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SYMPLECTIC GENERIC COMPLEX STRUCTURES ON FOUR-MANIFOLDS WITH $b_+ = 1$

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We study symplectic structures on Kähler surfaces with $p_g = 0$. We give an example of a projective surface which admits a symplectic structure which is not compatible with any Kähler metric.

1. Introduction

The main purpose of this note is to give a negative answer to a question raised by Tian-Jun Li [**Li08**]:

Question 1.1. *Let X be a closed, smooth, oriented four-manifold which underlies a Kähler surface such that $p_g(X) = 0$. Does X admit a symplectic generic complex structure?*

A complex structure J on X is called *symplectic generic* if for any symplectic form ω of X such that $-c_1(X, \omega)$ coincides with the canonical class K_J of J, there exists a Kähler form ω' cohomologous to either ω or $-\omega$.

One of the main motivations for this question is the fact that, by a result of Biran [**Bir99**], the existence of a symplectic generic complex structure on any rational four-manifold implies the famous Nagata's conjecture which states that given very general points $p_1, \ldots, p_\ell \in \mathbb{CP}^2$, with $\ell \geq 9$, any curve C in $\mathbb{CP}^2$ must satisfy

$$\deg C \geq \frac{\sum_{i=1}^{\ell} \operatorname{mult}_{p_i} C}{\sqrt{\ell}}$$

(see [**Li08**] for more details). that a smooth four-manifold X is said to be *rational* if it is diffeomorphic to either $S^2 \times S^2$ or $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, for some $k \geq 0$.

On the other hand, if X is the four-manifold underlying a smooth minimal projective surface of general type (i.e., with big and nef canonical line bundle) then there exists a symplectic form inside the class of the canonical line bundle of X (see [**Cat09, STY02**]). Therefore, if $p_g(X) = 0$,

the existence of a symplectic generic complex structure on X would, in particular, imply the existence of a Kähler–Einstein metric with negative curvature on X, by the result of Aubin and Yau. For example, Catanese and LeBrun [**CL97**] showed the existence of a Kähler–Einstein metric with negative curvature on the generic Barlow surface, which is a projective surface of general type homeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$. But the question remains a hard problem in general, as a classification of the projective surfaces with zero genus is still beyond our reach (see the recent survey [**BCP10**] for an updated
account).

Our example is obtained by considering the four-manifold $X = (\Sigma \times S^2)\#\overline{\mathbb{CP}^2}$, where Σ is a Riemannian surface of genus one. We show the existence of a symplectic form on X which is not cohomologous to any Kähler form on X, with respect to any complex structure J. From an algebraic geometric point of view, this corresponds to saying that the Seshardi constant of a suitable ample class on any uniruled projective surface over an elliptic curve is not maximal (e.g., see [**Gar06**]). In particular, it follows that X does not admit a symplectic generic complex structure.

Moreover, we describe a minimal surface of general type, for which the underlying manifold does not admit a symplectic generic complex structure. The construction relies on a recent result by Bauer and Catanese [**BC11**].

Note that both these examples have infinite fundamental group.

2. Preliminary results

In this section, we recall some basic definition and well-known facts about the space of symplectic forms on a smooth four-manifold.

Given a closed smooth oriented four-manifold X, we consider the *positive cone* of X, which is defined as the set

$$\mathcal{P}_X = \{a \in H^2(X,\mathbb{R}) \mid a^2 > 0\}.$$

Moreover, we denote by Ω_X the space of orientation-compatible symplectic forms on X. Let

$$\mathcal{C}_X = \{[\omega] \mid \omega \in \Omega_X\} \subseteq H^2(X,\mathbb{R}),$$

and let $K_\omega = -c_1(X,\omega)$ be the canonical class of $\omega \in \Omega_X$. We denote by $\mathcal{K}_X$ the union of all elements K_ω in $H^2(X,\mathbb{Z})$, where $\omega \in \Omega_X$. For any $K \in \mathcal{K}_X$, let

$$\mathcal{C}_{(X,K)} = \{[\omega] \in \mathcal{C}_X \mid K_\omega = K\}.$$

Let $\mathcal{E}_X$ be the set of cohomology classes whose Poincaré dual are represented by smoothly embedded spheres of self-intersection -1. In particular, X is said to be *minimal* if $\mathcal{E}_X$ is empty. Moreover, for any $K \in H^2(X,\mathbb{Z})$, we denote

$$\mathcal{E}_{(X,K)} = \{E \in \mathcal{E}_X \mid E \cdot K = -1\}.$$

The following result by Li and Liu [**LL01**] will play an important role:

Theorem 2.1. *Let X be a closed, smooth, oriented four-manifold with $b_+(X) = 1$.*

Then

$$\mathcal{C}_X = \{a \in \mathcal{P}_X \mid a \cdot E \neq 0, \quad \textit{for all } E \in \mathcal{E}_X\}.$$

Moreover,

(1) *If $K \in \mathcal{K}_X$ is not a torsion class, then $\mathcal{C}_{(X,K)}$ is contained in one of the two components of $\mathcal{P}_X$, denoted by $\mathcal{P}_{(X,K)}$, and*

$$\mathcal{C}_{(X,K)} = \{a \in \mathcal{P}_{(X,K)} \mid a \cdot E > 0 \quad \textit{for all } E \in \mathcal{E}_{(X,K)}\}.$$

(2) *If $K \in \mathcal{K}_X$ is a torsion class, then $C_{(X,K)} = P_X$.*

Proof. See [**LL01,** Theorem 4] and [**Li08,** Theorem 3.11]. □

Remark 2.1. Let (X, ω) be a symplectic four-manifold with $b_+(X) = 1$. It follows from [**Li06,** Proposition 6.3], that if its canonical class K_ω is torsion then $2K_\omega = 0$ in $H^2(X, \mathbb{Z})$. In particular, $K_{-\omega} = K_\omega$ and $\mathcal{C}_{(X,K_\omega)}$ is not contained in one of the two components of $\mathcal{P}_X$. On the other hand, if K_ω is not a torsion class, then $K_\omega \neq K_{-\omega}$.

We say that a complex structure J on X is *symplectic generic* if $\mathcal{C}_J$ is a connected component of $\mathcal{C}_{(X,K_J)}$, where $\mathcal{C}_J$ denotes the Kähler cone of J and K_J is the canonical class of J. In particular, if K_J is not a torsion class, then J is symplectic generic if $C_{(X,K_J)} = C_J$.

Lemma 2.1. *Let (X, J) be a minimal complex surface with $b_+(X) = 1$ and which admits a Kähler class $[\omega] \in \mathcal{C}_J$. Then J is a symplectic generic complex structure if and only if any J-holomorphic curve in X has non-negative self-intersection.*

Proof. By the Kähler Nakai–Moishezon criterion [**Buc99, Lam99**], if the Kähler cone $\mathcal{C}_J$ is not empty then it coincides with the set of elements in $\mathcal{P}_{(X,K_J)}$ which are positive on every J-holomorphic curve with negative self-intersection. Thus, if there is no such curve on X, it follows that J is a symplectic generic complex structure.

Let us assume now that C is a J-holomorphic curve with negative self-intersection. Let $v = \omega(C)$ and $m = -C^2$ and define $a(t) = [\omega] + tPD(C) \in H^2(X, \mathbb{R})$ for any $t \geq 0$. Then, since

$$a(t)^2 = [\omega]^2 + 2t\omega(C) + t^2C^2 > 2tv - t^2m,$$

it follows that there exists $T > v/m$ such that $a(T) \in \mathcal{P}_{(X,K_J)}$. Since X is minimal, Theorem 2.1 implies that $a(T)$ is represented by a symplectic form ω_T such that $K_{\omega_T} = K_J$. On the other hand, $\omega_T(C) = v - Tm < 0$, thus $a(T)$ is not a Kähler class. In particular, J is not a symplectic generic complex structure. $\square$

