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LOCAL STRUCTURE OF GENERALIZED COMPLEX MANIFOLDS

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We study generalized complex (GC) manifolds from the point of view of symplectic and Poisson geometry. We start by recalling that every GC manifold admits a canonical Poisson structure. We use this fact, together with Weinstein's classical result on the local normal form of Poisson, to prove a local structure theorem for GC, complex manifolds, which extends the result Gualtieri has obtained in the "regular" case. Finally, we begin a study of the local structure of a GC manifold in a neighborhood of a point where the associated Poisson tensor vanishes. In particular, we show that in such a neighborhood, a "first-order approximation" to the GC structure is encoded in the data of a constant B-field and a complex Lie algebra.

1. Introduction and main results

The main objects of study in this paper are irregular generalized complex (GC) structures on manifolds (the terminology is explained below). In this section, we state and discuss our main results. The rest of the paper is devoted to their proofs.

1.1. Background on GC geometry. We begin by recalling the setup of GC geometry. We use [**1**] as the main source for most basic results and definitions, a notable exception being the notion of a GC submanifold of a GC manifold, which is taken from [**2**].

The notion of a GC manifold was introduced by Hitchin (cf. [**3–5**]) and developed by Gualtieri [**1**]. If M is a manifold (by which we mean a finite-dimensional real C^∞ manifold), specifying a GC structure on M amounts to specifying either of the following two objects:

- an $\mathbb{R}$-linear bundle automorphism $\mathcal{J}$ of $TM \oplus T^*M$ which preserves the standard symmetric bilinear pairing $\langle (X,\xi),(Y,\eta)\rangle = \xi(Y)+\eta(X)$ and satisfies $\mathcal{J}^2 = -1$ or

- a complex vector subbundle $L \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ such that $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M = L \oplus \bar{L}$ and L is isotropic with respect to the $\mathbb{C}$-bilinear extension of $\langle \cdot, \cdot \rangle$ to $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$,

which is required to satisfy a certain integrability condition that is similar to the standard integrability condition for an almost complex structure on a real manifold. A bijection between the two types of structure defined above is obtained by associating to an automorphism $\mathcal{J}$ its $+i$-eigenbundle. In terms of L, the integrability condition is that the sheaf of sections of L is closed under the *Courant bracket* [**6**]

$$\big[(X,\xi),(Y,\eta)\big]_{\mathrm{cou}} = \left([X,Y],\, \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\cdot d\big(\iota_X\eta - \iota_Y\xi\big)\right). \tag{1.1}$$

One can check that this condition is equivalent to the vanishing of the *Courant–Nijenhuis tensor*

$$\mathcal{N}_{\mathcal{J}}(A,B) = \big[\mathcal{J}A,\mathcal{J}B\big]_{\mathrm{cou}} - \mathcal{J}\big[\mathcal{J}A,B\big]_{\mathrm{cou}} - \mathcal{J}\big[A,\mathcal{J}B\big]_{\mathrm{cou}} - \big[A,B\big]_{\mathrm{cou}}, \tag{1.2}$$

where A, B are sections of $TM \oplus T^*M$.

The two main examples of GC structures arise from complex and symplectic manifolds. If M is a real manifold equipped with an integrable almost complex structure $J\colon TM \to TM$, it is easy to check that the automorphism

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$$

defines a GC structure on M; such a GC structure is said to be *complex*. Similarly, if ω is a symplectic form on M, we can view it as a skew-symmetric map $\omega\colon TM \to T^*M$, and then the automorphism

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

also defines a GC structure on M; such a GC structure is said to be *symplectic*.

A GC structure on a manifold M induces a distribution $E \subseteq T_{\mathbb{C}}M$ which is smooth in the sense of [**7**]. Namely, E is the image of L under the projection map $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M \to T_{\mathbb{C}}M$. Note that E may not have constant rank, and in general, not every smooth section of E lifts to a smooth section of L. In order to avoid this problem, we define the *sheaf of section of* E to be the sheafification of the presheaf of sections of the bundle $T_{\mathbb{C}}M$ that can be lifted to a smooth section of L. In other words, it is the sheaf-theoretic image of the sheaf of sections of L in the sheaf of sections of $T_{\mathbb{C}}M$. Note that the sheaf of sections of E still determines E as a distribution, since for each $m \in M$ and each element in the fiber $e \in E_m$, there exists a section of L in a neighborhood of m whose image in $T_{\mathbb{C},m}M$ equals e.

The sheaf of sections of E is closed under the Lie bracket (i.e., E is involutive), as follows trivially from the definition of the Courant bracket.

Moreover, there is a (complex) 2-form ϵ on E defined as follows: if X, Y are sections of E, choose a section ξ of $T^*_{\mathbb{C}}M$ such that $(X,\xi) \in L$ and set $\epsilon(X,Y) = \xi(Y)$. If η is a section of $T^*_{\mathbb{C}}M$ such that $(Y,\eta) \in L$, then $\xi(Y) = -\eta(X)$ because L is isotropic with respect to the pairing $\langle\cdot,\cdot\rangle$, which implies that $\epsilon(X,Y)$ is independent of the choice of ξ; thus ϵ is well defined. Furthermore, one can define the tensor $d\epsilon \in \bigwedge^3(E^*)$ by the Cartan formula, which makes sense since E is involutive.

Proposition 1.1 (see [**1**]). *The data (E,ϵ) determines the GC structure L uniquely. Moreover, $d\epsilon = 0$.*

A special type of operation defined for GC structures, which plays an important role in our discussion, is the *transformation by a B-field*. Specifically, if B is a *real closed* 2-form on M, we define an orthogonal automorphism $\exp(B)$ of the bundle $TM \oplus T^*M$ via

$$\exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

where we view B as a skew-symmetric map $TM \to T^*M$. If $\mathcal{J}$ defines a GC structure on M, and the associated pair (E,ϵ) is constructed as above, then $\mathcal{J}' = \exp(B)\mathcal{J}\exp(-B)$ is another GC structure on M, which follows from the fact that $\exp(B)$ preserves the Courant bracket on $TM \oplus T^*M$, see [**1**]. Moreover, in this case, the $+i$-eigenbundle of $\mathcal{J}'$ is given by $L' = \exp(B)(L)$, and the associated pair (E',ϵ') is determined by $E' = E$, $\epsilon' = \epsilon + B\big|_E$, where, by a slight abuse of notation, we also denote by B the $\mathbb{C}$-bilinear extension of B to $T_{\mathbb{C}}M$. In our paper, a B-field transformation will always mean a transformation of the form $\mathcal{J} \mapsto \exp(B)\mathcal{J}\exp(-B)$, where B is a closed real 2-form. For a more detailed discussion and a more general notion of B-fields, see [**1**, **5**] and references therein.

Another important construction is that of the canonical symplectic foliation on a GC manifold. Namely, let us consider $E \cap \bar{E}$; this is a distribution in $T_{\mathbb{C}}M$ which is stable under complex conjugation, and hence has the form $\mathscr{S}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathscr{S}$ for some distribution $\mathscr{S} \subseteq TM$. Gualtieri [**1**] proves that $\mathscr{S}$ is a smooth distribution in the sense of [**7**] and that the 2-form ω on $\mathscr{S}$ defined by $\omega = \operatorname{Im}(\epsilon\big|_{\mathscr{S}})$ is (pointwise) nondegenerate. Moreover, it is now clear that the sheaf of sections of $\mathscr{S}$ is closed under the Lie bracket and that ω is a closed 2-form on $\mathscr{S}$, in the same sense as in Proposition 1.1. It follows from the results of [**7**] that through every point of M there is a maximal integral manifold of $\mathscr{S}$, which, by construction, inherits a natural symplectic structure.

For example, if the GC structure on M is complex, then $\mathscr{S} = 0$, whereas if the GC structure on M is symplectic, then $\mathscr{S} = TM$ and the canonical symplectic form on $\mathscr{S}$ coincides with the symplectic form defining the GC structure on M.

