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THE RELATIVE SYMPLECTIC CONE AND T^2-FIBRATIONS

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In this note we introduce the notion of the relative symplectic cone $\mathcal{C}_M^V$. As an application, we determine the *symplectic cone* $\mathcal{C}_M$ of certain T^2-fibrations. In particular, for some elliptic surfaces we verify the conjecture in [**17**]: If M underlies a minimal Kähler surface with $p_g > 0$, the symplectic cone $\mathcal{C}_M$ is equal to $\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$, where $\mathcal{P}^\alpha = \{e \in H^2(M;\mathbb{R}) | e \cdot e > 0 \text{ and } e \cdot \alpha > 0\}$ for nonzero $\alpha \in H^2(M;\mathbb{R})$ and $\mathcal{P}^0 = \{e \in H^2(M;\mathbb{R}) | e \cdot e > 0\}$.

Contents

1. Introduction

Given an oriented smooth manifold M known to admit symplectic structures, one would like to know which cohomology classes $\alpha \in H^2(M, \mathbb{R})$ can be represented by an orientation compatible symplectic form $\omega \in \Omega^2(M)$. We shall always restrict ourselves to symplectic forms which are compatible with the fixed orientation of the manifold M. This leads naturally to the definition of the symplectic cone:

$$\mathcal{C}_M = \{\alpha \in H^2(M) | [\omega] = \alpha, \omega \text{ is a symplectic form on } M\}. \tag{1.1}$$

In dimension 4, the symplectic cone has been determined in the following cases: S^2-bundles [**26**], T^2-bundles over T^2 [**9**], all $b^+ = 1$ manifolds ([**19**], see also [**2**, **25**]), and minimal manifolds underlying a Kähler surface with Kodaira dimension 0 [**17**]. A smooth 4-manifold M is said to be minimal if it contains no exceptional class, i.e., a degree 2 homology class represented by a smoothly embedded sphere of self-intersection -1.

Clearly, $\mathcal{C}_M$ is contained in $\mathcal{P}_M$, the cone of classes of positive squares in $H^2(M, \mathbb{R})$. Amazingly, when M is a minimal and in the list above, the symplectic cone $\mathcal{C}_M$ is actually equal to $\mathcal{P}_M$. In particular, it holds for any M underlying a minimal Kähler surface with $p_g = 0$ or Kodaira dimension 0.

In general, $\mathcal{C}_M$ is smaller than $\mathcal{P}_M$, as there are constraints coming from the Seiberg–Witten basic classes. This is a consequence of Taubes' remarkable equivalence between Seiberg–Witten invariants SW and Gromov invariants Gr [**33**]. As exceptional classes and the canonical class of any symplectic structure all give rise to SW basic classes, there are corresponding constraints on $\mathcal{C}_M$.

If the smooth manifold M underlies a minimal Kähler surface, a basic fact [**7**, **36**] is that all symplectic structures on M have the same canonical class up to sign. Denote and fix one such choice $-c_1(M)$. Due to the SW constraints, for any minimal Kähler surface with $p_g > 0$, we have $\mathcal{C}_M \subset \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$. In light of this beautiful fact the following conjecture was raised:

Conjecture 1.1 ([17, Question 4.9]). *If M underlies a minimal Kähler surface with $p_g > 0$, the symplectic cone $\mathcal{C}_M$ is equal to $\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$.*

We define $\mathcal{P}^\alpha = \{e \in \mathcal{P}_M | e \cdot \alpha > 0\}$ for nonzero $\alpha \in H^2(M; \mathbb{R})$ and $\mathcal{P}^0 = \mathcal{P}_M$. As $\mathcal{P}^0 = \mathcal{P}_M = \mathcal{C}_M$ this conjecture is known to be true when M

underlies a minimal Kähler surface with Kodaira dimension 0 (Prop. 4.10, [**17**], see also [**23**]).

In this note we will show that this conjecture holds for certain manifolds underlying minimal Kähler manifolds with $p_g > 0$ and Kodaira dimension 1. Many such manifolds are T^2-fibrations and can be written as a T^2-fiber sum of manifolds with $p_g = 0$ or Kodaira dimension 0.

There are many ways to explicitly construct new symplectic manifolds. Common among most of these methods is that some type of surgery is performed with respect to a codimension 2 symplectic submanifold V. It is natural to ask how the symplectic forms on the new manifolds relate to those on the constituent manifolds. This leads naturally to the notion of the relative symplectic cone $\mathcal{C}_M^V$ defined in Section 2. As examples, we consider T^2-fibrations over T^2 (see [**9**]) and manifolds with $b^+ = 1$. These are of interest, as we will consider T^2 fiber sums in the following sections.

The fiber sum of symplectic manifolds X and Y along symplectic embeddings of a codimension 2 symplectic manifold V, denoted $M = X\#_V Y$, as defined by Gompf [**10**] and McCarthy-Wolfson [**24**], and its inverse operation, the symplectic cut, defined by Lerman [**16**], are briefly described in Section 3.

We then proceed to show that the sum and cut operations naturally describe a cone $\mathcal{C}^{\mathrm{sum}}$ of sum forms in terms of the relative cones of X and Y with respect to V. We also observe that in the case V having trivial normal bundle, $\mathcal{C}^{\mathrm{sum}}$ is actually a subcone of the relative cone $\mathcal{C}_M^V$.

Furthermore, under some topological restrictions on the sum $M = X\#_V Y$ and the respective relative cones $\mathcal{C}_*^V$, we show that the *relative* symplectic cone $\mathcal{C}_M^V$ is actually equal to this subcone.

What does this imply for the symplectic cone of M? Notice that for a minimal T^2-fibration, the canonical class is proportional to the fiber class. Thus the relative symplectic cone, which is of course contained in the symplectic cone, is essentially equal to the symplectic cone. This strategy applies perfectly to fiber sums where one summand is a product T^2-fibration, hence verifying the conjecture for such T^2 fibrations (possibly with singular or multiple fibers). During the preparation of this paper, Friedl and Vidussi (see [**4**, **5**]) determined the symplectic cone of a product S^1-bundle over any 3-manifold or a S^1-bundle over a graph manifold in terms of the Thurston norm ball of the 3-manifold. Their results overlap ours for the product T^2-fibrations $T^2 \times \Sigma_g$.

We include an appendix concerning genericity results for almost complex structures J which make V pseudoholomorphic. These results are needed to determine the relative cone in the $b^+ = 1$ case of Section 2. They show that the set $\mathcal{J}_V$ of such almost complex structures J is rich enough to allow deformations of pseudoholomorphic curves. These results should be known to experts in the field, see [**3**] or [**35**].

One purpose of this note is to introduce the relative symplectic cone and prove a version of the gluing formula for fiber sums along T^2. Missing from the examples in Section 2 is the K3 surface. Hopefully this could be understood in a further paper, thus rounding off the known examples of symplectic manifolds with Kodaira dimension 0. Furthermore, the notion of the relative symplectic cone is useful in the symplectic birational geometry program (see the survey paper [**21**]). Moreover, it is an important ingredient in understanding the symplectic blow-down procedure in dimension 6 [**22**].

2. The relative symplectic cone

This section comprises two distinct parts. The first subsection introduces notation and general concepts of use in the following subsections. The latter subsections contain the definition of the relative symplectic cone and examples.

2.1. Preliminaries. Let M be an oriented manifold and V an oriented codimension 2 submanifold, not necessarily connected. Throughout this section, it will be necessary to carefully distinguish the class of V, denoted $\mathfrak{V} \in H_2(M)$, and the specific submanifold V. Throughout this paper we will not distinguish between $\mathfrak{V}$ and its Poincaré dual. Moreover, $\mathfrak{V}$ will always be a nonzero class and g will denote the genus of V.

We introduce two sets: The set of symplectic classes which evaluate positively on $\mathfrak{V}$:

$$\mathcal{C}_M^{\mathfrak{V}} = \{\alpha \in \mathcal{C}_M | \alpha \cdot \mathfrak{V} > 0\}$$

and the larger set of classes with positive square which evaluate positively on $\mathfrak{V}$:

$$\mathcal{P}^{\mathfrak{V}} = \{\alpha \in \mathcal{P}_M | \alpha \cdot \mathfrak{V} > 0\}.$$

Clearly $\mathcal{C}_M^{\mathfrak{V}} \subset \mathcal{P}^{\mathfrak{V}}$.

The structure of the set $\mathcal{P}^{\mathfrak{V}}$ as a subset of $\mathcal{P}_M$ will be important in the following sections. To this end, recall the "light cone lemma": Let (M, ω) be a symplectic 4-manifold with $b^+ = 1$. Then the set $\mathcal{P}_M$ consists of two connected components, separated by the hyperplane of classes with $[\omega] \cdot A = 0$. The component which contains $[\omega]$ is called the forward cone and is denoted by $\mathcal{P}^+$. Its complement will be denoted by $\mathcal{P}^- = \mathcal{P}_M \backslash P^+$.

Lemma 2.1 ((Light Cone Lemma) [27, Lemma 3.7]). *Suppose that (M, ω) is a symplectic 4-manifold with $b^+ = 1$. Let $a, b \in H^2(M)$ both lie in the closure $\overline{\mathcal{P}^+}$. Then $a \cdot b > 0$ unless $a = \lambda b$ and $a^2 = 0$.*

Note that if $a \in \overline{\mathcal{P}^-}$, then $-a \in \overline{\mathcal{P}^+}$. In particular, this means that any two classes in the same connected component of $\mathcal{P}_M$ have $a \cdot b > 0$. Moreover, if a, b lie in different components, then $-a, b$ lie in the same component and hence $a \cdot b < 0$.

Lemma 2.2. *If* $b^+ = 1$, *then* $\mathcal{P}^{\mathfrak{V}}$ *is connected if* $\mathfrak{V} \cdot \mathfrak{V} \geq 0$; *and* $\mathcal{P}^{\mathfrak{V}}$ *has two connected components if* $\mathfrak{V} \cdot \mathfrak{V} < 0$.

Proof. Assume that $\mathfrak{V} \cdot \mathfrak{V} > 0$. Then $\mathfrak{V}$ is in one of the components of $\mathcal{P}_M$ and, in particular, $\mathfrak{V} \in \mathcal{P}^{\mathfrak{V}}$. Let $a \in \mathcal{P}^{\mathfrak{V}}$, then $a \cdot \mathfrak{V} > 0$ by definition, hence by the light cone lemma a must lie in the same component of $\mathcal{P}_M$ as $\mathfrak{V}$. Hence $\mathcal{P}^{\mathfrak{V}}$ has only one component.

If $\mathfrak{V} \cdot \mathfrak{V} = 0$, then $\mathfrak{V}$ lies in the closure of a component of $\mathcal{P}_M$. As $\mathcal{P}^{\mathfrak{V}}$ contains no elements of square 0, the previous argument again shows that $\mathcal{P}^{\mathfrak{V}}$ has one component unless $\mathfrak{V} = 0$, which our assumptions exclude.

To understand the case $\mathfrak{V} \cdot \mathfrak{V} < 0$, note that the statement of the light cone lemma can be viewed as follows: If a lies in closure of the cone $\mathcal{P}_M$, then the hyperplane of classes defined by $a \cdot A = 0$ does not intersect the cone $\mathcal{P}_M$ at any point. However, in the case $\mathfrak{V} \cdot \mathfrak{V} < 0$ this can no longer be guaranteed. Hence we may have one or two connected components for $\mathcal{P}^{\mathfrak{V}}$. However, if we assume that we have only one connected component, then this implies that the hyperplane of classes with $\mathfrak{V} \cdot A = 0$ does not intersect $\mathcal{P}_M$, meaning all classes in the hyperplane have nonpositive square. Moreover, $\mathfrak{V}$ is not in this hyperplane. Hence the span of the generators of the hyperplane and $\mathfrak{V}$ generate a space of the same dimension as $H^2(M)$. Furthermore, this basis is negative semi-definite, which contradicts $b^+ = 1$. $\square$

2.1.1. Minimality.

Definition 2.3. Let $\mathcal{E}_M$ be the set of cohomology classes whose Poincaré dual are represented by smoothly embedded spheres of self-intersection -1. M is said to be (smoothly) minimal if $\mathcal{E}_M$ is the empty set.

Equivalently, M is minimal if it is not the connected sum of another manifold with $\overline{\mathbb{CP}^2}$. We say that N is a minimal model of M if N is minimal and M is the connected sum of N and a number of $\overline{\mathbb{CP}^2}$.

We also recall the notion of minimality for a symplectic manifold (M, ω): (M, ω) is said to be (symplectically) minimal if $\mathcal{E}_\omega$ is the empty set, where

$$\mathcal{E}_\omega = \{E \in \mathcal{E}_M |\ E \text{ is represented by an embedded } \omega\text{-symplectic sphere}\}.$$

A basic fact proved using SW theory [**18**, **20**, **31**], is: $\mathcal{E}_\omega$ is empty if and only if $\mathcal{E}_M$ is empty. In other words, (M, ω) is symplectically minimal if and only if M is smoothly minimal.

