SIMPLICITY OF Homeo($\mathbb{D}^2, \partial\mathbb{D}^2,$ Area) AND FRAGMENTATION OF SYMPLECTIC DIFFEOMORPHISMS

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In 1980, Albert Fathi asked whether the group of area-preserving homeomorphisms of the 2-disc that are the identity near the boundary is a simple group. In this paper, we show that the simplicity of this group is equivalent to the following fragmentation property in the group of compactly supported, area preserving diffeomorphisms of the plane: *there exists a constant m such that every element supported on a disc D is the product of at most m elements supported on topological discs whose area are half the area of D.*

En 1980, Albert Fathi pose la question de la simplicité du groupe des homéomorphismes du disque qui préservent l'aire et sont l'identité près du bord. Dans cet article, nous montrons que la simplicité de ce groupe est équivalente à une propriété de fragmentation dans le groupe des difféomorphismes du plan, préservant l'aire et à support compact, à savoir : *il existe une constante m telle que tout élément à support dans un disque D est le produit d'au plus m éléments dont les supports sont inclus dans des disques topologiques dont l'aire est la moitié de celle de D.*

0. Introduction

This paper is concerned with the algebraic study of the group

$$G = \text{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \text{Area})$$

of area-preserving homeomorphisms of the 2-disc that are the identity near the boundary. The central open question is the following.

Question 0.1 ([Fa80]). *Is G a simple group?*

The study of the simplicity of groups of homeomorphisms goes back as far as 1935. Indeed in the famous Scottish Book [**SB57**], Ulam asked if the identity component in the group of homeomorphisms of the n-sphere is a simple

group. This question was answered in the affirmative by Anderson and Fisher in the late 1950s [**An58**, **Fi60**]. In the 1970s lots of (smooth) transformation groups were studied by Epstein, Herman, Thurston, Mather, Banyaga, and proved to be simple (see the books [**Ba97**] or [**Bo08**]). Let us give some details on the group $G^{\mathrm{diff}} = \mathrm{Diffeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \mathrm{Area})$, which is the smooth analog of our group G. This group is not simple, since there exists a morphism from G^{diff} to $\mathbb{R}$, called the *Calabi invariant*. But Banyaga proved that the kernel of the Calabi invariant coincides with the subgroup $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$ generated by commutators, and is a simple group. Thus the normal subgroups of G^{diff} are exactly the inverse images of the subgroups of $\mathbb{R}$ under the Calabi morphism. The analog of Question 0.1 is also solved in higher dimensions. Indeed Fathi proved the simplicity of the group of volume-preserving homeomorphisms of the n-ball which are the identity near the boundary, when $n \geq 3$. However Question 0.1 remains unsolved so far.

Actually, some normal subgroups of G have been defined by Ghys ([**Gh07**], see [**Bo08**]), and by Oh and Müller [**MO07**]. But so far no one has been able to prove that these are proper subgroups: they might turn out to be equal to G. In this text, I propose to define still another family of normal subgroups $\{N_\varphi\}$ of G. I have not been able to prove that these subgroups are proper either, but we can prove that they are good candidates.

Theorem 0.1. *If some normal subgroup of the family $\{N_\varphi\}$ is equal to G, then G is simple.*

The present work has its origin in Fathi's proof of the simplicity in higher dimensions. Fathi's argument has two steps. The first step is a fragmentation result: any element of the group can be written as a product of two elements, each of which is supported on a topological ball whose volume is $\frac{3}{4}$ of the total volume. The second step shows how this fragmentation property, let us call it (P_1), implies the perfectness (and simplicity) of the group. While the second step is still valid in dimension 2, the first one fails. In the sequel we propose to generalize the fragmentation property (P_1) by considering a family of fragmentation properties (P_ρ) depending on the parameter $\rho \in (0, 1]$ (a precise definition is provided in Section 1). A straightforward generalization of Fathi's second step will prove that if property (P_ρ) holds for some ρ, then G is simple (Lemma 3.1 below). On the other hand, we notice that if none of properties (P_ρ) holds, then the subgroups N_φ are proper, and thus G is not simple (Lemma 2.1). Thus we see firstly that Theorem 0.1 holds, and secondly that Question 0.1 is translated into a fragmentation problem, namely the existence of some ρ such that property (P_ρ) holds. Christian Bonatti drew my attention to the possibility of formulating this fragmentation problem in terms of a single property (P_0). This property, which may be seen as the limit of properties (P_ρ) as ρ tends to

zero, is the following: *there exists a constant m such that any homeomorphism of the plane, supported on a disc having area equal to one, is the composition of m homeomorphisms supported on some topological discs having area equal to one half.* This discussion is summarized by the following statement.

Theorem 0.2. *The following properties are equivalent:*

(1) *the group G is simple,*
(2) *there exists some $\rho \in (0, 1]$ such that property (P_ρ) holds,*
(3) *property (P_0) holds.*

Furthermore, we will prove that the simplicity of G is also equivalent to the similar fragmentation property on the smooth subgroup G^{diff} (see Lemma 4.1 and Theorem 4.1 in Section 4 below). We will see in Section 6 that Entov–Polterovich quasi-morphisms, coming from Floer homology, implies that the fragmentation property (P_ρ) do not hold for $\rho \in (\frac{1}{2}, 1]$. Whether it holds or not for $\rho \in [0, \frac{1}{2}]$ remains an open question.

The definitions and precise statements are given in Section 1, as well as the links between properties (P_0) and (P_ρ) for $\rho > 0$. The proofs of Theorem 0.1 and 0.2 are given in Sections 2 and 3. Sections 4 and 5 provide the link with diffeomorphisms. Some more remarks, in particular the connection with other surfaces, are mentioned in Section 7. Sections 5–7 are independent.

1. The fragmentation norms

In the whole text, the disc $\mathbb{D}^2$ is endowed with the normalized Lebesgue measure, denoted by Area, so that $\text{Area}(\mathbb{D}^2) = 1$. The group G is endowed with the topology of uniform convergence (also called the C^0 topology), that turns it into a topological group. We recall that G is arcwise connected: an elementary proof is provided by the famous Alexander trick [**Al23**]. We will use the term *topological disc* to denote any image of a euclidean closed disc under an element of the group G. As a consequence of the classical theorems by Schönflies and Oxtoby–Ulam, any Jordan curve of null area bounds a topological disc (see [**OU41**]). Remember that the *support* of some $g \in G$ is the closure of the set of non-fixed points. For any topological disc D, denote by G_D the subgroup of G consisting of the elements whose support is included in the interior of D. Then each group G_D is isomorphic to G, as shown by the following "re-scaling" process. Let $\Phi \in G$ be such that $D = \Phi^{-1}(D_0)$ where D_0 is a euclidean disc. Then the map $g \mapsto \Phi g \Phi^{-1}$ provides an isomorphism between the groups G_{D_0} and G_D. We may now choose a homothecy Ψ that sends the whole disc $\mathbb{D}^2$ onto D_0, and similarly get an isomorphism $g \mapsto \Psi g \Psi^{-1}$ between G and G_{D_0}.

