

where $\mathcal{Z} = \mathbb{R}Z$ is the kernel of $d\alpha$, $\mathcal{Z}^0$ and $\mathcal{H}^0 = \mathbb{R}\alpha$ are the annihilators of $\mathcal{Z}$ and $\mathcal{H}$, respectively. The sections of $\mathcal{Z}^0$ are called *semi-basic forms*.

According to (2.1), we have decompositions of vector fields and 1-forms

$$X = (i_X \alpha) Z + \hat{X}, \quad \eta = (i_Z \eta)\alpha + \hat{\eta}, \tag{2.2}$$

where $\hat{X}$ is horizontal and $\hat{\eta}$ is semi-basic.

The mapping $\alpha^\flat : X \mapsto -i_X d\alpha$ carries X onto a semi-basic form. The restriction of $\alpha^\flat$ to horizontal vector fields is an isomorphism whose inverse will be denoted by $\alpha^\sharp$. The mapping

$$\Phi : \mathcal{N} \longrightarrow C^\infty(M), \quad \Phi(X) = i_X \alpha \tag{2.3}$$

establish the isomorphism between the vector space $\mathcal{N}$ of infinitesimal contact automorphisms onto the set $C^\infty(M)$ of smooth functions on M, with the inverse (see [**24, 25**])

$$\Phi^{-1}(f) = fZ + \alpha^\sharp(\widehat{df}).$$

The vector field $X_f = \Phi^{-1}(f)$ is called the *contact Hamiltonian vector field* and

$$\dot{x} = X_f \tag{2.4}$$

contact Hamiltonian equation corresponding to f. Note that

$$\mathcal{L}_{X_f}\alpha = df(Z)\alpha$$

and X_f is an *infinitesimal automorphism of* α ($\mathcal{L}_{X_f}\alpha = 0$) if and only if df is semi-basic. Note that $\Phi(Z) = 1$, i.e., $Z = X_1$.

The mapping (2.3) is a Lie algebra isomorphism, where on $\mathcal{N}$ we have the usual bracket and the *Jacobi bracket* on $C^\infty(M)$ defined by $[f, g] = \Phi[X_f, X_g]$:

$$X_{[f,g]} = [X_f, X_g], \quad X_{[1,f]} = [Z, X_f].$$

Note that df is semi-basic, if and only if $[1, f] = [Z, X_f] = 0$.

Together with the Jacobi bracket, we have the associated *Jacobi bi-vector field* Λ:

$$\Lambda(\eta, \xi) = d\alpha(\alpha^\sharp\hat{\eta}, \alpha^\sharp\hat{\xi}).$$

Let $\Lambda^\sharp : T^*M \to TM$ be the morphism defined by $\langle \Lambda^\sharp_x(\eta_x), \xi_x\rangle = \Lambda_x(\eta_x, \xi_x)$, for all $x \in M$, $\eta_x, \xi_x \in T^*_x M$. Then X_f may be written as $X_f = fZ + \Lambda^\sharp(df)$.

It can be easily checked that

$$[f, g] = d\alpha(X_f, X_g) + f\mathcal{L}_Z g - g\mathcal{L}_Z f = \Lambda(df, dg) + f\mathcal{L}_Z g - g\mathcal{L}_Z f.$$

The derivation of functions along the contact vector field X_f can be described by the use of the Jacobi bracket

$$\mathcal{L}_{X_f} g = [f, g] + g\mathcal{L}_Z f. \tag{2.5}$$

Thus, if df and dg are semi-basic, we have the following important property of the Jacobi bracket $[f, g]$.

Lemma 2.1. *Suppose that df and dg are semi-basic. Then*

$$[f, g] = d\alpha(X_f, X_g) = \Lambda(df, dg)$$

and the following statements are equivalent:

(i) f and g are in involution: $[f, g] = 0$, i.e., Hamiltonian contact vector fields X_f and X_g commute: $[X_f, X_g] = 0$.

(ii) g is the integral of the contact vector field X_f: $\mathcal{L}_{X_f} g = 0$.

(iii) f is the integral of the contact vector field X_g: $\mathcal{L}_{X_g} f = 0$.

Moreover, if $\mathcal{Z}$ is a simple foliation, i.e., there exists a surjective submersion $\pi : M \to P$ and the distribution $\mathcal{Z}$ consist of vertical spaces of the submersion: $\mathcal{Z} = \ker \pi_*$, then the base manifold P has a nondegenerate Poisson structure $\{\cdot,\cdot\}$ such that $[f,g] = \pi^*\{\bar{f},\bar{g}\}$, $f = \bar{f}\circ\pi$, $g = \bar{g}\circ\pi$ [**6,25**].

3. α-Complete pre-isotropic foliations

Let $\mathcal{F}$ be a foliation on a co-oriented contact manifold (M^{2n+1}, α). The *pseudo-orthogonal distribution* $\mathcal{F}^\perp$ is defined by

$$\mathcal{F}^\perp = \mathcal{Z} \oplus \Lambda^\sharp(\mathcal{F}^0).$$

where $\mathcal{F}^0$ is the annihilator of $\mathcal{F}$. It is locally generated by the Reeb vector field Z and the contact Hamiltonian vector fields, that correspond to the first integrals of $\mathcal{F}$.

A foliation $\mathcal{F}$ is said to be *α-complete* if for any pair f_1, f_2 of first integrals of $\mathcal{F}$ (where f_i may be a constant), the bracket $[f_1, f_2]$ is also a first integral of $\mathcal{F}$ (eventually a constant).

Theorem 3.1 (Libermann [26]). *A foliation $\mathcal{F}$ on (M^{2n+1},α) containing the Reeb vector field Z is α-complete if and only if the pseudo-orthogonal subbundle $\mathcal{F}^\perp$ is integrable, defining a foliation that is also α-complete and $(\mathcal{F}^\perp)^\perp = \mathcal{F}$. Then for any pair of integrals f, g of $\mathcal{F}$ and $\mathcal{F}^\perp$, respectively, we have $[f,g] = 0$.*

Let p be the rank of $\mathcal{F}^0$ and $f_1, \dots, f_p$ be a set of independent integrals of $\mathcal{F}$ in an open set U. Since $\ker \Lambda^\sharp_x = \mathbb{R}\alpha_x$, $\dim \mathcal{F}^\perp_x$ is equal to $p+1$ or p, depending the forms $\alpha, df_1, \dots, df_p$ are linearly independent or not. In the later case, the form induced by α on the leaf passing through x vanishes at x. Conversely, if $\alpha|_{\mathcal{F}} = 0$, i.e., $\mathcal{F} \subset \mathcal{H}$, then $\dim \mathcal{F}^\perp = p$.

A foliation $\mathcal{G}$ is *pseudo-isotropic* if $\mathcal{G} \subset \mathcal{H}$ [**27**].[2] Then α is a section of $\mathcal{G}^0$, the distribution $\mathcal{G}^\perp$ has the constant rank p and $\mathcal{G}^\perp$ is a vector bundle. A *Legendre foliation* is a pseudo-isotropic foliation of maximum rank n. Then $\dim \mathcal{G}^0 = \dim \mathcal{G}^\perp = n+1$.

By the analogy with a pre-isotropic embedding (see Lerman [**22**]), we introduce:

Definition 3.1. A foliation $\mathcal{F}$ is *pre-isotropic* if

(i) $\mathcal{F}$ is transversal to $\mathcal{H}$.
(ii) $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ is an isotropic subbundle of $\mathcal{H}$.

Lemma 3.1. *Condition (ii) is equivalent to the condition that $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ is a pseudo-isotropic foliation.*

[2]Submanifolds $G \subset M$ that are integral manifolds of $\mathcal{H}$ are also called isotropic submanifolds, e.g., see [**14**]. Here we keep Libermann's notation.

Proof. Let $(f_1, \dots, f_p)$ be a set of local integrals of $\mathcal{F}$ and let X, Y be sections of $\mathcal{G}$. Then α, $df_1, \dots, df_p$ are linearly independent and we have

$$\begin{aligned} &df_i(X) = df_i(Y) = \alpha(X) = \alpha(Y) = 0, \\ &df_i([X,Y]) = \mathcal{L}_X\mathcal{L}_Y f_i - \mathcal{L}_Y\mathcal{L}_X f_i = 0, \\ &d\alpha(X,Y) = \mathcal{L}_X\alpha(Y) - \mathcal{L}_Y\alpha(X) - \alpha([X,Y]) = -\alpha([X,Y]). \end{aligned}$$

Therefore, $\mathcal{G}$ is an isotropic subbundle of $\mathcal{H}$ if and only if it is integrable. □

Theorem 3.2. *Let $\mathcal{F}$ be a pre-isotropic foliation containing the Reeb vector field Z.*

(i) We have the flag of distributions $(\mathcal{G}, \mathcal{F}, \mathcal{E})$:

$$\mathcal{G} = \mathcal{F} \cap \mathcal{H} \ \subset \ \mathcal{F} \ \subset \ \mathcal{E} = \mathcal{G}^{\perp} = \mathcal{F}^{\perp}. \tag{3.1}$$

On the contrary, if $\mathcal{F}$ is a foliation containing the Reeb vector field Z and (3.1) *holds, then $\mathcal{F}$ is a pre-isotropic foliation.*

(ii) The foliation $\mathcal{F}$ (or $\mathcal{G}$) is α-complete, if and only if $\mathcal{E}$ is completely integrable. Assume $\mathcal{E}$ is integrable and let $f_1, \dots, f_p$ and $y_1, \dots, y_r$, $2n-p = r$ be any sets of local integrals of $\mathcal{F}$ and $\mathcal{E}$, respectively. Then:

$$[f_i, y_j] = 0, \quad [y_j, y_k] = 0, \quad [f_i, 1] = 0, \quad [y_i, 1] = 0.$$

(iii) Each leaf of an α-complete pre-isotropic foliation $\mathcal{F}$ as well as each leaf of the corresponding pseudo-isotropic foliation $\mathcal{G}$ has an affine structure.