We now recall the notion of a GC submanifold of a GC manifold. Let L be a GC structure on a manifold M, and let $N \subset M$ be a (locally closed) submanifold. We define a (not necessarily smooth) distribution L_N on N as follows. Set

$$\tilde{L}_N = L\big|_N \cap \Big(T_{\mathbb{C}}N \oplus (T^*_{\mathbb{C}}M\big|_N)\Big) \quad \text{and} \quad L_N = \mathrm{pr}(\tilde{L}_N),$$

where pr: $T_{\mathbb{C}}N \oplus (T^*_{\mathbb{C}}M\big|_N) \to T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$ denotes the natural projection map, $(X, \xi) \mapsto (X, \xi\big|_{T_{\mathbb{C}}N})$. It is proved in [**2**] that $\dim_{\mathbb{C}} L_{N,n} = \dim_{\mathbb{R}} N$ for all $n \in N$. However, L_N may *not* be a subbundle of $T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$. We say that N is a *GC submanifold* of M provided L_N is smooth, and defines a GC structure on N. It can be shown (cf. [**2**]) that a necessary and sufficient condition for this is that L_N is smooth and $L_N \cap \overline{L_N} = 0$ (integrability is then automatic).

In conclusion, we would like to mention that there exists a way of describing GC structures on manifolds in terms of spinors. In fact, most of [**1**] is written in the language of spinors. However, in our paper, we have made a conscious effort to state and prove all of our results in a spinor-free language. We hope that this approach helps illuminate the simple geometric ideas that underlie our main constructions.

1.2. The canonical Poisson structure on a GC manifold. From now on, we fix a manifold M equipped with a GC structure which, whenever convenient, we will think of in terms of either the automorphism $\mathcal{J}$ or the subbundle $L \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$. The starting point for our work is the observation that the canonical symplectic foliation $(\mathscr{S}, \omega)$ defined in Section 1.1 is in fact the symplectic foliation associated to a certain Poisson structure on M. The existence of a canonical Poisson structure on a GC manifold was also independently noticed by Gualtieri [**8**], and Lyakhovich and Zabzine [**9**].

Let us briefly explain why one could expect the existence of a natural Poisson structure on general grounds. Recall the definition of integrability as the vanishing of the Courant–Nijenhuis tensor (1.2). The condition $\mathcal{N}_{\mathcal{J}}(A, B) = 0$ can be naturally rewritten as a collection of four equations corresponding to the possibilities of either A or B being a section of TM or a section of T^*M. Let us also write $\mathcal{J}$ as a matrix

$$\mathcal{J} = \begin{pmatrix} J & \pi \\ \sigma & K \end{pmatrix}, \tag{1.3}$$

where J: $TM \to TM$, π: $T^*M \to TM$, σ: $TM \to T^*M$ and K: $T^*M \to T^*M$ are bundle morphisms. The requirements that $\mathcal{J}^2 = -1$ and $\mathcal{J}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$ force $K = -J^*$, $\pi = -\pi^*$, $\sigma = -\sigma^*$; in particular, π can be viewed as a bivector on M, i.e., a section of $\bigwedge^2 TM$. Moreover, it is a straightforward computation that in the case when $A = (0, \xi)$ and $B = (0, \eta)$, where ξ, η are sections of T^*M, the TM-component of $\mathcal{N}_{\mathcal{J}}(A, B)$ is the following expression:

$$[\pi\xi, \pi\eta] - \pi\left(\mathcal{L}_{\pi\xi}\eta - \frac{1}{2}d(\iota_{\pi\xi}\eta)\right) + \pi\left(\mathcal{L}_{\pi\eta}\xi - \frac{1}{2}d(\iota_{\pi\eta}\xi)\right). \tag{1.4}$$

Observe that this expression depends only on π and not on the other components of the matrix defining $\mathcal{J}$. However, one can check that no other entry of the matrix can be separated from the rest in this way. This suggests that π must play a special role in the theory. In fact, one has the following theorem.

Theorem 1.2. *The bivector π defines a Poisson structure on M. Moreover, the canonical symplectic foliation associated to this Poisson structure coincides with $(\mathscr{S}, \omega)$.*

The proof consists of a straightforward verification of the fact that the vanishing of expression (1.4) for all 1-forms ξ and η on M is equivalent to the vanishing of the Schouten bracket $[\pi, \pi]$; see [**8**, **10**] for details. The latter condition means precisely that π is a Poisson bivector.

Given a real-valued $f \in C^\infty(M)$, let us write

$$\mathcal{J}(0, df) = (X_f, \xi_f).$$

By construction, $X_f = \pi(df)$ is the Hamiltonian vector field on M associated to f. On the other hand, ξ_f is a certain differential 1-form on M. The following result complements Theorem 1.2.

Proposition 1.3. *The map $f \mapsto \xi_f$ has the following properties.*

(1) *For all $f, g \in C^\infty(M)$, we have*

$$\xi_{f \cdot g} = f \cdot \xi_g + g \cdot \xi_f.$$

(2) *If $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(M)$ defined by π, then*

$$\xi_{\{f,g\}} = \mathcal{L}_{X_f}(\xi_g) - \iota_{X_g}(d\xi_f).$$

(3) *If (E, ϵ) is associated to the GC structure $\mathcal{J}$ as in Section* 1.1, *then for all $f \in C^\infty(M)$, we have*

$$\mathcal{L}_{X_f}(\epsilon) = (d\xi_f)\big|_E.$$

We omit the proof, since it amounts to a few lines of straightforward computations using only Cartan's formula for the Lie derivative and the definition of the Poisson bracket, $\{f, g\} = X_f(g) = -X_g(f)$. The properties of the map $f \mapsto \xi_f$ turn out to be crucial in our proof of the local normal form for GC manifolds. Moreover, these results raise the question of whether one can give an explicit description of GC manifolds as Poisson manifolds equipped with additional structure. In other words, consider a GC structure on a manifold M defined by matrix (1.3). By Theorem 1.2, the pair (M, π) is a Poisson manifold. Then the problem is to describe, in the language of

Poisson geometry, the extra data on (M, π) that are needed to recover all of $\mathcal{J}$. Part (2) of Proposition 1.3 is a first step in this direction.

1.3. The local structure theorem for GC manifolds. We say that a GC structure on a manifold M is *regular* if the distribution $\mathscr{S}$ (equivalently, E) has locally constant rank. The structure is said to be *irregular* otherwise. The original motivation for our work came from trying to extend the local structure theorem proved in [**1**] for regular GC structures to the irregular case. Gualtieri proved that if $m \in M$ is a regular point of a given GC structure on M (i.e., the structure is regular in an open neighborhood of M), then there exists a neighborhood U of m in M such that the induced GC structure on U is a B-field transform of the product of a symplectic GC manifold and a complex GC manifold.

Let us fix a GC manifold M and a point $m_0 \in M$. We define the *rank*, $\mathrm{rk}_{m_0} M$, of M at m_0 to be the rank of the associated Poisson tensor π at m_0. The central result of our paper is the following.

Theorem 1.4. *There exists an open neighborhood U of m_0 in M, a real closed 2-form B on U, a symplectic GC manifold S and a GC manifold N with marked points $s_0 \in S$, $n_0 \in N$ such that $\mathrm{rk}_{n_0} N = 0$, and a diffeomorphism $S \times N \to U$ which takes (s_0, n_0) to m_0 and induces an isomorphism between the product GC structure on $S \times N$ and the transform of the induced GC structure on U via the 2-form B.*

This theorem is proved in Section 3.

Remark 1.5. It is easy to recover the result of Gualtieri from Theorem 1.4. Namely, if, with the notation of the theorem, the GC structure on M is regular in a neighborhood of m_0, then the rank of N must be zero in a neighborhood of n_0. It then follows by linear algebra that the GC structure on N must be B-complex in a neighborhood of n_0, and the fact that this structure can be written as the transform of a complex structure by a *closed* real 2-form follows from the local vanishing of Dolbeault cohomology (cf. [**1**]).