1.1. Definition of the fragmentation metrics. Let g be any element of G. We define the *size* of g as follows:

$$\mathrm{Size}(g) = \inf\{\mathrm{Area}(D), D \text{ is a topological disc that contains}$$
$$\text{the support of } g\}.$$

Let us emphasize the importance of the word *disc*: an element g which is supported on an annulus of small area surrounding a disc of large area has a large size. Also note that if g has size less than the area of some disc D, then g is conjugate to an element supported in D.

The following proposition says that the group G is generated by elements of arbitrarily small size. It is an immediate consequence of Lemma 6.5 in [**Fa80**] (where the size is replaced by the diameter).

Proposition 1.1 (Fathi). *Let $g \in G$, and $\rho \in (0,1]$. Then there exists some positive integer m, and elements $g_1, \dots g_m \in G$ of size less than ρ, such that*

$$g = g_m \cdots g_1.$$

We now define the family of "fragmentation norms."[1] For any element $g \in G$ and any $\rho \in (0,1]$, we consider the least integer m such that g is equal to the product of m elements of size less than ρ. This number is called the ρ-*norm* of g and is denoted by $||g||_\rho$. The following properties are obvious.

Proposition 1.2.

$$||hgh^{-1}||_\rho = ||g||_\rho, \quad ||g^{-1}||_\rho = ||g||_\rho, \quad ||g_1 g_2||_\rho \le ||g_1||_\rho + ||g_2||_\rho.$$

As a consequence, the formula

$$d_\rho(g_1, g_2) = ||g_1 g_2^{-1}||_\rho$$

defines a bi-invariant metric on G.

1.2. The normal subgroups N_φ. Given some element $g \in G$, we consider the ρ-norm of g as a function of the size ρ:

$$\rho \mapsto ||g||_\rho,$$

and call it the *complexity profile* of g. Let $\varphi : (0,1] \to \mathbb{R}^+$ be any non-increasing function. We define the subset N_φ containing those elements of G whose complexity profile is essentially bounded by φ:

$$N_\varphi = \{g \in G, ||g||_\rho = O(\varphi(\rho))\},$$

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[1]The definition of the fragmentation norm is not new, see Example 1.24 in [**BIP07**].

where the notation $\psi(\rho) = O(\varphi(\rho))$ means that there exists some $K > 0$ such that $\psi(\rho) < K\varphi(\rho)$ for every small enough ρ. The following is an immediate consequence of Proposition 1.2.

Proposition 1.3. *For any non-increasing function $\varphi : (0,1] \to \mathbb{R}^+$, the set N_φ is a normal subgroup of G.*

The reader who wants some examples where we can estimate the complexity profile may jump to Section 5, where we will see that commutators of diffeomorphisms have a profile equivalent to the function $\varphi_0 : \rho \mapsto \rho^{-1}$. This will imply that N_{φ_0} is the smallest non-trivial subgroup of our family $\{N_\varphi\}$.

1.3. The fragmentation properties (P_ρ). Let $\rho \in (0,1]$. We now define our fragmentation property (P_ρ) by asking for a uniform bound in the fragmentation of elements of size less than ρ into elements of a smaller given size.

(P_ρ) There exists some number $s \in (0,\rho)$, and some positive integer m, such that any $g \in G$ of size less than ρ satisfies $||g||_s \leq m$.

Here are some easy remarks. Let us denote by $P(\rho, s)$ the property that there exists a bound m with $||g||_s \leq m$ for every element g of size less than ρ. Fix some $\rho \in (0,1]$ and some ratio $k \in (0,1)$. Assume that property $P(\rho, k\rho)$ holds. Then by re-scaling we get that property $P(\rho', k\rho')$ also holds for any $\rho' < \rho$ (with the same bound m). In particular, we can iterate the fragmentation to get, for every positive n, property $P(\rho, k^n\rho)$ (with the bound m^n). This shows that property $P(\rho, s)$ implies property $P(\rho, s')$ for every $s' < s$. The converse is clearly true, so that property $P(\rho, s)$ depends only on ρ and not on s. In particular, we see that property (P_ρ) is equivalent to the existence of a number m such that every g of size less than ρ satisfies

$$||g||_{\frac{\rho}{2}} \leq m.$$

Also note that property (P_{ρ_0}) implies property (P_{ρ_1}) if $\rho_1 < \rho_0$ (again by re-scaling). Thus property (P_ρ) is more and more likely to hold as ρ decreases from 1 to 0.

1.4. The fragmentation property (P_0). In this paragraph we introduce the fragmentation property (P_0), and prove that it is equivalent to the existence of some $\rho > 0$ such that property (P_ρ) holds. Consider, just for the duration of this section, the group

$$\mathrm{Homeo_c}(\mathbb{R}^2, \mathrm{Area})$$

of compactly supported, area-preserving homeomorphisms of the plane. Any image of a euclidean closed disc under some element of this bigger group will

again be called a topological disc. We also define the size of an element of
the group as in G. Property (P_0) is as follows.[2]

(P_0) There exists some positive integer m such that any $g \in \mathrm{Homeo_c}$
$(\mathbb{R}^2, \mathrm{Area})$ of size less than 1 is the composition of at most m ele-
ments of $\mathrm{Homeo_c}(\mathbb{R}^2, \mathrm{Area})$ of size less than $\frac{1}{2}$.

Since each piece of the fragmentation provided by property (P_0) is supported
on a disc with area $\frac{1}{2}$, the union of the supports has area at most $\frac{m}{2}$, but
this gives no bound on the area of a topological disc containing this union.
However, if the union of the supports surround a region with big area, then
we may find a new fragmentation by "bursting the bubble," i.e., conjugating
the situation by a map that contracts the areas of the surrounded regions
and preserves the area everywhere else. This is the key observation, due to
Christian Bonatti, to the following lemma.

Lemma 1.4. *Property (P_0) holds if and only if there exists some $\rho \in (0, 1]$
such that property (P_ρ) holds.*

Proof. Let us prove the easy part. Suppose (P_ρ) holds for some $\rho > 0$,
let m be a bound for $||g||_{\frac{\rho}{2}}$ for those $g \in G$ of size less than ρ. Let $g \in$
$\mathrm{Homeo_c}(\mathbb{R}^2, \mathrm{Area})$ of size less than 1. Choose some element of the group
that sends the support of g into the euclidean unit disc $\mathbb{D}^2$, and compose
it with the homothecy that sends $\mathbb{D}^2$ onto a disc of area ρ included in $\mathbb{D}^2$;
we denote by Ψ the resulting map. Then $\Psi g \Psi^{-1}$ is an element of G of size
less than ρ. According to hypothesis (P_ρ), we may write this element as a
composition of m elements of G of size less than $\frac{\rho}{2}$. We may conjugate these
elements by Ψ^{-1} and take the composition to get a fragmentation of g into
m elements of size $\frac{1}{2}$. Thus (P_0) holds.