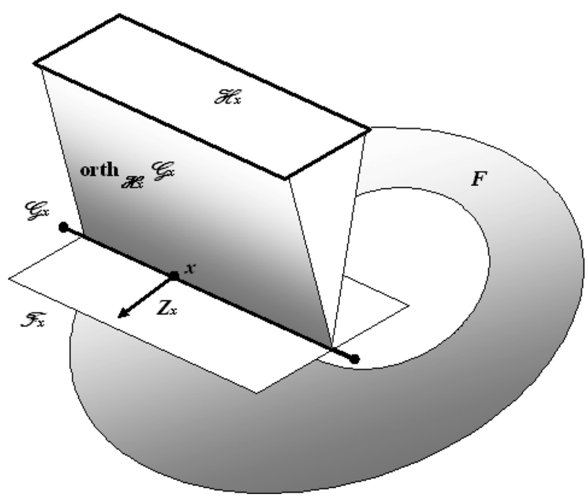

Figure 1. Illustration of Theorem 3.2: a torus F is a leaf through x.

Proof. (i) Let $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ be isotropic. We have $\mathcal{G}^0 = \langle \mathcal{F}^0, \alpha \rangle$ and $\ker \Lambda^\sharp = \mathbb{R}\alpha$. Thus:

$$\begin{aligned}\mathcal{F}^\perp = \mathcal{G}^\perp &= \mathcal{Z} \oplus \Lambda^\sharp(\mathcal{G}^0) = \mathcal{Z} \oplus \Lambda^\sharp(\mathcal{G}^0 \cap \mathcal{Z}^0)\\ &= \mathcal{Z} \oplus \alpha^\sharp(\mathcal{G}^0 \cap \mathcal{Z}^0) = \mathcal{Z} \oplus \operatorname{orth}_{\mathcal{H}} \mathcal{G} \supset \mathcal{Z} \oplus \mathcal{G} = \mathcal{F}.\end{aligned}$$

(ii) This item follows directly from Theorem 1 and the fact that integrals of $\mathcal{E}$ are also integrals of $\mathcal{F}$. Note that df_i and dy_j are semi-basic and X_{f_i}, X_{y_j} are infinitesimal automorphisms of α.

(iii) Let U be an open set where we have defined commuting integrals $y_1, \dots, y_r$ of $\mathcal{E}|_U$. Since $\mathcal{E}^\perp = \mathcal{F}$, the distribution $\mathcal{F}|_U$ is generated by a contact commuting vector fields $Z, X_{y_1}, \dots, X_{y_r}$:

$$[Z, X_{y_i}] = 0, \quad [X_{y_i}, X_{y_j}] = 0.$$

The distribution $\mathcal{G}|_U$ is generated by their horizontal parts $\hat{X}_{y_1}, \dots, \hat{X}_{y_r}$ which also commute. Indeed, since $\mathcal{G}$ is integrable $[\hat{X}_{y_i}, \hat{X}_{y_j}]$ is a section of $\mathcal{G}$, in particular it is horizontal. Further

$$\begin{aligned}0 = [X_{y_i}, X_{y_j}] &= [y_i Z + \hat{X}_{y_i}, y_j Z + \hat{X}_{y_j}]\\ &= [y_i Z, y_j Z] + [\hat{X}_{y_i}, \hat{X}_{y_j}] + y_i[Z, \hat{X}_{y_j}] + y_j[\hat{X}_{y_i}, Z]\\ &\quad - \mathcal{L}_{\hat{X}_{y_j}}(y_i)Z + \mathcal{L}_{\hat{X}_{y_i}}(y_j)Z.\end{aligned} \tag{3.2}$$

On the other hand, since $\mathcal{L}_Z y_i = 0$, we have

$$\begin{aligned}0 = [Z, X_{y_i}] &= [Z, y_i Z + \hat{X}_{y_i}] = [Z, y_i Z] + [Z, \hat{X}_{y_i}]\\ &= \mathcal{L}_Z(y_i)Z + [Z, \hat{X}_{y_i}] = [Z, \hat{X}_{y_i}].\end{aligned} \tag{3.3}$$

Therefore, taking the horizontal part in (3.2) we get

$$[\hat{X}_{y_i}, \hat{X}_{y_j}] = 0.$$

Thus, locally we have parallelism both on $\mathcal{F} = \langle Z, \hat{X}_{y_1}, \dots, \hat{X}_{y_r} \rangle$ and $\mathcal{G} = \langle \hat{X}_{y_1}, \dots, \hat{X}_{y_r} \rangle$. Now, let U' be an open set ($U \cap U' \neq \emptyset$) and let $y'_1, \dots, y'_r$ be commuting integrals of $\mathcal{E}|_{U'}$. Then, on $U \cap U'$ we have

$$\begin{aligned}y'_i &= \varphi_i(y_1, \dots, y_r), \quad i = 1, \dots, r\\ dy'_i &= \sum_j \frac{\partial \varphi_i}{\partial y_j} dy_j.\end{aligned}$$

From the definition $\hat{X}_{y'_i} = \alpha^\sharp(\widehat{dy'_i}) = \alpha^\sharp(dy'_i - (i_Z dy'_i)\alpha) = \alpha^\sharp(dy'_i)$, we get the fiber-wise linear transformation

$$\hat{X}_{y'_i} = \sum_j \frac{\partial \varphi_i}{\partial y_j} \hat{X}_{y'_j}, \quad i = 1, \dots, r,$$

which shows that the parallelism of $\mathcal{G}$ and $\mathcal{F}$ is independent of the chart. □

If $\mathcal{F}$ has the maximal dimension $n+1$ then $\mathcal{F}$ is pre-Legendrian, while $\mathcal{G}$ is a Legendrian foliation. The existence of an affine structure is already known for α-complete Legendre foliations [**19, 26, 27, 32**]. This imposes restrictions on the topology of the leaves. In particular, compact leaves of $\mathcal{G}$ and $\mathcal{F}$ are tori.

Of particular interest is the case when $\mathcal{F}$ is a simple foliation, i.e., the leaves of the foliation are fibers of the submersion. We will study such a situation in the next section.

4. Complete pre-isotropic contact structures

In this section, a contact structure does not need to be co-oriented.

Let $(M, \mathcal{H})$ be a $(2n+1)$-dimensional contact manifold and let

$$\pi : M \to W \tag{4.1}$$

be a proper submersion on p-dimensional manifold W, $p \geq n$. Define the distribution $\mathcal{F}$ as the kernel of $\pi_* : TM \to TW$, i.e., the leaves of $\mathcal{F}$ are fibers of π.

Definition 4.1. We shall say that $(M, \mathcal{H}, \mathcal{X})$ is a *complete pre-isotropic contact structure* if

(i) $\mathcal{F}$ is pre-isotropic, i.e., it is transversal to $\mathcal{H}$ and $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ is an isotropic subbundle of $\mathcal{H}$, or, equivalently $\mathcal{G}$ is a foliation.

(ii) $\mathcal{X}$ is an Abelian Lie algebra of infinitesimal contact automorphisms of $\mathcal{H}$, which has the fibers of π as orbits.

In the case $p = n$ (and connected fibers), we have a *regular completely integrable contact structure* $(M, \mathcal{H}, \mathcal{X})$ studied in Banyaga and Molino [**2**].

Suppose $\mathcal{F}$ is an α-complete foliation with compact leaves (according to the presence of the affine structure, the leaves are tori). Locally, in a neighborhood U of any fixed torus F the foliation is simple. There is a surjective submersion $\pi : U \to W = U/\mathcal{F}$, $\mathcal{F} = \ker \pi_*$. We can define an Abelian Lie algebra $\mathcal{X}$ of infinitesimal automorphisms of $\mathcal{H}$ by $Z, X_{y_1}, \dots, X_{y_r}$ where, $y_1, \dots, y_r$ are integrals of $\mathcal{E} = \mathcal{F}^\perp$. Thus, we have well-defined complete pre-isotropic contact structure $(U, \mathcal{H}, \mathcal{X})$.

