

## ON LEGENDRIAN SURGERIES BETWEEN LENS SPACES

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We obtain some obstructions to existence of Legendrian surgeries between tight lens spaces. We also study Legendrian surgeries between overtwisted contact manifolds.

### 1. Introduction

The focus of this paper is the following basic question: given two closed contact 3-manifolds  $(Y_1, \xi_1)$  and  $(Y_2, \xi_2)$ , can a Legendrian surgery on a link in  $(Y_1, \xi_1)$  produce  $(Y_2, \xi_2)$ ? If so, what can be said about the number of components of the surgery link? What happens if we do not fix contact structures  $\xi_1$  and  $\xi_2$ , but just want to obtain, say, a tight  $Y_2$  from a tight  $Y_1$ ?

It is well known that if  $(Y_2, \xi_2)$  is obtained from  $(Y_1, \xi_1)$  by a surgery on Legendrian link, the smooth cobordism between  $(Y_1, \xi_1)$  and  $(Y_2, \xi_2)$  has a Stein structure [E11]. Such cobordisms were studied by Etnyre and Honda [EH]; in particular, they showed a Stein cobordism between  $(Y_1, \xi_1)$  and  $(Y_2, \xi_2)$  exists whenever  $(Y_1, \xi_1)$  is overtwisted. If  $(Y_1, \xi_1)$  is Stein fillable, a Legendrian surgery produces a Stein filling of  $(Y_2, \xi_2)$ , and one can try to find obstructions to Legendrian surgeries by considering Stein fillings. From known classification theorems [E12, McD, Li] we immediately see, for example, that standard contact structures on  $L(p, 1)$  cannot be obtained from any lens space except  $S^3$  (see Section 2 for more results of this sort). More interestingly, we examine Stein fillings of planar open books to prove

**Theorem 1.1.** *No tight lens space can be obtained from itself by Legendrian surgery on a link.*

By contrast, it is easy to show that every overtwisted contact manifold can be obtain from itself by a Legendrian surgery on an appropriate Legendrian unknot (see Proposition 5.2).

Theorem 1.1 applies more generally to fillable contact 3-manifolds admitting planar open books. To prove the theorem, we show that for any contact structure supported by a planar open book, there is a bound on the second Betti number of its fillings (see Corollary 2.1).

When a Stein cobordism from  $(Y_1, \xi_1)$  to  $(Y_2, \xi_2)$  exists, one can wonder about its Betti numbers (note that cobordisms we consider arise from surgeries, and consist of 2-handles only). In particular, we can ask whether  $(Y_2, \xi_2)$  can be obtained from  $(Y_1, \xi_1)$  by a Legendrian surgery on a single knot; for the question to be meaningful, we will assume that obvious obstructions to existence of such surgery, both from the topological viewpoint (e.g., from the singular homology of  $Y_1$  and  $Y_2$ ) and from the homotopy classes of contact structures all vanish. Of course, this question is closely related to the question of integral knot surgery: can  $Y_2$  be obtained from  $Y_1$  by an integral surgery on a single knot? Such knot surgery questions have been studied by means of many tools. An important classical result, the knot complement theorem of Gordon–Luecke [GL], asserts that if  $S^3$  is obtained by a non-trivial surgery on a knot  $K \subset S^3$ , then  $K$  is an unknot. More recently, tools from the Heegaard Floer theory led to substantial progress [He, OS2, Ra] on understanding classical knots with lens space surgeries, and enumerating lens spaces that can be obtained from  $S^3$  by a surgery on a single knot (Berge’s conjecture). These results, together with considerations of the Thurston–Bennequin number, give some immediate easy results for Legendrian knot surgeries (see Section 4). Further, one expects that there are additional obstructions for surgeries between tight contact structures. Tight lens spaces are the easiest example because they have a complete classification, are all Stein fillable, and have simplest possible Heegaard Floer homology. Using Heegaard Floer contact invariants, we get

**Theorem 1.2.** *No tight contact structure on  $L(p_2, q_2)$  can be obtained from a tight contact  $L(p_1, q_1)$  by a Legendrian surgery on a single knot if  $p_1, p_2$  are coprime,  $-p_1q_2$  is a square modulo  $p_2$  and  $-1$  is not.*

(We follow the notation convention where  $L(p, q)$  stands for the oriented manifold, which is a  $-p/q$  surgery on the unknot in  $S^3$ . Note that this is opposite to conventions in [OS2, Ra]. We always assume that  $p > q > 0$ .)

Note that Theorem 1.2 does not contain any information about the existence of topological surgery. In some cases, an integral surgery between two lens spaces exists but cannot be made into a Legendrian surgery between tight contact structures; in other cases, we can only claim that a Legendrian surgery between tight lens spaces cannot exist, and it is possible that there is no integral surgery whatsoever. (We give examples in Section 3.)

It is also useful to observe that for small lens spaces, one can rule out Legendrian knot surgeries between tight contact structures simply by computing all of their three-dimensional homotopy invariants [Go]. However,

there are examples where Theorem 1.2 gives an obstruction to the existence of Legendrian surgery, but the homotopy invariants do not.

Next, we turn attention to overtwisted manifolds. By [EH], any contact manifold  $(Y_2, \xi_2)$  can be obtained from an overtwisted  $(Y_1, \xi_1)$  by a Legendrian surgery on a link. A priori, this Legendrian link may have more components than a smallest link in  $Y_1$  with an integral surgery producing  $Y_2$ .

We have

**Theorem 1.3.** *Suppose  $Y_2$  can be obtained from  $Y_1$  by an integral surgery on  $m$ -component link. Let  $\zeta$  be a tight,  $\eta$  an overtwisted contact structure on  $Y_2$ . Then  $(Y_2, \eta)$  can be obtained from some overtwisted contact structure on  $Y_1$  by a Legendrian surgery on an  $m$ -component link;  $(Y_2, \zeta)$  can be obtained from some overtwisted  $Y_1$  by a Legendrian surgery on an  $m + 1$  component link.*

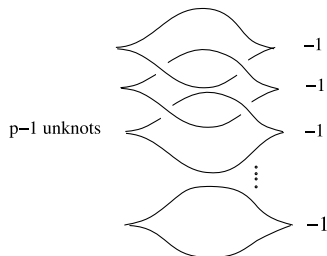
We establish a variety of other related results (many of them mere observations) in Section 5. We also give examples where  $Y_1, Y_2$  are related by an integral surgery on a single knot, but a two-component Legendrian link is needed to produce any tight  $Y_2$  from any overtwisted  $Y_1$  by a Legendrian surgery.