1.4. Linear GC structures. The term "linear GC structure" should not be confused with the notion of a *constant* GC structure on a real vector space discussed in Section 2. Rather, it is used in the same way as the term "linear Poisson structure" is used to describe the canonical Poisson structure on the dual space of a real Lie algebra.

Recall that if (M, π) is a Poisson manifold, and $m \in M$ is a point at which the Poisson tensor π vanishes, then a "first-order approximation" to π at m defines a real Lie algebra of dimension $\dim M$. Canonically, this Lie algebra can be identified with the quotient $\mathfrak{g} = \mathfrak{m}_m/\mathfrak{m}_m^2$, where $\mathfrak{m}_m$ denotes the ideal in the algebra of all real-valued C^∞ functions on M consisting of the functions that vanish at m. Since π vanishes at m, it is easy to check that $\mathfrak{m}_m$ is stable under the Poisson bracket, and $\mathfrak{m}_m^2$ is an ideal of $\mathfrak{m}_m$ in the sense of Lie algebras, and hence we obtain an induced Lie algebra bracket on $\mathfrak{g}$.

Therefore one expects that, near a point on a GC manifold where the associated Poisson tensor vanishes, the first-order approximation to the GC structure can be encoded in a real finite-dimensional Lie algebra equipped with additional structure. Indeed, we prove the following.

Theorem 1.6. *In a neighborhood of a point on a GC manifold where the associated Poisson tensor vanishes, the first-order approximation to the GC structure is encoded in a* complex *Lie algebra of complex dimension* $\frac{\dim M}{2}$, *and a B-field which is constant in appropriate local coordinates* (*and hence, a fortiori, is closed*).

The meaning of this statement is explained in Section 4.

A natural problem that arises is to give a local classification of GC manifolds near a point where the associated Poisson tensor vanishes. Together with our Theorem 1.4, a solution of this problem would yield a complete local classification of GC manifolds.

2. Linear algebra

2.1. In this section, we present the auxiliary results on linear algebra that are used in the proofs of our main theorems. We begin by recalling that the notion of a GC structure has an analog for vector spaces, which was studied in detail in [**1**, **2**]. Specifically, a *constant* GC structure on a real vector space V is defined either as an $\mathbb{R}$-linear automorphism $\mathcal{J}$ of $V \oplus V^*$, which preserves the standard symmetric bilinear pairing $\langle \cdot, \cdot \rangle$ and satisfies $\mathcal{J}^2 = -1$, or as a complex subspace $L \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$, which is isotropic with respect to the $\mathbb{C}$-bilinear extension of $\langle \cdot, \cdot \rangle$ and satisfies $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^* = L \oplus \overline{L}$. There is no integrability condition in this case. It is easy to see that constant GC structures on V correspond bijectively to GC structures on the underlying real manifold of V that are invariant under translations. Furthermore, it is obvious that if $\mathcal{J}$ is a GC structure on a manifold M, then for every point $m \in M$, the automorphism $\mathcal{J}_m$ of $T_mM \oplus T_m^*M$ induced by $\mathcal{J}$ defines a constant GC structure on T_mM. From now on, by a *GC vector space*, we will mean a real vector space equipped with a constant GC structure.

All notions and constructions discussed in Section 1.1 have obvious analogs for GC vector spaces. In particular, for a real vector space V, we let $\rho\colon V \oplus V^* \to V$, $\rho^*\colon V \oplus V^* \to V^*$ denote the natural projection maps. Given a GC structure on V defined by a subspace $L \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$, we let $E = \rho(L) \subseteq V_{\mathbb{C}}$. There is an induced $\mathbb{C}$-bilinear 2-form ϵ on E defined in the same way as in Section 1.1, and the pair (E, ϵ) determines the GC structure on V uniquely. Moreover, if $S \subseteq V$ is the real subspace satisfying $\mathbb{C} \otimes_{\mathbb{R}} S = E \cap \overline{E}$, then $\omega = \operatorname{Im}(\epsilon|_S)$ is a symplectic form on S. Finally, the notion of a GC subspace of a GC vector space V is defined in the obvious way: if $W \subseteq V$ is a real subspace, set

$$\tilde{L}_W = L \cap (W_{\mathbb{C}} \oplus V_{\mathbb{C}}^*) \quad \text{and} \quad L_W = \mathrm{pr}(\tilde{L}_W),$$

where pr: $W_{\mathbb{C}} \oplus V_{\mathbb{C}}^* \to W_{\mathbb{C}} \oplus W_{\mathbb{C}}^*$ is the projection map $(w, \lambda) \mapsto (w, \lambda\big|_{W_{\mathbb{C}}})$. We say that W is a *GC subspace* of V if $L_W \cap \overline{L_W} = (0)$; it is shown in [**2**] that in this case L_W is automatically a GC structure on W, called the *induced GC structure*.

The notion of a B-field transform is also defined in the obvious way. If $B \in \bigwedge^2 V^*$ is a skew-symmetric bilinear form on V, then the map

$$\exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

is a linear automorphism of $V \oplus V^*$ which preserves the standard pairing $\langle \cdot, \cdot \rangle$, and hence acts on constant GC structures on V via

$$L \mapsto \exp(B) \cdot L \quad \text{or} \quad \mathcal{J} \mapsto \exp(B) \cdot \mathcal{J} \cdot \exp(-B).$$

It is easy to check that, in terms of the pairs (E, ϵ), the transformation above is given by

$$(E, \epsilon) \longmapsto (E, \epsilon + B_{\mathbb{C}}\big|_E),$$

where $B_{\mathbb{C}}$ is the unique $\mathbb{C}$-bilinear extension of B to $V_{\mathbb{C}}$.

In what follows, we will occasionally need to consider GC structures on different vector spaces at the same time. Therefore, whenever a confusion may arise, we will use the notation $L_V \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$, $\mathcal{J}_V \in \mathrm{Aut}_{\mathbb{R}}(V \oplus V^*)$, $S_V \subseteq V$, $E_V \subseteq V_{\mathbb{C}}$, etc., to denote the objects $L, \mathcal{J}, S, E$, etc., that are associated to a given GC structure on a vector space V.

2.2. For future use, we make explicit the notions of an isomorphism and a product of GC structures. Given two real vector spaces, P and Q, equipped with GC structures L_P and L_Q, an *isomorphism of GC vector spaces* between P and Q is an $\mathbb{R}$-linear isomorphism ϕ: $P \to Q$ such that the induced map

$$(\phi_{\mathbb{C}}, (\phi_{\mathbb{C}}^*)^{-1})\colon P_{\mathbb{C}} \oplus P_{\mathbb{C}}^* \longrightarrow Q_{\mathbb{C}} \oplus Q_{\mathbb{C}}^*$$

carries L_P onto L_Q. The *direct sum* of the GC vector spaces P and Q is the vector space $P \oplus Q$ equipped with the GC structure $L_P \oplus L_Q$ (called the *product GC structure*), where we have made the natural identification

$$(P \oplus Q)_{\mathbb{C}} \oplus (P \oplus Q)_{\mathbb{C}}^* \cong P_{\mathbb{C}} \oplus P_{\mathbb{C}}^* \oplus Q_{\mathbb{C}} \oplus Q_{\mathbb{C}}^*.$$

Finally, if V is a GC vector space and $P, Q \subseteq V$ are two subspaces, we say that V is the direct sum of P and Q *as GC vector spaces* provided P, Q are GC subspaces of V, and if we equip P, Q with the induced GC structures and $P \oplus Q$ with the product GC structure, then the map $P \oplus Q \to V$ given by $(p, q) \mapsto p + q$ is an isomorphism of GC vector spaces.

The notions of an isomorphism and a product of GC structures have obvious extensions to GC manifolds, see [**2**].