On the contrary, we also have:

Theorem 4.1. *Let $(M, \mathcal{H}, \mathcal{X})$ be a complete pre-isotropic contact structure related to the submersion* (4.1). *Every point of M has an open, $\mathcal{X}$-invariant neighborhood U on which the contact structure can be represented by a local contact form α_U such that:*

(i) α_U is invariant by all elements of $\mathcal{X}$;

(ii) the restriction of $\mathcal{F}$ to U is α_U-complete.

Proof. (i) The proof of item (i) is a modification of the proof given in [**2**] for a regular completely integrable contact structure. From the definition, for every point x_0 of M, there exist $X \in \mathcal{X}$ transverse to $\mathcal{H}_{x_0}$. The vector field X is then transverse to $\mathcal{H}$ in some neighborhood U_2 of x_0. Let α_0 be a contact form defining $\mathcal{H}$ in $U_1 \subset U_2$. Then $\alpha_0(X) \neq 0$ on U_1 and define $\alpha = \alpha_0/\alpha_0(X)$.

Since $\mathcal{X}$ is Abelian, we have $i_{[Y,X]}\alpha = 0$, $Y \in \mathcal{X}$. Also, $i_X\alpha = 1$ and $\mathcal{L}_Y\alpha = \lambda\alpha$, for some function λ defined in U_1. Thus

$$0 = i_{[Y,X]}\alpha = \mathcal{L}_Y i_X\alpha - i_X\mathcal{L}_Y\alpha = \mathcal{L}_Y 1 - i_X(\lambda\alpha) = -\lambda,$$

i.e., Y is an infinitesimal automorphism of α. Since α is invariant by $\mathcal{X}$ and the orbits of $\mathcal{X}$ are the fibers of the submersion (4.1), the form α is well-defined on $U = \pi^{-1}(\pi(U_1))$ as well.

(ii) The foliation $\mathcal{F}|_U$ is α-complete if and only if $\mathcal{E}|_U = \mathcal{F}^\perp|_U$ is an integrable distribution.

From the identity

$$0 = \mathcal{L}_X\alpha = i_X d\alpha + d i_X\alpha = i_X d\alpha$$

we get that X is the Reeb vector field of α on U. Denote $Z = X$.

Let $X_1, \dots, X_r \in \mathcal{X}$ be vector fields such that $Z, X_1, \dots, X_r$ span the foliation $\mathcal{F}|_U$. Therefore, the corresponding contact Hamiltonians

$$y_i = \Phi(X_i) = i_{X_i}\alpha$$

are independent functions on U. Besides, y_i are π-vertical:

$$0 = i_{[X,X_i]}\alpha = \mathcal{L}_X i_{X_i}\alpha - i_{X_i}\mathcal{L}_X\alpha = \mathcal{L}_X y_i,$$

for all $X \in \mathcal{X}$.

The corank of the distribution $\mathcal{E}|_U$ is $r = 2n - p$. It is integrable and has $y_1, \dots, y_r$ as independent integrals. Indeed, by definition we have

$$\mathcal{E}_U = \langle X_f \mid f = \bar{f} \circ \pi,\ \bar{f} \in C^\infty(\pi(U))\rangle. \tag{4.2}$$

Since $f = \bar{f} \circ \pi$ and y_i are π-vertical we have, in particular, $\mathcal{L}_Z f = \mathcal{L}_Z y_i = 0$ (the differential df and dy_i are semi-basic on U). Now, by using $\mathcal{L}_{X_{y_i}} f = 0$ and Lemma 2.1 we get

$$\mathcal{L}_{X_f} y_i = 0, \quad i = 1, \dots, r. \tag{4.3}$$

The relations (4.2) and (4.3) prove the claim. □

5. Noncommutative contact integrability

Let us consider a contact vector field X and a *contact equation*

$$\dot{x} = X \tag{5.1}$$

on a $(2n+1)$-dimensional contact manifold $(M, \mathcal{H})$.

First, recall a general definition of non-Hamiltonian integrability (e.g., see [**3, 20, 36**]), slightly adopted with respect to the notations above. Equation

(5.1) is *(non-Hamiltonian) completely integrable* if there is an open dense subset $M_{\text{reg}} \subset M$ and a proper submersion

$$\pi : M_{\text{reg}} \to W \tag{5.2}$$

to a p-dimensional manifold W and an Abelian Lie algebra $\mathcal{X}$ of symmetries such that:

(i) the contact vector field X is tangent to the fibers of π;
(ii) the fibers of π are orbits of $\mathcal{X}$.

If (5.1) is completely integrable then M_{reg} is foliated on $(r+1)$-dimensional tori with a quasi-periodic dynamics. In nonholonomic mechanics, usually, an additional time reparametrization is required (e.g., see [**12, 17, 20**]).

However, the above definition does not reflect the underlying contact structure.

Definition 5.1. We shall say that the contact equation (5.1) is *noncommutatively contact completely integrable* if, in addition, $(M_{\text{reg}}, \mathcal{H}, \mathcal{X})$ is a complete pre-isotropic contact structure.

The regularity of the dynamics of integrable contact systems is described in the following statement.

Theorem 5.1. *Suppose that equation* (5.1) *is noncommutatively contact completely integrable by means of the submersion* (5.2) *and commuting symmetries $\mathcal{X}$. Let F be a connected component of the fiber $\pi^{-1}(w_0)$. Then F is diffeomorphic to a $r+1$-dimensional torus $\mathbb{T}^{r+1}$, $r = 2n - p$. There exists an open $\mathcal{X}$-invariant neighborhood U of F, an $\mathcal{X}$-invariant contact contact form α on U and a diffeomorphism $\phi : U \to \mathbb{T}^{r+1} \times D$,*

$$\phi(x) = (\theta, y, x) = (\theta_0, \theta_1, \dots, \theta_r, y_1, \dots, y_r, x_1, \dots, x_{2s}), \quad s = n - r, \tag{5.3}$$

where $D \subset \mathbb{R}^p$ is diffeomorphic to $W_U = \pi(U)$, such that

(i) $\mathcal{F}|_U$ is α-complete foliation with integrals $y_1, \dots, y_r, x_1, \dots, x_{2s}$, while the integrals of the pseudo-orthogonal foliation $\mathcal{E}|_U = \mathcal{F}|_U^{\perp}$ are $y_1, \dots, y_r$.

(ii) α has the following canonical form

$$\alpha_0 = (\phi^{-1})^*\alpha = y_0 d\theta_0 + y_1 d\theta_1 + \dots + y_r d\theta_r + g_1 dx_1 + \dots + g_{2s} dx_{2s}, \tag{5.4}$$

where y_0 is a smooth function of y and g_i are functions of (y, x).

(iii) the flow of X on invariant tori is quasi-periodic

$$(\theta_0, \theta_1, \dots, \theta_r) \longmapsto (\theta_0 + t\omega_0, \theta_1 + t\omega_2, \dots, \theta_r + t\omega_r), \quad t \in \mathbb{R}, \tag{5.5}$$

where frequencies $\omega_0, \dots, \omega_r$ depend only on y.

Definition 5.2. We refer to local coordinates (θ, y) stated in Theorem 5.1 as a *generalized contact action–angle coordinates.*

In the case when the contact manifold is co-oriented ($\mathcal{H} = \ker\alpha$) and we have the contact Hamiltonian equation (2.4), it is convenient to formulate noncommutative integrability in terms of the first integrals and the Jacobi bracket as well.

Theorem 5.2. *Suppose we have a collection of integrals $f_1, f_2, \dots, f_{2n-r}$ of equation* (2.4) *with the contact Hamiltonian either $f = f_1$ or $f = 1$, where:*

$$[1, f_i] = 0, \quad [f_i, f_j] = 0, \quad i = 1, \dots, 2n-r, \quad j = 1, \dots, r. \tag{5.6}$$

Let F be a compact connected component of the level set

$$\{x \mid f_1 = c_1, \dots, f_{2n-r} = c_{2n-r}\}$$

and assume

$$df_1 \wedge \dots \wedge df_{2n-r} \neq 0 \tag{5.7}$$

on F. Then F is diffeomorphic to a $r+1$-dimensional torus $\mathbb{T}^{r+1}$. There exists a neighborhood U of F with local generalized action–angle coordinates (5.3) *in which α has the form* (5.4) *and the dynamics is quasi-periodic* (5.5).

Proof. Consider the mapping

$$\pi = (f_1, \dots, f_{2n-r}): \quad M \to \mathbb{R}^{2n-r}.$$

From (5.7) there exists a neighborhood U of F such that $\pi|_U$ is a proper submersion to $\pi(U)$. Let $\mathcal{F}$ be a foliation with leaves that are fibers of π. Since df_i are semi-basic 1-forms, (5.7) implies $df_1 \wedge \dots \wedge df_{2n-r} \wedge \alpha \neq 0$. Thus, $\mathcal{F}$ is transversal to $\mathcal{H}|_U$ and the infinitesimal automorphisms of α

$$Z, X_{f_1}, \dots, X_{f_r} \tag{5.8}$$

are independent in U.

