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THE CALABI INVARIANT FOR SOME GROUPS OF HOMEOMORPHISMS

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We show that the Calabi homomorphism extends to some groups of homeomorphisms on exact symplectic manifolds. The proof is based on the uniqueness of the generating Hamiltonian (proved by Viterbo) of continuous Hamiltonian isotopies (introduced by Oh and Muller).

1. Introduction

1.1. The Calabi homomorphism. Let (M, ω) be a symplectic manifold, supposed to be *exact*, that is $\omega = d\lambda$ for some one-form λ called *Liouville form*. Equivalently, this also means that there exists a vector field X such that the Lie derivative satisfies: $\mathcal{L}_X \omega = \omega$. The vector field X is called the *Liouville vector field* and is related to the one-form λ by the relation $\iota_X \omega = \lambda$. For instance, cotangent bundles are exact symplectic manifolds.

Thanks to the work of Banyaga [**1**, **2**], the algebraic structure of the group $\mathrm{Ham_c}(M, \omega)$ of smooth compactly supported Hamiltonian diffeomorphisms of (M, ω) is quite well understood: there exists a group homomorphism, defined by Calabi [**3**]

$$\mathrm{Cal} : \mathrm{Ham_c}(M, \omega) \to \mathbb{R},$$

whose kernel ker(Cal) is a simple group.

The Calabi homomorphism is defined as follows. Let $\phi \in \mathrm{Ham_c}(M, \omega)$ and let H be a compactly supported Hamiltonian function generating ϕ, i.e., a smooth function $[0, 1] \times M \to \mathbb{R}$ such that:

- ϕ is the time one map of the flow $(\phi_H^t)_{t \in [0,1]}$ of the only time-dependent vector field X_H satisfying at any time $t \in [0, 1]$,

$$\iota_{X_{H(t,\cdot)}} \omega = dH(t, \cdot),$$

- there exists a compact set in M that contains all the supports of the functions $H_t = H(t, \cdot)$, for $t \in [0, 1]$.

Then, by definition,

$$\mathrm{Cal}(\phi) = \int_0^1 \int_M H(t,x)\omega^d dt, \tag{1.1}$$

where d is half the dimension of M. This expression does not depend on the choice of the generating function H, and gives a group homomorphism.

1.2. Question and results. We consider the following question.

Question 1.1. *To which groups of homeomorphisms does the Calabi homomorphism extend?*

Note that it is known (see, e.g., [**5**]) that the Calabi homomorphism does not behave continuously with respect to the C^0-topology. For instance, one can consider the following example.

Example 1.1. Let $\phi \in \mathrm{Ham_c}(\mathbb{R}^2, rdr \wedge d\theta)$, and consider the sequence (ϕ_n) in $\mathrm{Ham_c}(\mathbb{R}^2, rdr \wedge d\theta)$ given by

$$\phi_n(r,\theta) = \frac{1}{n}\phi^{n^4}(nr,\theta).$$

This sequence converges in the C^0-sense to Id, but one can easily check that its Calabi invariant remains constant.

Let us consider the group G of all homeomorphisms ϕ such that (on some interval where it is well-defined) the isotopy $t \mapsto [\mu_t, \phi]$ is a C^0-Hamiltonian isotopy (in the sense of [**11**], see Section 2.1 for the precise definition of G). Here, μ_t denotes the flow generated by the Liouville vector field X, and $[\mu_t, \phi] = \mu_t \circ \phi \circ \mu_t^{-1} \circ \phi^{-1}$. We will prove the following extension result for the Calabi invariant.

Theorem 1.1. *The Calabi homomorphism extends to a group homomorphism $G \to \mathbb{R}$.*

Let us now consider the special case where (M, ω) is the standard symplectic vector space $(\mathbb{R}^{2d}, \omega_0 = \sum_{i=1}^d dp_i \wedge dq_i)$, where we denote by $(q_1, \ldots, q_d, p_1, \ldots, p_d)$ the coordinates in $\mathbb{R}^{2d}$. In this case, we can show that G has an interesting subgroup.

