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SYMPLECTIC 4-MANIFOLDS WITH ARBITRARY FUNDAMENTAL GROUP NEAR THE BOGOMOLOV–MIYAOKA–YAU LINE

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In this paper, we construct a family of symplectic 4-manifolds with positive signature for any given fundamental group G that approaches the BMY line. The family is used to show that one cannot hope to do better than the BMY inequality in finding a lower bound for the function $f = \chi + b\sigma$ on the class of all minimal symplectic 4-manifolds with a given fundamental group.

1. Introduction

Let $\chi(S)$ and $\sigma(S)$ denote the Euler characteristic and signature of a closed 4-manifold, respectively. Minimal complex surfaces S of general type satisfy $c_1^2(S) > 0$, $\chi(S) > 0$ and

$$2\chi_h(S) - 6 \leq c_1^2(S) \leq 9\chi_h(S),$$

where $c_1^2(S) = 2\chi(S) + 3\sigma(S)$ and $\chi_h(S) = \frac{1}{4}(\chi(S) + \sigma(S))$. The second inequality is usually referred to as the Bogomolov–Miyaoka–Yau inequality. Finding symplectic (or Kähler) 4-manifolds on or near the BMY line has a long and interesting history [**2**, **3**, **5**, **8**–**11**].

All known examples of symplectic 4-manifolds on the BMY line, except $\mathbb{CP}^2$, have large fundamental groups. In fact, if S is a complex surface differing from $\mathbb{CP}^2$, the equality $c_1^2(S) = 9\chi_h(S)$ holds if and only if the unit disk $D^4 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 \leq 1\}$ covers S [**6**, **7**, **13**] and hence $|\pi_1(S)| = \infty$. One goal of 4-dimensional symplectic topology is to produce examples that fill in the geography with respect to (c_1^2, χ_h). In this article, we are interested in what can be said for a given fundamental group.

Stipsicz [**11**] constructed simply connected symplectic 4-manifolds C_n for which $c_1^2(C_n)/\chi_h(C_n) \to 9$ as $n \to \infty$. Our main theorem generalizes this result to any fundamental group.

Theorem 1.1. *Let G have a presentation with g generators $x_1, \dots, x_g$ and r relations $w_1, \dots, w_r$. For each integer $n > 1$, there exists a symplectic 4-manifold $M(G,n)$ with fundamental group G with Euler characteristic*

$$\chi(M(G,n)) = 75n^2 + 256n + 130 + 12(g + r + 1),$$

and signature

$$\sigma(M(G,n)) = 25n^2 - 68n - 78 - 8(g + r + 1).$$

Our interest in this question developed while investigating pairs $(a, b) \in \mathbb{R}^2$ for which the function $f = a\chi + b\sigma$ has a lower bound on the class of symplectic manifolds with a given fundamental group [**1**]. In that article, we considered the following.

Fix a finitely presented group G and let $\mathfrak{M}$ denote either the class $\mathfrak{M}(G)$ of closed symplectic 4-manifolds with fundamental group G or the class $\mathfrak{M}^{\min}(G)$ of minimal, closed symplectic 4-manifolds with fundamental group G.

For $b \in \mathbb{R}$, define $f_{\mathfrak{M}}(b) \in \mathbb{R} \cup \{-\infty\}$ to be the infimum

$$f_{\mathfrak{M}}(b) = \inf_{M \in \mathfrak{M}} \{\chi(M) + b\sigma(M)\}.$$

(In [**1**], we considered the infimum $f_{\mathfrak{M}}(a, b)$ of $a\chi + b\sigma$ on $\mathfrak{M}$ and showed that if $a \leq 0$, the infimum is $-\infty$. Thus we restrict to $f_{\mathfrak{M}}(1, b)$, which we more compactly denote by $f_{\mathfrak{M}}(b)$ in the present article.)

We showed in [**1**] that the set

$$D_{\mathfrak{M}} = \{b \mid f_{\mathfrak{M}}(b) \neq -\infty\}$$

(the *domain of* $f_{\mathfrak{M}}$) is an interval satisfying

$$[-1, 1] \subset D_{\mathfrak{M}(G)} \subset (-\infty, 1] \text{ and } \left[-1, \frac{3}{2}\right] \subset D_{\mathfrak{M}^{\min}(G)} \subset \left(-\infty, \frac{3}{2}\right].$$

The upper bounds are sharp; in fact $1 \in D_{\mathfrak{M}(G)}$ and $\frac{3}{2} \in D_{\mathfrak{M}^{\min}(G)}$.