2.3. The main results of GC linear algebra that we need are summarized in the following theorem.

Theorem 2.1. *Let V be any GC vector space, and let (S, ω) be defined as above.*

(a) *The notion of being a GC subspace is transitive; in fact, the following stronger statement holds: if $W_1 \subseteq V$ is a GC subspace and $W_2 \subseteq W_1$ is any real subspace, then W_2 is a GC subspace of V if and only if it is a GC subspace W_1 with respect to the induced GC structure on W_1.*[1] *Moreover, if this is the case, then the induced GC structure on W_2 is the same in both cases.*

(b) *A subspace $W \subseteq V$ is a GC subspace if and only if $W \cap S$ is a symplectic subspace of S (in the sense that $\omega\big|_{W\cap S}$ is nondegenerate) and $W_{\mathbb{C}} = (W_{\mathbb{C}} \cap E) + (W_{\mathbb{C}} \cap \overline{E})$.*

(c) *In particular, S itself is a GC subspace of V; the induced GC structure on S is B-symplectic, and moreover, S is the largest GC subspace of V with this property. The underlying symplectic structure on S is given by ω.*

(d) *The notion of being a GC subspace is invariant under B-field transformations of the GC structure on V.*

(e) *If $W \subseteq V$ is a real subspace such that $W + S = V$ (the sum is not necessarily direct), then W is a GC subspace of V if and only if $W \cap S$ is a symplectic subspace of S. In particular, any subspace of V that is complementary to S in the sense of linear algebra is automatically a GC subspace of V.*

(f) *Let $W \subseteq V$ be a real subspace such that $W + S = V$, and let S_0 denote any real subspace of S such that $S = S_0 \oplus (S \cap W)$, so that $V = S_0 \oplus W$. Then the following two conditions are equivalent:*
   (i) *S_0 and $S \cap W$ are orthogonal with respect to ω.*
   (ii) *W and S_0 are GC subspaces of V, and there exists a B-field $B \in \bigwedge^2 V^*$ which transforms the GC structure on V into the direct sum of the induced GC structures on S_0 and W.*

(g) *If the equivalent conditions of part* (f) *hold, then the choice of B is unique provided we insist that $B\big|_{S_0} = 0$ and $B\big|_{W} = 0$.*

Remark 2.2. As a byproduct of our discussion, we obtain an alternate proof of the structure theorem for constant GC structures (see [**1**, **2**]) which does not use spinors. Indeed, if $S \subseteq V$ is as above and $W \subseteq V$ is any complementary subspace to S, then parts (e) and (f) of the theorem imply that W is a GC subspace of V and the GC structure on V is a B-field transform of the direct product GC structure on $S \oplus W$. It is then easy to

[1] In general, however, GC subspaces do not behave well with respect to taking sums and intersections.

check that the induced GC structure on S (resp., W) is B-symplectic (resp., B-complex), see, e.g., [**2**].

Proof of Theorem 2.1. (a) It is trivial to check that the two definitions of L_{W_2} we obtain by viewing W_2 either as a subspace of V or as a subspace of W_1 coincide, whence the claim.
(b) We first show the necessity of the two conditions. It follows from the results of [**2**] that a subspace of S is a GC subspace if and only if it is a symplectic subspace with respect to the form ω. Now if W is any GC subspace of V, then $W \cap S = S_W$, whence $W \cap S$ is a GC subspace of W by the results of [**2**]. By part (a), it follows that $W \cap S$ is also a GC subspace of V, and hence a GC subspace of S.

Suppose now that W is a GC subspace of V, yet $(W_{\mathbb{C}} \cap E) + (W_{\mathbb{C}} \cap \overline{E}) \subsetneq W_{\mathbb{C}}$. Then there exists a nonzero real subspace $U \subset W$ with

$$U_{\mathbb{C}} \oplus \left[(W_{\mathbb{C}} \cap E) + (W_{\mathbb{C}} \cap \overline{E})\right] = W_{\mathbb{C}}.$$

This implies that

$$U_{\mathbb{C}} \cap \left[E + (W_{\mathbb{C}} \cap \overline{E})\right] = (0) \quad \text{and} \quad U_{\mathbb{C}} \cap \left[\overline{E} + (W_{\mathbb{C}} \cap E)\right] = (0).$$

Hence we can find $\ell, \ell' \in V_{\mathbb{C}}^*$ with $\ell\big|_{U_{\mathbb{C}}} = \ell'\big|_{U_{\mathbb{C}}} \not\equiv 0$ and $\ell\big|_{E+(W_{\mathbb{C}}\cap\overline{E})} \equiv 0 \equiv \ell'\big|_{\overline{E}+(W_{\mathbb{C}}\cap E)}$. This forces $\ell \in L \cap V_{\mathbb{C}}^*$, $\ell' \in \overline{L} \cap V_{\mathbb{C}}^*$ and $\ell\big|_{W_{\mathbb{C}}} = \ell'\big|_{W_{\mathbb{C}}} \not\equiv 0$, which means that

$$\left(\rho(\ell), \rho^*(\ell)\big|_{W_{\mathbb{C}}}\right) = \left(0, \ell\big|_{W_{\mathbb{C}}}\right) = \left(0, \ell'\big|_{W_{\mathbb{C}}}\right) = \left(\rho(\ell'), \rho^*(\ell')\big|_{W_{\mathbb{C}}}\right) \neq 0,$$

contradicting the assumption that W is a GC subspace of V.

Conversely, suppose that $W \subseteq V$ is a subspace such that $W_{\mathbb{C}} = (W_{\mathbb{C}} \cap E) + (W_{\mathbb{C}} \cap \overline{E})$ and $W \cap S$ is a GC subspace (equivalently, a symplectic subspace) of S. We will prove that W is a GC subspace of V. Assume that $\ell \in L$, $\ell' \in \overline{L}$ and $\rho(\ell) = \rho(\ell') \in W_{\mathbb{C}}$, $\rho^*(\ell)\big|_{W_{\mathbb{C}}} = \rho^*(\ell')\big|_{W_{\mathbb{C}}}$. Then, in particular, $\rho(\ell) = \rho(\ell') \in (W \cap S)_{\mathbb{C}}$ and $\rho^*(\ell)\big|_{(W\cap S)_{\mathbb{C}}} = \rho^*(\ell')\big|_{(W\cap S)_{\mathbb{C}}}$, so we deduce from the second assumption that $\rho(\ell) = \rho(\ell') = 0$ and $\rho^*(\ell)\big|_{(W\cap S)_{\mathbb{C}}} = \rho^*(\ell')\big|_{(W\cap S)_{\mathbb{C}}} = 0$. It remains to check that $\rho^*(\ell)\big|_{W_{\mathbb{C}}} = \rho^*(\ell')\big|_{W_{\mathbb{C}}} = 0$. But

$$\rho^*(\ell) = \ell \in L \cap V_{\mathbb{C}}^* = \mathrm{Ann}_{V_{\mathbb{C}}^*}(E) \quad \text{and} \quad \rho^*(\ell') = \ell' \in \overline{L} \cap V_{\mathbb{C}}^* = \mathrm{Ann}_{V_{\mathbb{C}}^*}(\overline{E}),$$

whence $\ell\big|_{W_{\mathbb{C}}\cap E} = 0 = \ell'\big|_{W_{\mathbb{C}}\cap\overline{E}}$, and also, since $\ell\big|_{W_{\mathbb{C}}} = \ell'\big|_{W_{\mathbb{C}}}$, we find from our first assumption that $\ell\big|_{W_{\mathbb{C}}} = \ell'\big|_{W_{\mathbb{C}}} = 0$, completing the proof.
(c) This is easy. We omit the proof since the straightforward argument is presented in [**2**].
(d) It follows from the remarks of Section 2.1 that a B-field transform changes neither E, nor S, nor $\omega = \mathrm{Im}(\epsilon\big|_S)$. Hence the claim follows from the characterization of GC subspaces given in part (b).