Now assume that (P_0) holds, and let $m \geq 2$ be given by this property.
We will prove that property (P_ρ) holds for $\rho = \frac{2}{m}$. Consider, in the plane, a
euclidean disc D of area $\frac{1}{\rho}$. By the same re-scaling trick as before, it suffices
to prove that any $g \in \mathrm{Homeo_c}(\mathbb{R}^2, \mathrm{Area})$ with size less than 1 and supported
in the interior of D may be fragmented as a product of m elements of
$\mathrm{Homeo_c}(\mathbb{R}^2, \mathrm{Area})$ with size less than one half and supported in the interior
of D. Property (P_0) provides us with a fragmentation $g = g'_m \circ \cdots \circ g'_1$ by
elements of size less than one half, but maybe not supported in D. Now
comes the "bursting the bubbles" trick. Let D' be a topological disc whose

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[2]Note that this property can be expressed in terms of another fragmentation norm.
Namely, we can identify the disc $\mathbb{D}^2$ with a euclidean disc in the plane of area 1, in such a
way that the Lebesgue measures correspond. Thus the group G is identified to a subgroup
of $\mathrm{Homeo_c}(\mathbb{R}^2, \mathrm{Area})$. For any element $g \in G$, denote by $||g||_{1/2}^\infty$ the smallest number N
such that there exist elements $g_1, \ldots, g_N$ of the group $\mathrm{Homeo_c}(\mathbb{R}^2, \mathrm{Area})$ of size less than
$1/2$, such that $g = g_N \cdots g_1$. Then property (P_0) says that the group G, equipped with
this norm, is bounded.

interior contains all the supports of the g_i''s. The union of the supports has area less than $\frac{1}{\rho}$. Thus we may find some topological discs $K_1', \ldots K_\ell'$, included in the interior of D', that are pairwise disjoint and disjoint from the supports of the g_i''s, such that

$$\text{Area}\left(D' \setminus \bigcup_{j=1}^{\ell} K_j'\right) < \frac{1}{\rho}.$$

Denote by D_0 the support of our original map g. Note that D_0 is included in the union of the supports of the g_i''s, thus it is disjoint from the K_j''s. Since D has area $\frac{1}{\rho}$, the previous inequality ensures the existence of some pairwise disjoint discs $K_1, \ldots K_\ell$ in the interior of D, disjoint from D_0, such that

$$\text{Area}\left(D \setminus \bigcup_{j=1}^{\ell} K_j\right) = \text{Area}\left(D' \setminus \bigcup_{j=1}^{\ell} K_j'\right).$$

Using Schönflies and Oxtoby–Ulam theorems, we can construct a homeomorphism Ψ of the plane satisfying the following properties:

(1) Ψ is the identity on D_0,
(2) $\Psi(D') = D$, $\Psi(K_j') = K_j$ for each j, and
(3) the restriction of Ψ to the set $D' \setminus \cup_{j=1}^{\ell} K_j'$ preserves the area.[3]

The first item shows that $\Psi g \Psi^{-1} = g$. Now for each i we define $g_i = \Psi g_i' \Psi^{-1}$. Then the second item guarantees that the g_i's are supported in D, and the third item entails that they preserve area and have size less than one half. The product of the g_i's is equal to g, which provides the desired fragmentation. $\qquad\square$

2. Simplicity implies fragmentation

Lemma 2.1. *Assume that none of the properties $(P_\rho), \rho \in (0,1]$ holds. Let $\varphi : (0,1] \to \mathbb{R}^+$ be any function. Then the normal subgroup N_φ is proper, i.e., it is not equal to G. In this case the group G is not simple.*

If we consider any element $f \neq \text{Id}$ in G, and the function $\varphi_f : \rho \mapsto ||f||_\rho$, then the normal subgroup N_{φ_f} contains f and thus is not equal to $\{\text{Id}\}$. Hence the non-simplicity of G will be a consequence of the non-triviality of the subgroups N_φ.

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[3] Having in mind the smooth case (Lemma 4.1 below), we notice that we may further demand that the map Ψ is a C^∞-diffeomorphism on the interior of $D' \setminus \cup_{j=1}^{\ell} K_j'$. Actually, we may even choose the sets D', K_j and K_j' to be smooth discs, and then the map Ψ may be chosen to be a C^∞-diffeomorphism of the plane.

Proof of Lemma 2.1. According to the easy remarks following the definition of property (P_ρ), the hypothesis of the lemma reads the following way:

($\star$) for every $\rho \in (0, 1]$ and every positive integer m there exists some element g of size less than ρ such that $||g||_{\frac{\rho}{2}} > m$.

We fix any function $\varphi : (0, 1] \to \mathbb{R}^+$, and we will construct some element g in G that does not belong to N_φ. Let us define $D_0 = \mathbb{D}^2$. We pick two sequences of discs $(C_i)_{i \geq 1}$ and $(D_i)_{i \geq 1}$ converging to a point, such that for every i (see Figure 1),

- C_i and D_i are disjoint and included in D_{i-1}, and
- the area of D_i is less than half the area of C_i.

We denote the area of C_i by ρ_i. We will construct a sequence $(g_i)_{i \geq 1}$, with each g_i supported in the interior of C_i, and then g will be defined as the (infinite) product of the g_i's. Note that since the discs C_i's are pairwise disjoint this product has a meaning, and since the sequence (C_i) converges to a point it actually defines an element of G. Since all the g_j's with $j > i$ will be supported in the interior of the disc D_i whose area is less than $\rho_i/2$ we will get

$$(2.1) \qquad ||g||_{\frac{\rho_i}{2}} \geq ||g_i \dots g_1||_{\frac{\rho_i}{2}} - 1.$$

The sequence (g_i) is constructed by induction. Assume $g_1, \dots, g_{i-1}$ have been constructed. Using hypothesis ($\star$), we may choose g_i supported on C_i such that $||g_i||_{\frac{\rho_i}{2}}$ is arbitrarily high, more precisely we demand the following inequality:

$$(2.2) \qquad ||g_i||_{\frac{\rho_i}{2}} \geq \frac{1}{\rho_i} \varphi\left(\frac{\rho_i}{2}\right) \quad + \quad ||g_{i-1} \dots g_1||_{\frac{\rho_i}{2}} \quad + \quad 1.$$
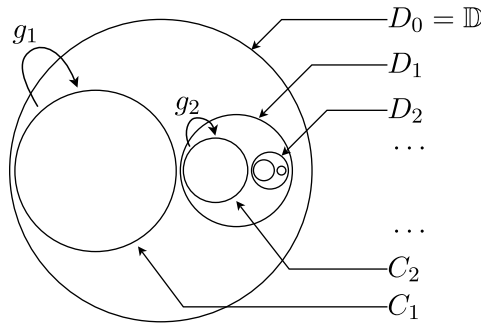
Figure 1. Construction of g.

Using inequality (2.1), the triangular inequality and inequality (2.2) we get

$$
\begin{aligned}
||g||_{\frac{\rho_i}{2}} &\geq ||g_i \ldots g_1||_{\frac{\rho_i}{2}} - 1 \\
&\geq ||g_i||_{\frac{\rho_i}{2}} \quad - \quad ||g_{i-1} \ldots g_1||_{\frac{\rho_i}{2}} - 1 \\
&\geq \frac{1}{\rho_i} \varphi\left(\frac{\rho_i}{2}\right).
\end{aligned}
$$

This proves that the complexity profile of g is not equal to $O(\varphi)$. In other words g does not belong to N_φ. $\qquad\square$

3. Fragmentation implies simplicity

Lemma 3.1. *Assume that property (P_ρ) holds for some $\rho \in (0,1]$. Then G is simple.*

This lemma is just a slight generalization of Fathi's argument showing that, under property (P_1), the group G is perfect: any element decomposes as a product of commutators. Then perfectness implies simplicity: this is due to "Thurston's trick," for completeness the argument is included in the proof below.