A class $K \in H^2(M, \mathbb{Z})$ is called a symplectic canonical class if there exists a symplectic form ω on M such that for any almost complex structure J tamed by ω,

$$K = K_\omega = -c_1(M, J).$$

Let $\mathcal{K}$ be the set of symplectic canonical classes of M. For any $K \in \mathcal{K}$ define

$$\mathcal{E}_K = \{E \in \mathcal{E}_M | K \cdot E = -1\}.$$

It is shown in Lemma 3.5 in [**19**] that for any ω,

$$\mathcal{E}_{K_\omega} = \mathcal{E}_\omega.$$

Let $S \subset \mathcal{K}$, then define the S-symplectic cone

$$\mathcal{C}_{M,S} = \{\alpha \in \mathcal{C}_M | \alpha = [\omega],\ (M,\omega) \text{ symplectic},\ K_\omega \in S\}. \tag{2.1}$$

2.1.2. Relative inflation. The relative inflation procedure allows one to deform a symplectic form in a smooth family while keeping a fixed submanifold symplectic. Moreover, this deformation is explicitly given. The following is the precise statement:

Lemma 2.4 ([3, Lemma 2.1.A]). *Let $V, C \subset (M,\omega)$ be two distinct 2-dimensional symplectic submanifolds. The submanifold V may be disconnected with pairwise disjoint components. Assume that $C \cdot C \geq 0$ and that C and V intersect positively and transversally in a finite number of points. Then there exists a two form ρ, supported in an arbitrarily small neighborhood of C, with the following properties:*

- *$[\rho]$ is Poincaré dual to $[C]$,*
- *$\omega(s,t) = s\omega + t\rho$ for every $s > 0$ and $t \geq 0$ and*
- *V is a symplectic submanifold with respect to $\omega(s,t)$ for any choice of (s,t).*

Note that the second statement of the lemma concerns symplectic forms not only symplectic classes. With these preparations, we can now proceed to the relative symplectic cone.

2.2. Definition and properties. We make the following definition:

Definition 2.5. A relative symplectic form on the pair (M,V) is an orientation compatible symplectic form on M such that $\omega|_V$ is an orientation compatible symplectic form on V. The relative symplectic cone of (M,V) is

$$\mathcal{C}_M^V = \{\alpha \in H^2(M) |\ [\omega] = \alpha,\ \omega \text{ is a relative symplectic form on } (M,V)\}. \tag{2.2}$$

The submanifold V is embedded, hence the adjunction equality $K_\omega \cdot \mathfrak{V} = 2g - 2 - \mathfrak{V} \cdot \mathfrak{V}$ must hold for the canonical class K_ω of any relative symplectic form ω on (M,V). Let $\mathcal{K}(V) = \{K \in \mathcal{K} \mid K \cdot \mathfrak{V} = 2g - 2 - \mathfrak{V} \cdot \mathfrak{V}\}$. The following is a consequence of the definition of $\mathcal{K}$:

Lemma 2.6. *If $\mathcal{K}(V) = \emptyset$, then there exists no symplectic form $\omega \in \mathcal{C}_M$ such that V is a ω-symplectic submanifold.*

For any $S \subset \mathcal{K}(V)$, define the S-relative symplectic cone

$$\mathcal{C}_{M,S}^{\mathfrak{V}} = \{\alpha \in \mathcal{C}_M^{\mathfrak{V}} | \alpha = [\omega], (M,\omega) \text{ symplectic}, K_\omega \in S\}. \tag{2.3}$$

Note that

$$\mathcal{C}_{M,S}^{\mathfrak{V}} = \mathcal{C}_{M,S} \cap \mathcal{P}^{\mathfrak{V}}. \tag{2.4}$$

The following lemma follows directly from the definition of the relative symplectic cone and the $\mathcal{K}(V)$-relative symplectic cone:

Lemma 2.7.

$$\mathcal{C}_M^V \subset \mathcal{C}_{M,\mathcal{K}(V)}^{\mathfrak{V}} \subset \mathcal{C}_M^{\mathfrak{V}} \subset \mathcal{P}^{\mathfrak{V}}$$

The inclusion $\mathcal{C}_M^V \subset \mathcal{P}^{\mathfrak{V}}$ can be strict, see Lemma 2.15.

Obviously there are maps

$$\mathcal{C}_M^V \hookrightarrow \mathcal{C}_M, \quad \mathcal{C}_M^V \to \mathcal{C}_V. \tag{2.5}$$

In fact, if V is the disjoint union of V_0 and V_1, then there is also a map $\mathcal{C}_M^V \to \mathcal{C}_M^{V_i}$. Note that the restriction mapping $\mathcal{C}_M^V \to \mathcal{C}_V$ is by no means generically injective. The following fact relates the relative cones to the symplectic cone:

Lemma 2.8. *Let $\mathcal{V}$ denote the set of oriented codimension 2 submanifolds of M. Then*

$$\bigcup_{V\in\mathcal{V}} \mathcal{C}_M^V = \mathcal{C}_M$$

Proof. The inclusion $\bigcup_{V\in\mathcal{V}} \mathcal{C}_M^V \subset \mathcal{C}_M$ follows from (2.5). Consider now a symplectic class $\alpha \in \mathcal{C}_M$ and denote by ω a symplectic form representing the class α . We distinguish two cases: $b^+ = 1$ and $b^+ > 1$.

Let $b^+ = 1$. Then α is trivially in the forward cone $\mathcal{P}^+$ of (M, ω). Moreover, if $\mathcal{E}_\omega \neq \emptyset$, $\alpha \cdot E > 0$ for all $E \in \mathcal{E}_\omega$. It now follows from Prop. 4.2 or Prop 4.3 in [**19**] that for k large enough the class $k\alpha$ is represented by a ω-symplectic surface. Therefore, if V represents the class $k\alpha$, it follows that $\alpha \in \mathcal{C}_M^V$.

If $b^+ > 1$, then the canonical class of (M, ω) for some almost complex structure J taming ω is represented by a ω-symplectic surface, see [**31**, Thm 0.2]. Hence, if V represents the canonical class K_ω, then $\alpha \in \mathcal{C}_M^V$. $\square$

The proof shows, that if $b^+ > 1$, we need only consider submanifolds V which are representatives of a canonical class K_ω of (M, ω) if we wish to understand $\mathcal{C}_M$ with respect to the relative cone. In particular, this shows that $\mathcal{C}_M \subset \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$ if $b^+ > 1$ and M is minimal Kähler, which is of interest in connection with Conjecture 1.1. Furthermore, it seems natural to wonder, whether there exist a finite set of submanifolds $\mathcal{V}_{\mathrm{f}} \subset \mathcal{V}$, such that they completely determine the symplectic cone of M. With respect to this question, a trivial but key observation connecting the relative symplectic cone and the symplectic cone is

Lemma 2.9. *Denote the submanifold V with opposite orientation by $\overline{V}$. Then*

$$\mathcal{C}_M^{\overline{V}} = -\mathcal{C}_M^V.$$

The following corollary will be useful in our applications in Section 4:

Corollary 2.10. *Suppose M underlies a minimal Kähler surface with $b^+ > 1$. If $c_1(M,\omega) = a\mathfrak{V}$ with $a \neq 0$ for some symplectic form ω and $\mathcal{C}_M^V = \mathcal{P}^{\mathfrak{V}}$, then $\mathcal{C}_M = \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$.*

We now proceed to calculate the relative cone for certain submanifolds V for two classes of symplectic manifolds: T^2 bundles over T^2 and manifolds with $b^+ = 1$.

2.3. T^2-bundles over T^2. The total spaces M of such bundles have been studied and classified by Sakamoto-Fukuhara [**8**], Ue [**34**] and Geiges [**9**]. In particular, with one exception, they all admit symplectic structures compatible with the bundle structure; in the case of a primary Kodaira surface this bundle structure must be specified as it is not unique. Moreover, the relative symplectic cone with respect to the fiber torus T^2_{f} has been determined explicitly by Geiges.

In [**8**], the manifolds M are classified according to the monodromy A, B of the bundle and the Euler class (x,y). A manifold M is determined by the tuple $(A,B,(x,y))$. In [**34**], the total spaces are classified according to their geometric type as defined by Thurston. Furthermore, an explicit representation of each is given in terms of generators of Γ such that $M = \mathbb{R}^4 \backslash \Gamma$. For example, the four torus T^4 is given by the following data: $(Id, Id, (0,0))$ (Id is the 2×2 identity matrix) with geometric type E^4 and $\Gamma = \mathbb{Z}^4$, i.e., $T^4 = \mathbb{R}^4 \backslash \mathbb{Z}^4$. From the explicit presentation of the generators of Γ, Geiges constructs symplectic forms, thereby determining the symplectic cones as well as the relative cones with respect to the fiber torus T^2_{f}. In the following we denote the class of a fiber torus T^2_{f} by F. We collect the data in the following table, details can be found in [**34**] and [**9**]:

Type	b_1	$\mathcal{C}_M$	$\mathcal{C}_M^{T^2_{\mathrm{f}}}$
T^4	4	$\mathcal{P}_M$	$\mathcal{P}_M^F$
Primary Kodaira surface	3	$\mathcal{P}_M$	$\mathcal{P}_M^F$
Hyperelliptic surface	2	$\mathcal{P}_M$	$\mathcal{P}_M$
(d)	2	$\mathcal{P}_M$	$\emptyset$
$(e)-(h)$	2	$\mathcal{P}_M$	$\mathcal{P}_M$

Note that the class of T^2-fibrations over T^2 provides a full range of possible relative cones, from $\emptyset$ to the maximal possible cone, see Lemma 2.7.

2.4. Manifolds with $b^+ = 1$. In this section, we will study the relative cone with respect to a submanifold V for manifolds with $b^+ = 1$. Particularly we will completely determine $\mathcal{C}_M^V$ when M is minimal.

The symplectic cone in this case is determined in [**19**, Thm. 4]:

Theorem 2.11. *Let M be a 4-manifold with $b^+ = 1$ and $\mathcal{C}_M$ nonempty. Let $\mathcal{E}_M$ denote the set of all exceptional classes of M. Then*

$$\mathcal{C}_M = \{e \in \mathcal{P}_M | 0 < |e \cdot E| \text{ for all } E \in \mathcal{E}_M\}.$$

In particular, if M is minimal, then $\mathcal{C}_M = \mathcal{P}_M$.

Consider the K-symplectic cone for $K \in \mathcal{K}$, see equation (2.1).

Theorem 2.12 ([19, Theorem 3]). *Let M be a 4-manifold with $b^+ = 1$. Then $\mathcal{C}_M$ is the disjoint union of $\mathcal{C}_{M,K}$ over $K \in \mathcal{K}$. For each $K \in \mathcal{K}$, $\mathcal{C}_{M,K}$ is contained in a component of $\mathcal{P}_M$, which we call the K-forward cone $\mathcal{P}^+(K)$. Moreover, for each $K \in \mathcal{K}$,*

$$\mathcal{C}_{M,K} = \{e \in \mathcal{P}^+(K) | e \cdot E > 0 \text{ for all } E \in \mathcal{E}_K\}.$$

The following theorem is the main result of this section:

Theorem 2.13. *Let M be a 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}_M^V \neq \emptyset$. If ω is a relative symplectic form on (M, V), then*

$$\mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \subset \mathcal{C}^V_M.$$

Proof. The following lemma will be central to the proof. We defer the proof of the lemma to Section 2.5.

Lemma 2.14. *Let us fix a relative symplectic form ω on (M, V). For any $A \in H_2(M; \mathbb{Z})$ with*

$$A \cdot E > 0 \text{ for all } E \in \mathcal{E}_\omega,$$
$$A \cdot \mathfrak{V} > 0, \quad A \cdot A > 0, \quad A \cdot [\omega] > 0$$
$$(A - K_\omega) \cdot [\omega] > 0, \quad (A - K_\omega) \cdot (A - K_\omega) > 0, (A - K_\omega) \cdot \mathfrak{V} > 0,$$

there exists an ω-symplectic submanifold C in the class A, intersecting V transversally and positively.

Fix a relative symplectic form ω, we may assume that $[\omega] \in \mathcal{C}^V_M$ is an integral class. Let $e \in \mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \cap H^2(M, \mathbb{Z})$. Then

$$e \cdot E > 0 \text{ for all } E \in \mathcal{E}_\omega, \quad e \cdot e > 0, \quad e \cdot \mathfrak{V} > 0.$$

Moreover, by [**19**, Prop. 4.1], $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$ is contained in one component of $\mathcal{P}_M$. Hence, noting that $[\omega] \in \mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$, it follows that $e \cdot [\omega] > 0$. Thus for large $l > 0$, the class $A = le - [\omega]$ will satisfy the assumptions of Lemma 2.14.

Apply Lemma 2.14 to the class $A = le - [\omega]$ for $l >> 0$ and Lemma 2.4 to the pair (V, C) with C the symplectic surface of class A obtained by Lemma 2.14. This proves that $s[\omega] + t(le - [\omega]) \in \mathcal{C}^V_M$, hence in particular le is in the relative cone and therefore also e, i.e.,

$$\mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \cap H^2(M, \mathbb{Z}) \subset \mathcal{C}^V_M.$$

It also follows that any real multiple of an integral class e in $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$ is in the relative cone $\mathcal{C}^V_M$.