(e) We will show that if $W \subseteq V$ is a subspace such that $V = W + S$, then we automatically have $W_{\mathbb{C}} = (W_{\mathbb{C}} \cap E) + (W_{\mathbb{C}} \cap \overline{E})$. The claim then follows from part (b). Let $w \in W_{\mathbb{C}}$, and write $w = e_1 + \overline{e_2}$, with $e_j \in E$ for $j = 1, 2$. Further, we can write $e_j = w_j + s_j$, where $w_j \in W_{\mathbb{C}}$ and $s_j \in S_{\mathbb{C}}$. *A fortiori*, $s_j \in E$, so $w_j \in E \cap W_{\mathbb{C}}$. Hence

$$w = w_1 + \overline{w_2} + (s_1 + \overline{s_2}),$$

where $w_1, w_2 \in W_{\mathbb{C}} \cap E$. This forces $s_1 + \overline{s_2} \in W_{\mathbb{C}}$, and since we also have $s_j \in S_{\mathbb{C}}$, it follows that $s_1 + \overline{s_2} \in W_{\mathbb{C}} \cap S_{\mathbb{C}} \subseteq W_{\mathbb{C}} \cap E$. Finally, we conclude that

$$w = (w_1 + s_1 + \overline{s_2}) + \overline{w_2},$$

where $w_1 + s_1 + \overline{s_2} \in W_{\mathbb{C}} \cap E$ and $\overline{w_2} \in W_{\mathbb{C}} \cap \overline{E}$, as desired.

(f), (g) First, it is clear that (ii) implies (i), since B-field transforms cannot change the imaginary part of ϵ. Conversely, assume that S_0 and $S \cap W$ are orthogonal with respect to ω. We will show that there exists exactly one B-field $B \in \bigwedge^2 V^*$ such that $B|_{S_0} = B|_W = 0$ and B transforms the given GC structure on V into the direct sum of the induced GC structures on S_0 and W.

Observe that $E_V = E_{S_0} \oplus E_W$. Indeed, it is clear that $E_{S_0} \oplus E_W \subseteq E_V$. Conversely, let $e \in E_V$ and write $e = e_1 + e_2$, where $e_1 \in (S_0)_{\mathbb{C}}$ and $e_2 \in W_{\mathbb{C}}$. Then, *a fortiori*, $e_1 \in E_V$, so we also have $e_2 \in E_V$, whence $e_1 \in E_V \cap (S_0)_{\mathbb{C}} = E_{S_0}$ and $e_2 \in E_V \cap W_{\mathbb{C}} = E_W$, proving the claim.

Note now that if the original GC structure on V is determined by (E_V, ϵ), then the product GC structure on $S_0 \oplus W$ is determined by

$$\Big(E_{S_0} \oplus E_W, \epsilon|_{E_{S_0}} + \epsilon|_{E_W}\Big).$$

To complete the proof, we must therefore show that there exists exactly one $B \in \bigwedge^2 V^*$ such that $B|_{S_0} = B|_W = 0$ and the pairing between E_{S_0} and E_W induced by (the complexification of) B is the same as the one induced by ϵ.

Suppose that such a B exists. Let $s \in S_0$, $w \in W$. Since w is real, we can write $w = e + \overline{e}$, where $e \in E \cap W_{\mathbb{C}}$. Then we must have

$$B(s, w) = B(s, e) + \overline{B(s, e)} = 2 \cdot \mathrm{Re}\ \epsilon(s, e),$$

which proves that B is unique if it exists. Conversely, let us define B on $S_0 \times W$ by this formula, and define B to be zero on S_0 and on W. We claim that B is well defined. Indeed, consider a different representation $w = e' + \overline{e'}$, where $e' \in E \cap W_{\mathbb{C}}$. Then

$$e - e' = \overline{e' - e} \in (W \cap S)_{\mathbb{C}},$$

which implies that $e - e' = i \cdot t$ for some $t \in W \cap S$, where $i = \sqrt{-1}$. Hence

$$\mathrm{Re}\, \epsilon(s, e - e') = - \mathrm{Im}\, \epsilon(s, t) = -\omega(s, t) = 0 \text{ by assumption},$$

which implies that B is well defined.

Finally, to show that B satisfies the required condition, it is enough to check (by linearity) that if $s \in S_0$ and $e \in E_W = W_{\mathbb{C}} \cap E$, then $B(s,e) = \epsilon(s,e)$. We have

$$e = \frac{e+\overline{e}}{2} + i \cdot \frac{e-\overline{e}}{2i} \quad \text{and} \quad \frac{e+\overline{e}}{2}, \frac{e-\overline{e}}{2i} \in W.$$

By construction,

$$B\left(s, \frac{e+\overline{e}}{2}\right) = 2\,\mathrm{Re}\ \epsilon\left(s, \frac{e}{2}\right) = \mathrm{Re}\ \epsilon(s,e),$$

and similarly $B(s,(e-\overline{e})/(2i)) = \mathrm{Im}\ \epsilon(s,e)$, which completes the proof. □

Remark 2.3. The following comment will be used in our proof of the local structure theorem for GC manifolds. Consider a variation of GC linear algebra where the vector space V is replaced by a smooth real vector bundle $\mathcal{V}$ over a base manifold $\mathfrak{B}$, and a GC structure on $\mathcal{V}$ is a subbundle $\mathcal{L} \subseteq \mathcal{V}_{\mathbb{C}} \oplus \mathcal{V}^*_{\mathbb{C}}$ such that for every point $b \in \mathfrak{B}$, the subspace $\mathcal{L}_b \subseteq \mathcal{V}_{b,\mathbb{C}} \oplus \mathcal{V}^*_{b,\mathbb{C}}$ defines a constant GC structure on the real vector space $\mathcal{V}_b$. Then we have the subdistributions $\mathcal{E} \subseteq \mathcal{V}_{\mathbb{C}}$ and $\mathcal{S} \subseteq \mathcal{V}$ which are the global analogs of E and S, respectively, which may have nonconstant rank, but are nevertheless smooth in the sense of [**7**], by the argument given in [**1**]. It is easy to check that, in fact, the proofs of parts (e), (f) and (g) of Theorem 2.1 go through in this setup with appropriate modifications that ensure smooth dependence on the point $b \in \mathfrak{B}$.

3. Local normal form

3.1. Strategy of the proof. We begin by outlining the strategy of our proof of Theorem 1.4. Our argument is an extension of the inductive argument of [**11**]. If $\mathrm{rk}_{m_0}\ M = 0$, then there is nothing to prove. Otherwise, following loc. cit., we can split M, locally near m_0, as a product $M = S \times N$ in the sense of Poisson manifolds, $M = S \times N$, where S is an open neighborhood of 0 in $\mathbb{R}^2$ with the induced standard symplectic form ω_0, and $m_0 \in M$ corresponds to $(0, n_0) \in S \times N$. By abuse of notation, we identify N with the submanifold $\{0\} \times N$ of M. It is clear that each "horizontal leaf" $S \times \{n\}$ is a GC submanifold of M.

Lemma 3.1. *The "transverse slice" N is a GC submanifold of M.*

The proof of this lemma is given at the end of the section. We equip N with the induced GC structure. It is clear that $\mathrm{rk}_{n_0}\ N = \mathrm{rk}_{m_0}\ M - 2$. Hence, by induction, it suffices to show that in a neighborhood of m_0, the GC structure on M is a B-field transform of the product of the symplectic structure on S and the induced GC structure on N. The proof of this fact consists of three steps, each involving a transformation by a closed 2-form and possibly replacing M by a smaller open neighborhood of m_0.

To save space, we will still use M to denote any of these sufficiently small neighborhoods. The steps are listed below.

(1) After a transformation by a closed 2-form B'' on M, the induced GC structure on each horizontal leaf $S \times \{n\}$ is the symplectic GC structure defined by ω_0 via the obvious identification $S \cong S \times \{n\}$.
(2) After a transformation by a closed 2-form B' on M that restricts to zero on the horizontal leaves $S \times \{n\}$ and on the transverse slice $\{0\} \times N$, we have that for each $n \in N$, the induced constant GC structure on $T_{(0,n)}M$ is the direct sum of the induced constant GC structures on $T_{(0,n)}(S \times \{n\})$ and on $T_n N$.
(3) After a transformation by a closed 2-form B on M that vanishes along N, the GC structure on all of M is the product of the symplectic GC structure on S and the induced GC structure on N.