We are interested in the value of the left endpoint e_G of $D_{\mathfrak{M}(G)}$, which is an intriguing invariant of a group G. (It may or may not be contained in $D_{\mathfrak{M}(G)}$.) Since $e_G \leq -1$, a straightforward argument shows that e_G is also the left endpoint of $D_{\mathfrak{M}^{\min}(G)}$.

In [**1**], we observed that the results of Stipsicz gives a better lower bound (than $-\infty$) when G is the trivial group, and so a consequence of the result of this article is an extension to all G. In fact Theorem 1.1 easily implies the following corollary.

Corollary 1.2. *For any finitely presented group G,*

$$D_{\mathfrak{M}(G)} \subset [-3, 1] \text{ and } D_{\mathfrak{M}^{\min}(G)} \subset \left[-3, \frac{3}{2}\right].$$

The BMY inequality $c_1^2 \leq 9\chi_h$ is equivalent to $f_{\mathfrak{M}^{\min}(G)}(-3) \geq 0$ provided G is not a surface group. Hence, the BMY conjecture and Corollary 1.2 together imply that $e_G = -3$. Thus, a weaker form of the BMY conjecture could be stated as follows.

Conjecture 1.3 (Weak BMY Conjecture)**.** *For each finitely presented group G, $e_G = -3$.*

2. The construction

We use the following notation. If X and Y are symplectic 4-manifolds containing closed genus g symplectic surfaces $F_X \subset X$ and $F_Y \subset Y$ such that $F_X^2 + F_Y^2 = 0$, then the symplectic sum [**4**] of X and Y along F_X and F_Y will be denoted by

$$X \#_{F_X, F_Y} Y.$$

Recall that topologically $X \#_{F_X, F_Y} Y$ is obtained by removing tubular neighborhoods of F_X and F_Y and identifying the resulting boundaries, which are S^1 bundles over a genus g surface, by a fiber-preserving, orientation reversing diffeomorphism.

The symplectic sum admits a symplectic structure so that any symplectic surface in $X - F_X$ or $Y - F_Y$ remains symplectic in $X \#_{F_X, F_Y} Y$. Moreover, if $E_X \subset X$ (resp. $E_Y \subset Y$) is a symplectic surface intersecting F_X once transversally (resp. intersecting F_Y transversally), then the symplectic sum can be constructed so that (the connected sum) $E_X \# E_Y$ is a symplectic surface in $X \#_{F_X, F_Y} Y$.

2.1. The first piece: symplectic manifolds with given fundamental group.

The following theorem was proven in [**1**].

Theorem 2.1. *Let G have a presentation with g generators $x_1, \ldots, x_g$ and r relations $w_1, \ldots, w_r$. Then there exists a symplectic 4-manifold $M(G)$ with $\pi_1(M(G)) \cong G$, Euler characteristic $\chi(M(G)) = 12(g + r + 1)$, and signature $\sigma(M(G)) = -8(g + r + 1)$.*

We will use the following fact. The manifold $M(G)$ constructed in Theorem 2.1 is obtained by taking symplectic sums of a certain base manifold with $g + r + 1$ copies of the basic elliptic surface $E(1)$. Since $E(1)$ admits a singular fibration with symplectic generic fibers and six cusp fibers (which are simply connected), so does $E(1) - F$, where F denotes the generic fiber in $E(1)$ along which the symplectic sum giving M is constructed. Thus each $M(G)$ contains a symplectic torus T_0 such that the induced homomorphism $\pi_1(T_0) \to \pi_1(M(G))$ is trivial.

2.2. The second piece: symplectic manifolds near the BMY line.

In [**11**], Stipsicz proved the following theorem.