We will assume that the reader is familiar with basic contact topology constructions and with Heegaard Floer contact invariants, and state the necessary facts only very briefly.

## 2. Obstructions from Stein fillings

The first way to find some obstructions for Stein cobordisms between lens spaces is provided by examining Stein fillings. Let  $(L(p, q), \xi_{\text{std}})$  denote the (tight) contact lens space obtained as a quotient of  $(S^3, \xi_{\text{std}})$ . Symplectic fillings of  $(L(p, q), \xi_{\text{std}})$  were classified by Lisca (up to orientation-preserving diffeomorphism and blow-up) [Li]; earlier results were obtained by Eliashberg for fillings of  $(S^3, \xi_{\text{std}})$  [E12], and by McDuff for  $(L(p, 1), \xi_{\text{std}})$  [McD]. We collect the statements about lens spaces with few fillings below; note that Lisca's results yield classification of Stein fillings up to diffeomorphism, since Stein surfaces are minimal.

- $(S^3, \xi_{\text{std}})$  has a unique Stein filling, namely  $D^4$ .
- For  $p \neq 4$ ,  $(L(p, 1), \xi_{\text{std}})$  has a unique filling, namely  $D_{-p}$ , the disc bundle over  $S^2$  with Euler number  $-p$ .  $(L(4, 1), \xi_{\text{std}})$  has two fillings,  $D_{-4}$  and another which is a rational homology ball.
- $(L(p, p - 1), \xi_{\text{std}})$  has a unique filling given by the diagram in Figure 1; also note that  $L(p, p - 1)$  carries a unique tight contact structure [Ho].
- All lens spaces  $(L(p, q), \xi_{\text{std}})$  have finitely many Stein fillings.



**Figure 1.** The unique Stein filling of the tight  $L(p, p - 1)$ . (The numbers indicate surgery coefficients relative to the contact framing.)

Since a Stein filling of  $L(p_1, q_1)$  together with a Stein cobordism from  $L(p_1, q_1)$  to  $L(p_2, q_2)$  forms a Stein filling of  $L(p_2, q_2)$ , the above results immediately give some obstructions to existence of Stein cobordisms. We can state them for both the standard contact structures and their conjugates, since a Stein filling  $(X, J)$  of  $(Y, \xi)$  induces a Stein filling  $(X, \bar{J})$  of  $(Y, \bar{\xi})$ . We also recall that there are exactly two universally tight contact structures on  $L(p, 1)$ :  $\xi_{\text{std}}$  and its conjugate. The proposition below holds because otherwise there would be non-standard fillings (with the second Betti number not allowed by the classification results above); to construct these non-standard fillings in (1) and (2), start with the fillings for  $L(p, q)$  that have no 1-handles.

**Proposition 2.1.** (1) *A tight  $S^3$  cannot be obtained from any other tight lens space by a Legendrian surgery on a link.*

(2) *Universally tight  $L(p, 1)$  can be obtained only from  $(S^3, \xi_{\text{std}})$  (by a Legendrian surgery on a knot); it cannot be obtained from any other tight lens space by a Legendrian surgery on any link.*

(3) *A tight  $L(n, n - 1)$  can be obtained from a tight  $L(m, m - 1)$  by a Legendrian surgery on a knot if and only if  $m = n - 1$ ; more generally, a tight  $L(n, n - 1)$  can be obtained from a tight  $L(m, m - 1)$  by a Legendrian surgery on a link if and only if  $m < n$ .*

In fact, part (2) can be extended: in a joint work with Jeremy Van Horn-Morris [PV] we show that every virtually overtwisted tight contact structure on  $L(p, 1)$  has a unique filling; it then follows that *any* tight  $L(p, 1)$  can be obtained only from  $(S^3, \xi_{\text{std}})$ .

We now establish the main result of this section.

**Theorem 2.1.** *Let  $(Y, \xi)$  be a Stein fillable contact manifold that admits a compatible planar open book decomposition. Then  $(Y, \xi)$  cannot be obtained from itself by Legendrian surgery on any link.*

We recall that all tight contact structures on lens spaces admit planar open books [Sch]. Thus, Theorem 1.1 is an immediate corollary of Theorem 2.1.

*Proof.* We will use a theorem of Wendl [We, Theorem 1] as our key tool. Wendl essentially proves that every Stein filling of a contact structure with a planar open book decomposition admits (after an “enlargement” of the symplectic structure) a compatible positive allowable Lefschetz fibration that induces the given open book on the boundary. Topologically, this means that every Stein filling is diffeomorphic to a Lefschetz fibration that corresponds to the product of positive Dehn twists in some factorization of the monodromy for a *given* open book. We show that this leads to an upper bound on Betti numbers of fillings. Indeed, suppose the page of the open book is a disc with  $m$  holes,  $P_m = D - (D_1 \cup D_2 \cdots \cup D_m)$ ; the monodromy  $f$  is an element of the mapping class group  $\text{Map}(P_m)$  of the disc with  $m$  holes.

Shrinking the holes and thinking of them as punctured points, we get a homomorphism  $\Phi : \text{Map}(P_m) \rightarrow \text{PBraid}_m$  into the pure braid group on  $m$  strings. (Dehn twists around the holes become trivial in  $\text{PBraid}_m$ .) Let  $\Sigma : \text{PBraid}_m \rightarrow \mathbb{Z}$  be given by the algebraic crossing number of braids; then  $\Sigma \circ \Phi$  will be an invariant of the monodromy of our open book. Now, suppose that the monodromy  $f$  is expressed as a product of positive Dehn twists around some simple closed curves in  $D - (D_1 \cup D_2 \cdots \cup D_m)$ . We can consider the number of punctures enclosed by each curve, since these are simple closed curves in the disc, so the complement of each has the “inside” and the “outside” component. Observe that each Dehn twist that encloses more than one puncture contributes positively to the algebraic crossing number  $(\Sigma \circ \Phi)(f)$  of the corresponding pure braid; this gives an upper bound on the number of such Dehn twists. To find an upper bound on the number of Dehn twists around the holes, pick a hole  $D_i$  and consider a self-homeomorphism of the  $m$ -holed disc that exchanges the roles of  $\partial D_i$  and  $\partial D$ . The previous argument now yields a bound on the number of Dehn twists around the outer component of the boundary, i.e., around the hole  $D_i$ . (The image of the monodromy in  $\text{PBraid}_m$  depends on the choice of the outer component; so we will get different bounds for different  $i$ .) We now have a bound on the total number of positive Dehn twists in decompositions of the monodromy, i.e., on the second Betti number of fillings of the contact structure compatible with  $(P_m, f)$ .