3.2. Step 1. We begin by introducing notation that will be used in the rest of the section. Let (p, q) denote the standard coordinates on S, so that $\omega_0 = dp \wedge dq$; we will also view them as part of a coordinate system $(p, q, r_1, \cdots, r_d)$ on M, where $r_1, \cdots, r_d$ are local coordinates on N centered at n_0. Note that for any such coordinate system on M, we have

$$X_p = -\frac{\partial}{\partial q} \quad \text{and} \quad X_q = \frac{\partial}{\partial p}, \tag{3.1}$$

where X_p and X_q denote the Hamiltonian vector fields on M associated to the functions p and q, as in Section 1.2.

Without loss of generality, we may assume that S is the open square on $\mathbb{R}^2$ defined by the inequalities $-1 < p < 1$, $-1 < q < 1$. A point $(s, n) \in S \times N = M$ will from now on be written as (a, b, n), where $a = p(s)$, $b = q(s) \in (-1, 1)$. We will denote by $\phi_s\colon M \to M$ and $\psi_t\colon M \to M$ the flows of the vector field X_p and X_q, respectively. Of course, these flows are not defined everywhere. Explicitly, we have, from equation (3.1),

$$\phi_s(a, b, n) = (a, b - s, n) \quad \text{and} \quad \psi_t(a, b, n) = (a + t, b, n). \tag{3.2}$$

It is clear that the flows ϕ_s and ψ_t commute with each other.

Furthermore, we define $\mathscr{S}_0$ (resp., $\mathscr{N}$) to be the distribution on M which is tangent to the horizontal leaves $S \times \{n\}$ (resp., to the transverse slices $\{s\} \times N$); note that $\mathscr{S}_0$ is spanned by the vector fields X_p, X_q.

We now prove statement (1) of Section 3.1. Since $M = S \times N$ as Poisson manifolds, it follows that for each $n \in N$, the induced GC structure on $S \times \{n\}$ is B-symplectic, with the underlying symplectic structure being given by ω_0. A general fact, proved in [**1**], is that on a B-symplectic GC manifold, both the underlying symplectic structure and the B-field are uniquely determined and, moreover, depend smoothly on the original GC structure. In our situation, we obtain a family $\{B''_n\}$ of closed 2-forms on the leaves $S \times \{n\}$, depending smoothly on n, such that for every $n \in N$, the B-field

B''_n transforms the induced GC structure on $S \times \{n\}$ into the symplectic structure on $S \times \{n\}$ defined by ω_0.

The usual proof of the Poincaré lemma shows that, after possibly shrinking S and N, we can find a smooth family $\{\sigma_n\}_{n\in N}$ of 1-forms on the leaves $S \times \{n\}$ such that $d\sigma_n = B''_n$ for each $n \in N$. Now let σ be an arbitrary 1-form on M such that $\sigma\big|_{S\times\{n\}} = \sigma_n$ for each $n \in N$; such a σ exists simply because $TM = \mathscr{S}_0 \oplus \mathscr{N}$ as vector bundles. By construction, the 2-form $B'' = d\sigma$ satisfies the requirement of statement (1) of Section 3.1.

3.3. Step 2. It follows now from parts (f) and (g) of Theorem 2.1, together with Remark 2.3, that for every point $n \in N$, there exists a unique 2-form $B'_n \in \bigwedge^2 T^*_{(0,0,n)}M$ with the following properties:

- $B'_n\big|_{T_nN} = 0$;
- $B'_n\big|_{T_{(0,0,n)}(S\times\{n\})} = 0$;
- B'_n transforms the constant GC structure on $T_{(0,0,n)}M$ into the direct sum of the induced GC structures on T_nN and $T_{(0,0,n)}(S \times \{n\})$;

and moreover, B'_n depends smoothly on n. We must show that there exists a closed 2-form B' on M such that for each $n \in N$, we have $B'\big|_{S\times\{n\}} = 0$ and $B'\big|_{T_{(0,0,n)}M} = B'_n$. In fact, we will define B' by an explicit formula.

Let us choose a coordinate system $\{x_i\}$ on S centered at $(0,0)$ (one can take $\{x_i\} = \{p, q\}$, but this is not important in this step), and a coordinate system $\{y_j\}$ on N centered at n_0, so that $\{x_i, y_j\}$ is a coordinate system on M centered at m_0. We denote the corresponding coordinate vector fields by $\mathfrak{s}_i = \partial/\partial x_i$, $\mathfrak{n}_j = \partial/\partial y_j$. We then define B' by the formulas

$$B'(\mathfrak{s}_i, \mathfrak{s}_k) = 0; \quad B'(\mathfrak{s}_i, \mathfrak{n}_j)(a, b, n) = B_n\big((\mathfrak{s}_i)_{(0,0,n)}, (\mathfrak{n}_j)_{(0,0,n)}\big)$$

(in particular, note that $B'(\mathfrak{s}_i, \mathfrak{n}_j)$ does not depend on the coordinates x_k);

$$B'(\mathfrak{n}_j, \mathfrak{n}_l) = \sum_i \Big[x_i \cdot \big(\mathfrak{n}_j B'(\mathfrak{s}_i, \mathfrak{n}_l) - \mathfrak{n}_l B'(\mathfrak{s}_i, \mathfrak{n}_j)\big)\Big].$$

By construction, B' satisfies all the required pointwise conditions, so we only have to check that B' is closed. However, it is straightforward to check, using the definition of B', that dB' annihilates any triple of vector fields chosen among the $\mathfrak{s}_i$'s and the $\mathfrak{n}_j$'s.

3.4. Step 3. We now complete the proof outlined in Section 3.1. Let us begin by exploring the consequence of the fundamental theorem of calculus in the context of Lie derivatives. With the notation of Section 3.2, let τ be a differential form on M of arbitrary degree.

Lemma 3.2. *For all $(a, b, n) \in S \times N = M$, we have*

$$\tau_{(a,b,n)} = \phi_b^* \tau_{(a,0,n)} - \int_0^b \big(\phi^*_{b-s}(\mathcal{L}_{X_p}\tau)\big)_{(a,b,n)} ds \tag{3.3}$$

and

$$\tau_{(a,b,n)} = \psi^*_{-a}\tau_{(0,b,n)} + \int_0^a \left(\psi^*_{t-a}(\mathcal{L}_{X_q}\tau)\right)_{(a,b,n)} dt. \tag{3.4}$$

The proof of this lemma is straightforward from the definition of Lie derivative and the fundamental theorem of calculus. Combining equations (3.3) and (3.4), we deduce that

$$\begin{aligned} \tau_{(a,b,n)} &= \phi^*_b\psi^*_{-a}(\tau_{(0,0,n)}) + \phi^*_b \int_0^a \left(\psi^*_{t-a}(\mathcal{L}_{X_q}\tau)\right)_{(a,0,n)} dt \qquad (3.5)\\ &\quad - \int_0^b \left(\phi^*_{b-s}(\mathcal{L}_{X_p}\tau)\right)_{(a,b,n)} ds \\ &= \psi^*_{-a}\phi^*_b(\tau_{(0,0,n)}) - \psi^*_{-a} \int_0^b \left(\phi^*_{b-s}(\mathcal{L}_{X_p}\tau)\right)_{(0,b,n)} ds \qquad (3.6)\\ &\quad + \int_0^a \left(\psi^*_{t-a}(\mathcal{L}_{X_q}\tau)\right)_{(a,b,n)} dt. \end{aligned}$$

In particular, Proposition 1.3(3) now implies that

$$\begin{aligned} \epsilon_{(a,b,n)} &= \phi^*_b\psi^*_{-a}(\epsilon_{(0,0,n)}) + \phi^*_b \int_0^a \left(\psi^*_{t-a}(d\xi_q)\right)_{(a,0,n)} dt \Big|_{E_{(a,b,n)}} \qquad (3.7)\\ &\quad - \int_0^b \left(\phi^*_{b-s}(d\xi_p)\right)_{(a,b,n)} ds \Big|_{E_{(a,b,n)}}. \end{aligned}$$

We now note that, due to the preparations of Sections 3.2 and 3.3, the GC structure on $S \times N$ defined as the product of the symplectic structure on S and the induced GC structure on N corresponds to the 2-form ϵ' on E defined by

$$\epsilon'_{(a,b,n)} = \phi^*_b\psi^*_{-a}(\epsilon_{(0,0,n)}).$$

The proof will therefore be complete if we show that the (real) 2-form B on $S \times N$ defined by

$$B_{(a,b,n)} = \phi^*_b \int_0^a \left(\psi^*_{t-a}(d\xi_q)\right)_{(a,0,n)} dt - \int_0^b \left(\phi^*_{b-s}(d\xi_p)\right)_{(a,b,n)} ds$$

is closed.

