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WONG'S EQUATIONS IN POISSON GEOMETRY

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We show that the Hamiltonian systems on Sternberg-Weinstein phase spaces which yield Wong's equations of motion for a classical particle in a gravitational and a Yang-Mills field, naturally arise as the first order approximation of generic Hamiltonian systems on Poisson manifolds at a critical Lagrangian submanifold. We further define a second order approximated system involving scalar fields which first appeared in Einstein-Mayer theory. Reduction and symplectic realization of this system are interpreted in terms of dimensional reduction and Kaluza-Klein theory.

1. Introduction

In 1970, when quantized non-abelian gauge theory was developing into a cornerstone of today's theoretical physics, Wong ([**29**]) wrote down his famous equations of motion generalizing Lorentz' equations for classical particles in gravitational and Yang-Mills fields. In 1977/78, Sternberg and Weinstein ([**23**, **28**]) showed that Wong's equations could be understood as a Hamiltonian system. From a modern point of view, the underlying *Sternberg-Weinstein phase space* is a Poisson manifold, obtained as the quotient of the cotangent bundle of a principal fiber bundle $P \to B$ by the lifted action of the structure group ([**19**, **20**]). As usual, the gravitational and Yang-Mills fields are modelled by a metric on B and a principal connection form on P, respectively. Further work has been done to study the structure and properties of the Sternberg-Weinstein phase space and generalizations (see [**8**], and the overview below).

On the other hand, Weinstein's splitting theorem ([**27**]) implies that a local linear approximation to any Poisson manifold is given by the product $T^*X \times \mathfrak{g}^*$ of a cotangent bundle with the dual of a Lie algebra endowed with the natural Poisson structure, which is locally equivalent to a Sternberg-Weinstein phase space. Our aim is to describe a global version of such an approximation which extends to the dynamical level.

1.1. Notation. For a manifold M, we denote by $\tau_M : TM \to M$ ($\pi_M : T^*M \to M$) the (co)tangent projection, $\Omega(M) := \oplus_{k=0}^{\dim M}\Omega^k(M)$ ($\mathfrak{X}(M) := \oplus_{k=0}^{\dim M}\mathfrak{X}^k(M)$) the (contravariant) exterior algebra, $[\cdot,\cdot]$ the Schouten-Nijenhuis bracket on $\mathfrak{X}(M)$, and $\mathrm{i}_{\mathcal{X}}$ ($\mathrm{L}_{\mathcal{X}}$) the contraction (Lie derivative) with respect to $\mathcal{X} \in \mathfrak{X}^1(M)$. The tangent map of $f : M \to \tilde{M}$ is denoted by Tf. For a fiber bundle $p : E \to M$ over M, $\Gamma(E)$ denotes the global sections of E, and VE the vertical tangent bundle. The product bundle of E and $\tilde{E}$ over M is denoted by $E \times_M \tilde{E}$, and we write $(pr_1, pr_2) : E \times_M \tilde{E} \to E \times \tilde{E}$ for the component projections. The dual of a linear map of *vector bundles* $\phi : E \to \tilde{E}$ over $f : M \to \tilde{M}$ is denoted by $\phi^* : S^k\tilde{E}^*|_{f(M)} \to S^kE^*$, in consistency with the pull-back since $\phi^*\bar{s} = \overline{\phi^* s} = \overline{\phi^* \circ s \circ f}$; by $\bar{u} \in C^\infty(E)$ we denote the fiber-polynomial function defined by a section $u \in \Gamma(S^\bullet E^*)$, $S^\bullet E^* = \oplus_{k=0}^{\infty} S^kE^*$, where S^k denotes the k-th symmetric tensor product. If $M = \tilde{M}$, we write $E \oplus_M \tilde{E}$ ($E \otimes_M \tilde{E}$) for the Whitney sum (fiberwise tensor product). For $q \in \Gamma(E \otimes_M \tilde{E})$ ($\sigma \in \Gamma(E^* \otimes_M \tilde{E}^*)$), we denote $q^\sharp : \tilde{E}^* \to E$ ($\sigma^\flat : E \to \tilde{E}^*$) the corresponding bundle morphism. Given a Poisson manifold (Z, ϖ), we denote by $X_f = \varpi^\sharp \circ df$ the Hamiltonian vector field of $f \in C^\infty(Z)$. For a principal bundle $P \to M$, the associated fiber bundle to P with standard fiber F will be denoted by $F(P)$. Scalar products and metrics are understood to be pseudo-euclidean in general.