To complete the proof, observe that if a contact manifold can be obtained from itself by some Legendrian surgery, we could generate fillings with arbitrary large  $b_2$  by starting with an arbitrary filling and adding multiple copies of the collection of 2-handles that has trivial effect on the contact boundary.  $\square$

The proof of Theorem 2.1 yields the following:

**Corollary 2.1.** *If  $(Y, \xi)$  admits a planar open book, there is a number  $M = M(Y, \xi)$  such that for any Stein filling  $X$  of  $(Y, \xi)$ , the bound  $b_2(X) < M$  holds.*

**Remark 2.1.** It is known [Ro] that for lens spaces, the only true cosmetic Dehn surgeries (i.e., rational knot surgeries that preserve the oriented manifold) are those that “switch” the roles of the solid tori in a genus 1 Heegaard decomposition of the lens space. (More precisely, one of these solid tori in  $L(p, q)$  is reglued to produce  $L(p, q')$  with  $qq' \equiv 1$  modulo  $p_2$ .) One can easily check that these surgeries can never be integral; thus the claim of Theorem 1.1 is moot for Legendrian surgeries on *knots*.

We also state an obvious corollary of Theorem 1.1:

**Corollary 2.2.** *If a Legendrian surgery on a link in a tight  $(L(p_1, q_1), \xi_1)$  produces a tight  $(L(p_2, q_2), \xi_2)$ , then no Legendrian surgery in  $(L(p_2, q_2), \xi_2)$  can produce  $(L(p_1, q_1), \xi_1)$ .*

### 3. Obstructions from Heegaard Floer contact invariants

In this section, we use the Heegaard Floer contact invariants [OS1] to find obstructions to existence of a Legendrian knot surgery between two tight lens spaces. We work with  $\mathbb{Z}/2\mathbb{Z}$  coefficients throughout. Recall that the invariant  $c(\xi)$  is an element of  $\widehat{HF}(-Y)$  for a contact 3-manifold  $(Y, \xi)$ ; if  $\xi$  is Stein-fillable,  $c(\xi) \neq 0$ . (Thus the contact invariant is non-zero for all tight lens spaces.)

If  $(Y_2, \xi_2)$  is obtained from  $(Y_1, \xi_1)$  by Legendrian surgery, the surgery cobordism carries a canonical  $\text{Spin}^c$  structure  $\mathfrak{t}$  induced by the Stein structure on the 2-handle. We can consider the map  $F_{\mathfrak{t}} : \widehat{HF}(-Y_2, \mathfrak{t}|_{-Y_2}) \rightarrow \widehat{HF}(-Y_1, \mathfrak{t}|_{-Y_2})$  induced by the  $\text{Spin}^c$  cobordism, as well as the map  $F : \widehat{HF}(-Y_2) \rightarrow \widehat{HF}(-Y_1)$  obtained by summing over all  $\text{Spin}^c$  structures on  $W$ . Then by [OS1, Gh, LS1],

$$F(c(\xi_2)) = F_{\mathfrak{t}}(c(\xi_2)) = c(\xi_1).$$

Thus, we can show that there is no Legendrian knot surgery from Stein-fillable  $(Y_1, \xi_1)$  to  $(Y_2, \xi_2)$  if we prove that any single 2-handle attachment producing  $-Y_1$  from  $-Y_2$  induces a vanishing map  $F$  on Heegaard Floer homology. We will follow this strategy to prove Theorem 1.2.

To prove that certain maps associated to surgeries vanish, we will make use of the surgery exact triangle. Recall that the exact triangle relates the Heegaard Floer homologies of a 3-manifold  $Y$  and the following surgeries. For a choice of longitude  $l$  for a knot  $K \subset Y$ , let  $Y_l$  be the manifold obtained by the surgery that attaches the meridian of the surgery torus to  $l$ ; for  $k \in \mathbb{Z}$ ,

let  $Y_{l+km}$  the manifold obtained by sending the meridian to  $l + km$ . Then the homologies of the three manifolds  $Y, Y_l, Y_{m+l}$  fit into an exact triangle:

$$\cdots \rightarrow \widehat{HF}(Y) \rightarrow \widehat{HF}(Y_l) \rightarrow \widehat{HF}(Y_{m+l}) \rightarrow \widehat{HF}(Y) \rightarrow \cdots$$

We are interested in lens spaces, which have the simplest possible Heegaard Floer homology; indeed, if  $Y$  is a lens space,  $\widehat{HF}(Y, \mathfrak{s}) = \mathbb{Z}/2\mathbb{Z}$  in each  $\text{Spin}^c$  structure  $\mathfrak{s}$  on  $Y$ . (A rational homology sphere with this property is called an  $L$ -space.) Recall the following:

**Lemma 3.1.** [OS2] *If  $Y, Y_l$ , and  $Y_{m+l}$  are members of a surgery triple such that  $Y$  and  $Y_l$  are  $L$ -spaces, and*

$$(3.1) \quad |H_1(Y_{l+m})| = |H_1(Y_l)| + |H_1(Y)|,$$

*then  $Y_{m+l}$  is also an  $L$ -space. Moreover, the map  $F : \widehat{HF}(Y) \rightarrow \widehat{HF}(Y_l)$ , induced by the surgery cobordism, is identically zero.*

*Proof of Theorem 1.2.* Suppose that the space  $Y_2 = L(p_2, q_2)$  is obtained by integral surgery on a knot in  $Y_1 = L(p_1, q_1)$ . The surgery cobordism from  $Y_1$  to  $Y_2$ , consisting of one 2-handle, can be considered as cobordism from  $-Y_2$  to  $-Y_1$ ; equivalently, we can say that  $-Y_1 = -L(p_1, q_1)$  is obtained by integral surgery on a knot  $K \subset -Y_2 = -L(p_2, q_2)$ . Let  $l$  be the choice of longitude for  $K$  corresponding to the surgery framing. Since  $p_1$  and  $p_2$  are assumed coprime,  $[K]$  generates  $H_1(-Y_2)$ .