By the Kähler Nakai–Moishezon criterion and Theorem 2.1, it also follows that a positive answer to Question 1.1 in the case of rational four-manifolds is equivalent to the following conjecture: any integral curve with negative self-intersection on the blow-up of $\mathbb{C}P^2$ at a set of points in very general position is a smooth rational curve with self-intersection -1. In fact, both the conjectures are equivalent to the following:

Conjecture 2.1 (Harbourne–Hirschowitz). *Let X be the blow-up of $\mathbb{C}P^2$ at a set of $n \geq 10$ points in very general position. Then, the closed cone of curves of X is spanned by the smooth rational curves with self-intersection -1 and the round positive cone of $x \in N_1(X)$ such that*

$$x^2 \geq 0 \qquad \text{and} \qquad H \cdot x \geq 0$$

for some fixed ample class H on X.

3. Ruled manifolds

In this section, we show the existence of a smooth uniruled complex manifold, which does admit a symplectic generic complex structure.

Lemma 3.1. *Let Σ be an elliptic curve, and let $p\colon Y \to \Sigma$ be a minimal ruled surface over Σ, such that the parity of the intersection pairing on $H^2(Y, \mathbb{Z})$ is odd. Let X be the blow-up of Y at one point $\eta \in Y$. Let k be the canonical class of X, and let e be the class of the exceptional divisor.*

Then the class $e - 2k$ contains an effective curve.

Proof. By Atiyah's classification [**Ati57**] of rank 2 vector bundles on an elliptic curve, it follows that $Y = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is either the indecomposable vector bundle contained in the sequence

$$0 \to \mathcal{O}_\Sigma \to \mathcal{E} \to \mathcal{O}_\Sigma(p) \to 0,$$

for some $p \in \Sigma$ or $\mathcal{E} = \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(L)$, where L is a line bundle of odd degree $m < 0$.

Let us consider first the case of the indecomposable vector bundle. It is known (e.g., see [**CC93**]) that in this case $\mathbb{P}(\mathcal{E})$ is isomorphic to the symmetric product $S^2\Sigma$ of the elliptic curve Σ, i.e., the quotient of $\Sigma \times \Sigma$ by the natural action of $\mathbb{Z}/2\mathbb{Z}$. We will denote by $[x, y] \in S^2\Sigma$ the class of an element $(x, y) \in \Sigma \times \Sigma$. Note that the projection $p\colon S^2\Sigma \to \Sigma$ is defined by $p([x, y]) = x + y$. Consider the family of curves

$$C_t = \{[x, t + x] \mid x \in \Sigma\}, \quad \text{for any } t \in \Sigma.$$

If $t \in \Sigma$ is not a two-torsion point, then the curve C_t is a smooth elliptic curve. Otherwise, C_t is a non-reduced elliptic curve. Note that, for any $s, t \in \Sigma$, we have $C_t = C_s$ if and only if $t = s$ or $t = -s$ and C_t and C_s are disjoint otherwise. It follows that $C_t^2 = 0$. Moreover, given $s, t \in \Sigma$, there exist exactly four points $x \in \Sigma$ such that $2x + t = s$. Thus, if t is a general point in Σ, then the general fiber of p meets C_t in exactly four points. Let f be the numerical class of the pull-back of the general fiber of p in X and let δ be the numerical class of the pull-back of C_t. Then

$$\delta^2 = C_t^2 = 0, \quad \delta \cdot e = 0 \quad \text{and} \quad \delta \cdot f = 4.$$

By adjunction, we have that $k \cdot \delta = -\delta^2 = 0$. Similarly, we have $k \cdot e = -1$ and $k \cdot f = -2$. Moreover, since e, f and k are a basis of $H^2(X, \mathbb{Q})$, it follows easily that $\delta = 2e - 2k$. For any point $\eta \in S^2\Sigma$ there exists $t \in \Sigma$ such that $\eta \in C_t$. If X is the blow-up of Y at η and C_t' is the proper transform of C_t in X, then C_t' is in the class of $(2-q)e - 2k$, where $q \geq 1$ is the multiplicity of C_t at η. In particular, the class $e - 2k$ contains an effective curve, as claimed.