Proof. We assume that there exists a number $\rho \in (0,1]$ and a positive integer m such that any element of size less than ρ may be written as the product of m elements of size less than $\frac{\rho}{2}$.

Let C_1 be a small disc. By usual fragmentation (Proposition 1.1), any element of G is a product of elements supported in a disc of area less than that of C_1, and any such element is conjugate to an element supported in the interior of C_1. Thus to prove perfectness it is enough to consider some element g supported in the interior of C_1 and to prove that g is a product of commutators.

Let us first prove that such a g is a product of two commutators when considered in the group $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$, that is, let us forget for a while about the area (this is a "pedagogical" step). Choose two sequences of discs $(C_i)_{i \geq 1}$ and $(D_i)_{i \geq 1}$ converging to a point, such that (see Figure 2)

- the interior of D_i contains both C_i and C_{i+1},
- the C_i's are pairwise disjoint, and
- the D_{2i}'s (resp. the D_{2i+1}'s) are pairwise disjoint.

For any $i \geq 1$ choose some $h_i \in \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$, supported on D_i, that sends C_i onto C_{i+1}. We let $g_1 := g$, thus g_1 is supported on C_1, and define inductively $g_{i+1} := h_i g_i h_i^{-1}$; thus g_i is a "copy" of g, supported on C_i, and the g_i's are pairwise commuting. Let

$$
K := g_2 g_3^{-1} g_4 g_5^{-1} \cdots, \quad K' := g_1 g_2^{-1} g_3 g_4^{-1} \cdots
$$
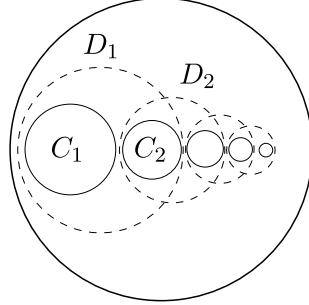
Figure 2. The sequences $(C_i)_{i\geq 1}$ and $(D_i)_{i\geq 1}$.

so that $KK' = K'K = g$. The map $K = [g_2, h_2][g_4, h_4]\cdots$ may be seen as an infinite product of commutators, but we need a finite product. Now define

$$G := g_2 g_4 \ldots, \quad H := h_2 h_4 \cdots, \quad G' := g_1 g_3 \cdots, \quad H' := h_1 h_3 \cdots$$

and observe that $K = [G, H]$ and $K' = [G', H']$: indeed these equalities may be checked independently on each disc D_i. Thus $g = [G, H][G', H']$ is a product of two commutators in $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$.

Now let us take care of the area. We will use sequences (C_i) and (D_i) as before, and we will get around the impossibility of shrinking C_i onto C_{i+1} inside the group G by using the fragmentation hypothesis. We may assume, for every i, the equality

$$\mathrm{Area}(C_{i+1}) = \tfrac{1}{2}\mathrm{Area}(C_i).$$

Moreover, by fragmentation, we may assume this time that g is supported in the interior of a disc $C_1' \subset C_1$ of area $\rho\,\mathrm{Area}(C_1)$. We use the fragmentation hypothesis re-scaled on C_1 to write

$$g = f_{1,1} \ldots f_{1,m}$$

(see Figure 3) with each $f_{1,j}$ supported in the interior of a topological disc included in C_1 and whose area is

$$\frac{\rho}{2}\mathrm{Area}(C_1) = \rho\,\mathrm{Area}(C_2).$$

We choose a disc $C_2' \subset C_2$ whose area also equals $\rho\,\mathrm{Area}(C_2)$ and, for each $j = 1, \ldots, m$, some $h_{1,j} \in G$ supported on D_1 and sending the support of $f_{1,j}$ inside C_2'. We define

$$g_2 := \prod_{j=1,\ldots,m} h_{1,j} f_{1,j} h_{1,j}^{-1}$$

which is supported on C_2' (see Figure 3). We apply recursively the (re-scaled) fragmentation hypothesis to get a sequence $(g_i)_{i\geq 1}$ with each g_i supported in the interior of a disc $C_i' \subset C_i$ having area $\rho\,\mathrm{Area}(C_i)$ and sequences
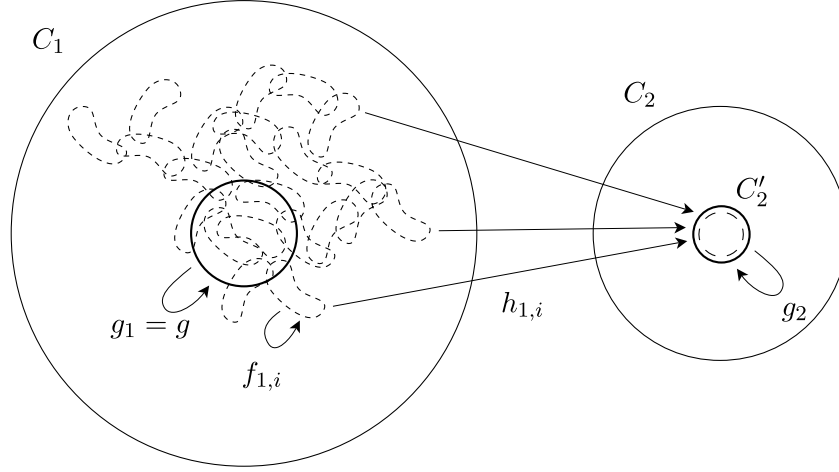
Figure 3. Fragment and push every piece inside the small disc....

$(f_{i,j})_{i\geq 1, j=1,\ldots,m}$ and $(h_{i,j})_{i\geq 1, j=1,\ldots,m}$ with $f_{i,j}$ supported on C_i and $h_{i,j}$ supported on D_i, such that

$$g_i = \prod_{j=1,\ldots,m} f_{i,j} \text{ and } g_{i+1} = \prod_{j=1,\ldots,m} h_{i,j} f_{i,j} h_{i,j}^{-1}.$$

Obviously g_i and g_{i+1} are equal up to a product of commutators *whose number of terms depends only on m*. More precisely, we have

$$g_i g_{i+1}^{-1} = \prod_{j=1,\ldots,m} f_{i,j} \prod_{j=m,\ldots,1} h_{i,j} f_{i,j}^{-1} h_{i,j}^{-1}$$

$$= [f_{i,1}, P] \left(\prod_{j=2,\ldots,m} f_{i,j} \prod_{j=m,\ldots,2} h_{i,j} f_{i,j}^{-1} h_{i,j}^{-1} \right) [f_{i,1}, h_{i,1}],$$

where P is equal to the term between parentheses, and we see recursively that $g_i g_{i+1}^{-1}$ is a product of $2m$ commutators of elements supported in D_i; we write

$$g_i g_{i+1}^{-1} = \prod_{j=1,\ldots,2m} [s_{i,j}, t_{i,j}].$$

It remains to define the infinite commutative products

$$K := g_2 g_3^{-1} g_4 g_5^{-1} \ldots, \quad K' := g_1 g_2^{-1} g_3 g_4^{-1} \ldots,$$

$$S_j := s_{2,j} s_{4,j} \ldots, \quad T_j := t_{2,j} t_{4,j} \ldots, \quad S_j' := s_{1,j} s_{3,j} \ldots, \quad T_j' := t_{1,j} t_{3,j} \ldots$$

and to check that

$$K = \prod_{j=1,\ldots 2m} [S_j, T_j], \quad K' = \prod_{j=1,\ldots 2m} [S_j', T_j'], \quad \text{and } g = KK'$$

is a product of $4m$ commutators. This proves that G is perfect.