We now want to show that $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \subset \mathcal{C}^V_M$. Observe that $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$ is an open convex cone. Therefore, for any α in $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$, we can write $\alpha = \sum_{i=1}^p \alpha_i$, where the rays of α_i are in $\mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$, arbitrarily close to that of α, and each $\alpha_i = s_i\beta_i$ for some positive real number s_i and an integral class $\beta_i \in \mathcal{C}^{\mathfrak{V}}_{M,K_\omega}$. Note that $\beta_i \cdot \beta_j > 0$ for all i, j by Lemma 2.1. Inductively it can be shown that for each $q \leq p$, $\sum_{i=1}^q \alpha_i$ is in the relative cone $\mathcal{C}^V_M$:

First we choose a relative symplectic form ω_1 for the pair (M, V) with $[\omega_1] = \alpha_1$, which we can do by the procedure described above. For a large integer l, since $\beta_2 \in \mathcal{C}^{\mathfrak{V}}_{M,K_\omega} \cap H^2(M, \mathbb{Z})$, we can apply Lemma 2.14 to $A = l\beta_2$ to obtain a submanifold C_2. Lemma 2.4 applied to the pair (V, C_2) on (M, ω_1) then shows that $s[\omega_1] + tA \in \mathcal{C}^V_M$. Choosing $s = 1$ and $t = \frac{s_2}{l}$ shows that $\alpha_1 + \alpha_2 = \alpha_1 + \frac{s_2}{l}A$ is in the relative cone.

Now choose a symplectic form ω_2 for the pair (M, V) with $[\omega_2] = \alpha_1 + \alpha_2$. This completes the argument. $\square$

Notice that when M is minimal and $\mathfrak{V} \cdot \mathfrak{V} < 0$, $\mathcal{C}^V_M$ contains a component of $\mathcal{P}^{\mathfrak{V}}$ whenever there is a relative symplectic form whose class lies in that component.

In the rest of this subsection we assume that M is minimal. The general case is more complicated and not needed in the application in Section 4, so it will be studied elsewhere.

Lemma 2.15. *Let M be a smoothly minimal 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}^V_M \neq \emptyset$. If $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} \neq 0$, then $\mathcal{C}^{\mathfrak{V}}_{M,\mathcal{K}(V)}$ is contained in one component of $\mathcal{P}_M$. In particular, if $\mathfrak{V} \cdot \mathfrak{V} < 0$, the inclusion $\mathcal{C}^V_M \subset \mathcal{P}^{\mathfrak{V}}$ is strict.*

Proof. Theorem 1 in [**19**] states that under the assumptions on M there is a unique symplectic canonical class up to sign. Denote this class by K. Theorem 3 and Proposition 4.1 of [**19**] show that symplectic forms with canonical class K lie in one component of the cone $\mathcal{P}_M$ while those with canonical class $-K$ lie in the other. Hence $\mathcal{C}^{\mathfrak{V}}_{M,\,\mathcal{K}(V)}$, which contains only symplectic classes representable by symplectic forms with canonical class K, is contained in one component of $\mathcal{P}_M$. Lemmas 2.2 and 2.7 complete the proof. $\square$

The following result follows directly from [**19**, Thm 1]:

Lemma 2.16. *Let M be a smoothly minimal 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}^V_M \neq \emptyset$. If $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} \neq 0$ or $\mathfrak{V} \cdot \mathfrak{V} \geq 0$, then $\mathcal{K}(V)$ is a point.*

Denote the class in $\mathcal{K}(V)$ by K. Together with Theorem 2.13, this leads to the following statement in the minimal case, notice that this is equality in Theorem 2.13:

Corollary 2.17. *Let M be a smoothly minimal 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}_M^V \neq \emptyset$. If $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} \neq 0$ or $\mathfrak{V} \cdot \mathfrak{V} \geq 0$, then*

$$\mathcal{C}_M^V = \mathcal{C}_{M,K}^{\mathfrak{V}}$$

is one connected component of $\mathcal{P}$. Moreover, if $\mathfrak{V} \cdot \mathfrak{V} \geq 0$, then $\mathcal{C}_M^V = \mathcal{P}^{\mathfrak{V}}$.

Proof. Assuming $\mathcal{C}_M^V \neq \emptyset$, we may choose a relative symplectic form ω on (M, V). Then $K_\omega = K$, hence by Theorem 2.13

$$\mathcal{C}_{M,\,K}^{\mathfrak{V}} = \mathcal{C}_{M,\,K_\omega}^{\mathfrak{V}} \subset \mathcal{C}_M^V.$$

Moreover, by Lemma 2.7 $\mathcal{C}_M^V \subset \mathcal{C}_{M,\mathcal{K}(V)}^{\mathfrak{V}} = \mathcal{C}_{M,\,K}^{\mathfrak{V}}$.

If $\mathfrak{V} \cdot \mathfrak{V} \geq 0$, then by Lemma 2.15, $\mathcal{C}_{M,\,\mathcal{K}(V)}^{\mathfrak{V}} = \mathcal{C}_{M,\,K}^{\mathfrak{V}}$ is contained in one component of $\mathcal{P}_M$. The minimal case of Theorem 2.12 and equation (2.4) imply that

$$\mathcal{C}_{M,\,K}^{\mathfrak{V}} = \mathcal{P}^+(K) \cap \mathcal{P}^{\mathfrak{V}}$$

and hence by Lemma 2.2, that $\mathcal{C}_M^V = \mathcal{P}^{\mathfrak{V}}$. □

Remark. If M is a minimal symplectic 4-manifold with $b^+ = 1$ and K is the unique canonical class, then it was shown in [**19**, Thm. 1 and Prop. 4.1], that $\mathcal{C}_M = \mathcal{C}_{M,K} \cup \mathcal{C}_{M,-K}$. This decomposition can be restated in terms of the relative cone of a submanifold V with $\mathfrak{V} \cdot \mathfrak{V} \geq 0$: $\mathcal{C}_M = \mathcal{C}_M^V \cup \mathcal{C}_M^{\overline{V}}$, see Corollary 2.9.

Missing from the corollary is the case $\mathfrak{V} \cdot \mathfrak{V} < 0$ and $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} = 0$. In this case V is an embedded sphere of self-intersection -2 or -1. This follows from $2g - 2 - \mathfrak{V} \cdot \mathfrak{V} = 0$, as then $\mathfrak{V} \cdot \mathfrak{V} < 0$ can only hold if $g = 0$. We will just deal with the case $\mathfrak{V} \cdot \mathfrak{V} = -2$ since otherwise M is not minimal. In the minimal $b^+ = 1$ case, the corollary shows that only if V is a sphere of self-intersection -2 can $\mathcal{C}_M^V$ possibly consist of two disjoint components:

Lemma 2.18. *Let M be a smoothly minimal 4-manifold with $b^+ = 1$ and V an oriented submanifold for which $\mathcal{C}_M^V \neq \emptyset$. If V is an embedded sphere of self-intersection -2, then $\mathcal{C}_M^V$ has two connected components and the inclusion $\mathcal{C}_M^V \cup \mathcal{C}_M^{\overline{V}} \subset \mathcal{C}_M$ is strict.*

Proof. The proof relies on [**6**, Prop. 2.4]: This result provides a self-diffeomorphism of M which gives rise to a reflection on cohomology. The diffeomorphism ϕ can be chosen such that $\phi(V) = V$, though ϕ does not fix V pointwise. In fact, it is an antipodal map on V, reversing the orientation of V. The effect of ϕ on cohomology is the reflection:

$$\phi : x \mapsto x + (x \cdot \mathfrak{V})\mathfrak{V},$$

which is precisely the reflection across the hyperplane of classes A with $A \cdot \mathfrak{V} = 0$.

Suppose ω is a relative symplectic form for (M, V) and K_ω is its canonical class. Notice that ϕ^* preserves the canonical class K_ω since $K_\omega \cdot \mathfrak{V} = 0$. Moreover, $\phi^*(\omega)$ is a symplectic form for $(M, \overline{V})$ since ϕ preserves the submanifold V but reverses the orientation of V and the canonical class of $\phi^*(\omega)$ is $\phi^*(K_\omega) = K_\omega$. This implies that $-\phi^*(\omega)$ is a relative symplectic form for (M, V). Notice however that the canonical class of $-\phi^*(\omega)$ is $-K_\omega$, which means that $-\phi^*(\omega)$ and ω lie in different components of $\mathcal{P}_M$. $\square$

Example. Consider the manifold $M = S^2 \times S^2$. Let V be a -2 sphere in the class $S_1 - S_2 \in H_2(M)$. Choose ω to be in the class $s_1 + 2s_2 \in \mathcal{C}_M^V$ where $s_i(S_j) = 0$ if $i = j$ and 1 if $i \neq j$. Then $\phi^*(\omega)$ is in the class

$$[\omega] + ([\omega] \cdot \mathfrak{V})\mathfrak{V} = [\omega] + \mathfrak{V} = 2s_1 + s_2.$$

Hence $-\phi^*(\omega)$ is in the class $-2s_1 - s_2$.

2.5. Proof of Lemma 2.14.

Proof. This relies on Proposition 4.3 in [**19**] and the genericity results of the Appendix.

Let us first recall the notion of A being J-effective and simple considered in [**3**]. A is J-effective and simple if, for a generic choice of $k(A) = \frac{-K_\omega(A) + A \cdot A}{2} \geq 0$ distinct points $\Omega_{k(A)}$ in M, there exists a connected J-holomorphic submanifold $C \subset M$ which represents A and passes through all the $k(A)$ points.

Proposition 4.3 in [**19**] implies that any A with

$$A \cdot E \geq 0 \text{ for all } E \in \mathcal{E},$$
$$A \cdot A > 0, \quad A \cdot [\omega] > 0,$$
$$(A - K_\omega) \cdot (A - K_\omega) > 0, \quad (A - K_\omega) \cdot [\omega] > 0,$$

is J-effective for any J tamed by ω.

Let $\mathcal{J}_V$ be the space of ω-tamed almost complex structures making V a pseudo-holomorphic submanifold. Then for every $J \in \mathcal{J}_V$ there exists a J-holomorphic curve C in class A by the previous considerations. We consider connected nodal J-holomorphic curves C representing A with multiple components each having as their image a ω-symplectic submanifold. However, for the purposes of this lemma, we need to exclude components which lie in V. Therefore consider a connected J-holomorphic curve C representing A: The curve C is given by a collection $\{(\phi_i, \Sigma_i)\}$ of maps ϕ_i and Riemann surfaces Σ_i. We want to reformulate this as a collection $\{(\varphi_i, C_i, m_i)\}$ of simple maps, submanifolds and multiplicities. If ϕ_i is a multiple cover, we replace it by a simple map φ_i with the same image and an integer m_i tracking the multiplicity. We also combine maps which have same image, adding the multiplicities and keeping only one copy of the map. Ultimately, we replace all the maps ϕ_i by simple embeddings φ_i with image C_i. We can therefore decompose the class A as $\sum_i m_i[C_i]$. Note that we are only interested in the submanifolds C_i, so we do allow the class $[C_i]$ to be divisible. However, we want the class to represent a submanifold, hence correspond to an embedding φ_i. We denote

(1) components with $[C_i]^2 < 0$ by B_i and
(2) those with $[C_i]^2 \geq 0$ which do not lie in V by A_i.

The class A_i could be a multiple of the class $\mathfrak{V}$, however, due to our decomposition above we consider only maps which are not multiple covers of V. Furthermore, we could have a component $[C_0] = \mathfrak{V}$, with a multiplicity $m \geq 1$, which is a (multiple) cover of V. Thus, $A = m\mathfrak{V} + \sum_i m_i B_i + r_i A_i$.

We begin with the negative square components: Let $B \cdot B < 0$. Consider the moduli space of $\mathcal{M}(B, J, g)$ of pairs (u, j), where $j \in \mathcal{T}_g$, the Teichmüller space of a closed oriented surface Σ_g of genus g, and $u : (\Sigma_g, j) \to (M, J)$ is a somewhere injective (j, J)-holomorphic curve in the class B. If $B \neq \mathfrak{V}$, then arguments similar to those in the proof of Lemma A.1 show that for generic $J \in \mathcal{J}_V$, the spaces $\mathcal{M}(B, J, g)$ are smooth manifolds of dimension $2(-K_\omega \cdot B + g - 1) + \dim G_g$.

By the adjunction formula the space of non-parametrized J-holomorphic curves has dimension

$$2(-K_\omega \cdot B + g - 1) \leq 2B \cdot B - (2g - 2).$$

Thus if $B \cdot B < 0$, $\mathcal{M}(B, J, g)$ is nonempty only if $g = 0$ and $B \cdot B = -1$. We conclude, that for a generic $J \in \mathcal{J}_V$, the only irreducible components of a cusp A-curve with negative self-intersection (except possibly the component C_0) have $B^2 = -1$. In particular, note that all $B_i \in \mathcal{E}_\omega$ and $k(B_i) = 0$. Hence, our assumptions imply that $A \cdot B_i > 0$ for all i.

Now let us divide the proof into two cases, in both we shall use the genericity results proven in the Appendix:

Case 1. $k(\mathfrak{V}) \geq 0$. The condition $k(\mathfrak{V}) \geq 0$ can be rewritten using the adjunction formula to state that $\mathfrak{V}^2 \geq 0$ if $g(\mathfrak{V}) \geq 1$ and $\mathfrak{V}^2 \geq -1$ at worst, if $g(\mathfrak{V}) = 0$. In the following we will allow the case $A = m\mathfrak{V}$.

The results of the Appendix, in particular Lemma A.1 and Lemma A.3, show, that we can find a generic set of pairs $(J, \Omega_{k(A)})$ such that $k(A_i) \geq 0$, each curve in class A_i resp. $\mathfrak{V}$ meets at most $k(A_i)$ resp. $k(\mathfrak{V})$ generic points and $\sum k(A_i) + k(\mathfrak{V}) \geq k(A)$.