Proposition 2.2 (Stipsicz). *For each non-negative integer n, there exists a symplectic 4–manifold $X(n)$ which admits a genus-$(15n+1)$ Lefschetz fibration with a section T_{n+2} of genus $(n+2)$ and self-intersection $-(n+1)$. Furthermore, $X(n)$ can be equipped with a symplectic structure such that T_{n+2} is a symplectic submanifold. The projection map $X(n) \to T_{n+2}$ induces an isomorphism on fundamental groups. The Euler characteristic of $X(n)$ is $\chi(X(n)) = 75n^2 + 180n + 12$ and the signature is $\sigma(X(n)) = 25n^2 - 60n - 8$.*

Denote by $F_{15n+1} \subset X(n)$ a fixed generic fiber of $X(n)$. This is a symplectic surface with trivial normal bundle.

2.3. The third piece: a simply connected manifold. Gompf constructs a symplectic 4-manifold $S_{1,1}$ in [**4**, Lemma 5.5] which contains a disjoint pair T, F of symplectically embedded surfaces T of genus one and F of genus two, with trivial normal bundles such that $S_{1,1} - (T \cup F)$ is simply connected. Thus, the symplectic sum A of two copies $S_{1,1}$ along the genus two surfaces

$$A = S_{1,1} \#_{F,F} S_{1,1}$$

contains a pair of disjointly embedded symplectic tori $T_1 \cup T_2 \subset A$ with trivial normal bundles so that the complement $A - (T_1 \cup T_2)$ is simply connected. Since $S_{1,1}$ has Euler characteristic 23 and signature -15, $\chi(A) = 50$ and $\sigma(A) = -30$.

The manifold A has a useful property, whose proof is a simple application of the Seifert–Van Kampen theorem.

Proposition 2.3. *Suppose B and C are symplectic 4-manifolds containing symplectic tori $i_B : T_B \subset B$ and $i_C : T_C \subset C$ with trivial normal bundles.*

Let $D = B \#_{T_B,T_1} A \#_{T_2,T_C} C$ be the symplectic sum of B, A, and C. Then

$$\pi_1(D) = \left(\frac{\pi_1(B)}{N((i_B)_*(\pi_1(T_B))} \right) \star \left(\frac{\pi_1(C)}{N((i_C)_*(\pi_1(T_C))} \right)$$

where $\star$ denotes free product and $N(H)$ denotes the normal closure of a subgroup H.

2.4. The fourth piece: an elbow. Let T be a torus and $\{a, b\}$ a pair of smoothly embedded loops forming a symplectic basis of $\pi_1 T$. Let $\varphi : T \to T$ be the Dehn twist around a. The mapping torus Y_ϕ fibers over S^1 with fiber T. Let $t_1 : S^1 \to Y_\phi$ denote a section. Taking a product of Y_ϕ with S^1 yields a symplectic 4-manifold $Y_\phi \times S^1$ (this is just Thurston's manifold from [**12**]) which fibers over a torus with symplectic torus fibers. Moreover, the symplectic structure can be chosen so that the section $t_1 \times \mathrm{id} : S^1 \times S^1 \to Y_\phi \times S^1$ is symplectic. Denote by $s_1 : S^1 \to \{p\} \times S^1 \subset Y_\phi \times S^1$ the loop representing the second factor.

Note that $Y_\phi \times S^1$ contains a torus $T' = b \times s_1$, where b is the curve described above in the fiber of Y_ϕ. The torus T' is homologically non-trivial by the Kunneth theorem, since b is non-trivial in $H_1(Y_\phi)$. Moreover, T' is Lagrangian with respect to the symplectic structure on $Y_\phi \times S^1$. Thus, the symplectic structure on $Y_\phi \times S^1$ can be perturbed slightly to make T' symplectic by adding a small closed 2-form that restricts to a volume form on T'. Note moreover that T' is disjoint from the section $t_1 \times s_1 : S^1 \times S^1 \to Y_\phi \times S^1$ since we can assume that t_1 intersects the fiber containing b in a point which does not lie on b. The tubular neighborhood of T' in $Y_\phi \times S^1$ is trivial since b can isotoped off itself in a fiber of $Y_\phi \to S^1$. Similarly the tubular neighborhood of the section $t_1 \times s_1$ is trivial since t_1 can be pushed off itself in Y_ϕ.