Let us recall briefly, according to Thurston, how perfectness implies simplicity. Let D be a disc and $g, h \in G$ be such that the discs $D, g(D), h(D)$ are pairwise disjoint. Let $u, v \in G$ be supported in D. In this situation the identity

$$[u, v] = [[u, g], [v, h]]$$

may easily be checked, and shows that $[u, v]$ belongs to the normal subgroup generated by g. Now given any $g \neq \mathrm{Id}$ in G, one can find an $h \in G$ and a disc D such that the above situation takes place. If G is perfect then so is the isomorphic group G_D, hence every f supported in D is a product of commutators supported in D, and by the above equality such an f belongs to the normal subgroup generated by g. By fragmentation this subgroup is thus equal to G. This proves that G is simple, and completes the proof of the lemma. $\square$

4. Fragmentation of diffeomorphisms

Here we further translate Question 0.1 into the diffeomorphisms subgroup G^{diff}.

Let $G^{\mathrm{diff}} = \mathrm{Diffeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \mathrm{Area})$ be the group of elements of G that are C^∞-diffeomorphisms. Note that for every topological disc D the group of elements supported in the interior of D,

$$G_D^{\mathrm{diff}} := G^{\mathrm{diff}} \cap G_D,$$

is isomorphic to G^{diff}, even when D is not smooth: indeed we may use Riemann conformal mapping theorem and Moser's lemma to find a smooth diffeomorphism Φ between the interiors of $\mathbb{D}^2$ and D with constant Jacobian, and the conjugacy by Φ provides an isomorphism (see [**GS79**] for the non-compact version of Moser's lemma). As in the continuous case, we define the ρ-norm $||g||_\rho^{\mathrm{diff}}$ of any element $g \in G^{\mathrm{diff}}$ as the minimum number m of elements $g_1, \ldots, g_m$ of G^{diff}, having size less than ρ, whose composition is equal to g. The fragmentation properties (P_ρ^{diff}) are defined as in the continuous setting.

(P_ρ^{diff}) (for $\rho \in (0, 1]$) There exists some number $s \in (0, \rho)$, and some positive integer m, such that any $g \in G^{\mathrm{diff}}$ of size less than ρ satisfies $||g||_s^{\mathrm{diff}} \leq m$.

(P_0^{diff}) There exists some positive integer m such that any $g \in \mathrm{Diffeo}_\mathrm{c}(\mathbb{R}^2, \mathrm{Area})$ of size less than 1 is the composition of at most m elements of $\mathrm{Diffeo}_\mathrm{c}(\mathbb{R}^2, \mathrm{Area})$ of size less than $\frac{1}{2}$.

Here $\mathrm{Diffeo}_\mathrm{c}(\mathbb{R}^2, \mathrm{Area})$ is the group of C^∞-diffeomorphisms of the plane that are compactly supported and preserve the area. The smooth version of Lemma 1.4 holds, with the same proof (using footnote 3).

Lemma 4.1. *Property* (P_0^{diff}) *holds if and only if there exists some* $\rho \in (0,1]$ *such that property* (P_ρ^{diff}) *holds.*

We now turn to the equivalence between fragmentation properties for homeomorphisms and diffeomorphisms.

Theorem 4.1. *For any* $\rho \in (0,1]$, *properties* (P_ρ) *and* (P_ρ^{diff}) *are equivalent.*

The proof of this equivalence requires two ingredients. The first one is the density of G^{diff} in G; this is a classical result, see for example [**Si**]. The second one is the uniformity of the fragmentation in G, and in G^{diff}, inside some C^0-neighborhood of the identity. This is provided by the following proposition, which is proved below.

Proposition 4.2. *For any* $\rho \in (0,1)$, *there exists a neighborhood* $\mathcal{V}_\rho$ *of the identity in G with the following properties.*

- *any g in $\mathcal{V}_\rho$ satisfies* $||g||_\rho \le \frac{2}{\rho}$; *and*
- *any g in $\mathcal{V}_\rho \cap G^{\mathrm{diff}}$ satisfies* $||g||_\rho^{\mathrm{diff}} \le \frac{2}{\rho}$.

As a consequence of this proposition we get the following comparison between the norms $||.||_\rho^{\mathrm{diff}}$ and $||.||_\rho$ on G^{diff}.

Corollary 4.3. *Any* $g \in G^{\mathrm{diff}}$ *satisfies*

$$||g||_\rho \le ||g||_\rho^{\mathrm{diff}} \le ||g||_\rho + \frac{2}{\rho}.$$

Proof. The first inequality is clear. To prove the second one consider some $g \in G^{\mathrm{diff}}$, and let $m = ||g||_\rho$. By definition there exists some elements $g_1, \ldots, g_m$ in G of size less than ρ such that $g = g_m \cdots g_1$. Since G^{diff} is C^0-dense in G, for any topological disc D the subgroup G_D^{diff} is also C^0-dense in G_D. Thus we can find elements $g'_1, \ldots, g'_m$ in G^{diff}, still having size less than ρ, whose product $g' = g'_m \cdots g'_1$ is a diffeomorphism arbitrarily C^0-close to g. According to the second item in Proposition 4.2 we get

$$||g'g^{-1}||_\rho^{\mathrm{diff}} \le \frac{2}{\rho}.$$

Since $||g'||_\rho^{\mathrm{diff}} \le m$ the triangular inequality gives $||g||_\rho^{\mathrm{diff}} \le m + \frac{2}{\rho}$, as wanted. $\square$

Proof of Theorem 4.1. Let us fix $\rho \in (0,1]$. The fact that (P_ρ) implies (P_ρ^{diff}) immediately follows from the corollary. We prove the converse implication. Assume that (P_ρ^{diff}) holds: there exists some $s \in (0,\rho)$ and $m > 0$ such that the quantity $||g'||_s^{\mathrm{diff}}$ is bounded by m on the elements g' of G^{diff} of size less than ρ. Choose any $g \in G$ of size less than ρ. According to the first item in Proposition 4.2, and using the density of G^{diff} in G, we get some $g' \in G^{\mathrm{diff}}$ having size less than ρ, and sufficiently close to g so that $||g'g^{-1}||_s \le \frac{2}{s}$. Since

(P_ρ^{diff}) holds we also have $||g'||_s^{\mathrm{diff}} \leq m$, and thus $||g'||_s \leq m$, from which we get $||g||_s \leq m + \frac{2}{s}$. Thus (P_ρ) also holds. $\qquad\qquad\qquad\qquad\qquad\square$

We now turn to the proof of Proposition 4.2. The classical proof of fragmentation for diffeomorphisms relies on the inverse mapping theorem and would only gives uniformity in a C^1-neighborhood of the identity (see [**Ba97**, **Bo08**]). Thus we will rather try to mimic the proof of the fragmentation for homeomorphisms.

Proof of Proposition 4.2. The proof of both items (uniformity of local fragmentation for homeomorphisms and diffeomorphisms) are very similar; we will only provide details for the diffeomorphisms case.