Further, if $k(A_i) \geq 0$ and $A_i^2 \geq 0$, then $k(r_i A_i) \geq 0$ for any positive integer r_i:

$$\begin{aligned} 0 \leq 2k(A_i) \leq 2r_i k(A_i) &= -K_\omega(r_i A_i) + r_i A_i \cdot A_i \\ &\leq -K_\omega(r_i A_i) + r_i^2 A_i \cdot A_i = 2k(r_i A_i). \end{aligned}$$

Note that this holds in particular for $m\mathfrak{V}$.

For such a generic choice of $(J, \Omega_{k(A)})$, let C be a connected curve representing A, which contains the $k(A)$ distinct points of $\Omega_{k(A)}$. Then

$$\begin{aligned}
2k(A) = & -K_\omega(m\mathfrak{V}) + \sum_i -K_\omega(m_iB_i) + \sum_i -K_\omega(r_iA_i) \\
& + m^2\mathfrak{V}^2 + \sum_i m_i^2B_i^2 + \sum_i r_i^2A_i^2 + 2\sum_i m\mathfrak{V}m_iB_i + 2\sum_i m\mathfrak{V}r_iA_i \\
& + 2\sum_{i>j} m_im_jB_i\cdot B_j + 2\sum_{i>j} m_ir_jB_i\cdot A_j + 2\sum_{i>j} r_ir_jA_i\cdot A_j \\
\geq & \, 2mk(\mathfrak{V}) + 2\sum_i r_ik(A_i) + (m^2-m)\mathfrak{V}^2 + 2\sum_i m\mathfrak{V}r_iA_i \\
& + 2\sum_{i>j} r_ir_jA_i\cdot A_j + \sum_i (m_i^2-m_i)B_i^2 + 2\sum_{i>j} m_im_jB_i\cdot B_j \\
& + 2\sum_{i>j} m_ir_jB_i\cdot A_j + 2\sum_i m\mathfrak{V}m_iB_i
\end{aligned}$$

If $\mathfrak{V}^2 = -1$, then denote $B_0 = \mathfrak{V}$ and include it in the following estimate. Fix an i and consider the terms in the last line. They can be rewritten as

$$2m_iA\cdot B_i - 2m_i^2B_i^2 + (m_i^2-m_i)B_i^2 \;=\; 2m_iA\cdot B_i + m_i^2 + m_i \;\geq 0$$

and thus we obtain the estimate

$$2k(A) \geq 2k(\mathfrak{V}) + 2\sum_i k(A_i).$$

Hence either $k(A) > k(\mathfrak{V}) + \sum_i k(A_i)$ or the following hold:

- $m_i = 0$ for all i, i.e., there are no components of negative square,
- $A_i\cdot A_j = 0 = A_i\cdot\mathfrak{V}$ for $i\neq j$,
- if $\mathfrak{V}^2\geq 0$, then $m=1$ or $k(\mathfrak{V})=0$ and $\mathfrak{V}^2=0$, and
- $r_i = 1$ or $k(A_i)=0$.

Therefore, the curve C representing A is an embedded J-holomorphic submanifold with a single nonmultiply covered component containing the set $\Omega_{k(A)}$ with $J\in\mathcal{J}_V$ or $k(A)=0$.

In the following cases we are done:

(1) $A\neq m\mathfrak{V}$
(2) $A = m\mathfrak{V}$ and $k(\mathfrak{V})>0$: The results above imply that $m=1$. Choose $\Omega_{k(A)}$ such that it contains a point not in V. Then C does not lie in V and intersects V locally positively.

We are left with the following case: $A = m\mathfrak{V}$ and $k(\mathfrak{V})=0$. However, in this case the previous results show that either $m=1$ or $\mathfrak{V}^2=0$. The latter is excluded by the assumption $A^2>0$. The former would mean $A=\mathfrak{V}$ and thus

$$0 = 2k(\mathfrak{V}) = \mathfrak{V}^2 - K_\omega \mathfrak{V} = \mathfrak{V}(A - K_\omega),$$

which is also excluded by assumption.

In all cases, we can perturb C to be transverse to V, see [**28**, **29**].

Case 2. $k(\mathfrak{V}) < 0$. In this case, the results of the Appendix show, that we can find a generic set of almost complex structures, such that V is rigid and there are no other curves in class $\mathfrak{V}$. In the following, we choose only complex structures from this set.

Even though we are working in the case $k(\mathfrak{V}) < 0$, it is possible for a multiple class $m\mathfrak{V}$ to have $k(m\mathfrak{V}) \geq 0$. For this reason, we will distinguish the following two objects:

(1) Classes $A_i = m_i\mathfrak{V}$ which correspond to components of the curve C in class $m\mathfrak{V}$, but which are NOT multiple covers of a submanifold in class $\mathfrak{V}$. If $\mathfrak{V}^2 < 0$, then positivity of intersections shows that any curve C can contain at most one component in class $m\mathfrak{V}$ for all m and this component must coincide with the manifold V. This situation was studied in greater generality in [**3**]. Furthermore, if $\mathfrak{V}^2 \geq 0$ and a class $A_i = m\mathfrak{V}$ occurs, then the results of Lemma A.1 apply. We may therefore assume , that $A_i^2 \geq 0$ in the following.
(2) The specific "class" mV which corresponds to components that have as their image the submanifold V.

Note further that we can choose our almost complex structures such that the components corresponding to mV are rigid, while those in $m_i\mathfrak{V}$ are not. Such a decomposition is not necessary in the case $k(\mathfrak{V}) \geq 0$, as we can choose a generic set of pairs such that $\mathcal{K}_V^{\mathfrak{V}}(J, \Omega)$ is smooth (see the Appendix for details); however, we do not know if V is an element in this set, nor does this matter for the calculation. In the current situation, the specific submanifold V acts differently than other elements in the class $\mathfrak{V}$.

We now proceed as in [**3**]: Consider the class $\tilde{A} = A - mV = \sum r_i A_i + m_i B_i$. We assume that such a decomposition is possible, i.e., there exists a not necessarily connected pseudoholomorphic curve in class $\tilde{A}$. The case $A = mV$ will be considered afterwards. We need to show that there exists a generic set of pairs $(J, \Omega_{k(A)})$ such that $k(A) > \sum k(A_i)$. Proceeding exactly as in [**3**], we obtain the following two estimates: $\sum k(A_i) \leq k'(\tilde{A})$, where k' is the modified count defined by McDuff [**27**]. Furthermore

$$2k(A) - 2k'(\tilde{A}) = m \underbrace{(A - K_\omega) \cdot V}_{>0 \text{ by assumption}} + \text{non-negative terms}$$

and hence, combining all these inequalities, for generic pairs $(J, \Omega_{k(A)})$ we obtain $k(A) > \sum k(A_i)$. We therefore conclude that we can rule out such a decomposition, hence "$A = mV$" or $A = \sum r_i A_i + m_i B_i$. In the latter case we are done by the same line of argument as in the $k(\mathfrak{V}) \geq 0$ case, albeit with the added restriction on the almost complex structures that V is rigid and no further curves in class $\mathfrak{V}$ exist. The case "$A = mV$" corresponds to the case $A = m\mathfrak{V}$ and $k(A) \geq 0$, more precisely,

$$0 < m(A - K_\omega) \cdot V = mA \cdot V - K_\omega \cdot V = A^2 - K_\omega \cdot A = k(A).$$

Thus, applying Lemma A.1, we can find a generic set of pairs $(J, \Omega_{k(A)})$ such that $A = m\mathfrak{V}$ is represented by an embedded curve with deformations. Choosing $\Omega_{k(A)}$ such that it contains a point not in V ensures that a representative of A in this case does not lie in V.

As before, we can make the curve C transverse to V. $\square$

3. The gluing formula

We now return to Conjecture 1.1 which was stated in the Introduction: Is the symplectic cone $\mathcal{C}_M$ equal to the $\mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$ for minimal Kähler manifolds? In this section we provide the theoretical framework necessary to answer this question for so-called "good" sums. We first review the symplectic sum and cut operations. This leads to the definition of a good sum and the subsequent homological reformulation of these operations.

3.1. Smooth fiber sum. Let X, Y be $2n$-dimensional smooth manifolds. Suppose we are given codimension 2 embeddings $j_* : V \to *$ into X and Y of a smooth closed oriented manifold V with normal bundles N_*V. Assume that the Euler classes of the normal bundle of the embedding of V in X resp. Y satisfy $e(N_XV) + e(N_YV) = 0$ and fix a fiber-orientation reversing bundle isomorphism $\Theta : N_XV \to N_YV$. By canonically identifying the normal bundles with a tubular neighborhood ν_* of $j_*(V)$, we obtain an orientation preserving diffeomorphism $\varphi : \nu_X \backslash j_X(V) \to \nu_Y \backslash j_Y(V)$ by composing Θ with the diffeomorphism that turns each punctured fiber inside out. This defines a gluing of X to Y along the embeddings of V denoted $M = X\#_{(V,\varphi)}Y$. The diffeomorphism type of this manifold is determined by the embeddings (j_X, j_Y) and the map Θ. Note also that if V has trivial normal bundle, then this construction should actually be viewed as a sum along $V \times S^1$.

In the rest of the paper, whenever we consider a fiber sum, we fix V, the embeddings j_* and the bundle isomorphism Θ without necessarily explicitly denoting either. This fixes the homology of the manifold $M = X\#_{(V,\varphi)}Y$.

Example 3.1. Consider for example the torus $T^4 = T^2_{\mathrm{f}} \times T^2$, where the first factor is the fiber direction. Let $M = T^4\#_{T^2_{\mathrm{f}}}T^4$ be the sum along the fiber T^2_{f}. Then M is actually $T^2_{\mathrm{f}} \times \Sigma_2$, as can be seen from the following:

$$\begin{tikzcd} T^2_{\mathrm{f}} \arrow[r] & T^4 \arrow[d] & T^4 \arrow[d] & T^2_{\mathrm{f}} \arrow[l] & & T^2_{\mathrm{f}} \arrow[r] & T^4\#_{T^2_{\mathrm{f}}}T^4 = M \arrow[d] \\ & T^2 & T^2 & & \Longrightarrow & & T^2\#T^2 = \Sigma_2 \end{tikzcd}$$

3.2. Symplectic sum and symplectic cut. We briefly describe the symplectic sum construction $M = X\#_V Y$ as defined by Gompf [**10**] (see also McCarthy–Wolfson [**24**]). Assume X and Y admit symplectic forms ω_X, ω_Y resp. If the embeddings j_* are symplectic with respect to these forms, then we obtain $M = X\#_{(V,\varphi)}Y$ together with a symplectic form ω created from ω_X and ω_Y. It was shown in [**10**] that this can be done without loss of symplectic volume.

Furthermore, Gompf showed that the symplectic form ω thus constructed on $M = X\#_{(V,\varphi)}Y$ from ω_X, ω_Y is unique up to isotopy. This result allows one to construct a smooth family of isotopic symplectic sums $M = X\#_{(V,\varphi_\lambda)}Y$ parametrized by $\lambda \in D^2\backslash\{0\}$ as deformations with a singular fiber $X \sqcup_V Y$ over $\lambda = 0$ (see [**14**, Sect. 2]). Therefore, we suppress φ_λ from the notation, choosing instead to work with an isotopy class where necessary.

Thus, a symplectic sum will be denoted by $M = X\#_V Y$, a symplectic class ω on the sum will denote an isotopy class.

The symplectic cut operation of Lerman [**16**] functions as follows: Consider a symplectic manifold (M, ω) with a Hamiltonian circle action and a corresponding moment map $\mu : M \to \mathbb{R}$. We can assume that 0 is a regular value, if necessary by adding a constant. We can thus cut M along $\mu^{-1}(0)$ into two manifolds $M_{\mu>0}$ and $M_{\mu<0}$, both of which have boundary $\mu^{-1}(0)$. If we collapse the S^1-action on the boundary, we obtain manifolds $\overline{M_{\mu>0}}$ and $\overline{M_{\mu<0}}$ which contain a real codimension 2 submanifold $V = \mu^{-1}(0)/S^1$. If we symplectically glue $\overline{M_{\mu>0}}$ and $\overline{M_{\mu<0}}$ along V we obtain again M.

Note that the above construction is local in nature, thus the moment map and the S^1 action need only be defined in a neighborhood of the cut.

The symplectic structure ω restricted to $M_{\mu>0}$ and $M_{\mu<0}$ reduces to a symplectic structure on $\overline{M_{\mu>0}}$ and $\overline{M_{\mu<0}}$ which have the same value on V. This motivates the sum decomposition of the symplectic cones in Section 3.3.

A symplectic cut is only possible on a symplectic manifold; thus, when discussing a symplectic cut on $M = X\#_V Y$, we implicitly consider only those isotopy classes allowing moment maps μ with $V = \mu^{-1}(0)\backslash S^1$.