Let us consider now the case of a decomposable vector bundle $\mathcal{E} = \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(L)$ where L is a line bundle on Σ of odd degree $m < 0$. Then, there exists an holomorphic section C in Y such that $C^2 = m$. If ξ is the numerical class of the pull-back of C in X, it follows easily that $2\xi = e + mf - k$, where f is the pull-back of the general fiber of p. In particular, $e - 2k = 4\xi + (-2mf - e)$ is the class of a (possibly not irreducible) effective curve in X. □

Remark 3.1. Note that the uniruled surface which is the projectivization of the decomposable vector bundle can be obtained as a deformation of the projectivization of the indecomposable one. Thus, in the proof of the previous lemma, the second case would follow immediately from the first one.

Lemma 3.2. *A complex surface X homeomorphic to $(\Sigma \times S^2)\#\overline{\mathbb{CP}^2}$, is bi-holomorphic to a blow up at a single point of a minimal ruled surface Y over an elliptic curve, such that the intersection pairing on $H^2(Y, \mathbb{Z})$ is odd.*

Proof. Recall that from the Enriques–Kodaira classification of complex surfaces, it follows that each complex surface with odd b_+ is Kähler, and that any algebraic surface of non-negative Kodaira dimension and zero holomorphic Euler characteristics is bi-meromorphic to a torus or a bi-elliptic surface. Since $b_+(X) = 1$, it follows that X is Kähler and $p_{\mathrm{g}}(X) = 0$. Thus X is algebraic. Since $\pi_1(X) = \mathbb{Z}^2$ and $\chi(\mathcal{O}_X) = 0$, we conclude that X has Kodaira dimension $-\infty$.

By the classification of algebraic surfaces, it follows that if Y is the minimal model of X, i.e., the surface obtained after blowing-down all the holomorphic (-1) spheres on X, then Y is a uniruled surface over a Riemannian surface Σ. Since $b_1(Y) = b_1(X) = 2$, it follows that the genus of Σ is one. Moreover, since $b_2(X) = 3$, it follows that X is the blow-up of a ruled surface over an elliptic curve at a single point $p \in Y$. In particular X has exactly two holomorphic rational curves E_1 and E_2 with self-intersection -1: one is the exceptional divisor of the blow-up map and the other is the strict transform of the rational fiber passing through the blown-up point. Assume that the intersection form on $H^2(Y, \mathbb{Z})$ has even parity. Let C be a curve on Y which passes through p and which meets the fiber of the fibration $Y \to \Sigma$ transversally at p. Then the strict transform of C in X has odd self-intersection and it does not intersect E_2. Thus, after contracting E_2 we obtain a surface Y' such that the intersection form on $H^2(Y', \mathbb{Z})$ has odd parity. After replacing Y by Y', we may assume that $H^2(Y, \mathbb{Z})$ has odd parity. □

Lemma 3.3. *Let $\pi\colon Y \to \Sigma$ be a ruled projective surface over an elliptic curve Σ, such that $H^2(Y, \mathbb{Z})$ has odd parity. Let X be the blow up of Y at a single point. Let k be the class of the canonical class of X and let e_1, e_2 be the classes of the two rational curves of self-intersection -1 on X.*

Then $\mathcal{E}_{(X,k)} = \{e_1, e_2\}$.