1.2. Overview. Below we describe the construction of Sternberg and Weinstein, and the transitive Lie algebroid structure on the restriction of the cotangent bundle of every Poisson manifold Z to a symplectic leaf $S \subset Z$. The last allows to define the notion of an E-connection form, intimately related to gauge fields. Similar results have earlier been obtained by Y. Vorobjev [**26**]. In Section 2, using the general fact that the dual bundle to any Lie algebroid carries a natural Poisson structure (see [**2**]), we construct, for any Lagrangian submanifold $X \subset S$ such that $dH|_X = 0$ for a generic Hamiltonian H on Z, a Poisson manifold Z' and a canonical Hamiltonian system defined by a Hamiltonian H_1' on Z', which can be considered as a linear approximation of Z and H, respectively. Locally, it is determined by a metric and a Yang-Mills field on X, as for the system yielding the Wong equations. Furthermore, we show that another system on Z', related to Einstein-Mayer theory ([**7**]), can be seen as a natural quadratic approximation, containing additional scalar fields. In Section 3 we constrain the construction to a coisotropic submanifold $Q \subset Z$, which leads to an approximated system well-known from dimensional reduction ([**4**]). It allows to interpret reduced scalar fields as Higgs fields. Finally, we show in Section 4 that our constructions can be locally related to Kaluza-Klein and gauge theory via the choice of a locally minimal symplectic realization $\rho : W \to U \subset Z$. In particular, it allows to interpret the original system locally as describing particle motion in a Yang-Mills field with values in a general Poisson-algebra. This will be treated in another work, except for some comments in section 4.1. The Yang-Mills equations for classical gauge theories involving non-linear Poisson structures should be similar to those of the Poisson sigma model of N. Ikeda ([**10**]) and Schaller-Strobl ([**21**]). This will also be treated separately.

Since the Lie algebras of usual gauge theories lead to linearizable Poisson structures, classical gauge theory turns out to be a generic setting in the Poisson geometric framework. Notice also that the results of this work should partly fit into the more general technical framework developed in [**12**] for studying linearized dynamics near invariant isotropic submanifolds.

1.3. The Sternberg-Weinstein phase space. If (W, ω) is a symplectic manifold with a surjective submersive moment map $\lambda : W \to \mathfrak{g}_{\mathtt{L}}^*$ generating the right action of a Lie group G with Lie algebra $Lie(G) = \mathfrak{g}_{\mathtt{L}}$ on W, and if the quotient W/G is a manifold, we know that the natural projection $\rho : W \to W/G$ coinduces a Poisson structure on W/G, and the G-orbits are symplectically orthogonal to the fibers of λ ([**27**]). We consider the following special case of this situation.

Proposition 1.1. *Let $p : P \to B$ be a principal fiber bundle over the manifold B with connected structure group G and $Lie(G) = \mathfrak{g}_{\mathtt{L}}$. The canonical lift of the (right) G-action on P to (T^*P, ω), where ω is the canonical symplectic form, is generated by a moment map $\lambda : T^*P \to \mathfrak{g}_{\mathtt{L}}^*$ given by $\alpha \mapsto (\varphi_y)^*(\alpha|_{V_yP})$, where $y = \pi_P(\alpha)$ and $\varphi : P \times \mathfrak{g}_{\mathtt{L}} \to VP$, $(y, D) \mapsto (D_P)(y)$, D_P being the fundamental vector field of $D \in \mathfrak{g}_{\mathtt{L}}$. Assuming that the quotient T^*P/G is a manifold, there is a coinduced Poisson structure ϖ on it. If ρ denotes the projection, $(\mathfrak{g}_{\mathtt{L}}^*, w^{\mathfrak{g}_{\mathtt{L}}}) \overset{\lambda}{\leftarrow} (T^*P, \omega) \overset{\rho}{\to} (T^*P/G, \varpi)$ is a dual pair.*

Definition 1.2. We call the quotient space $(T^*P/G, \varpi)$ the *Sternberg-Weinstein phase space*. Since G was supposed connected, the leaves of the symplectic foliation of T^*P/G are in bijection with the symplectic leaves of $\mathfrak{g}_{\mathtt{L}}^*$, that is, the coadjoint orbits. A coadjoint orbit $O \subset \mathfrak{g}_{\mathtt{L}}^*$ endowed with its symplectic structure and moment map is called a *generalized charge*, in analogy with the special case $G = U(1)$ related to electrodynamics.

Remark 1.3. Sternberg ([**23**]) and Weinstein ([**28**]) directly constructed the phase space corresponding to a specific choice of a coadjoint orbit, performing a Marsden-Weinstein reduction. Notice that the charge 0 corresponds to a particular symplectic leaf of minimal dimension and which is naturally identified with the base phase space T^*B.