We choose an integer m bigger than $\frac{2}{\rho}$, and we cut the disc into m strips of area less than $\frac{\rho}{2}$: more precisely, we choose m topological discs $D_1, \ldots, D_m$ such that (see Figure 4)

(1) $\mathrm{Area}(D_i) \leq \frac{\rho}{2}$,
(2) $\mathbb{D}^2 = D_1 \cup \cdots \cup D_m$,
(3) $D_i \cap D_j = \emptyset$ if $|j - i| > 1$, and
(4) $D_i \cup D_{i+1} \cup \cdots \cup D_j$ is a topological disc for every $i \leq j$, and the intersection of $D_1 \cup \cdots \cup D_i$ and $D_i \cup \cdots \cup D_m$ with the boundary of $\mathbb{D}^2$ is non-empty and connected for every i.

Now we define the following set $\mathcal{V}_\rho$:

$$\mathcal{V}_\rho = \{g \in G \text{ such that } g(D_i) \cap D_j = \emptyset \text{ for every } i, j \text{ with } |j - i| > 1\}.$$

Note that, due to item 3, $\mathcal{V}_\rho$ is a C^0-neighborhood of the identity in G. We will prove that each element of $\mathcal{V}_\rho \cap G^{\mathrm{diff}}$ can be written as a product of $m - 1$ elements of G_ρ^{diff}.

Let $g \in \mathcal{V}_\rho \cap G^{\mathrm{diff}}$. By hypothesis D_1 and $g(D_1)$ are both disjoint from the topological disc $D_3 \cup \cdots \cup D_m$. By the classical Lemma 4.4 below, we can find $\Psi_1 \in G^{\mathrm{diff}}$ such that

- $\Psi_1 = g$ on some neighborhood of D_1, and
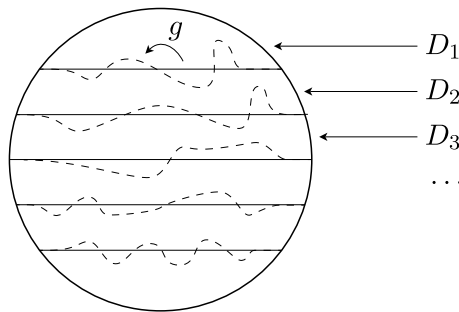
Figure 4. The discs D_i and the action of some g in $\mathcal{V}_\rho$.

- Ψ_1 is the identity on some neighborhood of $D_3 \cup \cdots \cup D_m$.

The diffeomorphism Ψ_1 is supported in the interior of the topological disc $D_1 \cup D_2$ whose area is less than or equal to ρ. Let $g_1 := \Psi_1^{-1} g$, thus g_1 is supported in the interior of $D_2 \cup \cdots \cup D_m$, and we easily check that this diffeomorphism is still in $\mathcal{V}_\rho$. In particular, $D_1 \cup D_2$ and its image under g_1 are both disjoint from $D_4 \cup \cdots \cup D_m$. We apply again the lemma to get some $\Psi_2 \in G$ such that

- $\Psi_2 = g_1$ on some neighborhood of $D_1 \cup D_2$, and
- Ψ_2 is the identity on some neighborhood of $D_4 \cup \cdots \cup D_m$.

Thus Ψ_2 is supported in the interior of $D_2 \cup D_3$. Let $g_2 := \Psi_2^{-1} g_1$; this diffeomorphism is in $\mathcal{V}_\rho$ and is supported in the interior of $D_3 \cup \cdots \cup D_m$. In the same way we construct diffeomorphisms $\Psi_1, \ldots, \Psi_{m-1}$, such that each Ψ_i is supported in the interior of $D_i \cup D_{i+1}$, and such that $g = \Psi_1 \circ \cdots \circ \Psi_{m-1}$. This completes the proof for the diffeomorphisms case. $\square$

Lemma 4.4. *Let D_1', D_2' be two disjoint topological discs in $\mathbb{D}^2$, and assume that the intersection of D_1' (resp. D_2') with the boundary of $\partial \mathbb{D}^2$ is non-empty and connected (and thus $\mathbb{D}^2 \setminus \mathrm{Int}(D_1' \cup D_2')$ is again a topological disc). Let $\Phi \in G^{\mathrm{diff}}$, and suppose that $\Phi(D_1')$ is disjoint from D_2'. Then there exists $\Psi \in G^{\mathrm{diff}}$ such that $\Psi = \Phi$ on some neighborhood of D_1' and $\Psi = \mathrm{Id}$ on some neighborhood of D_2'.*

Proof. By Smale's [**Sm59**] theorem and Moser's lemma (see, e.g., [**Ba97**] or [**Bo08**]), the group G is arcwise connected. Let $(\Phi_t)_{t \in [0,1]}$ be a smooth isotopy from the identity to Φ in G. It is easy to find another smooth isotopy $(g_t)_{t \in [0,1]}$ (that does not preserve area), supported in the interior of $\mathbb{D}^2$, such that

- for every t, $g_t(\Phi_t(D_1'))$ is disjoint from D_2', and
- $g_0 = g_1 = \mathrm{Id}$.

The isotopy $(\Phi_t') = (g_t \Phi_t)$ still goes from the identity to Φ. Consider the vector field tangent to this isotopy, and multiply it by some smooth function that is equal to 1 on some neighborhood of $\cup_t \Phi_t'(D_1')$ and vanishes on D_2'. By integrating this truncated vector field we get another isotopy (Ψ_t) such that

- on some neighborhood of D_1' we have $\Psi_t = g_t \Phi_t$ for every t, and in particular $\Psi_1 = \Phi$, and
- the support of Ψ_1 is disjoint from D_2'.

Thus Ψ_1 satisfies the conclusion of the lemma, except that it does not preserve the area. Let ω_0 be the Area form on $\mathbb{D}^2$, and ω_1 be the pre-image of ω_0 under Ψ_1. Then $\omega_1 = \omega_0$ on some neighborhood of $\partial \mathbb{D}^2 \cup D_1' \cup D_2'$. By Moser's lemma we may find some $\Psi_2 \in G$, whose support is disjoint from D_1' and D_2', and that sends ω_1 to ω_0. The diffeomorphism $\Psi = \Psi_1 \Psi_2^{-1}$ suits our needs. $\square$

Note that this lemma has a C^0-version, which is proved by replacing Smale's theorem by Alexander's trick, the truncation of vector fields by Schönflies's theorem, and Moser's lemma by Oxtoby–Ulam's theorem.

5. Profiles of diffeomorphisms

One can wonder what the complexity profile looks like for a diffeomorphism, both inside the group G^{diff} and inside the group G. The following proposition only partially solves this problem.

Proposition 5.1.