Further, from (2.5) and (5.6), we conclude

$$\begin{aligned} &[Z, X_{f_i}] = 0, \quad [X_{f_i}, X_{f_j}] = 0, \quad i = 1, \dots, 2n-r, \quad j = 1, \dots, r, \\ &\mathcal{L}_Z f_i = 0, \quad L_{X_{f_j}} f_i = 0, \quad \mathcal{L}_{X_{f_i}} f_j = 0, \end{aligned} \tag{5.9}$$

The relations (5.9) provide that the commuting vector fields (5.8) belong to $\mathcal{F}$. From the dimensional reason, they span $\mathcal{F}$. From (5.9) we also get that $f_1, \dots, f_r$ are integrals of the pseudo-orthogonal distribution $\mathcal{E} = \mathcal{F}^\perp$. When $\mathcal{E}$ is integrable. On the other hand, $\mathcal{F} \subset \mathcal{E}$ implies that the distribution $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ is isotropic (item (i) of Theorem 3.2).

Therefore, $\mathcal{F}$ is a complete pre-isotropic foliation with commuting symmetries (5.8). Now, the statement follows from Theorem 5.1. □

Proof of Theorem 5.1. *Step 1 (local bi-fibrations).* Since on each connected component of the fiber $\pi^{-1}(w_0)$, $\mathcal{X}$ induces a transitive action of $\mathbb{R}^{r+1}$ ($r = 2n-p$), the connected components of $\pi^{-1}(w_0)$ are $r+1$-dimensional tori $\mathbb{T}^{r+1}$ (e.g., see Arnold [**1**]).

Let us fix some connected component F of $\pi^{-1}(w_0)$. Consider some $\mathcal{X}$-invariant connected neighborhood U of F and a $\mathcal{X}$-invariant contact form α defining the distribution $\mathcal{H}|_U = \ker\alpha$ such that the corresponding Reeb vector field Z belongs to $\mathcal{X}$ (see the construction given in Theorem 4.1).

Let $y'_i = i_{X_i}\alpha$ be contact Hamiltonians of r independent contact vector fields $X_i \in \mathcal{X}$, $\mathcal{F}|_U = \langle Z, X_1, \dots, X_r\rangle$. The functions $y'_1, \dots, y'_r$ are then integrals of the pseudo-orthogonal foliation as well (see the proof of Theorem 3). They are π-vertical, and by $\bar{y}'_i$ we denote the corresponding functions on $W_U = \pi(U)$. Locally, for U small enough, the foliation $\mathcal{E}|_U$ is also a fibration ρ_U over an open set V_U diffeomorphic to a ball in $\mathbb{R}^r$ with local coordinates $\bar{y}' = (\bar{y}'_1, \dots, \bar{y}'_r)$ (the using of $\bar{y}'_i$ will be clear from the contexts). Therefore, we have a bi-fibration

$$\begin{array}{ccc} & U & \\ \swarrow{\scriptstyle \pi_U} & & {\scriptstyle \rho_U}\searrow \\ W_U & & V_U \end{array}$$

with pseudo-orthogonal fibers $\mathcal{F}|_U$ and $\mathcal{E}|_U$.

Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{2s})$ be any collection of independent functions, where $(\bar{y}', \bar{x})$ are local coordinates on W_U. Let $x_a = \bar{x}_a \circ \pi_U$, $a = 1, \dots, 2s$. By the use of the methods developed by Arnold [**1**], it follows that locally we have a trivial toric fibration $U \cong \mathbb{T}^{r+1} \times W_U$ with coordinates

$$(\varphi_0, \dots, \varphi_r, y'_1, \dots, y'_r, x_1, \dots, x_{2s}).$$

The angular variables $(\varphi_0, \dots, \varphi_r)$ are chosen such that

$$Y_\nu = \partial/\partial\varphi_\nu = \sum_{\mu=0}^{r} \Lambda_{\nu\mu} X_\mu,$$

where the Reeb vector field Z of α is denoted by X_0 and the invertible matrix $(\Lambda_{\nu\mu}) \in GL(r+1)$ depends only on (y', x).

Step 2 (description of α). By construction, the functions $y'_j = i_{X_j}\alpha$ are ρ_U-basic. Since $\mathcal{L}_{X_j}\alpha = 0$, the 1-forms

$$i_{X_j} d\alpha = -d(\alpha(X_j)) = -dy'_j, \quad j = 1, \dots, r$$

are also ρ_U-basic.[3] Besides, $i_{X_0} d\alpha = i_Z d\alpha = 0$. Therefore

$$i_{Y_\nu} d\alpha = \sum_{\mu=0}^{r} \Lambda_{\nu\mu} i_{X_\mu} d\alpha = -\sum_{\mu=0}^{r} \Lambda_{\nu\mu} dy'_\mu, \quad \nu = 0, 1, \dots, r \tag{5.10}$$

[3]Let $\pi : M \to P$ be a surjective submersion. A 1-form ω is *semi-basic* if $i_X\omega = 0$ for all vertical vector fields X. It is *basic* if $\omega = \pi^*\mu$, where μ is a 1-form on P. In particular, a basic form is semi-basic as well [**25**].

are ρ_U-semi basic 1-forms. Here $y'_0 \equiv 1$. In particular, $d\alpha$ does not contain the terms with $d\varphi_\nu \wedge d\varphi_\mu$. So α takes the form

$$\alpha = \sum_{\nu=0}^{r} y_\nu d\varphi_\nu + \sum_{i=1}^{r} \tilde{f}_i dy'_i + \sum_{a=1}^{2s} \tilde{g}_a dx_a, \tag{5.11}$$

where $y_\nu = y_\nu(y', x)$, $\nu = 0, \dots, r$. Thus, it follows:

$$i_{Y_\nu} d\alpha = -dy_\nu + \sum_{i=1}^{r} \frac{\partial \tilde{f}_i}{\partial \varphi_\nu} dy'_i + \sum_{a=1}^{2s} \frac{\partial \tilde{g}_a}{\partial \varphi_\nu} dx_a. \tag{5.12}$$

By combining (5.10), (5.12) and the fact that the matrix $(\Lambda_{\nu\mu})$ does not depend on φ, we obtain that $\tilde{f}_i$ and $\tilde{g}_a$ are linear in angular variables. Since they are periodic in φ_ν, they only depend on (y', x) and

$$i_{Y_\nu} d\alpha = -dy_\nu. \tag{5.13}$$

From (5.11) and (5.13) we find the Lie derivatives

$$\mathcal{L}_{Y_\nu}\alpha = i_{Y_\nu} d\alpha + d i_{Y_\nu}\alpha = -dy_\nu + dy_\nu = 0, \quad \nu = 1, \dots, r$$

and conclude that α is invariant with respect to the angle coordinates vector fields $\partial/\partial\varphi_\nu = Y_\nu$.

Now, according to Lemma 5.1, the matrix $(\Lambda_{\nu\mu})$ depends only on y'-variables. Therefore, the 1-forms $i_{Y_\nu} d\alpha$ (see (5.10)) as well as the functions y_ν (see (5.13)) are ρ_U-basic. Note that $y_\nu = i_{Y_\nu}\alpha$ are contact Hamiltonians of the contact vector fields Y_ν.

Among y_ν there are r independent functions at every point in U. With eventually shrinking of U and a permutation of indexes, we can assume that $y_1, \dots, y_r$ are independent and $y_0 = y_0(y_1, \dots, y_r)$ (i.e., $\bar{y}_1, \dots, \bar{y}_r$ are new coordinates on V_U). As a result, the contact form reads

$$\alpha = \sum_{\nu=0}^{r} y_\nu d\varphi_\nu + \sum_{i=1}^{r} f_i(y, x) dy_i + \sum_{a=1}^{2s} g_a(y, x) dx_a. \tag{5.14}$$

Introducing the new angle variables

$$(\theta_0, \theta_1, \dots, \theta_r) = (\varphi_0, \varphi_1 - f_1(y, x), \dots, \varphi_r - f_r(y, x)), \tag{5.15}$$

the form (5.14) becomes

$$\alpha = \sum_{i=0}^{r} y_i d\theta_i + \sum_{a=1}^{2s} g_a(y, x) dx_a + df,$$

where $f = f(y, x) = \sum_{i=1}^{r} y_i f_i(y, x)$ is a π_U-basic function. Due to the translation (5.15), the coordinate vector fields of θ and φ coincide: $\partial/\partial\theta_\nu = \partial/\partial\varphi_\nu = Y_\nu$.