In the case $M = T^4\#_{T^2_{\mathrm{f}}}T^4$ it is possible to understand the geometric construction underlying the cut: Consider $M = T^4\#_{T^2_{\mathrm{f}}}T^4 = T^2 \times \Sigma_2$ and view Σ_2 such that we have the holes on either end and a cylindrical connecting piece S in between. Furthermore, in M this copy of Σ_2 is transverse to the fiber T^2. Choose local coordinates (λ, θ, t) on $S \times T^2$, $(\lambda, \theta) \in [-1, 1] \times [0, 2\pi]$ coordinates on S, t a coordinate on T^2. Consider an S^1 action on the second coordinate stemming from the Hamiltonian $\mu : S \to \mathbb{R}$ given by $\mu(\lambda, \theta, t) = \lambda$. Locally, any symplectic form is given by $\omega = a\,\mathrm{d}\lambda \wedge \mathrm{d}\theta + b\,\mathrm{d}t + \Omega$, with $\Omega(\frac{\partial}{\partial\theta}, \cdot) = 0$ and $a \in \mathbb{R}$ nonzero.

The symplectic cut defined by μ produces $M_{\mu<0} = T_b^2 \times T^2$ with T_b^2 a punctured torus with boundary S^1. Over each point of the boundary, there is a fiber T_f^2, hence $\partial M_{\mu<0} = T^3$. Collapsing this boundary under the S^1 action produces $T^4 = T_\mathrm{f}^2 \times T^2$. In particular, the action maps $\mathrm{d}\lambda \wedge \mathrm{d}\theta$ to local coordinates on a neighborhood of the collapsed boundary on T_b^2 without loss of volume.

3.3. $\mathcal{C}^V_{X\#_V Y}$ from $\mathcal{C}^V_X$ and $\mathcal{C}^V_Y$.

3.3.1. The cone of sum forms. We are interested in the symplectic cone $\mathcal{C}_M$ of the 4-manifold M. Suppose this manifold can be obtained as a symplectic sum $M = X\#_V Y$. Let us fix the symplectic embeddings as well as the map Θ. In the following, we will distinguish between the manifold M and the specific viewpoint as a symplectic sum from X and Y along V by explicitly denoting $M = X\#_V Y$. Accordingly, we define the following symplectic cone associated to the symplectic sum:

Definition 3.2. Suppose that $M = X\#_V Y$. Define the cone of sum forms, $\mathcal{C}^{sum}_{X\#_V Y}$, to be the set of classes of symplectic forms on M which can be obtained by summing X and Y with symplectic embeddings $\tilde{j}_*$ and bundle map $\tilde{\Theta}$ isotopic to the fixed choice j_* and Θ.

We obtain the following result:

Theorem 3.3. *For a symplectic manifold $M = X\#_V Y$,*

$$\mathcal{C}^{sum}_{X\#_V Y} = \Psi(\Phi^{-1}(\mathcal{C}^V_X \oplus \mathcal{C}^V_Y)), \tag{3.1}$$

where Φ, Ψ are the maps on cohomology corresponding to the inclusion of X, Y into $X \sqcup_V Y$ and the projection of $X\#_V Y$ onto the singular manifold $X \sqcup_V Y$ respectively (see (3.2) below).

Proof. Consider the following maps on cohomology:

$$\begin{tikzcd} H^2(X \sqcup_V Y) \arrow[r, "\Psi"] \arrow[d, "\Phi"] & H^2(X\#_V Y) \\ H^2(X) \oplus H^2(Y) & \end{tikzcd} \tag{3.2}$$

Let $C_{X\sqcup_V Y} := \Phi^{-1}(\mathcal{C}^V_X \oplus \mathcal{C}^V_Y) \in H^2(X \sqcup_V Y)$. We can view this set as the collection of classes of symplectic forms in X and Y which are symplectic and equal on V. More precisely, we obtain $C_{X\sqcup_V Y}$ by pulling back of $\mathcal{C}_V$ under the restriction maps r_X, r_Y from (2.5):

$$\begin{tikzcd} C_{X\sqcup_V Y} \arrow[r] \arrow[d] & \mathcal{C}_X^V \arrow[d, "r_X"'] \\ \mathcal{C}_Y^V \arrow[r, "r_Y"] & \mathcal{C}_V \end{tikzcd}$$

The symplectic sum takes (X, ω_X) and (Y, ω_Y) and produces a symplectic manifold $(X\#_V Y, \omega)$. This will work for any relative symplectic forms ω_X and ω_Y identical on V. (By identical we mean that the symplectomorphism used to produce the symplectic singular manifold $X \sqcup_V Y$ maps these two forms symplectically to each other along V.) Thus any symplectic class $(\alpha_X, \alpha_Y) \in C_{X\sqcup_V Y}$ can be summed to produce a symplectic class $\alpha \in \mathcal{C}^{sum}_{X\#_V Y}$. Therefore $\Psi(\Phi^{-1}(\mathcal{C}_X^V \oplus \mathcal{C}_Y^V)) \subset \mathcal{C}^{sum}_{X\#_V Y}$.

On the other hand, given any symplectic class in $\mathcal{C}^{sum}_{X\#_V Y}$, any symplectic representative ω of such a class can be symplectically cut, such that the manifolds (X, V) and (Y, V) result with symplectic forms ω_X and ω_Y which agree on V. Hence, $\Psi^{-1}\mathcal{C}^{sum}_{X\#_V Y} \subset \Phi^{-1}(\mathcal{C}_X^V \oplus \mathcal{C}_Y^V)$. $\square$

Remark.

(1) Theorem 3.3 is valid for any dimension.
(2) In general, the cone of sum forms will be a strict subset of the relative cone $\mathcal{C}_M^V$. For example, consider $M = K3\#_{T^2_{\mathrm{f}}}K3$, the fiber sum of two K3 surfaces along a fiber torus. This has $b_2(M) = 45$, which is also the dimension of the relative cone $\mathcal{C}_M^{T^2_{\mathrm{f}}}$. On the other hand, $\mathcal{C}_{K3}^{T^2_{\mathrm{f}}}$ has dimension 22. Hence the cone of sum forms must be a strict subset of the relative cone. This indicates that a precise study of the second homology of the symplectic sum $M = X\#_V Y$ should be interesting, and we dedicate the rest of the section to this analysis.

3.3.2. The second homology of $M = X\#_V Y$. We assume that X, Y are 4-manifolds and that V has trivial normal bundle. The latter ensures that the class of V will exist in $H_2(X\#_V Y)$ after summing, albeit not the particular copy of V along which was summed. Denote the class of V by $f \in H_2(X\#_V Y)$. In this section, we shall describe a "natural" basis of the second homology with respect to the fiber sum operation, which will allow us to efficiently construct and deconstruct cohomology classes on $X\#_V Y$.

We begin by detailing the role of the maps involved in the symplectic sum in the structure of the second homology of $M = X\#_V Y$. The homology of M can be analyzed by the Mayer–Vietoris sequences for the triples $(X \sqcup_V Y, X, Y)$ and $(X\#_V Y, X\backslash V, Y\backslash V)$:

$$\begin{tikzcd} H_2(V) \arrow[r, "(j_X, j_Y)_*"] & H_2(X) \oplus H_2(Y) \arrow[r, "\phi"] & H_2(X \sqcup_V Y) \\ H_2(S_V) \arrow[u] \arrow[r, "\lambda"] & H_2(X\backslash V) \oplus H_2(Y\backslash V) \arrow[u] \arrow[r, "\rho"] & H_2(X\#_V Y) \arrow[u, "\psi"'] \\ R_V \arrow[u] & & \mathcal{R}_{X\#_V Y} \arrow[u] \end{tikzcd} \tag{3.3}$$

The map λ, defined on classes in the homology of the normal unit circle bundle S_V, is induced by the canonical identification of the tubular neighborhoods and the normal bundles. The map ρ is identity on the classes which are supported away from V and on classes supported near V is defined by the gluing map φ, in particular by Θ. The map $\phi : H_2(X) \oplus H_2(Y) \to H_2(X \sqcup_V Y)$ produces classes with the appropriate matching conditions on V as determined by Θ in preparation for summing along V. The map $\psi : H_2(X\#_V Y) \to H_2(X \sqcup_V Y)$ is induced by the gluing map φ, in particular by the embeddings j_* and the isomorphism Θ. Then ψ correctly decomposes classes in $X\#_V Y$ in accordance with the symplectic gluing. The set $\mathcal{R}_{X\#_V Y}$ is completely determined by R_V and an understanding of how these classes bound in M. The outer columns are exact, for a detailed discussion of the kernel R_V see [**13**, Sect. 5].

We are in the four-dimensional setting, thus when we consider the Poincaré dual diagram to (3.3), we obtain in particular the following component:

$$\begin{tikzcd} H^2(X \sqcup_V Y) \arrow[r, "\Psi"] \arrow[d, "\Phi"'] & H^2(X\#_V Y) \arrow[r] & \mathcal{R}^D_{X\#_V Y} \arrow[r] & 0 \\ H^2(X) \oplus H^2(Y) & & & \end{tikzcd} \tag{3.4}$$

This is precisely the diagram used in the proof of Theorem 3.3. This motivates the detailed discussion of the generators of the second homology which follows.

To explicitly describe the second homology of $M = X\#_V Y$, we consider the following part of the Mayer–Vietoris sequences as before:
(3.5)
$$
\begin{tikzcd}
H_2(X)\oplus H_2(Y) \arrow[r, "\phi"] & H_2(X\sqcup_V Y) \arrow[r, "\delta"] & H_1(V) \\
H_2(X\backslash V)\oplus H_2(Y\backslash V) \arrow[u] \arrow[r, "\rho"] & H_2(M) \arrow[u, "\psi"] \arrow[r, "{(\gamma,t)}"] & H_1(S_V)\simeq H_1(S^1)\oplus H_1(V) \arrow[u, "\mu"] \\
 & \mathcal{R}_{X\#_V Y} \arrow[u] &
\end{tikzcd}
$$

Define the subgroups $x = \rho(H_2(X\backslash V), 0)$ and $y = \rho(0, H_2(Y\backslash V))$ of $H_2(M)$. Elements in x and y are representable by submanifolds in X, Y resp. which are supported away from the submanifold V. Denote generators of x and y by x_i and y_i. Note that $x_i\cdot f = 0 = y_i \cdot f$ in the intersection form.

Define the subgroup $\Gamma = (\gamma,t)^{-1}(H_1(S^1),0)\simeq\mathbb{Z}$. Representatives γ^M of this subgroup are submanifolds formed from submanifolds $\gamma^X\in X$ and $\gamma^Y\in Y$, these being supported in any neighborhood of V and thus affected by the sum construction. The submanifolds γ^* intersect V nontrivially and $\gamma^*\cdot f = 1$. We denote the generator of this subgroup by γ^M. Note that $\psi(\gamma^M)$ is always nontrivial: $\psi(\gamma^M) = (\gamma^X, \gamma^Y)$.

Define $\tau = (\gamma,t)^{-1}(0, H_1(V))$ and $\mathcal{R}_M = \ker\psi$. The following holds.

Lemma 3.4. $\psi(\tau) = \mathrm{coker}(\phi)$

Proof. Let g_i be generators of $H_1(V)$. Then τ is generated by

$$\tau_i = (\gamma,t)^{-1}(0,g_i).$$

The commutativity of (3.5) shows

$$\delta\psi(\tau_i) = (\mu)(\gamma,t)\tau_i = \mu(0,t(\tau_i)) = t(\tau_i) = g_i. \tag{3.6}$$

Thus it follows from $g_i\neq 0$ that $\psi(\tau_i)\notin\ker\delta$. Thus

$$\psi(\tau_i)\in H_2(X\sqcup_V Y)/\ker\delta = H_2(X\sqcup_V Y)/\mathrm{im}\phi = \mathrm{coker}(\phi). \tag{3.7}$$

Hence $\psi(\tau)\subset\mathrm{coker}(\phi)$.