Let  $C$  be the complement of a tubular neighborhood of  $K$  in  $-Y_2$ . The homology  $H_1(\partial C)$  of its torus boundary is generated by the chosen longitude  $l$  and the meridian  $m$  of the knot  $K$ . There exists an essential simple closed curve in  $\partial C$  that bounds in  $C$ . After an appropriate choice of orientations, such curve belongs to a (uniquely determined) homology class of the form  $am + p_2l \in H_1(\partial C)$ . The quantity  $a \bmod p_2$  is the self-linking number  $K \cdot K$ , which is an invariant of  $K$ . We will, however, consider  $a$  as an integer; note that  $a$  is well defined once the longitude  $l$  is fixed. Also observe that  $a$  and  $p_2$  are coprime.

We perform surgery on  $K \subset -Y_2$  and consider the surgery triple  $Y = -Y_2, Y_l = -Y_1$ , and  $Y_{l+m}$ . The integer  $a$  determines the size of homology of integral surgeries on  $K$ . Indeed, we have  $|H_1(Y_{l+km})| = |-a + kp_2|$ , where we assume that  $a$  is computed with respect to the chosen longitude  $l$ . (This is a standard calculation; see e.g., [Ra] for a quick review with proofs.) Thus,  $|H_1(Y_l)| = |-a|$  and  $|H_1(Y_{l+m})| = |-a + p_2|$ .

In our setup,  $Y_l = -L(p_1, q_1)$ , so  $a = \pm p_1$ . We next determine the sign of  $a$ . Indeed, recall that for a knot  $K \subset -L(p_2, q_2)$  the self-linking number  $K \cdot K$  can take values  $k^2q' \bmod p_2$ , where  $q'$  is the multiplicative inverse of  $q_2$  modulo  $p_2$ , and  $k$  is an integer. Then,  $aq_2$  is a square modulo  $p_2$ . The hypotheses of Theorem 1.2 imply that  $-p_1q_2$  is a square mod  $p_2$  but  $p_1q_2$  is not; thus,  $a = -p_1$ .

Since  $p_1, p_2 > 0$ , we now have  $|H_1(Y_{l+m})| = p_1 + p_2$ ,  $|H_1(Y)| = p_2$ ,  $|H_1(Y_l)| = p_1$ . So the condition (3.1) is satisfied, and Lemma 3.1 implies that the map  $F : \widehat{HF}(-L(p_2, q_2)) \rightarrow \widehat{HF}(-L(p_1, q_1))$ , induced by the cobordism, is trivial. As explained in the beginning of the section, it follows that a tight  $L(p_2, q_2)$  cannot be obtained from a tight  $L(p_1, q_1)$  by a Legendrian surgery.  $\square$

**Remark 3.1.** When  $p_2$  is prime, the hypotheses of Theorem 1.2 are equivalent to requiring that  $p_2$  be congruent to 3 mod 4, and exactly one of the numbers  $p_1, q_2$  be a square mod  $p_2$ .

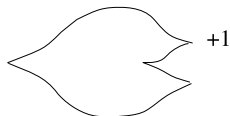
**Example 3.1.** For the lens spaces  $Y_1 = L(3r - 1, q)$  and  $Y_2 = L(3, 1)$  the hypotheses of the theorem hold for any  $r, q > 0$ , thus none of the two tight contact structures  $\zeta_1, \zeta_2$  on  $L(3, 1)$  can be obtained from any tight contact structure on  $L(3r - 1, q)$  by a Legendrian surgery. Note, however, that there is an obvious integral surgery between  $Y_1 = L(3r - 1, r)$  and  $L(3, 1)$ . In fact, by classification of tight contact structures [Ho], every tight contact structure on  $L(3r - 1, r)$  can be obtained by a Legendrian surgery on either  $(L(3, 1), \zeta_1)$  or  $(L(3, 1), \zeta_2)$ . If a tight  $(L(3r - 1, r), \xi)$  is obtained from a tight  $(L(3, 1), \zeta_1)$ , Corollary 2.2 rules out a Legendrian link surgery from  $(L(3r - 1, r), \xi)$  to  $(L(3, 1), \zeta_1)$ . Theorem 1.2 rules out a Legendrian knot surgery from  $(L(3r - 1, r), \xi)$  to both  $(L(3, 1), \xi_1)$  and  $(L(3, 1), \xi_2)$ .

#### 4. Obstructions from the Thurston–Bennequin framing and known knot surgery results

The Legendrian surgery coefficient is one less than the Thurston–Bennequin framing of the Legendrian knot. Thus, one can try to obtain obstructions to existence of Legendrian knot surgeries between two manifolds by enumerating knots with given integral surgeries, and then using bounds on the Thurston–Bennequin framing of these knots. While both of these steps are difficult in general, some special cases follow as immediate consequences of known results. We collect them in this section; it is interesting to compare these with the results from preceding sections. (We prove no new integral surgery results or Thurston–Bennequin bounds here.)

Here and in Section 5, we will use (+1) contact surgeries along with Legendrian surgeries. (See [DGS] for a detailed treatment of the material reviewed in this paragraph.) Recall that a +1 surgery is an operation that can be thought of as inverse to Legendrian surgery; more precisely, if  $K$  is a Legendrian knot and  $K'$  is its Legendrian push-off, then a Legendrian surgery on  $K$  and a +1 contact surgery on  $K'$  cancel each other. Every contact 3-manifold can be obtained from  $(S^3, \xi_{\text{std}})$  by Legendrian and +1 contact surgeries on components of some link. Unlike Legendrian surgery, +1 surgery does not always preserve Stein fillability or other similar properties





**Figure 2.** The  $+1$  surgery on the shark gives an overtwisted  $S^3$  with  $d_3 = \frac{1}{2}$ .

of contact structures, and may produce both tight and overtwisted results. For example,  $+1$  surgery on an unknot with  $\text{tb} = -2$  (Figure 2) in the standard tight  $S^3$  yields an overtwisted  $S^3$ . We will often encounter this particular overtwisted  $S^3$ , and will refer to it as the surgery on the “shark” (alluding to the shape of the stabilized unknot). A contact surgery diagram for a contact structure allows to compute its three-dimensional homotopy invariant  $d_3$  [Go] via the formula

$$(4.1) \quad d_3(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(X) - 3\text{sign}(X) + 2}{4} + m,$$

where  $X$  is a 4-manifold bounded by  $Y$  and obtained by adding 2-handles to  $B^4$  as dictated by the surgery diagram,  $\mathfrak{s}$  is the corresponding  $\text{Spin}^c$  structure on  $X$ , and  $m$  is the number of  $(+1)$ -surgeries in the diagram. The  $\text{Spin}^c$  structure  $\mathfrak{s}$  arises from an almost-complex structure defined in the complement of a finite set in  $X$ , and