Any metric γ on B naturally defines a Hamiltonian H_0 on T^*B, quadratic on the fibers and describing geodesic motion in a gravitational field modelled by γ on B. Given in addition a Yang-Mills field modelled by a G-equivariant principal connection 1-form $\hat{A}$ on P, we dispose of the projections

$$\mu_{\hat{A}} : T^*P \to T^*B \qquad \text{and} \qquad \check{\mu}_{\hat{A}} : T^*P/G \to T^*B. \tag{1.1}$$

Then, $H = \check{\mu}^*_{\hat{A}} H_0$ defines a Hamiltonian system on T^*P/G *(minimal coupling)*. Consider the bundle $\mathfrak{g}^*_{\mathtt{L}}(P^\pi)$ for $P^\pi = T^*B \times_B P \stackrel{p^\pi = pr_1}{\longrightarrow} T^*B$, with the induced and coadjoint right G-action. The map $\hat{\psi}_{\hat{A}}$ defined by

$$\hat{\psi}_{\hat{A}} = (\hat{\mu}_{\hat{A}}, \lambda) = (\mu_{\hat{A}} \times_B \pi_P, \lambda) : \quad T^*P \longrightarrow P^\pi \times \mathfrak{g}^*_{\mathtt{L}}$$

is a G-equivariant diffeomorphism. Thus, it induces a diffeomorphism of the quotient spaces $\psi_{\hat{A}} : T^*P/G \longrightarrow \mathfrak{g}^*_{\mathtt{L}}(P^\pi)$. Denoting $p^\pi_{\mathfrak{g}^*_{\mathtt{L}}} : \mathfrak{g}^*_{\mathtt{L}}(P^\pi) \to T^*B$ the natural projection, we obtain the commutative diagram:

$$\begin{array}{ccccc} T^*B \stackrel{\mu_{\hat{A}}}{\longleftarrow} T^*P & \xrightarrow{\quad\hat{\psi}_{\hat{A}}\quad} & P^\pi \times \mathfrak{g}^*_{\mathtt{L}} & & \\ \nwarrow \check{\mu}_{\hat{A}} \quad \downarrow \rho & & \downarrow & \searrow p^\pi \circ pr_1 & \\ T^*P/G & \xrightarrow{\quad\psi_{\hat{A}}\quad} & \mathfrak{g}^*_{\mathtt{L}}(P^\pi) & \stackrel{p^\pi_{\mathfrak{g}^*_{\mathtt{L}}}}{\longrightarrow} & T^*B \end{array}$$

If we define a Poisson structure and a Hamiltonian on $\mathfrak{g}^*_{\mathtt{L}}(P^\pi)$ by setting $\varpi_{\hat{A}} = (\psi_{\hat{A}})_* \varpi$ and $H_{\hat{0}} = (p^\pi_{\mathfrak{g}^*_{\mathtt{L}}})^* H_0$, then the systems $(T^*P/G, \varpi, H)$ and $(\mathfrak{g}^*_{\mathtt{L}}(P^\pi), \varpi_{\hat{A}}, H_{\hat{0}})$ are $\psi_{\hat{A}}$-related. In addition, $\mathfrak{g}^*_{\mathtt{L}}(P^\pi)$ is naturally fibred over T^*B, the phase space corresponding to the charge 0, while the gauge field influences the dynamics only via the Poisson structure. This is the natural phase space for writing down the equations of motion of particles in a gravitational and a Yang-Mills field, by calculating $\varpi_{\hat{A}}$ explicitly.

Definition 1.4. The systems $(T^*P/G,\ \varpi,\ H)$ and $(\mathfrak{g}^*_{\mathtt{L}}(P^\pi),\ \varpi_{\hat{A}},\ H_{\hat{0}})$ are called the *Wong system* and the *left gauged Wong system*, respectively.

It is well-known that *Kaluza-Klein theory* provides a description of Wong dynamics on the realization space T^*P ([**11**], [**14**], [**13**]).

Definition 1.5. Let $p : P \to B$ be a principal fiber bundle with structure group G. Let $\mathfrak{g}_{\mathtt{L}} = Lie(G)$, and let ι be a scalar product on $\mathfrak{g}_{\mathtt{L}}$. To any metric γ on B, and to any principal connection form $\hat{A}$ on P, we associate a *Kaluza-Klein metric* (or *bundle metric*) κ by setting

$$\kappa = p^*\gamma + \hat{A}^*\iota \tag{1.2}$$

where $(\hat{A}^*\iota)_s(U, V) = \iota(\hat{A}_s(U), \hat{A}_s(V))$ for all $s \in P; U, V \in T_sP$.