Definition 1.1. We denote by $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ the identity component of the group of compactly supported bilipschitz symplectic homeomorphisms.

Remark 1.1. Since Lipschitz maps are almost everywhere differentiable, the pull-back of a differential form by a bilipschitz map is well defined as a differential form with L^∞ coefficients. Therefore, as in the smooth case, a bilipschitz homeomorphism ϕ of M is symplectic if $\phi^*\omega = \omega$.

Note that a bilipschitz homeomorphism which is the C^0-limit of smooth symplectomorphisms is symplectic in this sense. This follows from the Gromov–Eliashberg rigidity theorem (see, e.g., [**9**, p. 59]).

Our result is then the following.

Theorem 1.2. *The following inclusions hold:*

$$\mathrm{Ham}(\mathbb{R}^{2d}, \omega_0) \subset \mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0) \subset G.$$

The inclusion $\mathrm{Ham}(\mathbb{R}^{2d}, \omega_0) \subset \mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ is obvious. The proof of the other inclusion will be made in two steps. First, we will show that elements of $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ close to the identity admit generating functions of class C^1. Second, we will remark that such generating functions allow to construct C^0-Hamiltonian isotopies, which will imply that those elements are in G.

We believe that Theorem 1.2 and its proof can be adapted to general exact symplectic manifolds. For the sake of simplicity we state it only in $\mathbb{R}^{2d}$.

Remark 1.2. In the special case of the (two-dimensional) open disk, the fact that the Calabi homomorphism extends to $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ was already proved by Haïssinsky [**6**].[1] His methods are completely different.

Let us also mention that Gambaudo and Ghys have proved that two diffeomorphisms of the disk that are conjugated by an area preserving homeomorphism have the same Calabi invariant [**5**].

1.3. Motivation. Our motivation for this work comes from two distinct problems. The first one is the following:

Question 1.2 (Fathi [**4**])**.** *Is the group* $\mathrm{Homeo}_c(\mathbb{D}_2, \mathrm{area})$ *of compactly supported area preserving homeomorphisms of the disk a simple group?*

Several non-trivial normal subgroups of $\mathrm{Homeo}_c(\mathbb{D}_2, \mathrm{area})$ have been defined by Ghys [**2**], Oh–Muller [**12**] and recently by Le Roux [**8**]. But so far, no one has been able to prove that any of them is a proper subgroup.

Our study is inspired by the work of Muller and Oh. They introduced on any symplectic manifold (M, ω) a group denoted $\mathrm{Hameo}(M, \omega)$, whose elements are homeomorphisms called *hameomorphisms* (as the contraction of "Hamiltonian homeomorphisms"). This group contains all compactly supported Hamiltonian diffeomorphisms and, in the case of the disk, forms a normal subgroup of $\mathrm{Homeo}_c(\mathbb{D}_2, \mathrm{area})$. Fathi noticed that if one could extend the Calabi homomorphism to the group of hameomorphisms, then it would be necessarily a proper subgroup, and $\mathrm{Homeo}_c(\mathbb{D}_2, \mathrm{area})$ would not be simple.

In the present paper, we propose a different approach: instead of constructing a group which is known to be normal but on which it is unknown

[1] Area preserving quasiconformal maps of the plane are bilipschitz. Therefore, Haïssinsky's result is precisely the fact that the Calabi homomorphism extends to $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$.

whether the Calabi homomorphism extends, we construct a group (namely G) to which the Calabi invariant extends but for which it is unknown whether it is normal.

Another motivation is a very natural general problem: how can one generalize Hamiltonian dynamics in a non-smooth context? or (less optimistic) which properties of Hamiltonian maps can be extended? The present paper concentrates on a particular aspect: the Calabi homomorphism.

Our interest in the group $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ comes from the fact that it gives a large family of examples of elements of G, but also from the fact that it is a quite natural generalization of the Hamiltonian group, which could be considered to study the extension of other aspects of Hamiltonian dynamics.