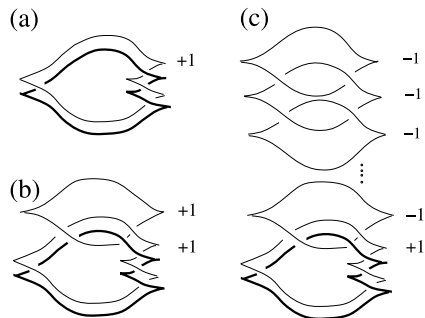
$$(4.2) \quad \langle c_1(\mathfrak{s}), h \rangle = \text{rot}(K),$$

where  $h$  is a homology generator of  $X$  corresponding to the handle attachment along an (oriented) Legendrian knot  $K$ .

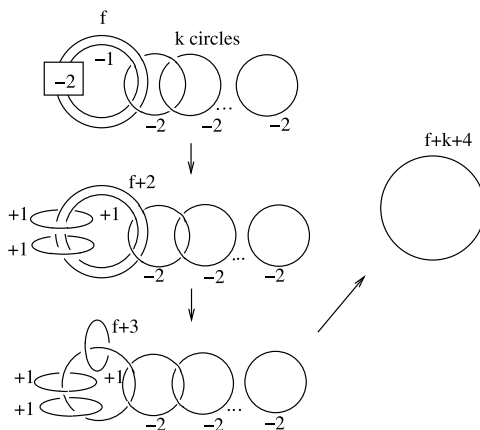
(Here we assume that  $c_1(\xi)$  is torsion; recall that in this case the invariant  $d_3$  and the  $\text{Spin}^c$  structure on  $Y$  induced by  $\xi$  determine the homotopy class of the plane field  $\xi$ .) For instance, for the surgery on the shark from Figure 2, we get  $d_3 = \frac{1}{2}$ .

We first consider surgeries on the unknot in a tight or overtwisted  $S^3$ . In the tight  $(S^3, \xi_{\text{std}})$ , Legendrian surgeries are restricted by the fact that  $\text{tb}(\text{unknot}) \leq -1$ . If we want to get a tight result by a surgery in an overtwisted  $S^3$ , we need to consider knots with tight complements (i.e., *non-loose* knots). Non-loose unknots in overtwisted  $S^3$  were classified in [EF]; they only exist in  $(S^3, \xi_o)$  with  $d_3(\xi_o) = \frac{1}{2}$ , can only have values of  $(\text{tb}, \text{rot})$  equal to  $(n, \pm(n-1))$  with  $n > 0$ , and are classified by  $\text{tb}$  and  $\text{rot}$ . An explicit construction of the non-loose knots in  $(S^3, \xi_o)$  was first given in [Dy]. We find it useful to give a contact surgery description of these unknots.

**Proposition 4.1.** *The contact surgery diagrams shown in Figure 3 represent  $(S^3, \xi_o)$ . The Legendrian knots (shown by thicker lines) represent non-loose unknots with  $\text{tb} = n$  and  $\text{rot} = \pm(n-1)$  for all  $n > 0$ .*



**Figure 3.** Non-loose unknots with (a)  $tb = +2$ , (b)  $tb = +1$ , (c)  $tb \geq 3$ .



**Figure 4.** Computing the framings. Unlike Figure 3, the numbers indicate Seifert framings, not those relative to the contact structure.

*Proof.* First, observe that the manifold shown is  $S^3$ , the contact structure is overtwisted and has  $d_3 = +\frac{1}{2}$ , thus we have  $(S^3, \xi_o)$ . Legendrian surgery on each thickly drawn knot cancels the  $+1$  contact surgery on the shark and produces a tight contact manifold; it follows that all of these knots are non-loose. We compute the Thurston–Bennequin number: if we ignore the contact surgeries for  $(S^3, \xi_o)$  (i.e., consider the thick knots as living in  $S^3$ ), the contact framing on each of these unknots is 2 less than the Seifert framing from the discs they bound in  $(S^3, \xi_{std})$  (in other words,  $tb = -2$ ). A few Kirby moves (see Figure 4 for the case of non-loose knots with  $tb \geq 3$ ) show that the framing increases in the surgered manifold, so that the unknot in the diagram with  $k$  Legendrian surgeries in Figure 3(c) has  $tb = -2 + k + 4 = k + 2$ . Cases (a) and (b) in Figure 3 are similar.  $\square$

**Proposition 4.2.** *If  $p > 2$ , a tight  $L(p, 1)$  cannot be obtained from an overtwisted  $S^3$  by a surgery on a Legendrian unknot.  $\mathbb{R}P^3$  with its unique tight contact structure  $\xi_t$  can be obtained from both tight and overtwisted  $S^3$ .*

*Proof.* The first part follows immediately from examination of the Thurston–Bennequin numbers of non-loose knots. As for the second part, a surgery on the unknot with  $\text{tb} = -1$  and  $\text{rot} = 0$  produces  $(\mathbb{R}P^3, \xi_t)$  from  $(S^3, \xi_{\text{std}})$ . To get  $(\mathbb{R}P^3, \xi_t)$  from an overtwisted sphere, do surgery on one of the two non-loose unknots with  $\text{tb} = +3$  in  $(S^3, \xi_o)$ . Note that these surgeries exhaust all possibilities: by [KMOS] the only surgery in  $S^3$  that produces  $\mathbb{R}P^3$  is  $\pm 2$  surgery on the unknot. The unknot then has to be non-loose and have  $\text{tb} = -1$  or  $+3$ , which means that the contact structure on  $S^3$  must be  $\xi_{\text{std}}$  or  $\xi_o$ .  $\square$

We also have

**Proposition 4.3.** *A tight  $-L(4n + 3, 4)$  and a tight  $S^3$  cannot be obtained from each other by a single Legendrian surgery (in any direction). In addition, a tight  $S^3$  cannot be obtained from an overtwisted  $-L(4n + 3, 4)$ .*