Recall first from Proposition 1.3 that $\xi_{\{f,g\}} = \mathcal{L}_{X_f}(\xi_g) - \iota_{X_g}(d\xi_f)$ for all C^∞ functions f, g on M; on the other hand, the definition of the map $f \mapsto \xi_f$ implies that if $\{f, g\}$ is a constant function on M, then $\xi_{\{f,g\}} = 0$. We deduce that

$$\mathcal{L}_{X_p}(\xi_q) = \iota_{X_q}(d\xi_p), \quad \mathcal{L}_{X_q}(\xi_p) = \iota_{X_p}(d\xi_q), \tag{3.8}$$

$$\mathcal{L}_{X_p}(\xi_p) = \iota_{X_p}(d\xi_p), \quad \mathcal{L}_{X_q}(\xi_q) = \iota_{X_q}(d\xi_q). \tag{3.9}$$

We now compute

$$(\mathcal{L}_{X_q}B)_{(a,b,n)} = \lim_{\gamma\to 0} \frac{1}{\gamma} \cdot \Big[\psi^*_\gamma(B_{(a+\gamma,b,n)}) - B_{(a,b,n)}\Big],$$

and

$$\psi_\gamma^* B_{(a+\gamma,b,n)} = \phi_b^* \int_0^{a+\gamma} \big(\psi_{t-a}^*(d\xi_q)\big)_{(a,0,n)} dt - \int_0^b \psi_\gamma^* \phi_{b-s}^* \big((d\xi_p)_{(a+\gamma,s,n)}\big) ds,$$

which leads to

$$(\mathcal{L}_{X_q} B)_{(a,b,n)} = \phi_b^*(d\xi_q)_{(a,0,n)} - \int_0^b \phi_{b-s}^* \Big(\big(\mathcal{L}_{X_q}(d\xi_p)\big)_{(a,s,n)}\Big) ds. \tag{3.10}$$

However, we have, from equation (3.8) and Cartan's formula for $\mathcal{L}_{X_q}$,

$$\mathcal{L}_{X_q}(d\xi_p) = d\iota_{X_q}(d\xi_p) = d\mathcal{L}_{X_p}(\xi_q) = \mathcal{L}_{X_p}(d\xi_q).$$

Substituting this into equation (3.10) and combining with Lemma 3.2, we obtain

$$\mathcal{L}_{X_q} B = d\xi_q. \tag{3.11}$$

A similar, but easier, computation shows that

$$\mathcal{L}_{X_p} B = d\xi_p. \tag{3.12}$$

We now compute $\iota_{X_p} B$. We use the fact that contraction commutes with integration of differential forms, and also that the vector field X_p is invariant under the flows ϕ_s and ψ_t:

$$\begin{aligned}
(\iota_{X_p} B)_{(a,b,n)} &= \phi_b^* \int_0^a \big(\psi_{t-a}^*(\iota_{X_p} d\xi_q)\big)_{(a,0,n)} dt - \int_0^b \big(\phi_{b-s}^*(\iota_{X_p} d\xi_p)\big)_{(a,b,n)} ds \\
&= \phi_b^* \int_0^a \big(\psi_{t-a}^*(\mathcal{L}_{X_q} \xi_p)\big)_{(a,0,n)} dt - \int_0^b \big(\phi_{b-s}^*(\mathcal{L}_{X_p} \xi_p)\big)_{(a,b,n)} ds \\
&= \phi_b^*\Big((\xi_p)_{(a,0,n)} - \psi_{-a}^*\big((\xi_p)_{(0,0,n)}\big)\Big) + (\xi_p)_{(a,b,n)} - \phi_b^*\big((\xi_p)_{(a,0,n)}\big) \\
&= (\xi_p)_{(a,b,n)} - \phi_b^* \psi_{-a}^*\big((\xi_p)_{(0,0,n)}\big),
\end{aligned}$$

where we have used equations (3.8) and (3.9) in the second equality and Lemma 3.2 in the third equality. However, $(\xi_p)_{(0,0,n)} = 0$. This follows from the fact that $(\xi_p)_{(0,0,n)}$ depends only on the value of dp at the point $(0,0,n)$ and on the induced constant GC structure on $T_{(0,0,n)}M$; on the other hand, after the preparations of Sections 3.2 and 3.3, the constant GC structure on $T_{(0,0,n)}M$ is the direct sum of the induced GC structure on $T_n N$ and the symplectic GC structure on $T_{(0,0,n)}(S \times \{n\})$. Therefore,

$$\iota_{X_p} B = \xi_p. \tag{3.13}$$

Likewise,

$$\iota_{X_q} B = \xi_q. \tag{3.14}$$

Let us compare equations (3.12) and (3.13). We can rewrite equation (3.12) as $d(\iota_{X_p} B) + \iota_{X_p}(dB) = d\xi_p$, whence equation (3.13) implies that $\iota_{X_p}(dB) = 0$. Likewise, equations (3.11) and (3.14) force $\iota_{X_q}(dB) = 0$. But X_p, X_q span the tangent space to every horizontal leaf $S \times \{n\}$ at every point.

Hence, to show that $dB = 0$, it remains to check that the restriction of dB to each transverse slice $\{s\}\times N$ is equal to zero. By construction, the restriction of B itself to $\{(0,0)\} \times N$ is zero. Let us pick three arbitrary sections Z_1, Z_2, Z_3 of $\mathscr{N}$ which commute with X_p and X_q. Then $(dB)(Z_1, Z_2, Z_3) = 0$ along $N = \{(0,0)\} \times N$, and furthermore

$$\begin{aligned}\mathcal{L}_{X_p}\big[(dB)(Z_1, Z_2, Z_3)\big] &= (\mathcal{L}_{X_p}(dB))(Z_1, Z_2, Z_3)\\ &= (d\mathcal{L}_{X_p}B)(Z_1, Z_2, Z_3)\\ &= (dd\xi_p)(Z_1, Z_2, Z_3) = 0,\end{aligned}$$

where we have used equation (3.12) and the fact that X_p commutes with each Z_j. Similarly, $\mathcal{L}_{X_q}\big[(dB)(Z_1, Z_2, Z_3)\big] = 0$. It follows that $(dB)(Z_1, Z_2, Z_3) = 0$ everywhere on M and completes the proof.

Proof of Lemma 3.1. Let $(\mathscr{S}, \omega)$ denote the canonical symplectic foliation associated to the GC structure on M, and recall from Section 3.2 that $\mathscr{S}_0 \subseteq \mathscr{S}$ denotes the foliation tangent to the leaves $S\times\{n\}$. Since $M = S\times N$ as Poisson manifolds, it follows that at each point $(s, n) \in N$, the tangent space $T_{(s,n)}(S \times \{n\}) = (\mathscr{S}_0)_{(s,n)}$ is orthogonal to $T_{s,n}(\{s\} \times N) \cap \mathscr{S}_{(s,n)}$ with respect to ω. In particular, by Theorem 2.1(e), the transverse slice N satisfies the pointwise condition for being a GC submanifold of M, and hence we must only show that L_N is a subbundle of $T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$.