From the definition of coker (ϕ) it follows that coker $(\phi) = H_2(X\sqcup_V Y)/\mathrm{im}\phi$ and hence any nontrivial element c of the cokernel is supported in a neighborhood of V, but is not generated out of elements in $H_2(X)$ and $H_2(Y)$. In particular, $c\cdot\gamma^* = 0$. Thus any lift $\tilde\tau$ of an element c in the cokernel by ψ to $H_2(M)$ has $\gamma(\tilde\tau) = 0$. Furthermore,

$$0\neq\delta(c) = \delta\psi(\psi^{-1}\tau) = (\mu)(\gamma,t)\tilde\tau = t(\tilde\tau), \tag{3.8}$$

so $(\gamma, t)\tilde{\tau} \in 0 \oplus H_1(V)$ is nontrivial. Therefore, $\tilde{\tau} \in \tau$ and thus coker $(\phi) \subset \psi(\tau)$. $\square$

Consider now the set $\mathcal{R}_{X\#_V Y}$. In particular, let us describe how objects in this set are generated from submanifolds of $X \backslash V$ and $Y \backslash V$. Define $\mathcal{R}_Y$ as the image of the map $i_*^Y \Delta : H_1(V) \to H_2(Y \backslash V)$ where i^Y is the inclusion of S_V into $Y \backslash V$ and $\Delta : H_1(V) \to H_2(S_V)$ stems from the Gysin sequence for the bundle $S_V \to V$. If V is two-dimensional, then the map Δ is an injection. Furthermore, consider for each simple closed curve l in V the preimage in $\partial N_Y V$, this is a torus. Such tori are called rim tori. We restate a result in [**13**]:

Lemma 3.5 ([**13**, **Lemma 5.2**]). *Each element $R \in \mathcal{R}_Y$ can be represented by a rim torus.*

Under symplectic gluing, rim tori glue and are the elements of $\mathcal{R}_M$, in particular the elements $i_*^X \Delta l$ and $-i_*^Y \Delta l$ for some loop $l \in H_1(V)$ glue. An example of this process is the generation of nonfiber tori in $K3$ when viewed as a sum $E(1)\#_{T^2_f} E(1)$ (see [**12**, Sect. 3.1]). These are then precisely the elements of $\mathcal{R}_{K3} = \{T_1^2, T_2^2\}$ and, in the same process, $\tau = \{S_1^2, S_2^2\}$ is produced. This accounts for the two new hyperbolic terms in the intersection form. We observe the following

Lemma 3.6. *Assume that $H_1(V) \to H_1(Y)$ is an injection and V has trivial normal bundle. Then Y has no rim tori and $\tau = 0 = \mathcal{R}_{X\#_V Y}$.*

Proof. To prove that Y has no rim tori, it will suffice to show that $i_* : H_2(S_V) \to H_2(Y \backslash V)$ is trivial on elements which are trivial under the map $\pi_* : H_2(S_V) \to H_2(V)$. Therefore, consider the map $\xi : H_3(Y) \to H_2(S_V)$ where $S_V = V \times S^1$. Let $W \in H_3(Y)$, then $\xi(W) = W \cap S_V = W \cap (V \times S^1)$. In particular, $W \cap V \in H_1(V)$, thus by the injectivity assumption, if this intersection is nontrivial, it is nontrivial in $H_1(Y)$. Therefore, the map ξ maps $H_3(Y)$ onto the space generated by $\alpha \times S^1$ and $\beta \times S^1$, where α, β are generators of $H_1(V)$. This space is the kernel of π_* and the map i_* is trivial on it.

Let us now consider $\mathcal{R}_{X\#_V Y}$. Elements of this set are constructed by symplectic gluing from elements in $X \backslash V$ and $Y \backslash V$, or equivalently, from classes in $H_2(X \backslash V)$ and $H_2(Y \backslash V)$. In particular, considering (3.3), only classes in R_V are relevant, these are precisely those classes that do not map trivially to $H_2(V)$. As we have seen above, our assumption implies that $H_2(S_V) \to H_2(Y \backslash V)$ is trivial. Hence every element in $\mathcal{R}_{X\#_V Y}$ would be trivial.

Thus $\mathcal{R}_{X\#_V Y} = 0$. Furthermore, $\tau = 0$ is clear from the assumption. $\square$

The previous discussion allows us to explicitly state a set of generators for $H_2(M)$:

- $\{f\}$ is the fiber class present in both X and Y, in our case this is the class of V;

- $\{x_i\}$, $\{y_i\}$;
- $\{k_i\} \subset \mathcal{R}_{X\#_V Y}$, generators that are represented by submanifolds mapping to 0 in $X \sqcup_V Y$;
- $\{\gamma\}$ generated out of elements of the homology of both X and Y, e.g $[\Sigma_2]$ from copies of T^2 in Example 3.1. Note that this is the origin for the nonsurjectivity of the map ψ: $\psi[\Sigma_2]$ will always have a fixed relative orientation of the two copies of T^2 into which Σ_2 degenerates. Thus the pairing of the tori with opposite orientation will not lie in the image of ψ;
- $\{\tau_i\} \subset \tau$; these objects will persist in $X \sqcup_V Y$ and hence contribute to its homology as well.

Given this set of generators, we can explicitly state how an element in the cone of sum forms decomposes: Given $\alpha = \sum_i a_i X_i + b_i Y_i + cF + g\Gamma + e_i\mathcal{R}_i + t_i T_i \in \mathcal{C}^{\mathrm{sum}}$ and taking the Poincaré dual basis of the one given above, we obtain two forms $\alpha_X = \sum_i a_i X_i + c^X F^X + g\Gamma^X$ and $\alpha_Y = \sum_i b_i Y_i + c^Y F^Y + g\Gamma^Y$ on X resp. Y. Note that this is ultimately a direct result of Theorem 3.3.

3.3.3. Good sums. If we know the relative cones of X and Y, then, considering (3.3), we should obtain information on the structure of the relative cone on $M = X\#_V Y$ by using the Poincaré duals of the maps ϕ and ψ. For this to work nicely, one needs ϕ to be surjective and ψ to be injective. We thus make the following definition:

Definition 3.7. A symplectic sum $M = X\#_V Y$ is called good if ϕ is surjective and ψ is injective.

This statement is equivalent to $\tau = 0 = \mathcal{R}$, and Lemma 3.6 provides a simple criterion to check this.

Theorem 3.8. *Suppose $M = X\#_V Y$ is good and V has trivial normal bundle. If for X, Y,*

$$\mathcal{C}_*^V = \{\alpha \in \mathcal{P}_* \mid \alpha \cdot \mathfrak{V} > 0\}, \tag{3.9}$$

then

$$\mathcal{C}^{\mathrm{sum}}_{X\#_V Y} = \{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\}. \tag{3.10}$$

Consequently, $\mathcal{C}_M^V = \{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\}$.

Proof. The second result is immediate: Theorem 3.3 and Lemma 2.7 show that $\mathcal{C}^{\mathrm{sum}}_{X\#_V Y} \subset \mathcal{C}_M^V \subset \{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\}$.

The first result follows, if we can show $\{\alpha \in \mathcal{P}_M \mid \alpha \cdot \mathfrak{V} > 0\} \subset \mathcal{C}^{\mathrm{sum}}_{X\#_V Y}$.

We proceed as remarked above, using Theorem 3.3: Taking the Poincaré dual basis of the one given above, we can write each $\alpha \in \{\alpha \in \mathcal{P}_M \mid \alpha{\cdot}\mathfrak{V} > 0\}$ as

$$\alpha = \sum_i a_i X_i + b_i Y_i + cF + g\Gamma + e_i\mathcal{R}_i + t_i T_i; \quad g > 0. \tag{3.11}$$

As $\mathcal{R} = 0 = \tau$, the last two terms drop.

We must now show, that $\alpha \in \mathcal{C}^{\text{sum}}_{X\#_V Y}$. We thus choose a possible pair of classes α_X and α_Y in $H^2(X)$ resp. $H^2(Y)$ as determined by Theorem 3.3 and show that this can be done in such a way as to ensure that they are representable by a relative symplectic form. We first determine a relation which preserves the volume. In the following, we show how to choose this pair, so that they lie in their respective relative cones $\mathcal{C}^V_*$. Hence the class α can be obtained by summing two classes representable by relative symplectic forms and thus, by Gompf's result, $\alpha \in \mathcal{C}^{\text{sum}}_{X\#_V Y}$.

Choose the candidates for classes summing to α as follows:

$$\begin{aligned} \alpha_X &= \sum_i a_i X_i + c^X F^X + g\Gamma^X \in H^2(X), \\ \alpha_Y &= \sum_i b_i Y_i + c^Y F^Y + g\Gamma^Y \in H^2(Y), \end{aligned} \tag{3.12}$$

where F^* and Γ^* are the Poincaré duals on X, Y. The coefficient g must be the same for both, as $g = \alpha(\mathfrak{V}) = \alpha_X(\mathfrak{V}) = \alpha_Y(\mathfrak{V})$. The class F has $F^2 = 0$ due to the triviality of the normal bundle of V, similarly $(F^*)^2 = 0$. The volume of each of these is

$$\alpha^2 = \left(\sum a_i X_i\right)^2 + \left(\sum b_i Y_i\right)^2 + (g\Gamma)^2 + 2\sum\left(a_i X_i g\Gamma + b_i Y_i g\Gamma\right) + cFg\Gamma \tag{3.13}$$

and

$$\alpha_X^2 = \left(\sum a_i X_i\right)^2 + g^2(\Gamma^X)^2 + 2\sum\left(a_i X_i g\Gamma^X\right) + c^X F^X g\Gamma^X. \tag{3.14}$$

Thus the difference of the volumes is calculated to be

$$\alpha^2 - \alpha_X^2 - \alpha_Y^2 = (g\Gamma)^2 - \left(g\Gamma^X\right)^2 - \left(g\Gamma^Y\right)^2 \tag{3.15}$$

$$+\, 2\left(\sum a_i X_i g\Gamma - \sum a_i X_i g\Gamma^X\right) \tag{3.16}$$

$$+\, 2\left(\sum b_i Y_i g\Gamma - \sum b_i Y_i g\Gamma^Y\right) \tag{3.17}$$

$$+\, 2\left(cFg\Gamma - c^X F^X g\Gamma^X - c^Y F^Y g\Gamma^Y\right) \tag{3.18}$$

Note the following: The morphism $\Psi : H^2(X \sqcup_V Y) \to H^2(M)$ relates the intersection forms, giving the following relations:

(1) $(\Gamma^X)^2 + (\Gamma^Y)^2 = (\Gamma^X \oplus \Gamma^Y)^2 = \Psi((\Gamma^X \oplus \Gamma^Y)^2) = \Psi(\Gamma^X \oplus \Gamma^Y)^2 = \Gamma^2$;
(2) $\alpha_i X_i \beta\Gamma = \Psi(\alpha_i X_i(\beta\Gamma^X \oplus \beta\Gamma^Y)) = \alpha_i X_i \beta\Gamma^X$.

Applying these relations, it follows immediately that 3.15 is trivial,

$$3.16 \Rightarrow a_i X_i g\Gamma - a_i X_i g\Gamma^X = a_i(g-g)X_i\Gamma^X = 0 \tag{3.19}$$

and analogously for 3.17 and Y. Furthermore, (3.18) becomes

$$\begin{aligned} cFg\Gamma - c^X F^X g\Gamma^X - c^Y F^Y g\Gamma^Y &= cFg\Gamma - \Psi(c^X F^X g\Gamma^X + c^Y F^Y g\Gamma^Y) \\ &= cFg\Gamma - \Psi(c^X F^X + c^Y F^Y)g\Gamma. \end{aligned} \tag{3.20}$$

The condition for this to vanish is

$$\Psi(c^X F^X + c^Y F^Y) = cF. \tag{3.21}$$

Thus, by choosing $c = c^X + c^Y$ we preserve the volume.

Now, we must show that this can be done in such a way as to ensure $\alpha_*^2 > 0$. Choose c^X, c^Y so that volume is preserved. Then $\alpha^2 = \alpha_X^2 + \alpha_Y^2 > 0$, and we may assume $\alpha_X^2 > 0$. This holds true for any choice of c^* satisfying 3.21.

Squaring α_X and denoting $B = \sum a_i X_i + g\Gamma^X$, we obtain

$$f(c^X) = \alpha_X^2 = B^2 + 2B \cdot F^X c^X + (c^X)^2 (F^X)^2 = B^2 + 2B \cdot F^X c^X. \tag{3.22}$$

We can always solve $f(c^X) = \rho$ for any $\rho > 0$. Thus we can ensure that $\alpha^2 > \alpha_X^2 > 0$ holds. Then also $\alpha_Y^2 = \alpha^2 - \alpha_X^2 > 0$ holds, and thus each α_* must lie in $\mathcal{C}_*^V$, hence $\tilde{\alpha} = (\alpha_X, \alpha_Y) \in \mathcal{C}_{X \sqcup_V Y}$ by definition of this set. Thus $\{\alpha \in \mathcal{P}_* \mid \alpha \cdot \mathfrak{V} > 0\} \subset \mathcal{C}^{sum}_{X\#_V Y}$. □

This result is of particular interest, as it shows that good sums preserve the structure of the relative cone. Thus, if X, Y have relative cones as assumed in the theorem and the sum is good, we can apply this result repeatedly to obtain the relative cone of $nX\#_V mY$:

$$\mathcal{C}^V_{nX\#_V mY} = \{\alpha \in \mathcal{P}_{nX\#_V mY} \mid \alpha \cdot \mathfrak{V} > 0\}. \tag{3.23}$$

4. Symplectic cone of certain T^2-fibrations

4.1. $T^2 \times \Sigma_g$. We now show that Theorem 3.8 can be applied to $T^2 \times \Sigma_g$. The results of the previous section assume two things: a certain form of the relative symplectic cone and that the sum be good.

Fix $Y = T^2 \times \Sigma_k$. Thus by Lemma 3.6 we do not need to verify the condition $\mathcal{R} = 0 = \tau$ when applying Theorem 3.8, i.e., all sums $X\#_V Y$ are good.

The following result follows immediately:

Theorem 4.1. *Let $M = T^2 \times \Sigma_k$. Then*

$$\mathcal{C}_M^{T^2_{\mathrm{f}}} = \{\alpha \in \mathcal{P}_M \mid \ \alpha \cdot [T^2_{\mathrm{f}}]^D > 0\}. \tag{4.1}$$

Consequently, $\mathcal{C}_M = \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}$.

Proof. We proceed by induction: Let $M = T^4$. Then the result holds due to Lemma 2.3 and Corollary 2.10. Summing repeatedly we obtain

$$M = T^2 \times \Sigma_k = T^4 \#_{T^2_{\mathrm{f}}} (T^2 \times \Sigma_{k-1}).$$

Using the induction hypothesis, which ensures that

$$\mathcal{C}_{T^2\times\Sigma_{k-1}}^{T^2_{\mathrm{f}}} = \{\alpha \in \mathcal{P}_{T^2\times\Sigma_{k-1}} \mid \ \alpha\cdot[T^2_{\mathrm{f}}]^D > 0\},$$

the result now follows from Theorem 3.8 (see equation (3.23)) and Corollary 2.10. Note that $b^+ \geq 3$ for any k, hence by Thm IV.2.7, [**1**], we have $p_g > 0$. □

As noted in the Introduction, this result also follows from results in [**4**, **5**].