Several other possible groups generalizing the Hamiltonian group have already been considered in literature. The group $\mathrm{Hameo}(M, \omega)$ mentioned above is one of them, another has been studied by Humiliere [**7**]. But this direction of research is still to be developed.

2. The group G and the Calabi invariant

2.1. The group G. To define the group G we first need the following notion.

Definition 2.1 (Oh–Muller [**12**])**.** A C^0-Hamiltonian isotopy is a path $(\phi^t)_{t\in[0,\delta]}$ of homeomorphisms of M for which there exist a compact set K and a sequence of smooth Hamiltonian functions H_n on M with support in K, such that

- (H_n) converges to some continuous function $H : [0, \delta] \times M \to \mathbb{R}$ in the C^0-sense,
- $(\phi^t_{H_n})$ converges to ϕ^t in the C^0-sense, uniformly in $t \in [0, \delta]$.

The function H is called a C^0-Hamiltonian function generating (ϕ^t).

Remark 2.1. The elements of C^0-Hamiltonian isotopies are *symplectic homeomorphisms*, i.e., homeomorphisms which are the C^0 limit of a sequence of symplectic diffeomorphisms supported in a common compact set.

It is not difficult to check that if (ϕ^t) and (ψ^t) are two C^0-Hamiltonian isotopies generated by F and G, then $((\phi^t)^{-1})$ and $(\phi^t \circ \psi^t)$ are C^0-Hamiltonian isotopies generated by $-F(t, (\phi^t)^{-1}(x))$ and $F(t, x) + G(t, \phi^t(x))$, and that if f is any symplectic homeomorphism, $(f^{-1} \circ \phi^t \circ f)$ is a C^0-Hamiltonian isotopy generated by $F(t, f(x))$. This means that the computations are the same as in the smooth case.

The main result concerning C^0-Hamiltonian isotopies is:

Theorem 2.1 (Viterbo [**13**])**.** *A given C^0-Hamiltonian isotopy is generated by a unique C^0-Hamiltonian function.*

This theorem is the only non-trivial result needed in this paper. Its proof needs at some point a (hard!) rigidity result in symplectic topology due to Gromov.

Definition 2.2. We denote by G the set of all compactly supported symplectic homeomorphisms ϕ for which there exists some $\delta > 0$ small enough, such that the isotopy $([\mu_t, \phi])_{t\in[0,\delta]}$ is a C^0-Hamiltonian isotopy.

Remark 2.2. As in the introduction, $\mu_t(x)$ denotes the flow (when it is defined) of the Liouville vector field X, at time t and point $x \in M$. Note that it satisfies $\mu_t^*\omega = e^t\omega$.

Let ϕ be a compactly supported homeomorphism of M. Then there exists a real number $\delta > 0$, such that for any $t \in [0, \delta]$, μ^t and $(\mu^t)^{-1}$ are well defined on the support of ϕ. Thus, the conjugation $\mu_t \circ \phi \circ \mu_t^{-1}$ is well defined on $\mu_t(\mathrm{Supp}(\phi))$. In the complement of this set, it is the identity where it is defined. Therefore, we can extend it to a well-defined homeomorphism still denoted $\mu_t \circ \phi \circ \mu_t^{-1}$ just by setting it to equal the identity where it is not defined.

Clearly, G contains $\mathrm{Ham_c}(M, \omega)$.

Proposition 2.1. *The set G is a group. Moreover, if the first cohomology group $H^1(M, \mathbb{R})$ vanishes, G does not depend on the choice of the Liouville vector field.*

Proof. Let $\phi, \psi \in G$. For δ small enough $([\mu_t, \phi])_{t\in[0,\delta]}$ and $([\mu_t, \psi])_{t\in[0,\delta]}$ are C^0-Hamiltonian isotopies. Then, note that

$$[\mu_t, \phi \circ \psi] = [\mu_t, \phi] \circ (\phi \circ [\mu_t, \psi] \circ \phi^{-1}),$$

and

$$[\mu_t, \phi^{-1}] = \phi^{-1} \circ [\mu_t, \phi]^{-1} \circ \phi.$$

We conclude with Remark 2.1 that G is a group.