Since L_N is the image of $\tilde{L}_N = L\big|_N \cap \Big(T_{\mathbb{C}}N \oplus \big(T^*_{\mathbb{C}}M\big|_N\big)\Big)$ under the projection map $T_{\mathbb{C}}N \oplus \big(T^*_{\mathbb{C}}M\big|_N\big) \to T_{\mathbb{C}}N \oplus T^*_{\mathbb{C}}N$, it suffices to show that $\tilde{L}_N$ is a subbundle of $T_{\mathbb{C}}N \oplus \big(T^*_{\mathbb{C}}M\big|_N\big)$. Further, since $\tilde{L}_N$ is defined as the intersection of two subbundles of $\big(T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M\big)\big|_N$, it suffices to show that $\tilde{L}_N$ has constant rank on N. Considering the projection of $\tilde{L}_N$ onto $T_{\mathbb{C}}N$, we obtain a short exact sequence

$$0 \longrightarrow \big(L \cap T^*_{\mathbb{C}}M\big)\big|_N \longrightarrow \tilde{L}_N \longrightarrow \big(E\big|_N \cap T_{\mathbb{C}}N\big) \longrightarrow 0.$$

Now $L \cap T^*_{\mathbb{C}}M = \mathrm{Ann}_{T^*_{\mathbb{C}}M}(E)$, so

$$\begin{aligned}\mathrm{rk}\,\tilde{L}_N &= \mathrm{rk}\big(\mathrm{Ann}_{T^*_{\mathbb{C}}M}(E)\big|_N\big) + \mathrm{rk}\big(E\big|_N\big) - \mathrm{rk}\left(\frac{(E|_N)}{(E|_N \cap T_{\mathbb{C}}N)}\right)\\ &= \dim M - \mathrm{rk}\,\frac{(E|_N + T_{\mathbb{C}}N)}{T_{\mathbb{C}}N}\\ &= \dim M - \mathrm{rk}\,\frac{T_{\mathbb{C}}M|_N}{T_{\mathbb{C}}N} = \dim N;\end{aligned}$$

we have used the fact that $E\big|_N + T_{\mathbb{C}}N = T_{\mathbb{C}}M\big|_N$, which follows from $\mathscr{S}\big|_N + TN = TM\big|_N$. Thus, in fact, not only is the rank of $\tilde{L}_N$ constant, but the projection map $\tilde{L}_N \to L_N$ is an isomorphism (since L_N has constant rank equal to dim N). $\square$

4. Linearization of GC structures

In this section, we consider a GC structure $\mathcal{J}$ on a manifold M such that the associated Poisson tensor has rank zero at a certain point $m \in M$. Our goal is to describe a "first-order approximation" to the GC structure in a neighborhood of m. We will use the notation $\mathfrak{m}_m^2 \subset \mathfrak{m}_m \subset C^\infty(M)$ in the same sense as in Section 1.4. Also, for $f \in C^\infty(M)$, we will use the notation (X_f, ξ_f) as defined in Section 1.2. Let us assume that $\mathcal{J}$ is given by matrix (1.3). Thus, by assumption, $\pi_m \colon T_m^*M \to T_mM$ is the zero map. Hence, if we consider the induced constant GC structure $\mathcal{J}_m$ on T_mM, its matrix has the form

$$\mathcal{J}_m = \begin{pmatrix} J_m & 0 \\ \sigma_m & -J_m^* \end{pmatrix}.$$

It is proved, for instance, in [**2**], that a constant GC structure of this form is a B-field transform of a complex GC structure on T_mM. If $B_m \in \bigwedge^2 T_m^*M$ is any 2-form which transforms $\mathcal{J}_m$ into a complex GC structure, we can extend B_m to a differential 2-form B on a neighborhood of m in M which is constant in the appropriate local coordinates, and hence, *a fortiori*, is closed. Applying the transformation defined by B to the structure $\mathcal{J}$ reduces us to the situation where $\sigma_m = 0$.

We now assume that $\sigma_m = 0$ and explain what we mean by the first-order approximation to $\mathcal{J}$ at the point m, proving Theorem 1.6 at the same time. Let $\mathfrak{g} = \mathfrak{m}_m/\mathfrak{m}_m^2$ be the real Lie algebra which encodes the first-order approximation to π at m, as defined in Section 1.4. Thus the Lie bracket on $\mathfrak{g}$ is induced by the Poisson bracket on $C^\infty(M)$ defined by π. We can also think of π as a $C^\infty(M)$-linear map from $\Gamma(M, T^*M)$ to $\mathfrak{m}_m \cdot \Gamma(M, TM)$, which induces an $\mathbb{R}$-linear map

$$\frac{\Gamma(M, T^*M)}{\mathfrak{m}_m} \cdot \Gamma(M, T^*M) \longrightarrow \mathfrak{m}_m \cdot \frac{\Gamma(M, TM)}{\mathfrak{m}_m^2} \cdot \Gamma(M, TM).$$

This map also encodes the first-order approximation to π. It is then natural to define the first-order approximation to $\mathcal{J}$ to be the $\mathbb{R}$-linear automorphism of

$$\left(\frac{\mathfrak{m}_m \cdot \Gamma(M, TM)}{\mathfrak{m}_m^2} \cdot \Gamma(M, TM)\right) \oplus \left(\frac{\Gamma(M, T^*M)}{\mathfrak{m}_m} \cdot \Gamma(M, T^*M)\right)$$

induced by $\mathcal{J}$. Note, however, that the map

$$\mathfrak{m}_m \cdot \frac{\Gamma(M, TM)}{\mathfrak{m}_m^2} \cdot \Gamma(M, TM) \longrightarrow \frac{\Gamma(M, T^*M)}{\mathfrak{m}_m} \cdot \Gamma(M, T^*M)$$

induced by σ clearly vanishes; moreover, since J and $K = -J^*$ determine each other, we can concentrate our attention on the map

$$\frac{\Gamma(M, T^*M)}{\mathfrak{m}_m} \cdot \Gamma(M, T^*M) \longrightarrow \frac{\Gamma(M, T^*M)}{\mathfrak{m}_m} \cdot \Gamma(M, T^*M)$$

induced by $-J^*$. Now the de Rham differential d induces an $\mathbb{R}$-linear isomorphism

$$\frac{\mathfrak{m}_m}{\mathfrak{m}_m^2} \xrightarrow{\simeq} \frac{\Gamma(M, T^*M)}{\mathfrak{m}_m} \cdot \Gamma(M, T^*M),$$

and we have, by definition $\xi_f = -J^*(df)$ for any $f \in \mathfrak{m}_m$. By transport of structure, $-J^*$ induces an $\mathbb{R}$-linear automorphism of the Lie algebra $\mathfrak{g}$ which we will denote by A; by construction, $A^2 = -1$. To obtain further information on A, we will study it from the point of view of the map $f \mapsto \xi_f$.

Let $f, g \in \mathfrak{m}_m$. Part (2) of Proposition 1.3 yields

$$\xi_{\{f,g\}} = \mathcal{L}_{X_f}(\xi_g) - \iota_{X_g}(d\xi_f) = \left[\iota_{X_f}(d\xi_g) - \iota_{X_g}(d\xi_f)\right] + d\big(\iota_{X_f}(\xi_g)\big).$$

But $X_f = \pi(df)$ and $X_g = \pi(dg)$, which implies that the first term vanishes modulo $\mathfrak{m}_m$. On the other hand, $\xi_g \equiv d(Ag)$ modulo $\mathfrak{m}_m$, whence $\iota_{X_f}(\xi_g) \equiv \{f, Ag\}$ modulo $\mathfrak{m}_m^2$. Thus we conclude that $A\{f,g\} \equiv \{f, Ag\}$ modulo $\mathfrak{m}_m^2$, which is precisely the condition for A to make $\mathfrak{g}$ a *complex* Lie algebra. This completes the proof of Theorem 1.6.

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