Proof. Let e be a class in $H_2(X, \mathbb{Z})$ which can be represented by a smoothly embedded sphere in X such that $e^2 = -1$. Then e belongs to the kernel of $\pi_* : H_2(X, \mathbb{Z}) \to H_2(\Sigma, \mathbb{Z})$. This kernel is spanned by e_1 and e_2 and we deduce $e = \pm(ne_1 + (n-1)e_2)$ for some integer n. At the same time $e_1 \cdot k = e_2 \cdot k = -1$, since e_1, e_2 are the classes of exceptional curves on X. Thus, if $e \in \mathcal{E}_{(X,k)}$, then $e \cdot k = -1$ which implies $e = e_1$ or $e = e_2$. □

Theorem 3.1. *Let Σ be a Riemann surface of genus 1, let $\Sigma \times S^2$ be the trivial S^2-bundle on Σ and let $X = (\Sigma \times S^2)\#\overline{\mathbb{CP}}^2$.*

Then, for any complex structure J on X, there exists a symplectic form ω on X such that ω is not Kähler with respect to J. Moreover, X does not admit any symplectic generic complex structure.

Proof. Let J be a complex structure on X, let k be the canonical class of (X, J) and let e be the class of the exceptional divisor E of the contraction $X \to Y$, whose existence is guaranteed by Lemma 3.2. Let a be the first Chern class of an ample line bundle on X. By Lemma 3.1, it follows that $v = a \cdot (e - 2k) > 0$. Let

$$a(t) = a + t(e - 2k) \in H^2(X, \mathbb{R}), \quad \text{for all } t > 0.$$

In particular, $a(t) \cdot (e - 2k) = v - t$ and $a(v)^2 = a^2 + v^2 > 0$. Thus, there exists $T > v$ such that $a(T)^2 > 0$. Moreover, if $E \in \mathcal{E}_{(X,k)}$, then

$$a \cdot E > 0 \qquad k \cdot E = -1 \quad \text{and by Lemma 3.3} \quad e \cdot E \geq -1.$$

Thus, $a(t) \cdot E > 0$ for all $t > 0$. Since $b_+(X) = 1$, Theorem 2.1 implies that the class $a(T)$ is represented by a symplectic form ω, such that $K_\omega = k$.

On the other hand, by Lemma 3.1, the class $e - 2k$ is represented by a J-holomorphic curve C such that $a(T) \cdot C < 0$, since $T > v$. Thus, the class

$a(T)$ does not contain a Kähler form. In particular, J is not a symplectic generic complex structure. □

4. Non-ruled manifolds

In this section, we study Question 1.1 in the case of smooth *minimal* four-manifolds with non-negative Kodaira dimension.

Question 4.1. *Let X be a minimal four-manifold which underlies a Kähler surface such that $p_{\mathrm{g}}(X) = 0$. Does X admit a symplectic generic complex structure?*

In particular, we show that the question has positive answer in the case of zero Kodaira dimension and we provide an example of a minimal surface of general type which does not admit a symplectic generic complex structure.

By the Seiberg–Witten theory, the Kodaira dimension of a Kähler surface is preserved under diffeomorphism [**FM97**]. As noted in [**Li08**], any uniruled four-manifold, i.e., a manifold which underlies a Kähler surfaces of Kodaira dimension $-\infty$, admits a symplectic generic complex structure.

We first consider the case of zero Kodaira dimension:

Proposition 4.1. *Let X be a four-manifold which underlies a Kähler surface such that $p_{\mathrm{g}}(X) = 0$ and $\mathrm{kod}(X) = 0$.*

Then X admits a symplectic generic complex structure.

Proof. By the classification of algebraic surfaces, it follows that the canonical class of X is numerically trivial. Thus, by the adjunction formula, the only holomorphic curves of negative self-intersection, are smooth rational curves C such that $C^2 = -2$. In particular, Lemma 2.1 implies that it is sufficient to show that there exists a complex structure on X which does not admit any of these curves.

By the classification of algebraic surfaces, we just need to consider two cases: Enriques surfaces and bi-elliptic surfaces. The moduli space of Enriques surfaces is irreducible and by a result of Barth and Peters [**BP83,** Proposition 2.8], the generic Enriques surface does not contain any smooth rational curve of self-intersection -2.

If X is a bi-elliptic surface, then $X = \Sigma_1 \times \Sigma_2/G$, where Σ_1 and Σ_2 are Riemannian surfaces of genus one and G is an abelian group acting by complex multiplication on Σ_1 and by translation on Σ_2. In particular, since the universal cover of X is $\mathbb{C}^2$, it follows that X does not contain any rational curve. Thus, X does not admit any negative self-intersection curve.