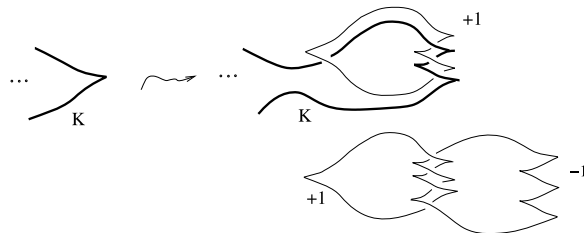
*Proof.* By [Ra], the only integral surgery that produces  $-L(4n + 3, 4)$  from  $S^3$  is the  $4n + 3$  surgery on the positive  $(2n + 1, 2)$  torus knot  $T_{2n+1,2}$ . The maximal Thurston–Bennequin number of  $T_{2n+1,2}$  in the tight  $S^3$  equals to  $2n - 1$ , thus no  $\pm 1$  contact surgery on this knot can produce  $-L(4n + 3, 4)$ .  $\square$

## 5. On overtwisted contact structures

A result from [EH] asserts that a Stein cobordism from  $(Y_1, \xi_1)$  to  $(Y_2, \xi_2)$  exists whenever  $(Y_1, \xi_1)$  is overtwisted. We now examine the relation between the second Betti numbers of such Stein cobordisms and smooth cobordisms between  $Y_1$  and  $Y_2$ . When working with overtwisted contact structures, we will often use Eliashberg’s classification result that says that two overtwisted contact structures are isotopic whenever they are homotopic as plane fields.

**Proposition 5.1.** *If  $Y_2$  can be obtained from  $Y_1$  by an integral surgery on an  $n$ -component link, then an overtwisted contact structure on  $Y_2$  can be obtained from an overtwisted contact structure on  $Y_1$  by a Legendrian surgery on a link with the same number of components. Moreover, one of the contact structures ( $\xi_1$  on  $Y_1$  or  $\xi_2$  on  $Y_2$ ) can be fixed.*

*Proof.* We first consider the case where  $Y_2$  is obtained from  $Y_1$  by a surgery on a single knot. Suppose that  $K$  is a knot in  $Y_1$  that has an integral surgery producing  $Y_2$ , and let  $l$  be the choice of longitude of  $K$  that corresponds to the surgery framing. More precisely, we think of  $l \in H_1(\partial(Y_1 - \nu K))$  as the homology class of a curve on the torus such that  $m \cdot l = 1$ , where



**Figure 5.** Matching the contact and the surgery framings: increasing  $\text{tb}$ .

$m \in H_1(\partial(Y_1 - \nu K))$  is the class of the meridian of the knot. (Here the torus  $\partial(Y_1 - \nu K)$  is oriented as the boundary of the tubular neighborhood  $\nu K$ .) The surgery producing  $Y_2$  is then given by attaching a solid torus to  $Y_1 - \nu K$  as dictated by the framing  $l$ . Now in the contact manifold  $(Y_1, \xi_1)$ , isotope  $K$  to make it Legendrian, and let  $\text{tb} \in H_1(\partial(Y_1 - \nu K))$  stand for the framing on  $K$  induced by  $\xi_1$ . To perform Legendrian surgery, we need to have  $l = \text{tb} - m$ . If  $l = \text{tb} - k m$  with  $k > 1$ , we can decrease the Thurston–Bennequin framing by stabilizing the knot. (To make sure that the result of the surgery will be overtwisted, we can connect-sum, away from the knot, with an overtwisted  $S^3$  with  $d_3 = -\frac{1}{2}$ . This won't change the homotopy, and thus the isotopy type of  $\xi_1$ .) If  $l = \text{tb} + k m$  with  $k \geq 0$ , we will increase the Thurston–Bennequin number as shown in Figure 5. As before we first take the connected sum of  $\xi_1$  and an overtwisted  $S^3$  with  $d_3 = -\frac{1}{2}$ , where the latter is shown in Figure 5. This  $S^3$  is in turn the connected sum of  $(S^3, \xi_o)$ , which is the  $(+1)$  contact surgery on the shark, and an  $S^3$  with  $d_3 = -\frac{3}{2}$ , which is the contact surgery on the two-component link shown in the figure. (See [DGS] for the detailed calculation of the homotopy invariants.) Taking a connected sum of  $K$  with an unknot with  $\text{tb} = +2$  that lives in  $(S^3, \xi_o)$  gives a knot with the Thurston–Bennequin framing  $\text{tb}(K) + 3m$ . We can repeat this procedure (together with extra stabilizations if needed) to get the required value of  $\text{tb}$ . (The fact that the Thurston–Bennequin number of a knot can be increased in overtwisted contact structures is well-known; in fact this can be achieved by taking connected sums with the boundary of the overtwisted disc; see [E11]. We included a contact surgery version of such construction for completeness of discussion.)

Observe that in this process we did not change the isotopy type of  $\xi_1$  (but have little control over the isotopy type of  $\xi_2$ ). To obtain a given overtwisted contact structure  $\xi_2$  on  $Y_2$ , we can apply a similar procedure for  $+1$  contact surgery on the knot dual to  $K$  in  $(Y_2, \xi_2)$ .

To prove the statement for links with multiple components, iterate the above argument.  $\square$

**Remark 5.1.** Note that we cannot hope to fix both  $\xi_1$  and  $\xi_2$ : suppose that  $Y_2$  is obtained from  $Y_1$  by a surgery on a knot, and observe that two arbitrary homotopy classes of contact structures cannot always be connected by a single-handle attachment. (Examples are easy to find on  $S^3$ .)

**Proposition 5.2.** *Every overtwisted contact manifold can be obtained from itself by a Legendrian surgery on an appropriate Legendrian unknot.*

*Proof.* First consider  $S^3$  with the overtwisted contact structure  $\xi_o$  that has  $d_3(\xi_o) = +\frac{1}{2}$ . There is a Legendrian unknot with  $\text{tb} = 0$  and  $\text{rot} = 1$ : for example, we can stabilize the non-loose unknot with  $\text{tb} = +2$  and  $\text{rot} = -1$  to decrease  $\text{tb}$  and increase  $\text{rot}$ . From (4.1), the Legendrian surgery on this unknot  $U$  results in a contact structure homotopic to  $\xi_o$  (and necessarily overtwisted, since  $d_3 = +\frac{1}{2}$ ); but then the resulting contact structure is isotopic to  $\xi_o$ . To find an unknot with  $\text{tb} = 0$  and  $\text{rot} = -1$  in any other overtwisted contact structure  $\xi$ , we represent  $\xi$  as the connected sum of itself and the overtwisted contact structure on  $S^3$  with  $d_3 = -\frac{1}{2}$  (as in Proposition 5.1 and Figure 5), and then find a copy of the required unknot in the  $(S^3, \xi_o)$  part of the contact structure.  $\square$

We now turn to the case where the convex end of the cobordism may be tight.