Remark (Fibered symplectic forms). Every class in $\mathcal{C}_M$ can be represented by a symplectic form which restricts to a symplectic form on the fibers of M. Denote the set of such forms by $\mathcal{S}$. Then this set is contractible (and nonempty) ([**11**, Thm. 1.4]). See also McDuff [**25**].

4.2. $X\#(T^2\times\Sigma_k)$. In the following we allow the fibration to have singular or multiply covered fibers. If we sum along a generic fiber, avoiding these special fibers, we find no obstruction to applying the methods developed above.

Theorem 4.2. *Let X be a minimal symplectic manifold with $b^+ = 1$. Let $V \subset X$ be a torus with trivial normal bundle and $\mathcal{C}_X^V \neq \emptyset$. Consider the manifold $M = X\#_V Y$. Then*

$$\mathcal{C}_M = \mathcal{P}^{\mathfrak{V}} \cup \mathcal{P}^{-\mathfrak{V}}.$$

Proof. We begin with the trivial case: Assume that $X = S^2\times T^2$. Then the fiber sum is a trivial sum and we obtain $M = Y$. The result was shown in Theorem 4.1.

Assume in the following that $X \neq S^2\times T^2$. Using the assumptions, we obtain from Corollary 2.17 that $\mathcal{C}_X^V = \mathcal{P}^{\mathfrak{V}}$. Lemma 3.6 and Theorem 3.8 now show that $C_M^V = \mathcal{P}^{\mathfrak{V}} \subset \mathcal{P}_M$, hence by Lemma 2.8 and Lemma 2.9 we obtain

$$\mathcal{P}^{\mathfrak{V}} \cup \mathcal{P}^{-\mathfrak{V}} \subset \mathcal{C}_M.$$

Let ω be a relative symplectic form on (X,V). Denote the corresponding canonical class by K_ω. The adjunction equality shows that $K_\omega\cdot\mathfrak{V} = 0$. Hence, if $K_\omega\cdot K_\omega \geq 0$, it follows from Lemma 2.1 that $K_\omega = a\mathfrak{V}$ for some $a\in\mathbb{R}$. If $K_\omega\cdot K_\omega < 0$, then X is a S^2-bundle over a Riemann surface of genus $g\geq 2$, hence contains no torus with trivial normal bundle.

Moreover, from the symplectic sum construction, it follows for any symplectic form ω on M obtained from relative symplectic forms ω_1 and ω_2 on (X,V) resp. (Y,V) that

$$K_\omega = K_{\omega_1} + K_{\omega_2} + 2\mathfrak{V}.$$

Thus K_ω is a multiple of $\mathfrak{V}$ for such sum symplectic forms. If K_ω is a nonzero multiple of $\mathfrak{V}$, then $\mathcal{C}_M \subset \mathcal{P}^{\mathfrak{V}}\cup\mathcal{P}^{-\mathfrak{V}}$ as K_ω is a SW basic class and we have proven the theorem.

We must show that K_ω is a nonzero multiple of $\mathfrak{V}$. The canonical class of Y is a positive multiple of $\mathfrak{V}$. Thus, if K_{ω_1} is torsion, we are done. Assume that K_{ω_1} is nontorsion. Then the classification in [**19**, Prop. 5.2], shows that K_{ω_1} is a negative multiple of $\mathfrak{V}$ only if $X = \mathbb{C}P^2$, $S^2 \times S^2$ or a S^2-bundle over T^2. Only the trivial S^2-bundle over T^2 admits a torus with trivial normal bundle and we have assumed that $X \neq S^2 \times T^2$. □

Corollary 4.3. *Let X be a minimal elliptic Kähler surface with $p_g = 0$ and $M = X\#_{T^2_{\mathrm{f}}}Y$. Then Conjecture* 1.1 *holds, i.e.,*

$$C_M = \mathcal{P}^{c_1(M)} \cup \mathcal{P}^{-c_1(M)}.$$

Proof. The condition $p_g = 0$ for the Kähler manifold X implies $b^+ = 1$, see [**1**, Thm IV.2.7]. The result now follows from the previous theorem and the uniqueness of the canonical class in the Kähler class. □

Remark. The manifold X could be an Enriques surface, a hyperelliptic surface or a Dolgachev surface.

5. Appendix

Let V be a fixed smooth codimension 2 submanifold of a symplectic manifold (X, ω). Let $\mathcal{J}_V$ be the set of almost complex structures compatible with ω such that V is pseudoholomorphic for each $j \in \mathcal{J}_V$. We wish to show that $\mathcal{J}_V$ has a rich enough structure to allow for genericity statements for J-holomorphic curves. These results are presumably known to experts in the field, the methods used can be found in [**30**, **35**]; we include them for completeness. Let $A \in H_2(X)$ be any class, except that in the case $A^2 = 0 = K_\omega(A)$ the class A should be indivisible. We begin by defining a universal space which we shall use throughout this section: Fix a closed compact Riemann surface Σ. The universal model $\mathcal{U}$ is defined as follows: This space will consist of Diff(Σ) orbits of a 4-tuple (i, u, J, Ω) with

(1) $u : \Sigma \to X$ an embedding of a finite set of points from a Riemann surface Σ such that $u_*[\Sigma] = A$ and $u \in W^{k,p}(\Sigma, X)$ with $kp > 2$,
(2) $\Omega \subset X$ a set of m distinct points (with $\Omega = \emptyset$ if $m \leq 0$) such that $\Omega \subset u(\Sigma)$,
(3) i a complex structure on Σ and $J \in \mathcal{J}_V$.

Note that every map u is locally injective.

In order to show the necessary genericity results, we will call upon the Sard-Smale Theorem. This will involve the following technical difficulty: The spaces $\mathcal{J}_V$ and any subsets thereof which we will consider are not Banach manifolds in the C^∞-topology. However, the results we wish to obtain are for smooth almost complex structures. In order to prove our results, we need to apply Taubes trick (see [**33**] or [**30**, Sect. 3.2]), which replaces the smooth spaces by C^l-almost complex structures and apply the Sard-Smale theorem in that setting. Then, one constructs a countable collection of sets, whose intersection is the generic set of smooth structures, and shows that each of these is dense and open by explicit argumentation in the space of smooth structures. We will not go through this technical step but implicitly assume this throughout the section, details can be found in [**30**, Ch. 3].

Lemma A.1. *Let $A \in H_2(X, \mathbb{Z})$, $A \neq \mathfrak{V}$ and $k(A) = \frac{1}{2}(A^2 - K_\omega(A)) \geq 0$. Let Ω denote a set of $k(A)$ distinct points in X. Denote the set of pairs $(J, \Omega) \in \mathcal{J}_V \times X^{k(A)}$ by $\mathcal{I}$. Let $\mathcal{J}_V^A$ be the subset of pairs (J, Ω) which are nondegenerate for the class A in the sense of Taubes* [**32**, Def. 2.1]. *Then $\mathcal{J}_V^A$ is a set of second category in $\mathcal{I}$.*

The term "nondegenerate as defined by Taubes" states that for the pair (J, Ω) the linearization of the operator $\overline{\partial}$ at any J-holomorphic submanifold of X representing the class A and containing the set Ω has trivial cokernel. Note that the universal model excludes multiple covers of the submanifold V in the case that $A = a\mathfrak{V}$ for $a \geq 2$, and we can thus assume that any map $u : \Sigma \to M$ with $[u(\Sigma)] = A$ satisfies $u(\Sigma) \not\subset V$.

Proof. To prove this statement, we will define a map $\mathcal{F}$ from a universal model $\mathcal{U}$ to a bundle with fiber $W^{k-1,p}(\Lambda^{0,1}T^*\Sigma \otimes u^*TX)$ and show that it is submersive at its zeroes. Then we can apply the Sard-Smale theorem to obtain that $\mathcal{J}_V^A$ is of second category.

Define the map $\mathcal{F}$ as $(i, u, J, \Omega) \mapsto \overline{\partial}_{i,J}u$. Then the linearization at a zero (i, u, J, Ω) is given as

$$\mathcal{F}_*(\xi, \alpha, Y) = D_u\xi + \frac{1}{2}(Y \circ du \circ i + J \circ du \circ \alpha) \tag{5.1}$$

where D_u is Fredholm, Y and α are variations of the respective almost complex structures.

Consider $u \in \mathcal{U}$ such that there exists a point $x_0 \in \Sigma$ with $u(x_0) \in X \backslash V$ and $du(x_0) \neq 0$. (The second condition is satisfied almost everywhere, as u is a J-holomorphic map.) Then there exists a neighborhood N of x_0 in Σ such that

(1) $du(x) \neq 0$,
(2) $u(x) \notin V$ for all $x \in N$.

In particular, we know that the map u is locally injective on N. Furthermore, we can find a neighborhood in N, such that there are no constraints on the almost complex structure $J \in \mathcal{J}_V$, i.e., this neighborhood does not intersect V. In particular, any variation Y with support in N leaves V J-holomorphic and is, therefore, admissable. Denote this open set by N as well.

Let $\eta \in \operatorname{coker} \mathcal{F}_*$. Consider any $x \in N$ with $\eta(x) \neq 0$. Then [**30**, Lemma 3.2.2], provides a matrix Y_0 with the properties

- $Y_0 = Y_0^T = J_0 Y_0 J_0$ with J_0 the standard almost complex structure in a local chart, and
- $Y_0[du(x) \circ i(x)] = \eta(x)$.

Choose any variation Y of J with support on N such that $Y(u(x)) = Y_0$. Then define the map $f : N \to \mathbb{R}$ by $\langle Y \circ du \circ i, \eta\rangle$. We can find an open set N_1 in N such that $f > 0$ on that open set. Using the local injectivity of the map u and arguing as in [**30**, Sect. 3.2], we can find a neighborhood $N_2 \subset N_1$ and a neighborhood $U \subset M$ of $u(x_0)$ such that $u^{-1}(U) \subset N_2$. Choose a cutoff function β supported in U such that $\beta(u(x)) = 1$. Hence in particular

$$\int_\Sigma \langle \mathcal{F}_*(0, 0, \beta Y), \eta\rangle > 0 \tag{5.2}$$

and therefore $\eta(x) = 0$. This result holds for any $x \in N$, therefore η vanishes on an open set.

As we have assumed $\eta \in \operatorname{coker} \mathcal{F}_*$, it follows that

$$0 = \int_\Sigma \langle \mathcal{F}_*(\xi, 0, 0), \eta\rangle \; = \; \int_\Sigma \langle D_u\xi, \eta\rangle$$

for any ξ. Then it follows that $D_u^*\eta = 0$ and $0 = \triangle\eta + l.o.t.$ Therefore Aronszajn's theorem allows us to conclude that $\eta = 0$ and hence $\mathcal{F}_*$ is surjective.

Thus we have the needed surjectivity for all maps admitting x_0 as described above: $u(x_0) \notin V$ and $du(x_0) \neq 0$. As stated before, this last condition is fulfilled off a finite set of points on Σ. The first holds for any map u in class A as we have assumed that $A \neq \mathfrak{V}$.

Now apply the Sard-Smale theorem to the projection onto the last two factors of (i, u, J, Ω). □

Given $J \in \mathcal{J}_V$ and $\Omega \in X^m$, define the set $\mathcal{K}_V^A(J, \Omega)$ to be the set of J-holomorphic submanifolds which are abstractly diffeomorphic to a Riemann surface Σ, contain the set Ω and represent the class A. Then the same methods as in the above proof together with index calculations of the projection operator onto the last two factors lead to the following results: If $m > k(A)$ or $m < 0$, then $\mathcal{K}_V^A(J, \Omega)$ is empty for generic (J, Ω), if $m = k(A)$, then $\mathcal{K}_V^A(J, \Omega)$ is a smooth 0-dimensional manifold for generic (J, Ω). In particular, there exists a set of second category in $\mathcal{J}_V$, such that if $k(A) \geq 0$ then any pseudoholomorphic submanifold in class A meets a generic set of at most $k(A)$ distinct points.

As we have seen in the above proof, for the class $A = \mathfrak{V}$ which may have representatives which do not lie outside of V, we must be careful. In particular, it is conceivable that the particular submanifold V chosen may not be generic in the sense of Taubes, i.e., the set $\mathcal{J}_V$ may contain almost complex structures for which the linearization of $\overline{\partial}_J$ at the embedding of V is not surjective. The rest of this section addresses this issue. We begin by showing that the cokernel of the linearization of the operator $\overline{\partial}_J$ at a J-holomorphic embedding of V has the expected dimension:

Let j be an almost complex structure on V. Define $\mathcal{J}_V^j = \{J \in \mathcal{J}_V | J|_V = j\}$ and call any J-holomorphic embedding of V for $J \in \mathcal{J}_V^j$ a j-holomorphic embedding.