Suppose now that $H^1(M, \mathbb{R}) = 0$, and that μ'_t is the flow of another Liouville vector field. Then, $\eta_t = \mu'_t \circ \mu_t^{-1}$ is a smooth symplectic isotopy which is Hamiltonian since $H^1(M, \mathbb{R}) = 0$. The Hamiltonian generating (η_t) is not compactly supported but Remark 2.1 still applies to the identity

$$[\mu'_t, \phi] = \eta_t \circ [\mu_t, \phi] \circ (\phi \circ \eta_t^{-1} \circ \phi^{-1}), \tag{2.1}$$

showing that G would be the same if it was defined with another Liouville vector field. □

2.2. Examples: fibered rotation in $\mathbb{R}^2$. In this section, we give a sufficient condition for a fibered rotation of $\mathbb{R}^2$ to be in G.

By definition, a *fibered rotation* is an homeomorphism ϕ of $\mathbb{R}^2$ described in polar coordinates (r,θ) by the formula

$$\phi(r,\theta) = (r, \theta + \rho(r)),$$

for some continuous *angular* function $\rho : (0,+\infty) \to \mathbb{R}$ with bounded support. It is easily checked that any fibered rotation lies in the identity component of the group of compactly supported area preserving homeomorphism of $\mathbb{R}^2$.

We consider μ_t the Liouville flow given by $\mu_t(r,\theta) = (e^{t/2}r, \theta)$. Its commutator with a fibered rotation is given by

$$[\mu_t, \phi](r,\theta) = (r, \theta - \rho(r) + \rho(e^{-t/2}r)).$$

If ϕ is moreover a diffeomorphism, the generating Hamiltonian of the isotopy $t \mapsto [\mu_t, \phi]$ is

$$H(t,r,\theta) = \frac{1}{2} r\rho(e^{-t/2}r) - \frac{1}{2}\int_0^r \rho(e^{-t/2}s)\, ds.$$

Now suppose that ρ is a continuous and integrable angular function, such that $r\rho(r)$ converges to 0 when r tends to 0. Suppose also that ρ_k is a sequence of smooth angular functions with bounded support and satisfying $\rho_k(r) = \rho(1/k)$ for $r \leqslant 1/k$ and $|\rho_k(r) - \rho(r)| \leqslant 1/k$ for $r > 1/k$. Then, the associated sequence of fibered rotations (ϕ_k) converges in the C^0-sense to ϕ, and the sequence of Hamiltonians (H_k) generating the isotopies $t \mapsto [\mu_t, \phi_k]$ also C^0-converges.

As a consequence, *any fibered rotation associated to an integrable angular function ρ such that $r\rho(r) \overset{r\to 0}{\longrightarrow} 0$, belongs to G.*

2.3. Extension of the Calabi homomorphism. In this section, we prove that the Calabi homomorphism extends to G. Let us first give a new formula for the Calabi invariant, for which we need to choose a Liouville form instead of choosing an isotopy.

Lemma 2.1. *Let $\phi \in \mathrm{Ham}_c(M,\omega)$ and let $H_{\lambda,\phi}$ be the generating Hamiltonian function of the smooth Hamiltonian isotopy $([\mu_t, \phi])$. Then,*

$$\mathrm{Cal}(\phi) = \frac{1}{d+1}\int_M H_{\lambda,\phi}(0,x)\omega^d.$$

Proof. First note that if ϕ is the time one map of some Hamiltonian function H, and if we suppose $\mu_\delta \circ \phi \circ \mu_\delta^{-1}$ to be well defined, then it can be generated by the Hamiltonian function $e^\delta H \circ \mu_\delta^{-1}$. After an easy change of variables in equation (1.1), one obtain

$$\mathrm{Cal}(\mu_\delta \circ \phi \circ \mu_\delta^{-1}) = e^{(d+1)\delta}\mathrm{Cal}(\phi),$$

where d is half the dimension of M. Thus,

$$\mathrm{Cal}([\mu_\delta, \phi]) = (e^{(d+1)\delta} - 1)\mathrm{Cal}(\phi).$$

Hence, applying formula (1.1) to $H_{\lambda,\phi}$,

$$\mathrm{Cal}(\phi) = \frac{1}{e^{(d+1)\delta} - 1} \int_0^\delta \int_M H_{\lambda,\phi}(t,x)\omega^d dt.$$