- *For any $g \in G^{\mathrm{diff}}$ we have $||g||_\rho^{\mathrm{diff}} = O(\frac{1}{\rho^2})$, and*
- *For any g in the commutator subgroup $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$ we have $||g||_\rho^{\mathrm{diff}} = O(\frac{1}{\rho})$.*

If the support of $g \in G$ has area A, then clearly for any ρ we need at least $\frac{A}{\rho}$ elements of G_ρ or G_ρ^{diff} to get g. Thus, according to the second point, the profile of any g in $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$ is bounded from above and below by multiples of the function $\varphi_0 : \rho \mapsto \frac{1}{\rho}$; and this holds both in G and G^{diff}. In particular, we see that N_{φ_0} is the smallest non-trivial subgroup of our family $\{N_\varphi\}$. I have not been able to decide whether the first point is optimal, nor whether $G^{\mathrm{diff}} \subset N_{\varphi_0}$ or not (if not, of course, then G is not simple). It might also happens that G^{diff} is included in N_{φ_0} but not in the analog smooth group $N_{\varphi_0}^{\mathrm{diff}}$.

We also notice that *every non-trivial normal subgroup of G contains the commutator subgroup $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$*, and thus the normal subgroup of G generated by $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$ is the only minimal non-trivial normal subgroup of G. This fact is an immediate consequence of Thurston's trick (see the last paragraph of Section 3) and Banyaga's theorem. Indeed let g be a non-trivial element in G, choose $h \in G$ and a disc D such that $D, g(D), h(D)$ are pairwise disjoint. Thurston's trick shows that the normal subgroup $N(g)$ generated by g in G contains some non-trivial commutator of diffeomorphisms, let us denote it by Φ. By Banyaga's theorem any element of $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$ is a product of conjugates (in G^{diff}) of Φ and Φ^{-1}, and thus $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$ is included in $N(g)$.

Proof of Proposition 5.1. Let $g \in G^{\mathrm{diff}}$, and fix some smooth isotopy $(g_t)_{t \in [0,1]}$ from the identity to g in G^{diff}. Let $M > 0$ be such that every trajectory of the isotopy has speed bounded from above by M.

We now fix some $\rho > 0$, and let m be the smallest integer such that $m \geq \frac{2}{\rho}$. We consider some discs $D_1, \ldots D_m$ as in the proof of Proposition 4.2; if we choose the D_i's to be horizontal slices, then for every i, j with $|i - j| > 1$ we get

$$d(D_i, D_j) > \frac{C}{m},$$

where d is the euclidean metric of the unit disc and C is some constant (maybe $C = \frac{\pi}{2}$). Due to the definition of M, within any interval of time less than $\frac{C}{mM}$, no point moves a distance more than $\frac{C}{m}$: for every $t, t' \in [0,1]$, for every $x \in \mathbb{D}^2$,

$$|t - t'| < \frac{C}{mM} \quad \implies \quad d(g_t(x), g_{t'}(x)) < \frac{C}{m}.$$

In particular, the topological disc $g_{t'}g_t^{-1}(D_i)$ remains disjoint from D_j for every $|i - j| > 1$; that is, the diffeomorphism $g_{t'}g_t^{-1}$ belongs to the neighborhood $\mathcal{V}_\rho$ defined in the proof of Proposition 4.2. Thus we can write g as the product of at most $\frac{mM}{C} + 1$ elements of $\mathcal{V}_\rho$,

$$g = g_1 = \left(g_1 g_{1-\frac{1}{k}}^{-1}\right)\left(g_{1-\frac{1}{k}}g_{1-\frac{2}{k}}^{-1}\right)\cdots\left(g_{\frac{1}{k}}g_0^{-1}\right)$$

(where k is the integer part of $\frac{mM}{C} + 1$). Each element in $\mathcal{V}_\rho$ is the product of at most $m - 1$ elements whose sizes are less than ρ, thus we get the estimate

$$||g||_\rho^{\mathrm{diff}} \leq (m-1)\left(\frac{mM}{C} + 1\right).$$

When ρ tends to 0, the right-hand side quantity is equivalent to $\frac{4M}{C}\frac{1}{\rho^2}$, which proves the first point of the proposition.

We turn to the second point. We first prove the result for some special commutator. Let D be any displaceable disc, say of area $\frac{1}{3}$, and let Φ be any non-trivial element of G^{diff} supported in the interior of D. Choose some $\Psi \in G^{\mathrm{diff}}$ such that $\Psi(D)$ is disjoint from D. Let us define $g := [\Phi, \Psi]$. We claim that for any ρ we have
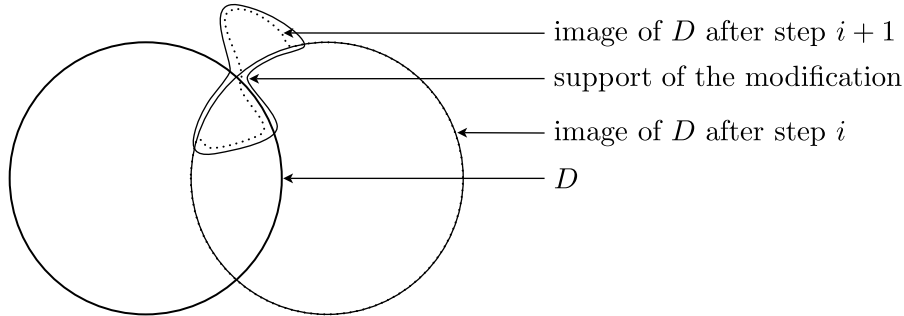
$$||g||_\rho^{\mathrm{diff}} \leq \frac{4}{3\rho}.$$

To prove the claim fix some positive ρ. It is easy to find, almost explicitly, some $\Psi_\rho \in G^{\mathrm{diff}}$ which is a product of less than $\frac{2}{3\rho}$ elements of size less than ρ and that moves D disjoint from itself (see Figure 5). Then

$$|| [\Phi, \Psi_\rho] ||_\rho^{\mathrm{diff}} \leq ||\Phi\Psi_\rho\Phi^{-1}||_\rho^{\mathrm{diff}} + ||\Psi_\rho^{-1}||_\rho^{\mathrm{diff}} = 2||\Psi_\rho||_\rho^{\mathrm{diff}} \leq \frac{4}{3\rho}.$$

We now notice that the map $g = [\Phi, \Psi]$ is conjugate to $[\Phi, \Psi_\rho]$: indeed we may find some $\Theta \in G^{\mathrm{diff}}$ that is the identity on D and equals $\Psi_\rho\Psi^{-1}$ on $\Psi(D)$ (this uses a variation on Lemma 4.4), and such a Θ provides the conjugacy. Since the fragmentation norm is a conjugacy invariant, this proves the claim.

We end the proof of the proposition by using Banyaga's theorem. Since the commutator subgroup $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$ is simple, the normal subgroup of G^{diff} generated by g in G^{diff} is equal to $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$. As in the C^0 case the set of elements $g' \in G^{\mathrm{diff}}$ satisfying $||g'||_\rho^{\mathrm{diff}} = O(\frac{1}{\rho})$ is a normal subgroup, since it

image of D after step $i+1$

support of the modification

image of D after step i

D

Figure 5. How to move a disc disjoint from itself within $\frac{2}{3\rho}$ modifications of size less than ρ: after each modification, the area of the intersection of D with its image has decreased by almost $\frac{\rho}{2}$.

contains g it has to contain $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$. This proves the second point of the proposition.   □

6. Solution of the fragmentation problem for $\rho > \frac{1}{2}$

In [**EPP**] the authors describe some quasi-morphisms on the group of symplectic diffeomorphisms on various surfaces that are continuous with respect to the C^0-topology and, as a consequence, extend continuously to the group of area-preserving homeomorphisms. It turns out that their family of quasi-morphisms on the disc provides a solution to the easiest half of our fragmentation problems: properties (P_ρ) and (P_ρ^{diff}) do not hold when $\rho > \frac{1}{2}$. We discuss this briefly.