Define Elb(n) to be the symplectic sum $\text{Elb}(n) = (Y_\phi \times S^1)\#_{T,T^2}(T^2 \times \Sigma_{n-1})$. The symplectic sum can be carried out so that the sections of $Y_\phi \times S^1 \to S^1 \times S^1$ and $T^2 \times \Sigma_{n-1} \to \Sigma_{n-1}$ yield a symplectic section of the resulting fibration $\text{Elb}(n) \to \Sigma_n$. Thus, Elb$(n)$ contains a disjoint pair of symplectic surfaces with trivial normal bundles, a torus $T' = b \times s_1$, and a genus n surface, the image of the section, which we denote by D_n.

Letting $t_2, s_2, \ldots, t_n, s_n$ denote the generators of $\pi_1(\Sigma_{n-1})$, one computes

$$\begin{aligned}\pi_1(\text{Elb}(n)) = \langle a, b, t_1, s_1, \ldots, t_n, s_n |\ & a \text{ central}, [b, t_1] = a,\\ & [b, t_i] = 1 \text{ for } i > 1, [b, s_i] = 1 \text{ for all } i,\\ & \textstyle\prod_{i=1}^n [t_i, s_i] = 1\rangle.\end{aligned}$$

The inclusion of T' into Elb(n) takes the generators of $\pi_1 T'$ to b and s_1, and the inclusion of D_n takes the standard surface group generators to $t_1, s_1, \ldots, t_n, s_n$. The Euler characteristic and signature of Elb(n) both vanish.

The manifold $\text{Elb}(n) - D_n$ is a punctured torus fibration over Σ_n and hence has a presentation with the same generators and all the same relations except that one no longer has a commuting with b, i.e., a commutes with all generators except b.

2.5. The fifth piece: an elliptic surface. We find a symplectically embedded surface J of genus $n+3$ and self-intersection $n+1$ in the elliptic surface $E(n+5)$ such that $E(n+5) - J$ is simply connected as follows. Consider $n+3$ copies of the generic fiber and one copy of the section in a fibration $E(n+5) \to \mathbb{CP}^1$ with $6(n+5)$ cusp fibers. The section and fibers are symplectic with regards to the symplectic structure on the elliptic fibration $E(n+5)$. Resolve the $n+3$ transverse double points [**4**] to get a symplectically embedded surface J of genus $n+3$ and self-intersection $n+1$ (the fiber hits the section once and that section has self-intersection $-(n+5)$). The complement $E(n+5)-J$ is simply connected because $E(n+5)$ has a simply connected fiber which intersects J in one point: the normal circle of a tubular neighborhood of J is nullhomotopic in $E(n+5) - J$.

2.6. Putting the pieces together. We begin by a modification of Stipsicz's construction. Let $Z(n)$ be the symplectic sum of Elb$(15n+1)$

and $X(n)$ along $D_{15n+1} \subset \text{Elb}(15n+1)$ and the fiber F_{15n+1} of the Lefschetz fibration $X(n) \to \Sigma_{n+2}$

$$Z(n) = \text{Elb}(15n+1) \#_{D_{15n+1}, F_{15n+1}} X(n).$$

The symplectic sum can be constructed so that the fiber $T \subset \text{Elb}(15n+1)$ and the section $T_{n+2} \subset X(n)$ add to yield a symplectic surface of genus $n+3$, $K_{n+3} = T \# T_{n+2} \subset Z(n)$ [**4**]. The important property of $Z(n)$ is that it contains a symplectic torus T', since D_{15n+1} and T' are disjoint.

The fundamental group of $Z(n)$ is easily computed, since $\text{Elb}(15n+1) - D_{15n+1}$ is a fiber bundle with punctured torus fibers and $X(n) - F_{15n+1}$ is a Lefschetz fibration over a punctured genus $n+2$ surface with at least one simply connected fiber. Using the Seifert–Van Kampen theorem and Novikov additivity one obtains the following.