Now, letting δ converge to 0, we obtain the desired formula. □

Once this formula obtained, extending the Calabi homomorphism to G is very easy, even though it relies on the "hard symplectic topology" uniqueness Theorem 2.1.

Proof. Let $\phi \in G$ and let H be the *unique* C^0-Hamiltonian function generating $([\mu_t, \phi])_{t\in[0,\delta]}$ for some small δ. We set

$$\widetilde{\mathrm{Cal}}(\phi) = \frac{1}{d+1} \int_M H(0,x)\omega^n.$$

By Lemma 2.1, $\widetilde{\mathrm{Cal}}$ coincide with Cal on $\mathrm{Ham_c}(M,\omega)$. Moreover using Remark 2.1 and the formulas in the proof of Proposition 2.1, one checks easily that $\widetilde{\mathrm{Cal}} : G \to \mathbb{R}$ is a group homomorphism. □

Remark 2.3. If $H^1(M,\mathbb{R}) = 0$, then $\widetilde{\mathrm{Cal}}$ does not depend on the choice of the Liouville vector field. This is an immediate consequence of equation (2.1).

3. The inclusion $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0) \subset G$

The space of bilipschitz compactly supported maps of $\mathbb{R}^{2d}$ carries the structure of a topological group induced by the distance defined as follows. Let f, g be two such maps. We endow $\mathbb{R}^{2d}$ with the standard euclidean norm $\|\cdot\|$ and we denote $d_{C^0}(f,g) = \sup_{x\in\mathbb{R}^{2d}} \|f(x) - g(x)\|$ and

$$\mathrm{dil}(f,g) = \sup_{x\neq y\in\mathbb{R}^{2d}} \frac{\|(f(x) - g(x)) - (f(y) - g(y))\|}{\|x - y\|}.$$

Then, the bilipschitz distance between f and g is given by

$$D(f,g) = d_{C^0}(f,g) + d_{C^0}(f^{-1}, g^{-1}) + \mathrm{dil}(f,g) + \mathrm{dil}(f^{-1}, g^{-1}).$$

As mentioned in the introduction, the proof will use the notion of generating function.

Lemma 3.1. *For some neighborhood of the identity $\mathcal{V}$ in* $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ *such that for any homeomorphism $f \in \mathcal{V}$, there is a unique (up to constant shift) $C^{1,1}$ function $S : \mathbb{R}^{2d} \to \mathbb{R}$ such that for any $x, y, \eta, \xi \in \mathbb{R}^d$,*

$$f(x,y) = (\xi,\eta) \quad \Longleftrightarrow \quad \begin{cases} \xi = x + \dfrac{\partial S}{\partial \eta}(x,\eta), \\ y = \eta + \dfrac{\partial S}{\partial x}(x,\eta). \end{cases} \tag{3.1}$$

The map S is called the generating function *associated to f.*

The map $\Psi : \mathcal{V} \to C^{1,1}(\mathbb{R}^{2d}, \mathbb{R})$ which associates to f its generating function $\Psi(f) = S$ is a homeomorphism onto a neighborhood of 0 in $C^{1,1}(\mathbb{R}^{2d}, \mathbb{R})$ endowed with its natural topology.

Remark 3.1. Relation (3.1) means that under the symplectic identification $j : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, $(x, y; \xi, \eta) \mapsto (x, \eta; y - \eta, \xi - x)$, one has $j(\mathrm{graph}(f)) = \mathrm{graph}(dS)$.

Proof of Lemma 3.1. We only give the proof of the first part and let the second to the reader. It is completely analogous to what happens in the smooth case which is well known.