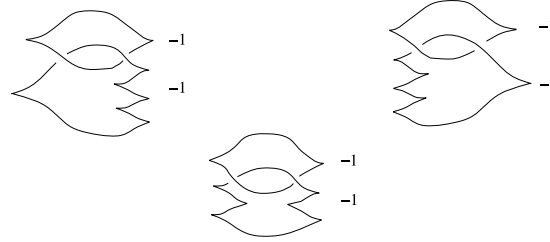
**Lemma 5.1.** *Suppose  $(Y_2, \xi_2)$  can be obtained from  $(Y_1, \xi_1)$  by a Legendrian surgery. Then there is an overtwisted contact structure  $\xi'_2$  on  $Y_2$  and a Legendrian surgery that produces  $(Y_1, \xi_1)$  from  $(Y_2, \xi'_2)$ .*

*Proof.* Suppose that  $K$  is a Legendrian knot in  $(Y_1, \xi_1)$  such that  $(-1)$  contact surgery on  $K$  produces  $(Y_2, \xi_2)$ . If we stabilize  $K$  twice, the same integral surgery becomes a  $+1$  contact surgery and results in  $Y_2$  with a different contact structure  $\xi'_2$ . By [LS2], a  $+1$  surgery on a stabilized Legendrian knot always yields an overtwisted contact structure. Thus  $\xi'_2$  will be overtwisted; canceling the  $+1$  surgery by a Legendrian surgery, we recover  $(Y_1, \xi_1)$ .  $\square$

**Corollary 5.1.** *If some contact structure on  $Y_2$  can be obtained from a tight contact  $Y_1$ , then the same tight contact structure on  $Y_1$  can be obtained by a single Legendrian surgery from an overtwisted contact structure on  $Y_2$ .*

**Proposition 5.3.** *Suppose that  $Y_2$  can be obtained from  $Y_1$  by an integral surgery on an  $n$ -component link. Then any tight contact structure  $\xi_2$  on  $Y_2$  can be obtained from some overtwisted  $(Y_1, \xi_1)$  by a Legendrian surgery on a link with  $n + 1$  components.*

*Proof.* First do a  $+1$  contact surgery on a shark in  $(Y_2, \xi_2)$  to obtain an overtwisted  $(Y_2, \xi'_2)$ . By Proposition 5.1, a Legendrian surgery on some  $n$ -component link in an overtwisted  $(Y_1, \xi_1)$  produces  $(Y_2, \xi'_2)$ . Now, do a Legendrian surgery on a push-off of the shark to undo the  $+1$  surgery and recover  $(Y_2, \xi_2)$ .  $\square$



**Figure 6.** Tight contact structures  $\Xi_0$  (middle),  $\Xi_1$  (left) and  $\Xi_2$  (right) on  $L(7, 4)$ .

Proposition 4.3 gives an example where the result of the previous proposition cannot be improved.

We conclude with a slightly more interesting example: Legendrian surgeries between various contact structures on  $S^3$  and  $L(7, 4)$ . Recall by [Ho] there are only three tight contact structures on  $L(7, 4)$ , and they are given by Legendrian surgeries depicted in Figure 6. The values of the  $d_3$ -invariant can be computed to be 0 for the contact structure  $\Xi_0$  given by the more symmetric surgery diagram, and  $-\frac{2}{7}$  for the other two contact structures  $\Xi_1, \Xi_2$ .

**Proposition 5.4.** *Although  $(L(7, 4), \Xi_0)$  can be obtained from  $(S^3, \xi_{\text{std}})$ , the contact manifolds  $(L(7, 4), \Xi_1)$  and  $(L(7, 4), \Xi_2)$  cannot be obtained by a surgery on Legendrian knot from any contact structure on  $S^3$ , tight or overtwisted.*

*Proof.* First, by [Ra] we know that the integral surgery that can produce  $L(7, 4)$  from  $S^3$  has to be  $-7$  surgery on the left-handed trefoil. If we start with a tight  $S^3$ , there is a unique Legendrian left-handed trefoil with  $\text{tb} = -6$  (by the classification of Legendrian torus knots [EH1]). This trefoil has  $\text{rot} = 0$ , so the Legendrian surgery produces  $(L(7, 4), \Xi_0)$ .

If we want to produce the contact structure  $\Xi_1$  or  $\Xi_2$  from an overtwisted  $S^3$ , we need to look for a non-loose left-handed trefoil with  $\text{tb} = -6$ . Note that for such a knot, the Thurston–Bennequin bound will be satisfied: indeed,  $\text{tb}$  and  $\text{rot}$  of a null-homologous non-loose knot are restricted by the inequality

$$-|\text{tb}(K)| + |\text{rot}(K)| \leq 2g - 1.$$

(see [Dy], Proposition 5.3 in the arXiv version).

It follows that  $|\text{rot}(\text{trefoil})| \leq 7$ . To pin down the value of  $\text{rot}$ , note that since contact structures on  $S^3$  have half-integer values of  $d_3$ , by (4.1) and (4.2), we have

$$\frac{m}{2} + \frac{c_1^2 - 2 + 3}{4} = -\frac{2}{7}, \quad c_1^2 = -\frac{\text{rot}^2}{7},$$

for some integer  $m$ . Thus,  $\frac{\text{rot}^2 - 8}{7} = 2m + 1$  is an integer,  $\text{rot}^2 \equiv 1 \pmod{7}$ , and then  $\text{rot} = \pm 1$ ,  $m = -1$ . This means that a Legendrian surgery in  $S^3$  can produce  $(L(7, 4), \Xi_1)$  or  $(L(7, 4), \Xi_2)$  only if  $S^3$  is equipped with the contact structure  $\xi_{\text{otw}}$  with  $d_3(\xi_{\text{otw}}) = -\frac{1}{2}$ .