Step 3 (Moser's deformation, see, e.g., [**2, 14**]*).* Let

$$\alpha_0 = \sum_{\nu=0}^{r} y_\nu d\theta_\nu + \sum_{a=1}^{2s} g_a(y,x)\, dx_a.$$

and $Z = X_0$ be the Reeb vector field of α. It is π_U-vertical and we have

$$i_Z\alpha = i_Z\alpha_0 = 1, \quad i_Z d\alpha = i_Z d\alpha_0 = 0,$$

implying $\mathcal{L}_Z\alpha = \mathcal{L}_Z\alpha_0 = 0$.

Following [**2**], consider the vector field $Y = -fZ$, where f is the π_U-basic function defined above. The flow ϕ_t of Y is a complete flow that preserves the toric fibration. Define $\alpha_t = \alpha_0 + tdf$. Then we have

$$\mathcal{L}_Y\alpha_t = \mathcal{L}_Y\alpha_0 + t\mathcal{L}_Y h = \mathcal{L}_Y\alpha_0 = i_Y d\alpha_0 + d(i_Y\alpha_0) = -df = -\partial\alpha_t/\partial t.$$

Thus

$$\frac{d}{dt}(\phi_t^*\alpha_t) = \phi_t^*\left(\mathcal{L}_Y\alpha_t + \frac{\partial\alpha_t}{\partial t}\right) = 0,$$

which implies that $\phi_1^*\alpha_1 = \phi_1^*\alpha = \alpha_0$. Finally, the required change of variables is $\phi = \phi_{-1}$.

Step 3 (linearization). Since the system is non-Hamiltonian completely integrable, we have a quasi-periodic motion on invariant tori [**3, 36**]. The special form of a linearization, where frequencies only depend on $y_1, \dots, y_r$ follows from Lemma 5.1 below. □

Remark 5.1. The action functions $y_\nu = i_{Y_\nu}\alpha$ constructed above have an another interesting interpretation. Let $\gamma_\nu(T)$ be a cycle homologous to the trajectories of the field $\partial/\partial\theta_\nu$ restricted to any invariant torus T within U. Then it follows

$$y_\nu|_T = \frac{1}{2\pi}\int_{\gamma_\nu(T)} \alpha. \tag{5.16}$$

Indeed, since $d\alpha|_T = 0$ (the tangent space of T splits into an isotropic horizontal part and $\mathbb{R}Z = \ker d\alpha$) the value of the integral (5.16) is the same for all $\gamma_\nu(T)$ in the same homology class. Then (5.16) simply follow from (5.4). In the opposite direction, we can use (5.16) as a definition of y_ν. By construction, the functions y_ν are π_U-vertical. As in the symplectic case (see Nehoroshev [**31**]), it can be proved that they are also ρ_U-vertical.

Remark 5.2. Let $Z = z_0(y)Y_0 + \dots + z_r(y)Y_r$ be the local expression of the Reeb vector field. It is uniquely determined from the conditions $i_Z\alpha_0 = 1$, $i_Z d\alpha_0 = 0$, i.e.,

$$z_0 y_0 + \dots + z_r y_r = 1, \quad z_0 dy_0 + \dots + z_r dy_r = 0. \tag{5.17}$$

If $z_0 = 0$ at some point $y = \tilde{y}$, then $z_1 dy_1 + \dots + z_r dy_r = 0$ at $\tilde{y}$. Since dy_i, $i = 1, \dots, r$ are independent 1-forms, we get $z_1 = \dots = z_r = 0$ at $\tilde{y}$ which contradict (5.17). Therefore $z_0 \neq 0$ on V_U. Now, by solving (5.17) we get

$$z_0 = \frac{1}{y_1 \frac{\partial y_0}{\partial y_1} + \dots + y_r \frac{\partial y_0}{\partial y_r} - y_0}, \quad z_i = -\frac{1}{z_0}\frac{\partial y_0}{\partial y_i}, \quad i = 1, \dots, r.$$

Therefore, typically, the flow of the Reeb vector field is quasi-periodic and everywhere dense in invariant tori. Also, typically, the induced pseudo-isotropic foliation $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$ has noncompact invariant manifolds.

Remark 5.3. Consider the 1-form $\gamma = \sum_{a=1}^{2s} g_a(y,x)dx_a = \alpha_0 - \sum_{\nu=0}^{r} y_\nu d\theta_\nu$. Since $d\alpha_0$ has the maximal rank, according to Darboux's theorem [**25**], there is a coordinate transformation $q_j = q_j(y,x), p_j = p_j(y,x)$, $j = 1, \dots, s$ such that $\gamma = p_1 dq_1 + \dots + p_s dq_s$, i.e.,

$$\alpha_0 = y_0 d\theta_0 + y_1 d\theta_1 + \dots + y_r d\theta_r + p_1 dq_1 + \dots + p_s dq_s.$$

Lemma 5.1. *Let $(M, \mathcal{H}, \mathcal{X})$ be a complete pre-isotropic contact structure and let $U \subset M$ be an $\mathcal{X}$-invariant set endowed with an $\mathcal{X}$-invariant contact form α. Suppose*

(i) The foliation $\mathcal{F}|_U = \ker \pi_*|_U$ is α-complete and there exist everywhere independent integrals $y_1, \dots, y_r : U \to \mathbb{R}$, of the pseudo-orthogonal foliation $\mathcal{E}|_U = \mathcal{F}|_U^\perp$.

(ii) Let X be a contact vector field tangent to the fibers of π_U, commuting with $\mathcal{X}$.

Then X can be written as a fiber-wise linear combination

$$X = f_0 Z + f_1 X_1 + \dots + f_r X_r,$$

where functions $f_0, \dots, f_r$ depend only on y, Z is the Reeb vector field of α and $X_i = X_{y_i}$ are contact Hamiltonian vector fields of y_i, $i = 1, \dots, r$.

Proof. Under the assumption (i), $Z, X_1, \dots, X_r$ are independent vector fields that generate α-complete pre-isotropic foliation $\mathcal{F}|_U$.

Next, we shall prove that X commute with Z. Firstly, note that Z commute with $\mathcal{X}$.[4] Indeed, let $Y \in \mathcal{X}$. We have

$$\Phi([Y,Z]) = i_{[Y,Z]}\alpha = \mathcal{L}_Y i_Z \alpha - i_Z \mathcal{L}_Y \alpha = 0. \tag{5.18}$$

Since (2.3) is an isomorphism we get $[Y,Z] = 0$.

Secondly, note that any π-vertical vector field K (not need to be contact field) that commute with $\mathcal{X}$, commute with X as well. Indeed, any point in U has a π-invariant neighborhood U' where K can be written as a linear combination $\sum_{\nu=1}^{r} g_\nu Y_\nu$ where g_ν are π-basic functions and $(Y_0, \dots, Y_r)$ is a

[4]Here we consider slightly more general situation then it is needed for Theorem 5.1, where, by construction of α, Z is already an element of $\mathcal{X}$. However, we shall use the above formulation for a proof of Proposition 6.2.

collection of vector fields in $\mathcal{X}$ that generate $\mathcal{F}|_U$. Therefore

$$[X,K]=\sum_{\nu=0}^{r}[X,g_\nu Y_\nu]=\sum_{\nu=0}^{r}(g_\nu[X,Y_\nu]+dg_\nu(X)Y_\nu)=0.$$

From the above considerations it follows that X commute with Z. Let $f=i_X\alpha$ be the contact Hamiltonian of X. Since $[Z,X_f]=0$ we have $[1,f]=0$ and df is a semi-basic form. Since X is π_U-vertical, we have $\mathcal{L}_{X_f}g=0$, where g is any local integral of $\mathcal{F}$. It is clear that dg is semi-basic and applying Lemma 2.1 again, it follows $\mathcal{L}_{X_g}f=0$. When f is an integral of the pseudo-orthogonal foliation $\mathcal{E}|_U$.

Under the assumptions of Lemma 5.1, integrals of $\mathcal{E}|_U$ are functions of y and we have $f=f(y)$. Let $f_i=\partial f/\partial y_i$, $i=1,\dots,r$. The forms $df,dy_1,\dots,dy_r$ are semi-basic, so

$$\begin{aligned} X &= \Phi^{-1}(f)=fZ+\alpha^\sharp(df)=fZ+\sum_{i=1}^{r}f_i\alpha^\sharp(dy_i)\\ &= fZ+\sum_{i=1}^{r}f_i(X_i-y_iZ)=f_0Z+f_1X_1+\dots+f_rX_r, \end{aligned}$$

where $f_0=f-(y_1f_1+\dots+y_rf_r)$. □

Remark 5.4. Let X be π_U-horizontal contact vector field. From the proof of the lemma, we see that commuting of X with $\mathcal{X}$ is equivalent to the commuting with the Reeb vector field Z, i.e., with the condition that X is an infinitesimal automorphisms of α. Also, the condition that $\mathcal{F}|_U$ is α-complete is equivalent to the condition that Z is a section of $\mathcal{F}|_U$, see Proposition 6.2 given below.