Lemma 2.4. *The fundamental group of $Z(n)$ is the free product of $\mathbb{Z}$ with generator b and a genus $n+2$ surface group generated by x_i, y_i:*

$$\pi_1(Z(n)) = \mathbb{Z}b \star \left\langle x_i, y_i, i = 1, \ldots, n+2 | \prod [x_i, y_i] = 1 \right\rangle .$$

The symplectic manifold $Z(n)$ contains a disjoint pair of symplectic surfaces, $T' \cup K_{n+3} \subset Z(n)$ satisfying $[T']^2 = 0$, and $[K_{n+3}]^2 = -n-1$. The induced homomorphism $\pi_1(T') \to \pi_1(Z(n))$ is the map

$$\langle a, s_1 \mid [a, s_1] \rangle \longrightarrow \pi_1 Z(n) \quad a \longmapsto a, \;\; s_1 \longmapsto 1.$$

The induced homomorphism $\pi_1(K_{n+3}) \to \pi_1(Z(n))$ is the map

$$\left\langle a, b, x_1, y_1, \ldots, x_{n+2}, y_{n+2} \mid [a, b] \prod [x_i, y_i] = 1 \right\rangle \longrightarrow \pi_1(Z(n))$$
$$a \longmapsto a, \quad b \longmapsto 1, \quad x_i \longmapsto x_i, \quad y_i \longmapsto y_i.$$

Moreover, $\chi(Z(n)) = 75n^2 + 240n + 12$ and $\sigma(Z(n)) = 25n^2 - 60n - 8$.

The symplectic sum of $Z(n)$ with $E(n+5)$ along J, $Z(n) \#_{K_{n+3}, J} E(n+5)$ is a simply connected symplectic 4-manifold containing a torus T_1 with trivial normal bundle and appropriate Euler characteristic and signature. We take symplectic sum of this manifold with A to obtain an example with a torus whose complement is simply connected.

Define $W(n)$ to be the symplectic sum

$$W(n) = A \#_{T_1, T'} Z(n) \#_{K_{n+3}, J} E(n+5).$$

Then since $\pi_1(A - (T_2 \cup T_2)) = 1$, the following proposition follows straightforwardly.

Proposition 2.5. *The symplectic manifold $W(n)$ is simply connected and contains a symplectic torus $T_2 \subset W(n)$ with trivial normal bundle so that*

$\pi_1(W(n)-T_2) = 1$. *It has Euler characteristic* $\chi(W(n)) = 75n^2+256n+130$ *and signature* $\sigma(W(n)) = 25n^2 - 68n - 78$.

We can now prove Theorem 1.1.

Proof of Theorem 1.1. The symplectic sum

$$M(G,n) = M(G)\#_{T_0,T_2}W(n)$$

has fundamental group G by Proposition 2.3. The calculations of $\chi(M(G,n))$ and $\sigma(M(G,n))$ are routine. □

References

[1] S. Baldridge and P. Kirk, *On symplectic 4-manifolds with prescribed fundamental group*, Commentarii Math. Helv. E-print GT0504345, in press.

[2] Z. Chen, *On the geography of surfaces—simply connected minimal surfaces with positive index*, Math. Ann. **126** (1987), 141–164.

[3] Z. Chen, *The existence of algebraic surfaces with preassigned Chern numbers*, Math. Z. **206** (1991), 241–254.

[4] R. Gompf, *A new construction of symplectic manifolds*, Ann. Math. **142** (1995), 527–595.

[5] R. Gompf and A. Stipsicz, 4–*Manifolds and Kirby calculus*, in 'Graduate studies in mathematics,' **20**, American Mathematical Society, Providence, Rhode Island, 1999.

[6] F. Hirzebruch, *The signature of ramified coverings,* in 'Global Analysis' (Papers in Honor of K. Kodaira), (D. Spencer and S. Iyanaga, eds.), Princeton University Press, 1969, 253–265.

[7] Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **268** (1984), 159–171.

[8] B. Moishezon and M. Teicher, *Simply connected algebraic surfaces of positive index*, Invent. Math. **89** (1987), 601–644.

[9] U. Persson, C. Peters and G. Xiao, *Geography of spin surfaces*, Topology **35** (1996), 845–862.

[10] A. Stipsicz, *Simply connected 4-manifolds near the Bogomolov–Miyaoka–Yau line*, Math. Res. Lett. **5** (1998), 723–730.

[11] A. Stipsicz, *Simply connected symplectic 4–manifolds with positive signature*, Turk. J. Math. **23** (1999), 145–150, Proceedings of 6th Gökova Geometry-Topology Conference.

[12] W.P. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55**(2) (1976), 467–468.

[13] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA **74** (1977), 1798–1799.

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