By Lemma 2.1, it follows that any complex structure on X is symplectic generic. □

If X is a minimal surface of general type with $p_{\mathrm{g}}(X) = 0$, it is well known that $q = 0$ and $1 \leq K_X^2 \leq 9$. Thus, the moduli space of X is a union of finitely many irreducible varieties. Nevertheless, it is still not clear what the topology for these surfaces is (see [**BCP10**] for a recent survey). As stated in the introduction, if X is the four-manifold underlying the surface X, a positive answer to Question 4.1 would imply the existence of a complex structure on X which admits a Kähler-Einstein metric. By the results in [**Bar84, LP07, PPS09a, PPS09b**], it follows that there exist a surface of general type which is homeomorphic to $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, for $5 \leq k \leq 8$. It follows by [**CL97, RŞ09**] that, on any of these surfaces, there exists a complex structure which admits a Kähler–Einstein metric with negative curvature.

In general, if X is a minimal surface of general type with $p_{\mathrm{g}}(X) = 0$, then $\chi(\mathcal{O}_X) = 1$ and by Noether's formula we have

$$b_2(X) = \chi(X) - 2 = 12\chi(\mathcal{O}_X) - K_X^2 - 2 = 10 - K_X^2.$$

Thus, if $K_X^2 = 9$, then any class in $\mathcal{P}_X$ is the multiple of an ample class and the answer to Question 4.1 is obvious.

Let us consider now the case of a surface of general type S with $p_{\mathrm{g}}(X) = 0$ and $K_X^2 = 8$. All the known examples have infinite fundamental group and their universal cover is the bidisk $\Delta_1 \times \Delta_2 \subseteq \mathbb{C}^2$ [**BCP10**], so we assume that S is of this type. Denote by w_1 and w_2 two semi-positive (1,1)-forms on $\Delta_1 \times \Delta_2$ obtained via pullbacks of Poincaré metrics from the projections of the bidisk to its factors. For any $a, b > 0$ the form $aw_1 + bw_2$ is Kähler on the bidisk and is invariant under the action of $\pi_1(X)$. Thus, it descends to a Kähler form $w_{a,b}$ on S. Since $b_2(X) = 2$, it follows that for $a, b > 0$ the forms $w_{a,b}$ span one of the two connected components of $\mathcal{P}_X$, and so the complex structure on X is symplectic generic.

On the other hand, the results in [**BC11**] immediately imply the existence of a minimal surface of general type which does not admit a symplectic generic complex structure. Burniat showed the existence of a minimal surface X of general type such that $K_X^2 = 6$, $p_{\mathrm{g}}(X) = 0$, and which is a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of $\mathbb{CP}^2$ blown-up at three points. We will call such a surface a *Burniat surface.*

Theorem 4.1. *Let X be a four-manifold which underlies a Burniat surface. Then X does not admit a symplectic generic complex structure.*

Proof. By [**BC11,** Theorem 0.2], any complex structure J on X is a Burniat surface. In particular, X admits a J-holomorphic curve C of negative self-intersection, which maps to a (-1)-curve on the blow-up of $\mathbb{CP}^2$ at three points. More specifically, C is an elliptic curve of self-intersection -1. Thus, by Lemma 2.1, it follows that J is not symplectic generic. □

Note that a Burniat surface has infinite fundamental group. We do not know any complex surface with $p_{\mathrm{g}} = 0$, finite fundamental group and which does not admit a symplectic generic complex structure.

Recall finally that there exist a wide class of minimal elliptic surfaces of Kodaira dimension 1 and with $p_{\mathrm{g}} = 0$. These surfaces have topological Euler characteristic equal to 12, the base of the corresponding elliptic fibration is $\mathbb{C}P^1$, and the fibration can have any number of multiple fibers greater than 1. It would be interesting to show that all such surfaces admit a symplectic generic complex structure.

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