5.1. Discrete systems. Khesin and Tabachnikov defined integrability of discrete

$$\Psi: M\to M, \tag{5.1}$$

and continuous contact systems (5.1) in terms of the existence of an invariant complete pre-Legendrian foliation $\mathcal{F}$, with additional property that on every leaf F of $\mathcal{F}$, the foliation $\mathcal{G}|_F$ has a holonomy invariant transverse smooth measure. It turns out that this condition implies the existence of a global contact form α and that $\mathcal{G}$ is an α-complete Legendrian foliation [**19**].

As in [**19**], we can say that a discrete contact system (5.1) that preserves the contact form α is *integrable in a noncommutative sense* if it possesses an α-complete pre-isotropic invariant foliation $\mathcal{F}$. Also, following the lines of the proof of Lemma 3.5 [**19**], one can prove that α determines a holonomy invariant transverse smooth measure of the foliation $\mathcal{G}=\mathcal{F}\cap\mathcal{H}$ restricted to the leaves of $\mathcal{F}$.

5.2. Examples. For $s = 0$, Theorem 4 recover contact action–angle coordinates given by Banyaga and Molino [**2**]. If M is a compact manifold with a regular effective contact action of $\mathbb{T}^{n+1}$, then W is the sphere S^n and for $n \geq 3$, M is diffeomorphic to $\mathbb{T}^{n+1} \times S^n$ (see Lutz [**28**]).

Besides noncommutatively integrable geodesic flow restricted to the unit co-sphere bundles [**4, 16**], a natural class of examples of contact flows integrable in a noncommutative sense are the Reeb flows on K-contact manifolds (M^{2n+1}, α) where the rank of the manifold is less then $n+1$ (see Yamazaki [**35**] and Lerman [**23**]).

The regular and almost regular contact manifolds studied by Boothby and Wang [**6**] and Thomas [**33**] provide the most degenerate examples with $\dim W = \dim M - 1$.

The billiard system within an ellipsoid in the Euclidean space $\mathbb{R}^n$ is one of the basic examples of integrable mappings (e.g., see [**9, 34**]). Similarly, the billiard system inside an ellipsoid in the pseudo-Euclidean space $\mathbb{R}^{k,n-k}$ is completely integrable as well. Here, the billiard system is described by a symplectic transformation on the spaces of space-like and time-like geodesics, while it is a contact transformation on the space of light-like geodesics (for more details, see Khesin and Tabachnikov [**18, 19**]). The considered billiard systems are defined within ellipsoids with different semi-axis. Further properties of ellipsoidal billiards in the pseudo-Euclidean spaces have been studied in [**10**], where description of periodical trajectories has been derived, including the cases of symmetric ellipsoids. It can be proved that the billiard systems, both in $\mathbb{R}^n$ and $\mathbb{R}^{k,n-k}$, within symmetric ellipsoids are completely integrable in the noncommutative sense (the geodesic flow on a symmetric ellipsoid is considered in [**8**]). In particular, the billiard maps restricted to the space of null geodesics are noncommutatively completely integrable contact transformations.

6. Complete pre-isotropic structures of the Reeb type

In this section, we consider some global properties of the fibration (4.1).

Proposition 6.1. *Let $(M, \mathcal{H}, \mathcal{X})$ be a complete pre-isotropic contact structure and assume that $\mathcal{H}$ is co-oriented. Then there exists a global contact form α representing $\mathcal{H}$ and invariant by elements of $\mathcal{X}$.*

Proof. We can cover W by open sets W_i such that we have contact 1-forms α_{U_i} invariant by $\mathcal{X}$ on every $U_i = \pi^{-1}(W_i)$ (Theorem 3). Let $\bar{\lambda}_i$ be the partition of unity subordinate to covering $\{W_i\}$. Since $\mathcal{H}$ is oriented, for all nonempty intersections $U_i \cap U_j$, we have smooth positive functions a_{ij}, $\alpha_{U_i} = f_{ij}\alpha_{U_j}|_{U_i \cap U_j}$.

Define the 1-form α by $\alpha = \sum_i \lambda_i \alpha_{U_i}$, $\lambda_i = \bar{\lambda}_i \circ \pi$. Then, on U_k we have

$$\alpha = a_k \alpha_{U_k}, \tag{6.1}$$

where $a_k = \sum_{i,U_i\cap U_k\neq\emptyset} \lambda_i f_{ki} > 0$ is a π-basic function. When α is a contact form that define $\mathcal{H}$.

It remains to prove $\mathcal{X}$-invariance of α. Let $X \in \mathcal{X}$. Then $\mathcal{L}_X\lambda_i = 0$. Further, by construction, X preserve all local contact forms α_{U_i}. Thus

$$\mathcal{L}_X\alpha = \sum_i (\mathcal{L}_X\lambda_i)\alpha_{U_i} + \lambda_i\mathcal{L}_X\alpha_{U_i} = 0. \qquad \square$$

Let Z be the Reeb vector field of the globally $\mathcal{X}$-invariant contact form α. Then, as in (5.18), we get $[Z, Y] = 0$, $Y \in \mathcal{X}$. However, it turns out that the foliation $\mathcal{F} = \ker \pi_*$ not need to be α-complete since Z not need be a section of $\mathcal{F}$.

Recall that a contact toric action on a co-oriented contact manifold (M, α) is of the *Reeb type* if the Reeb vector field corresponds to an element of the Lie algebra of the torus [**5**]. Similarly, we give the following definition.

Definition 6.1. Let (M, α) be a co-oriented contact manifold with a complete pre-isotropic contact structure defined by commuting infinitesimal automorphisms $\mathcal{X}$ of α, such that the associated Reeb vector field Z is a section of $\mathcal{F} = \ker \pi_*$. We refer to a triple $(M, \alpha, \mathcal{X})$ with the above property as a *complete pre-isotropic structure of the Reeb type.*

Proposition 6.2. *Let $(M, \alpha, \mathcal{X})$ be a complete pre-isotropic structure of the Reeb type. Then the associated foliation $\mathcal{F} = \ker \pi_*$ is α-complete.*

Proof. Locally, every leaf F of $\mathcal{F}$ has a π-invariant neighborhood U with local generalized contact action–angle coordinates (5.3) in which $\mathcal{H}$ is represented by the contact form $\alpha_0 = \sum_\nu y_\nu d\theta_\nu + \sum_a g(y, x)dx_a$ and $\mathcal{F}|_U$ is α_0-complete (Theorem 5.1). We need to prove that $\mathcal{F}$ is complete with respect to the contact form α as well.

We have $\alpha|_U = \frac{1}{a} \cdot \alpha_0$ for some nonvanishing function $a : U \to \mathbb{R}$. In what follows, by Z^α, Z^{α_0}, X^α_f, $X^{\alpha_0}_f$ and Φ_α, Φ_{α_0} we denote the Reeb vector fields, contact Hamiltonian vector fields and the isomorphisms (2.3) with respect to α and α_0, respectively. They are related by

$$X^\alpha_f = \Phi^{-1}_\alpha(f) = \Phi^{-1}_{\alpha_0}(af) = X^{\alpha_0}_{af}, \quad Z^\alpha = \Phi^{-1}_\alpha(1) = \Phi^{-1}_{\alpha_0}(a) = X^{\alpha_0}_a$$

(see Proposition 13.7, [**25**]).

On the other hand, by the argument used in (5.18), with Φ replaced by Φ_α, we get $[Z^\alpha, X] = 0$, $X \in \mathcal{X}$. Therefore, we can apply Lemma 5.1 with $Z^\alpha = X^{\alpha_0}_a$ and α_0, instead of X and α, concluding that a is a function of actions variables $y = (y_1, \dots, y_r)$ only.

Let f be an integral of $\mathcal{F}$. Since da and df are semi-basic, we get that the contact Hamiltonian vector field

$$\begin{aligned} X_f^\alpha &= \Phi_\alpha^{-1}(f) = \Phi_{\alpha_0}^{-1}(af) \\ &= (af)Z^{\alpha_0} + \alpha_0^\sharp(adf + fda) = (af)Z^{\alpha_0} + a\alpha_0^\sharp(df) + f\alpha_0^\sharp(da) \\ &= afZ^{\alpha_0} + a(X_f^{\alpha_0} - fZ^{\alpha_0}) + f(X_a^{\alpha_0} - aZ^{\alpha_0}) \\ &= aX_f^{\alpha_0} + fZ^\alpha - afZ^{\alpha_0}, \end{aligned}$$

is a section of pseudo-orthogonal complement of $\mathcal{F}$ with respect to α_0. Thus, the pseudo-orthogonal complements of $\mathcal{F}$ with respect to α and α_0 coincides. This completeness the proof. □

Remark 6.1. Let us return to the construction of an invariant contact form α given in Proposition 6.1. From the proof of Proposition 6.2, we obtain that $\mathcal{F} = \ker \pi_*$ is α-complete if the functions a_k defined by (6.1) depend only on actions variables. If this is not the case, suppose additionally that the Reeb vector field Z is transversal to $\mathcal{F}$ at every point. Then we can consider the foliation $\tilde{\mathcal{F}}$ generated by $\mathcal{X}$ and Z. It can be proved that $\tilde{\mathcal{F}}$ is α-complete. Note that if $n = p$, i.e., $(M, \mathcal{H}, \mathcal{X})$ is a regular completely integrable contact structure, then a_k depends only on action variables and $\mathcal{F}$ is α-complete.