Remember that a map $\phi : G^{\mathrm{diff}} \to \mathbb{R}$ is a *quasi-morphism* if the function

$$|\phi(gh) - \phi(g) - \phi(h)|$$

is bounded on $G^{\mathrm{diff}} \times G^{\mathrm{diff}}$ by some quantity $\Delta(\phi)$ called the *defect* of ϕ. A quasi-morphism is called *homogeneous* if it satisfies $\phi(g^n) = n\phi(g)$ for every g and every integer n. The construction of the continuous quasi-morphisms uses the quasi-morphisms of Entov and Polterovich on the 2-sphere [**EP03**]. This quasi-morphism is a *Calabi* quasi-morphism, that is, it coincides with the Calabi morphism when restricted to those diffeomorphims supported on any displaceable disc. By embedding $\mathbb{D}^2$ inside the two-sphere as a non-displaceable[4] disc ([**EP03**], Theorem 1.11 and Section 5.6), and subtracting the Calabi morphism ([**EPP**]), one can get a family $(\phi_{\rho'})_{\rho' \in (\frac{1}{2}, 1]}$ of homogeneous quasi-morphisms on G^{diff}. (As these quasi-morphisms extend to G,

---

[4]Note that when the area of the disc tends to the total area of the sphere, the parameter ρ' below tends to $\frac{1}{2}$; the bigger the disc, the more useful the quasi-morphism, as far as our problem is concerned.

see [**EPP**], we could alternatively choose to work with G and (P_ρ) instead of G^{diff} and (P_ρ^{diff}).) They satisfy the following properties:

(1) $\phi_{\rho'}(g') = 0$ for any $g' \in G^{\mathrm{diff}}$ whose size is less than ρ', and
(2) for every $\rho > \rho'$ there exists some $g \in G^{\mathrm{diff}}$ of size less than ρ with $\phi_{\rho'}(g) \neq 0$.

This entails that for every $\rho \in (\frac{1}{2}, 1]$, property (P_ρ^{diff}) do not hold. To see this, given some $\rho \in (\frac{1}{2}, 1]$, we fix some $\rho' \in (\frac{1}{2}, \rho)$, and we search for some $g \in G^{\mathrm{diff}}$ having size less than ρ and whose ρ'-norm is arbitrarily large (see the easy remarks at the end of Section 1). The first property of $\phi_{\rho'}$ entails, for every $g \in G^{\mathrm{diff}}$,

$$\phi_{\rho'}(g) \leq (||g||_{\rho'}^{\mathrm{diff}} - 1)\Delta(\phi_{\rho'}).$$

On the other hand the second property provides some g with size less than ρ and such that $\phi_{\rho'}(g) \neq 0$. Since $\phi_{\rho'}$ is homogeneous, the sequence $(\phi_{\rho'}(g^n))_{n \geq 0}$ is unbounded. Thus the sequence $(||g^n||_{\rho'}^{\mathrm{diff}})_{n \geq 0}$ is also unbounded, which proves that (P_ρ^{diff}) do not hold.

We are naturally led to the following question.

Question 6.1. *Does there exist, for every $\rho' \in (0, \frac{1}{2}]$, some homogeneous quasi-morphism $\phi_{\rho'}$ satisfying properties (1) and (2) as expressed above?*

A positive answer would imply a negative answer to Question 0.1.

7. Some more remarks

7.1. "Lots of" normal subgroups (if any!) The proof of Lemma 2.1 can easily be modified to show that, if none of the properties (P_ρ) holds, then there exists an uncountable family $\mathcal{F}$ of functions φ such that the corresponding family of normal subgroups $(N_\varphi)_{\varphi \in \mathcal{F}}$ is totally ordered by inclusion. The following is another attempt to express that if G is not simple, then it has to contain "lots of" normal subgroups.

Corollary 7.1. *Assume G is not simple. Then every compact subset K of G is included in a proper normal subgroup of G.*

Note that the situation is radically different for the diffeomorphisms group G^{diff}, since (by Banyaga's theorem [**Ba97**], and since the centralizer of G^{diff} is trivial) any one-parameter subgroup of diffeomorphisms that is not included in the commutator subgroup $[G^{\mathrm{diff}}, G^{\mathrm{diff}}]$ normally generates G^{diff}. However these are not purely algebraic statements since they involve the topology of the groups G and G^{diff}.

Proof of Corollary 7.1. Consider some $\rho \in (0, 1]$. Let $\mathcal{V}_\rho$ be the neighborhood of the identity given by Proposition 4.2: we have $||g||_\rho < \frac{2}{\rho}$ for every $g \in \mathcal{V}_\rho$. By compactness we may find a finite family $g_1, \ldots, g_k$ such that the

sets $g_i.\mathcal{V}_\rho$ cover K. Thus the fragmentation is also uniform on K, in other words the set K is bounded with respect to the norm $||g||_\rho$. Define

$$\varphi_K(\rho) := \sup\{||g||_\rho, g \in K\}.$$

This defines a non-increasing function, and clearly K is included in the normal subgroup N_{φ_K}. According to Theorem 0.1, if G is not simple then N_{φ_K} is a proper subgroup of G, which completes the proof of the corollary. $\square$

7.2. Other surfaces. Let S be any compact surface equipped with an area form. Consider the group of homeomorphisms that preserves the measure associated to the area form, and denote by $G_0(S)$ the (normal) subgroup generated by the homeomorphisms that are supported inside a topological disc. For example, $G_0(\mathbb{S}^2)$ is just the group of orientation and area preserving homeomorphisms of the sphere, and $G_0(\mathbb{T}^2)$ is the group of orientation and area preserving homeomorphisms of the torus with zero mean rotation vector. The group $G_0(S)$ may also be seen as the closure of the group of hamiltonian diffeomorphisms of S inside the group of homeomorphisms. For every surface S, it is an open question whether the group $G_0(S)$ is simple or not.

Exactly as before, on $G_0(S)$ we may define the size of an element supported in a topological disc, the family of fragmentation norms $||.||_\rho^S$, the family of normal subgroups N_φ^S, and the fragmentation properties (P_ρ^S). Then Lemma 2.1 still holds, with the same proof: the failure of all properties (P_ρ^S) would entail that every normal subgroup N_φ^S is proper, and that $G_0(S)$ is not simple. On the other hand, the proof of Lemma 3.1 shows that each property $(P_\rho^{\mathbb{D}^2})$ forces the simplicity of $G_0(S)$ (alternatively, we could use the original Lemma 3.1 and Thurston trick to see that the simplicity of $G = G_0(\mathbb{D}^2)$ implies that of $G_0(S)$). But property (P_ρ^S) is a priori weaker than $(P_\rho^{\mathbb{D}^2})$, because on a general surface one has more space than in the disc to perform the fragmentation. This prevents us from fully translating the simplicity question into a fragmentation problem on the other surfaces.

Using a different approach, one might hope to recover this equivalence by adapting to the C^0 context the homology machinery introduced by Thurston in the smooth category (see [**Ba97**] or [**Bo08**], Section 2.2).

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