Let $f \subset \mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ be close to the identity. Then, if we denote $q : \mathbb{R}^{2d} \to \mathbb{R}^d$, $(x, y) \mapsto x$ and $p : \mathbb{R}^{2d} \to \mathbb{R}^d$, $(x, y) \mapsto y$ the canonical projections, the maps

$$y \mapsto p \circ f(x, y) \quad \text{and} \quad \xi \mapsto q \circ f^{-1}(\xi, \eta)$$

are Lipschitz-close to the identity and thus (by standard arguments) are bilipschitz homeomorphisms of $\mathbb{R}^{2d}$, close to the identity, in any given points x,η. Now let $\alpha(x, \cdot)$ be the inverse of $p \circ f(x, \cdot)$ and $\beta(\cdot, \eta)$ be the inverse of $q \circ f^{-1}(\cdot, \eta)$ which are again close to the identity in the Lipschitz sense. It is not difficult to check that the maps α and β are Lipschitz.

For $i \in \{1, \dots, n\}$, we denote by α_i, β_i the ith components of α and β with respect to the canonical basis. Let us now check that the Lipschitz one-form $\sum_i(\alpha_i dx_i + \beta_i d\eta_i)$ is closed (i.e., its differential vanishes whenever it is defined). Let (x, y) be a point where f admits a differential, and let $(\xi, \eta) = f(x, y)$. Then, at the point (x, η),

$$\begin{aligned} d\left(\sum_i(\alpha_i dx_i + \beta_i d\eta_i)\right) &= \sum_{i,j}\left(\frac{\partial \alpha_i}{\partial \eta_j} - \frac{\partial \beta_i}{\partial x_j}\right) d\eta_j \wedge dx_i \\ &+ \sum_{i,j}\left(\frac{\partial \alpha_i}{\partial x_j}\right) dx_j \wedge dx_i + \sum_{i,j}\left(\frac{\partial \beta_i}{\partial \eta_j}\right) d\eta_j \wedge d\eta_i. \end{aligned} \tag{3.2}$$

Let us denote $D = \dfrac{\partial(p \circ f)}{\partial y}$ and $C = \dfrac{\partial(p \circ f)}{\partial x}$. Since df is symplectic, $\dfrac{\partial(q \circ f^{-1})}{\partial \xi}$ is the transpose D^{T} of D. Moreover, differentiating $\alpha(x, p \circ f(x, y)) = y$ and $\beta(q \circ f^{-1}(\xi, \eta), \eta) = \xi$, we obtain

$$\frac{\partial \alpha}{\partial \eta} = D^{-1} \quad \text{and} \quad \frac{\partial \beta}{\partial x} = (D^{\mathrm{T}})^{-1} = (D^{-1})^{\mathrm{T}}.$$

This implies that the first term on the right-hand side of (3.2) vanishes. From $\alpha(x, p \circ f(x, y)) = y$, we also get $\dfrac{\partial \alpha}{\partial x} + \dfrac{\partial \alpha}{\partial \eta} C = 0$, hence $\dfrac{\partial \alpha}{\partial x} = -D^{-1}C$. But since df^{-1} is symplectic, we have $DC^{\mathrm{T}} - CD^{\mathrm{T}} = 0$, hence

$(D^{-1}C)^{\mathrm{T}} - D^{-1}C = 0$. It follows that $\dfrac{\partial\alpha}{\partial x}$ is symmetric and therefore that the second term on the right-hand side of (3.2) vanishes. The third term also vanishes for similar reasons and the form $\sum_i(\alpha_i dx_i + \beta_i d\eta_i)$ is closed.