Let $(M, \alpha, \mathcal{X})$ be a complete pre-isotropic structure of the Reeb type and assume the fibers of (4.1) are connected. Theorem 5.1 and Proposition 6.2 provide that $\pi : M \to W$ is a toric fibration. There is an open covering W_i of W and local trivializations $\phi_i : U_i = \pi^{-1}(W_i) \to \mathbb{T}^{r+1} \times D_i$,

$$\phi_i(x) = (\theta^i, y^i, x^i) = (\theta_0^i, \theta_1^i, \dots, \theta_r^i, y_1^i, \dots, y_r^i, x_1^i, \dots, x_{2s}^i), \quad s = n - r,$$

where $D_i \subset \mathbb{R}^p$ is an open set diffeomorphic to W_i, such that

(i) the fibers of π are represented as the level sets of functions (y^i, x^i), where the action variables y^i are integrals of the pseudo-orthogonal foliation $\mathcal{E} = \mathcal{F}^\perp$ restricted to U_i;

(ii) α has the following canonical form:

$$\alpha_i = (\phi_i^{-1})^*\alpha = y_0^i d\theta_0^i + y_1^i d\theta_1^i + \dots + y_r^i d\theta_r^i + g_1^i dx_1^i + \dots + g_{2s}^i dx_{2s}^i,$$

where y_0^i is a smooth function of y^i and g_a^i are functions of (y^i, x^i).

Proposition 6.3. *Suppose that the intersection of W_i and W_j, i.e., of U_i and U_j is connected. Then on $U_i \cap U_j$ we have the following transition formulas:*

$$\theta^j_\nu = \sum_{\mu=0}^{r} M^{ij}_{\nu\mu}(\theta^i_\mu + F^{ij}_\mu(y^i, x^i)), \tag{6.2}$$

$$y^j_\nu = \sum_{\mu=0}^{r} K^{ij}_{\nu\mu} y^i_\mu, \quad \nu = 0, \dots, r, \tag{6.3}$$

$$x^j_a = X^{ij}_a(y^i, x^i), \quad a = 1, \dots, 2s, \tag{6.4}$$

where matrixes $K^{ij} = (K^{ij}_{\nu\mu})$ and $M^{ij} = (K^{ij}_{\nu\mu})$ belong to $GL(r+1, \mathbb{Z})$, $M = (K^T)^{-1}$, and functions $X^{ij}_a(y^i, x^i)$, $F^{ij}_\nu(y^i, x^i)$ satisfy

$$g^i_a = \sum_{b=1}^{2s} g^j_b \frac{\partial X^{ij}_b}{\partial x^i_a}, \quad \sum_{b=1}^{2s} g^j_b \frac{\partial X^{ij}_b}{\partial y^i_k} + \sum_{\nu=0}^{r} y^i_\nu \frac{\partial F^{ij}_\nu}{\partial y^i_k} = 0. \tag{6.5}$$

Proof. Since y^i and y^j (respectively, (y^i, x^i) and (y^j, x^j)) are integrals of the pseudo-orthogonal foliation $\mathcal{E}$ (respectively, of $\mathcal{F}$) we have:

$$\theta^j_\nu = \Theta^{ij}_\nu(\theta^i, y^i, x^i), \quad y^j_k = Y^{ij}_k(y^i), \quad x^j_a = X^{ij}_a(y^i, x^i), \tag{6.6}$$

$\nu = 0, \dots, r, k = 1, \dots, r, a = 1. \dots, 2s$.

Let us fix some invariant torus $T = \pi^{-1}(w_0)$ within $U_i \cap U_j$ ($w_0 \in W_i \cap W_j$). From (5.16), we have

$$y^j_\nu|_T = \int_{\gamma^j_\nu(T)} \alpha = \sum_{\mu=0}^{r} K^{ij}_{\nu\mu} \int_{\gamma^i_\mu(T)} \alpha = \sum_{\mu=0}^{r} K^{ij}_{\nu\mu} y^i_\mu|_T,$$

where $K^{ij} \in GL(r+1, \mathbb{Z})$ is a matrix which relates two different bases of cycles $(\gamma^j_0(T), \dots, \gamma^j_r(T))$ and $(\gamma^i_0(T), \dots, \gamma^i_r(T))$ defined in Remark 5.1. From (6.6) and the connectedness of $W_i \cap W_j$ the matrix K^{ij} is constant. This proves (6.3). Therefore

$$i_{\partial/\partial\theta^j_\nu} d\alpha = -dy^j_\nu = -\sum_\mu K^{ij}_{\nu\mu} dy^i_\mu = \sum_\mu K^{ij}_{\nu\mu} i_{\partial/\partial\theta^i_\mu} d\alpha,$$

implying that $\partial/\partial\theta^j_\nu - \sum_\mu K^{ij}_{\nu\mu}\partial/\partial\theta^i_\mu \in \ker d\alpha = \mathbb{R}Z$.

Let λZ be the difference of $\partial/\partial\theta^j_\nu$ and $\sum_\mu K^{ij}_{\nu\mu}\partial/\partial\theta^i_\mu$. Then

$$\lambda = \alpha(\lambda Z) = \alpha(\partial/\partial\theta^j_\nu - \sum_\mu K^{ij}_{\nu\mu}\partial/\partial\theta^i_\mu) = y^j_\nu - \sum_\mu K^{ij}_{\nu\mu} y^i_\mu = 0.$$

Thus, from (6.6), permuting the indexes i and j, we obtain

$$\frac{\partial}{\partial\theta^j_\nu} = \sum_\mu \frac{\partial\Theta^{ji}_\mu}{\partial\theta^j_\nu} \frac{\partial}{\partial\theta^i_\mu} = \sum_\mu K^{ij}_{\nu\mu} \frac{\partial}{\partial\theta^i_\mu},$$

leading to the fact that Θ^{ji}_μ is linear in θ^j_ν and that can be written into a form

$$\Theta^{ji}_\mu = \sum_\nu \left(K^{ij}_{\nu\mu}\theta^j_\nu + F^{ji}_\nu(y^j, x^j)\right).$$

From the above expression we get (6.2), where $\sum_{\lambda=0}^r K^{ij}_{\lambda\mu}M^{ij}_{\lambda\nu} = \delta_{\nu\mu}$.

Replacing (6.3) and the differentials of (6.2), (6.4) into the identity

$$\sum_{\nu=0}^{r} y^i_\nu d\theta^i_\nu + \sum_{a=1}^{2s} g^i_a(y^i, x^i)dx^i_a = \sum_{\lambda=0}^{r} y^j_\lambda d\theta^j_\lambda + \sum_{b=1}^{2s} g^j_b(y^j, x^j)dx^j_b, \tag{6.7}$$

and comparing the terms with dx^i_a and dy^i_k we get (6.5). □

The study of toric fibrations within the symplectic geometry framework is based on the papers of Duistermaat [**11**] (Lagrangian fibration) and Dazord and Delzant [**7**] (isotropic fibrations). On the other side, Banyaga and Molino defined characteristic invariants of *regular* and *singular* completely integrable contact structures and proved a classification theorem: two completely integrable contact structures with the same invariants are isomorphic [**2**]. For contact toric actions and singular completely integrable contact structures, see also [**5, 22, 29**], respectively.

Here, we consider the existence of global contact action–angle coordinates by using the arguments already used in the paper.

The possibility of taking all matrices K^{ij} and M^{ij} equal to the identity reflects the fact that the fibration by the invariant tori is a principal $\mathbb{T}^{r+1}$-bundle. When this does not happen, it is said that we have nontrivial monodromy [**11**].

Let $W' \subset W$, $\dim W' = \dim W$ be a connected compact submanifold (with a smooth boundary) and consider the fibration $\pi : M' \to W'$, $M' = \pi^{-1}(W')$. It is obvious that the necessary condition for the existence of global contact action–angle variables is that $M' \to W'$ is a trivial principal bundle.

The following sufficient, but not necessary, conditions for $M' \to W'$ to be trivial are well known (e.g., see [**13**]):

(i) If W' is simply connected then $\pi : M' \to W'$ is a principal $\mathbb{T}^{r+1}$ bundle.

(ii) In addition, if the second cohomology group $H^2(W', \mathbb{Z})$ vanish then the principal bundle is trivial and M' is diffeomorphic to $\mathbb{T}^{r+1} \times W'$.