Lemma A.2. *Fix a j-holomorphic embedding $u : (\Sigma, i) \to (X, J)$ for some $J \in \mathcal{J}_V^j$. If $k(\mathfrak{V}) \geq 0$, then there exists a set $\mathcal{J}_V^{g,j}$ of second category in $\mathcal{J}_V^j$ such that for any $J \in \mathcal{J}_V^{g,j}$ the linearization of $\overline{\partial}_{i,J}$ at the embedding u is surjective. If $k(\mathfrak{V}) < 0$, then then there exists a set $\mathcal{J}_V^{g,j}$ of second category in $\mathcal{J}_V^j$ such that the submanifold V is rigid in X.*

Let us first explain the structure of the proof before giving the exact proof. We follow ideas of [**35**, Sect. 4]. We need to show that for a fixed embedding $u : \Sigma \to X$ of V the linearization $\mathcal{F}_*$ of $\overline{\partial}_{i,J}$ at u has a cokernel of the correct dimension for generic $J \in \mathcal{J}_V^j$. To do so, we will consider the operator $\mathcal{G}(\xi, \alpha, J) := \mathcal{F}_*(\xi, \alpha, 0)$ at (i, u, J, Ω). We will show that the kernel of the linearization $\mathcal{F}_*$ for nonzero ξ has the expected dimension for generic J and hence the linearization of $\overline{\partial}_{i,J}$ at u also has the expected dimension. Note also that for any $J \in \mathcal{J}_V^j$, the map u is J-holomorphic.

What is really going on in this construction? The operator $\mathcal{F}$ is a section of a bundle over $\mathcal{U}$, as described above. Further, we consider a map $\mathcal{U} \to \mathcal{J}_V^j$. In this map, we fix a "constant section" (u, j), i.e., we consider the structure of the tangent spaces along a fixed map u where we do not let the almost complex structure along V vary. On the other hand, it is only this structure j which makes u pseudoholomorphic. Hence fixing (u, j) is akin to considering a constant section in the bundle $\mathcal{U} \to \mathcal{J}_V^j$. In particular, we are only interested in the component of the tangent space along this section which corresponds to the tangent space along the moduli space $\mathcal{M} = \mathcal{F}^{-1}(0)$, as this will give us insight into the dimension of $\mathcal{M}$. Along (u, j), this is precisely the component of the kernel of $\mathcal{F}_*$ with $Y = 0$ as the complex structure is fixed on V, i.e., the set of pairs (ξ, α) such that $\mathcal{F}_*(\xi, \alpha, 0) = 0$, which corresponds to exactly the zeroes of $\mathcal{G}$. When considering the zeroes of the map $\mathcal{G}$ viewed over $\mathcal{J}_V^j$, we find that this is a collection of finite dimensional vector spaces. We may remove any part of these spaces, so long as we leave an open set, which is enough to allow us to determine the dimension of the underlying vector spaces. Hence, removing $\xi = 0$, a component along which we cannot use our methods to determine the dimension of the kernel, still leaves a large enough set to be able to determine the dimension of the moduli space $\mathcal{M}$. We therefore want to show that the kernel of the linearization $\mathcal{F}_*$ for nonzero ξ which is the zero set of $\mathcal{G}$ for nonzero ξ has the expected dimension $\max\{k(\mathfrak{V}), 0\}$ for generic J.

Proof. The operator $\mathcal{G}$ is defined as

$$W^{1,p}(u^*TX) \times H_i^{0,1}(T_{\mathbb{C}}\Sigma) \times \mathcal{J}_V^j \to L^p(u^*TX \otimes T^{0,1}\Sigma)$$
$$(\xi, \alpha, J) \mapsto D_u^J \xi + \frac{1}{2} J \circ du \circ \alpha,$$

where the term $D_u^J = \frac{1}{2}(\nabla\xi + J\nabla\xi \circ i)$ for some J-hermitian connection ∇ on X, say for example the Levi–Civita connection associated to J. Note that we could define this operator also for a smooth embedding $u : \Sigma \to X$, but that we have fixed the almost complex structure on V and therefore $\overline{\partial}_{i,J} u = 0$ for any $J \in \mathcal{J}_V^j$.

Let (ξ, α, J) be a zero of $\mathcal{G}$. Linearize $\mathcal{G}$ at (ξ, α, J):

$$\mathcal{G}_*(\gamma, \mu, Y) = D_u^J \gamma + \frac{1}{2} \nabla_\xi Y \circ du \circ i + \frac{1}{2} J \circ du \circ \mu.$$

As stated above, we assume nonvanishing ξ, hence we can assume that $\xi \neq 0$ on any open subset. Let $\eta \in \operatorname{coker} \mathcal{G}_*$. Let $x_0 \in \Sigma$ be a point with $\eta(x_0) \neq 0 \neq \xi(x_0)$. In a neighborhood of $u(x_0) \in V$ the tangent bundle TX splits as $TX = N_V \oplus TV$ with N_V the normal bundle to V in X. With respect to this splitting, the map Y has the form

$$y = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

with all entries J-antilinear and $b|_V = 0$, thus ensuring that V is pseudoholomorphic and accounting for the fact that we have fixed the almost complex structure along V. Thus $\nabla_\xi Y$ can have a similar form, but with no restrictions on the vanishing of components along V. In particular, assuming η projected to N_V is nonvanishing, we can choose

$$\nabla_\xi Y = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

at x_0 such that $B(x_0)[du(x_0) \circ i(x_0)](v) = \eta^{N_V}(x_0)(v)$ and $B(x_0)[du(x_0) \circ i(x_0)](\overline{v}) = \eta^{N_V}(x_0)(\overline{v})$ for a generator $v \in T_{x_0}^{1,0}\Sigma$ and where η^{N_V} is the projection of η to N_V. Then, using the same universal model as in the previous lemma, we can choose neighborhoods of x_0 and a cutoff function β such that

$$\int_\Sigma \langle \mathcal{G}_*(0, 0, \beta Y), \eta \rangle > 0$$

and thus any element of the cokernel of $\mathcal{G}_*$ must have $\eta^{N_V} = 0$. An argument in [**35**] shows that the projection of η to TV must also vanish. Therefore the map $\mathcal{G}_*$ is surjective at the embedding $u : \Sigma \to V$.

Thus the set $\{(\xi, \alpha, J) | \mathcal{G}(\xi, \alpha, J) = 0,\ J \in \mathcal{J}_V^j,\ \xi \neq 0\}$ is a smooth manifold and we may project onto the last factor. Then applying Sard-Smale, we obtain a set $\mathcal{J}_V^{g,j}$ of second category in $\mathcal{J}_V^j$, such that for any $J \in \mathcal{J}_V^{g,j}$, the kernel of the linearization of $\overline{\partial}$ at nonzero perturbations ξ

of the map u is a smooth manifold of the expected dimension. In the case $k(\mathfrak{V}) \geq 0$, this however implies that $\mathcal{F}_*$ at (i, u, J, Ω) is surjective. Therefore, we have found a set $\mathcal{J}_V^{g,j}$ of second category in $\mathcal{J}_V^j$ such that the linearization of $\overline{\partial}_{i,J}$ at u is surjective at all elements of $\mathcal{J}_V^{g,j}$.

If however $k(\mathfrak{V}) < 0$, then this kernel is generically empty. This implies the rigidity of the embedding u of V. □

We have thus shown that for a fixed embedding we can find a generic set of almost complex structures among those making the embedding pseudoholomorphic, such that the linearization of $\overline{\partial}$ at u has cokernel of the expected dimension. We would like to state a similar result for the space $\mathcal{K}_V^{\mathfrak{V}}(J, \Omega)$ as we stated for the class A. In order to do so, note that in our model, we consider orbits under the action of $\mathrm{Diff}(\Sigma)$. Hence, given any two embeddings $u : (\Sigma, i) \to (X, J)$ and $v : (\Sigma, i) \to (X, \tilde{J})$ of V for $J, \tilde{J} \in \mathcal{J}_V^j$, there exists a $\phi \in \mathrm{Diff}(\Sigma)$ such that $u = v \circ \phi$, i.e., u and v correspond to the same point in the universal model $\mathcal{U}$.

For every almost complex structure j on V the previous results provide the following:

(1) A set $\mathcal{J}_V^{g,j}$ of second category in $\mathcal{J}_V^j$ with the property that the linearization of the operator $\overline{\partial}$ at a fixed j-holomorphic embedding of V is surjective ($k(\mathfrak{V}) \geq 0$) or is injective ($k(\mathfrak{V}) < 0$).
(2) Up to a map $\phi \in \mathrm{Diff}(\Sigma)$, there is a unique j-holomorphic embedding of V for all $J \in \mathcal{J}_V^j$.

Therefore, consider the following set:

$$\mathcal{J}_V^g = \bigcup_j \mathcal{J}_V^{g,j} \subset \bigcup_j \mathcal{J}_V^j = \mathcal{J}_V.$$

Note that we $\mathcal{J}_V^g$ is actually a disjoint union of sets. The following properties hold:

(1) The set $\mathcal{J}_V^g$ is dense in $\mathcal{J}_V$.
(2) The linearization of the operator $\overline{\partial}$ at a fixed j-holomorphic embedding of V is surjective ($k(\mathfrak{V}) \geq 0$) or is injective ($k(\mathfrak{V}) < 0$) for any $J \in \mathcal{J}_V^g$.
(3) Up to a map $\phi \in \mathrm{Diff}(\Sigma)$, there is a unique j-holomorphic embedding of V.

We can now state the final result concerning genericity that we will need:

Lemma A.3. *Let Ω denote a set of $k(\mathfrak{V})$ distinct points in X.*

(1) $k(\mathfrak{V}) \geq 0$*: Denote the set of pairs $(J, \Omega) \in \mathcal{J}_V \times X^{k(\mathfrak{V})}$ by $\mathcal{I}$ (with $\Omega = \emptyset$ if $k(\mathfrak{V}) \leq 0$). Let $\mathcal{J}_{\mathfrak{V}}$ be the subset of pairs (J, Ω) which are nondegenerate for the class A in the sense of Taubes* [**33**]*. Then $\mathcal{J}_{\mathfrak{V}}$ is dense in $\mathcal{I}$.*

(2) $k(\mathfrak{V}) < 0$*: There exists a dense set* $\mathcal{J}_{\mathfrak{V}} \subset \mathcal{J}_V$ *such that* V *is rigid, i.e., there exist no pseudoholomorphic deformations of* V *and there are no other pseudoholomorphic maps in class* $\mathfrak{V}$*.*

Proof. To begin, we will replace the set $\mathcal{J}_V \times X^{k(\mathfrak{V})}$ by $\mathcal{J}_V^g \times X^{k(\mathfrak{V})}$ which is a dense subset, as seen from the previous remarks. Further, for any $(J, \Omega) \in \mathcal{J}_V^g \times X^{k(\mathfrak{V})}$, we have surjectivity or injectivity of the linearization at the embedding of V.

Consider the case $k(\mathfrak{V}) \geq 0$. Fix a j on V. Then consider the set $\mathcal{J}_V^{g,j}$ provided by Lemma A.2. The linearization at the embedding of V is surjective for any $J \in \mathcal{J}_V^{g,j}$. For any element (i, u, J, Ω) of $\mathcal{U}$ with $u(\Sigma) \not\subset V$ representing the class $\mathfrak{V}$ and $J \in \mathcal{J}_V^{g,j}$, arguments as in the proof of Lemma A.1 provide the necessary surjectivity. Therefore, there exists a further set $\mathcal{J}_{\mathfrak{V}}^{g,j}$ of second category in $\mathcal{J}_V^{g,j} \times X^{k(\mathfrak{V})}$ such that any pair $(J, \Omega) \in \mathcal{J}_{\mathfrak{V}}^{g,j}$ is nondegenerate.

Define $\mathcal{J}_{\mathfrak{V}} = \bigcup_j \mathcal{J}_{\mathfrak{V}}^{g,j}$. This is a dense subset of $\mathcal{J}_V^g \times X^{k(\mathfrak{V})}$ such that any pair $(J, \Omega) \in \mathcal{J}_{\mathfrak{V}}$ is nondegenerate.

If $k(\mathfrak{V}) < 0$, then restrict to $\mathcal{J}_V^g$ as well. Thereby we have already ensured that V is rigid. Now apply the proof of Lemma A.1 to the universal model $\mathcal{U}$, which we modify to allow only maps $u : (\Sigma, i) \to (X, J)$ such that $u(\Sigma) \not\subset V$. Then we can find a set $\mathcal{J}_{\mathfrak{V}}$ of second category in $\mathcal{J}_V^g$ such that there exist no maps in class $\mathfrak{V}$ other than the embedding of V. $\square$

Note that by results of Taubes, if $k(\mathfrak{V}) < 0$, the set $\mathcal{J}_{\mathfrak{V}}$ is a set of first category in $\mathcal{J}$. Further, it is not clear whether it is possible to improve the denseness statement to include openness in $\mathcal{J}_V$. In the case $k(\mathfrak{V}) \geq 0$, this is also unclear.

Furthermore, if $k(\mathfrak{V}) \geq 0$, then we have shown that the set $\mathcal{K}_V^{\mathfrak{V}}(J, \Omega)$ has the desired properties, i.e., for a dense set of pairs (J, Ω) $\mathcal{K}_V^{\mathfrak{V}}(J, \Omega)$ is a smooth 0-dimensional manifold unless $m > k(\mathfrak{V})$, in which case it is generically empty.

Similar results have been proven by Jabuka in [**15**]. However, that result only provides an isotopic copy of V in the case $k(\mathfrak{V}) \geq 0$.

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