As a consequence (of the Poincaré lemma for currents, for example), $\sum_i(\alpha_i dx_i + \beta_i d\eta_i)$ is exact. Since it is a Lipschitz one-form, it has to be the differential of a $C^{1,1}$ function σ. Now, since f is compactly supported, $\alpha(x,\eta) = x$ and $\beta(x,\eta) = \eta$ out of a compact set, thus, up to a constant shift, $\sigma(\eta,\xi) = \langle x,\eta\rangle$ out of a compact set. Then, it is easily checked that the function $S : \mathbb{R}^{2d} \to \mathbb{R}$, $(x,\eta) \mapsto \sigma(x,\eta) - \langle x,\eta\rangle$ satisfies the relation (3.1). $\square$

The next step of the proof of the inclusion $\mathrm{Bilip}(\mathbb{R}^{2d},\omega_0) \subset G$ is to use the generating functions to show that any element $f \in \mathcal{V}$ belongs to G where $\mathcal{V}$ is a neighborhood like in the previous lemma.

Let $f \in \mathcal{V}$ and $S = \Psi(f)$ its generating function. Let us approximate S by convolution as follows. Let χ be a smooth non-negative function, defined on $\mathbb{R}^{2d}$, whose support is contained in a disk centered in 0 and with integral equal to 1. For any positive integer k, we set $\chi_k = k^{2d}\chi(\frac{\cdot}{k})$. Then, it is well known that the sequence of smooth functions (S_k) defined by

$$S_k(x,\eta) = \chi_k * S(x,\eta) = \int_{\mathbb{R}^{2d}} S(x-u,\eta-v)\chi_k(u,v)\,du\,dv,$$

converges in the $C^{1,1}$ sense to S as k goes to infinity. Moreover, there exists a compact set that contains the supports of every S_k.

For k large enough, S_k belongs to the open set $\Psi(\mathcal{V})$ so that we can set $f_k = \Psi^{-1}(S_k)$. Now remark that for t small enough, the conjugation $\mu_t \circ f \circ \mu_t^{-1}$ of f by the Liouville flow $\mu_t : x \mapsto e^{t/2}x$ is in $\mathcal{V}$ and is associated to the generating function $S_t(x,\eta) = e^t S(e^{-t/2}x, e^{-t/2}\eta)$. Let us denote $f_k^t = \mu_t \circ f_k \circ \mu_t^{-1}$ and $S_k^t(x,\eta) = e^t S_k(e^{-t/2}x, e^{-t/2}\eta) = \Psi(f_k^t)$.

The path $t \mapsto S_k^t$ is a smooth path of smooth generating functions. Therefore, $t \mapsto f_k^t$ is a smooth Hamiltonian isotopy (starting at f). By Lemma 3.1, since S_k^t converges $C^{1,1}$ to S_t, f_k^t converges in the Lipschitz sense hence C^0 to $\mu_t \circ f \circ \mu_t^{-1}$. Now, let H_k^t denote the Hamiltonian function generating the isotopy f_k^t. According to the classical Hamilton–Jacobi equation (see, e.g., [**10**, p. 283]) for any time t and any point $x, y \in \mathbb{R}^{2d}$,

$$H_k^t(x,y) = \frac{\partial S_k^t}{\partial t}(x, p \circ \Psi^{-1}(S_k^t)(x,y)).$$

This implies that H_k^t also converges uniformly. As a consequence, the isotopy $t \mapsto \mu_t \circ f \circ \mu_t^{-1}$ is a C^0-Hamiltonian isotopy starting at f, and $t \mapsto [\mu_t, f]$ is a C^0-Hamiltonian isotopy, starting at Id.

We have proved $\mathcal{V} \subset G$. Let us now finish the proof of the inclusion $\mathrm{Bilip}(\mathbb{R}^{2d},\omega_0) \subset G$. One consequence of Lemma 3.1 is that $\mathrm{Bilip}(\mathbb{R}^{2d},\omega_0)$ is locally arcwise connected. Since it is connected, it is also arcwise connected. As a consequence, any element f in $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ can be linked to the identity by a continuous path. Cutting this path into sufficiently small pieces, f can be written as a composition of elements in a neighborhood of the identity. It follows that any element in $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ is a product of elements in G. Thus, any element in $\mathrm{Bilip}(\mathbb{R}^{2d}, \omega_0)$ is in G.

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