Indeed, if W' is simply connected then the monodromy of the restricted fibration $\pi : M' \to W'$ is trivial providing that $\pi : M' \to W'$ is a principal $\mathbb{T}^{r+1}$ bundle. For the second assertion, note that the Chern class of $\mathbb{T}^{r+1} = U(1) \times \cdots U(1)$-bundle is equal to

$$c = c(L_0 \oplus \cdots \oplus L_r) = (1 + c_1(L_0)) \cdots (1 + c_1(L_r)),$$

where L_ν is the bundle associated to the νth factor $U(1)$. They are all trivial in the case $H^2(M, \mathbb{Z}) = 0$. When, $\mathbb{T}^{r+1}$-bundle is also trivial.

Now we can formulate the following statement.

Theorem 6.1 (Global contact action–angle variables). *Let $(M, \alpha, \mathcal{X})$ be a complete pre-isotropic structure of the Reeb type and let $W' \subset W$, $\dim W' = \dim W$ be a connected compact submanifold (with a smooth boundary) such that*

(i) $\pi : M' \to W'$ is a trivial principal $\mathbb{T}^{r+1}$ bundle, $M' = \pi^{-1}(W')$.

(ii) There exist everywhere independent functions $\bar{x}_1, \dots, \bar{x}_{2s}$ defined is some neighborhood of W' satisfying:

$$\langle dx_1, \dots, dx_{2s} \rangle \cap \mathcal{E}^0 = 0, \tag{6.8}$$

where $x_a = \bar{x}_a \circ \pi$ and $\mathcal{E} = \mathcal{F}^\perp$ is the pseudo-orthogonal foliation of $\mathcal{F}$.

Then there exist global action–angle variables $(\theta_0, \dots, \theta_r, y_0, \dots, y_r)$ and functions $\bar{g}_1, \dots, \bar{g}_{2s} : W' \to \mathbb{R}$ such that the contact form α on M' reads

$$\alpha_0 = y_0 d\theta_0 + \dots + y_r d\theta_r + \pi^*(\bar{g}_1 d\bar{x}_1 + \dots + \bar{g}_{2s} d\bar{x}_{2s}). \tag{6.9}$$

Remark 6.2. Proposition 3 and Theorem 6 are contact analogs of Proposition 1 and Theorem 2' in Nehoroshev [**31**], respectively. In Theorem 2' [**31**], instead of the condition (i), the condition that W' is a simply connected manifold with vanishing of the second cohomology class $H^2(W', \mathbb{R})$ is used. A variant of the statement with noncompact invariant manifolds is proved in [**13**].

Proof. Since $\pi : M' \to W'$ is a trivial principal $\mathbb{T}^{r+1}$ bundle, there exist global angles variables $(\varphi_1, \dots, \varphi_r)$. Repeating the arguments used in the proof of Theorem 5.1, we get that the coordinate vector fields $Y_\nu = \partial/\partial\varphi_\nu$ preserve α and we can define actions as their contact Hamiltonians:

$$y_\nu = \Phi(Y_\nu) = i_{Y_\nu}\alpha : \quad M' \to \mathbb{R}, \quad \nu = 0, \dots, r.$$

They are redundant integrals of the pseudo-orthogonal foliation that satisfy relations (5.17), where z_ν are the components of the Reeb vector field Z with respect to vector fields Y_ν. The functions y_ν are π-basic and let $\bar{y}_\nu$ be the corresponding functions on W', $y_\nu = \bar{y}_\nu \circ \pi$. They are subjected to the constrains

$$\bar{z}_0 \bar{y}_0 + \dots + \bar{z}_r y_r = 1, \quad \bar{z}_0 d\bar{y}_0 + \dots + \bar{z}_r d\bar{y}_y = 0, \tag{6.10}$$

where $z_\nu = \pi \circ \bar{z}_\nu$.

Moreover, according to the assumption (6.8), in a neighborhood of any point $w_0 \in W'$, we can take r independent functions among $\bar{y}_\nu$ that are independent of $\bar{x}_1, \dots, \bar{x}_{2s}$ providing a local coordinate chart.

Let $\{W_i\}$ be a finite covering of W' such that on every W_i we can take local coordinates $(\bar{y}^i, \bar{x})$, where $(\bar{y}^i_1, \dots, \bar{y}^i_r)$ is a subcollection of redundant actions $(\bar{y}_0, \dots, \bar{y}_r)$.

As in Theorem 5.1 we get that the contact form in $U_i = \pi^{-1}(W_i)$ reads

$$\alpha^i = \alpha_\theta + \pi^*\alpha^i_y + \pi^*\alpha^i_x, \tag{6.11}$$

where $\alpha_\theta = \sum_{\nu=0}^{r} y_\nu d\varphi_\nu$, $\alpha_y^i = \sum_{k=1}^{r} \bar f_k^i(\bar y^i, \bar x) d\bar y_k^i$, $\alpha_x^i = \sum_{a=1}^{2s} \bar g_a^i(\bar y^i, \bar x) d\bar x_a$.

Thus, on M' we have a unique decomposition $\alpha = \alpha_\theta + \pi^*\alpha_y + \pi^*\alpha_x$, locally given by (6.11). It is obvious that we can write α_x as $\alpha_x = \sum_{a=1}^{2s} \bar g_a d\bar x_a$, where $\bar g_a : W' \to \mathbb{R}$.

Next, consider the filtration

$$V_1 = W_1 \subset V_2 = W_1 \cup W_2 \subset \cdots \subset V_N = W_1 \cup \cdots \cup W_N = W'.$$

Applying Lemma 6.1 given below $(N-1)$ times we obtain functions $\bar f_0, \dots, \bar f_r : W' \to \mathbb{R}$, satisfying the identities

$$\bar f_0 d\bar y_0 + \bar f_1 d\bar y_1 + \cdots + \bar f_r d\bar y_r = \bar f_1^i d\bar y_1^i + \dots \bar f_r d\bar y_r^i$$

on every W_i.

Therefore, after globally defined transformation

$$(\theta_0, \theta_1, \dots, \theta_r) = (\varphi_0 - f_0, \varphi_1 - f_1, \dots, \varphi_r - f_r), \quad f_\nu = \bar f_\nu \circ \pi,$$

the form α becomes

$$\alpha = \sum_{\nu=0}^{r} y_\nu d\theta_\nu + \pi^* \sum_{a=1}^{2s} \bar g_a d\bar x_a + df,$$

where $f = \sum_\nu y_\nu f_\nu$ is a π-basic function. Now, as in Theorem 5.1, applying Moser's deformation for a compact manifold M' and family of forms $\alpha_t = \alpha_0 + t df$, we get the required statement. □

Lemma 6.1. *Suppose that on W' we have an open set U with local coordinates $(\bar y_1, \dots, \bar y_r)$ and an open set V endowed with 1-forms*

$$\gamma_U = F_1 d\bar y_1 + \cdots + F_\nu d\bar y_r, \quad \gamma_V = G_1 d\bar y_0 + \cdots + G_\nu d\bar y_r,$$

that are equal on the intersection $U \cap V$. Then there exist functions $E_0, \dots, E_r$ defined on $U \cup V$ satisfying

$$\gamma_U = E_0 d\bar y_0 + \cdots + E_r d\bar y_r|_U, \quad \gamma_V = E_0 d\bar y_0 + \cdots + E_r d\bar y_r|_V.$$

Proof. The statement is trivial if $U \cap V = \emptyset$. Assume $U \cap V \neq \emptyset$. According to constraints (6.10), the form γ_U does not change under the addition of terms proportional to $\bar z_0 d\bar y_0 + \cdots + \bar z_r d\bar y_y$. We are looking for a function $A : U \to \mathbb{R}$ that satisfies

$$A\bar z_0 = G_0, \quad F_1 + A\bar z_1 = G_1, \quad \dots \quad F_r + A\bar z_r = G_r \tag{6.12}$$

on $U \cap V$. Although it is overdetermined system, due to the condition that $\gamma_U = \gamma_V$ it has an unique solution. Indeed, on U we have $\bar z_0 \neq 0$ and $\partial \bar y_0 / \partial \bar y_i = -\bar z_i / \bar z_0$ (see Remark 5.2). Therefore, the equality $\gamma_U = \gamma_V$ implies the following compatibility conditions

$$F_1 = -G_0 \frac{\bar z_1}{\bar z_0} + G_1, \dots, F_r = -G_0 \frac{\bar z_r}{\bar z_0} + G_r, \quad \bar y \in U \cap V. \tag{6.13}$$

From (6.13), we obtain that $A = G_0/\bar{z}_0 : U \cap V \to \mathbb{R}$ is a solution of (6.12). Now we take an arbitrary extension of A from $U \cap V$ to U and define

$$E_0|U = A,\ E_0|_V = G_0, \quad E_i = F_i + A\bar{z}_i|_U,\ E_i = G_i|_V, \quad i = 1, \dots, r.$$

□

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