Our next claim is that the contact structure  $\xi_{\text{otw}}$  on  $S^3$  in this case would have to be tight rather than overtwisted. (This leads to a contradiction, proving the proposition.) Indeed, for  $\Xi = \Xi_1$  or  $\Xi_2$ , we show that Heegaard Floer contact invariant is non-zero for the contact structure obtained from  $(L(7, 4), \Xi)$  by a  $+1$  contact surgery on the knot dual to the left-handed trefoil. Consider the map

$$F_{X, \mathfrak{t}} : \widehat{HF}(-L(7, 4), \mathfrak{s}) \rightarrow \widehat{HF}(-S^3),$$

that is induced by the surgery cobordism  $X$  equipped with the  $\text{Spin}^c$  structure  $\mathfrak{t}$  induced by the Stein structure on the 2-handle. The oriented smooth 4-manifold  $X$  with boundary that gives this cobordism from  $-L(7, 4)$  to  $-S^3$  can also be viewed as the cobordism from  $S^3$  to  $L(7, 4)$  given by our Legendrian surgery, i.e., the topological  $-7$  surgery on the left-handed trefoil. Since  $H_2(X)$  is generated by a surface with self-intersection  $-7$ ,  $X$  is negative definite. Since  $S^3$  and  $L(7, 4)$  are both  $L$ -spaces, by [LS3, Lemma 2.5] the map  $\widehat{F}_{X, \mathfrak{t}}$  will be injective as long as its degree shift equals to  $d(-S^3) - d(-L(7, 4), \mathfrak{s})$ , where the  $\text{Spin}^c$  structure  $\mathfrak{s}$  is the restriction of  $\mathfrak{t}$  to  $L(7, 4)$ . But  $d(-S^3) - d(-L(7, 4), \mathfrak{s}) = d(L(7, 4), \mathfrak{s}) - d(S^3) = d_3(\Xi) - d_3(\xi_{\text{otw}})$ , since we proved that  $d_3(\xi_{\text{otw}}) = -\frac{1}{2} = d(S^3) - \frac{1}{2}$ , and  $d_3(\Xi) = d(L(7, 4), \mathfrak{s}) - \frac{1}{2}$  because  $\mathfrak{s}$  is induced by  $\Xi$ . We know that

$$d_3(\Xi) - d_3(\xi_{\text{otw}}) = \frac{(c_1(X, \mathfrak{t}))^2 - 2\chi(X) - 3\sigma(X)}{4}$$

since  $(X, \mathfrak{t})$  is a Stein cobordism between the two contact manifolds; on the other hand, the latter quantity is the degree shift of  $F_{X, \mathfrak{t}}$ . Therefore,  $c(\xi_{\text{otw}}) = F_{X, \mathfrak{t}}(c(\Xi))$  is non-zero.  $\square$

### References

[DGS] F. Ding, H. Geiges and A. Stipsicz, *Surgery diagrams for contact 3-manifolds*, Turkish J. Math. **28**(1) (2004), 41–74.

[Dy] K. Dymara, *Legendrian knots in overtwisted contact structures on  $S^3$* , Ann. Global Anal. Geom. **19**(3) (2001), 293–305; [arXiv:math.GT/0410122](https://arxiv.org/abs/math/0410122).

[El1] Ya. Eliashberg, *Topological characterization of Stein manifolds of dimension  $> 2$* , Int. J. Math. **1**(1) (1990), 29–46.

- [El2] Ya. Eliashberg, *Filling by holomorphic discs and its applications*, London Math. Soc. Lecture Notes Series **151** (1991), 45–67.
- [EF] Ya. Eliashberg and M. Fraser, *Topologically trivial Legendrian Knots*, J. Symplectic Geom. **7**(2) (2009), 77–127.
- [EH] J. Etnyre and K. Honda, *On symplectic cobordisms*, Math. Ann. **323**(1) (2002), 31–39.
- [EH1] J. Etnyre and K. Honda, *Knots and contact geometry. I. Torus knots and the figure eight knot* J. Symplectic Geom. **1**(1) (2001), 63–120.
- [Gh] P. Ghiggini, *Ozsváth–Szabó invariants and fillability of contact structures*, Math. Z. **253**(1) (2006), 159–175.
- [Go] R. Gompf, *Handlebody construction of Stein surfaces*, Ann. Math. **148**(2) (1998), 619–693.
- [GL] C. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2**(2) (1989), 371–415.
- [He] M. Hedden, *On Floer homology and the Berge conjecture on knots admitting lens space surgeries*, [arXiv:0710.0357](https://arxiv.org/abs/0710.0357).
- [Ho] K. Honda, *On the classification of tight contact structures. I*, Geom. Topol. **4** (2000), 309–368.
- [KMOS] P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó, *Monopoles and lens space surgeries*, Ann. Math. **165**(2) (2007), 457–546.
- [Li] P. Lisca, *On symplectic fillings of lens spaces*, Trans. Amer. Math. Soc. **360**(2) (2008), 765–799.
- [LS1] P. Lisca and A. Stipsicz, *Seifert fibered contact three-manifolds via surgery*, Alg. Geom. Top. **4** (2004), 199–217.
- [LS2] P. Lisca and A. Stipsicz, *Notes on the contact Ozsváth–Szabó invariants*, Pac. J. Math. **228**(2) (2006), 277–295.
- [LS3] P. Lisca and A. Stipsicz, *Ozsváth–Szabó invariants and tight contact 3-manifolds. III*, J. Symplectic Geom. **5**(4) (2007), 357–384.
- [McD] D. McDuff, *The structure of rational and ruled symplectic 4-manifolds*, J. Am. Math. Soc. **3** (1990), 679–712.
- [OS1] P. Ozsváth and Z. Szabó, *Heegaard Floer homologies and contact structures*, Duke Math. J. **129**(1) (2005), 39–61.
- [OS2] P. Ozsváth and Z. Szabó, *On Knot Floer homology and lens space surgeries*, Topology **44**(6) (2005), 1281–1300.
- [PV] O. Plamenevskaya and J. Van Horn-Morris, *Planar open books, monodromy factorizations, and symplectic fillings*; [arXiv:0912.1916](https://arxiv.org/abs/0912.1916).
- [Ra] J. Rasmussen, *Lens space surgeries and L-space homology spheres*, [arXiv:0710.2531](https://arxiv.org/abs/0710.2531).
- [Ro] Y.-W. Rong, *Some Knots not determined by their complements*, in ‘Quantum Topology’, Ser. Knots Everything, **3**, World Scientific Publ., NJ, 1993, 339–353.
- [Sch] S. Shöenberger, *Planar open books and symplectic fillings*, Ph.D. thesis, University of Pennsylvania, 2005.
- [We] C. Wendl, *Strongly fillable contact manifolds and J-holomorphic foliations*, [arXiv:0806.3193](https://arxiv.org/abs/0806.3193).



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