Theorem 1.6. *Let $p : P \to B$, γ, $\hat{A}$, ι and κ be as in definition* (1.5), *and let H_0 and K be the Hamiltonians defined by γ and κ on T^*B and T^*P, respectively. Let $\rho : (T^*P, \omega) \to (T^*P/G, \varpi)$ be the Poisson projection as in Proposition* 1.1, *define $\check{\mu}_{\hat{A}} : T^*P/G \to T^*B$ as in* (1.1) *and set $H :=$*

$\check{\mu}^*_{\hat{A}} H_0$. *Then, the Hamiltonian systems* (T^*P, ω, K) *and* $(T^*P/G, \varpi, H)$ *are* ρ_**-related. Furthermore, the projections by* p *of the geodesics of the Kaluza-Klein metric* (1.2) *coincide with the projections by* $\pi_B \circ \check{\mu}_{\hat{A}} : T^*P/G \to B$ *of the solutions of the Hamiltonian equations defined by* H *on* $(T^*P/G, \varpi)$.

1.4. The Lie algebroid over a symplectic leaf. Let (Z, ϖ) be a Poisson manifold. The following well-known observation implies that the triple $(T^*Z, -\varpi^\sharp, \{\cdot,\cdot\})$ is a Lie algebroid (see [**16**]) over Z.

Proposition 1.7. *There exists a unique* $\mathbb{R}$*-bilinear, skew-symmetric operation* $\{\cdot,\cdot\} : \Omega^1(Z) \times \Omega^1(Z) \to \Omega^1(Z)$ *such that* $\{df, dg\} = d\{f, g\}$ *and*

$$\{\alpha, f\beta\} = f\{\alpha, \beta\} - (\varpi^\sharp \circ \alpha)(f)\beta \tag{1.3}$$

for all $f, g \in C^\infty(Z), \alpha, \beta \in \Omega^1(Z)$. *This operation is given by the formulas*

$$\{\alpha, \beta\} = \mathrm{L}_{\varpi^\sharp \circ \beta}\alpha - \mathrm{L}_{\varpi^\sharp \circ \alpha}\beta - d\varpi(\alpha, \beta) = \mathrm{i}_{\varpi^\sharp \circ \beta} d\alpha - \mathrm{i}_{\varpi^\sharp \circ \alpha} d\beta + d\varpi(\alpha, \beta).$$

Furthermore, it provides $\Omega^1(Z)$ *with a Lie algebra structure such that* $-\varpi^\sharp \circ$ *is a Lie algebra homomorphism, i.e.,* $-\varpi^\sharp \circ \{\alpha, \beta\} = [-\varpi^\sharp \circ \alpha, -\varpi^\sharp \circ \beta]$.

The dual bundle to any Lie-algebroid carries a natural Poisson structure (see ([**2**]). The natural Poisson structure of the tangent bundle TZ is called the *tangent* Poisson structure on TZ, while $(T^*Z, -\varpi^\sharp, \{\cdot,\cdot\})$ is called the *tangent Lie algebroid* (see [**1**]). Recall that the pointwise *transversal Lie algebra* and *Poisson structures* arise as restrictions of the bracket to (arbitrary extensions of) elements of the annihilator spaces $(T_zS_z)^0$ and its natural dual T_zZ/T_zS_z, respectively, where S_z is the symplectic leaf through any $z \in Z$.

Let now $i_S : S \to Z$ be the inclusion of a fixed symplectic leaf (S, ω_S). The mutually dual vector bundles over S defined by

$$E = T^*Z|_S \longrightarrow S \qquad \text{and} \qquad E^* = TZ|_S \longrightarrow S.$$

are called the *Lie algebroid* and the *dual Lie algebroid* associated to S, respectively. Furthermore, the mutually dual vector bundles

$$L = (TS)^0 \longrightarrow S \qquad \text{and} \qquad L^* = TZ|_S/TS \longrightarrow S,$$

where $(TS)^0$ denotes the annihilator subbundle in $T^*Z|_S$, are called the *Lie algebra bundle* and the *dual Lie algebra bundle* to Z at S, respectively. Writing $\varpi^\sharp$ for $\varpi^\sharp|_S$, we have the exact sequence of vector bundles over S

$$0 \longrightarrow L \hookrightarrow E \xrightarrow{-\varpi^\sharp} TS \longrightarrow 0. \tag{1.4}$$

Lemma 1.8. *There is a well-defined bracket on the space of sections of* E

$$\{\cdot,\cdot\} : \Gamma(E) \times \Gamma(E) \to \Gamma(E) \qquad \{\alpha|_S, \beta|_S\} = \{\alpha, \beta\}|_S \tag{1.5}$$

for all $\alpha, \beta \in \Omega^1(Z)$, *where the bracket on the right is the bracket of Proposition* 1.7. *The restriction of this bracket to* $\Gamma(L)$ *is given by the fiberwise bracket of the transverse Lie algebras to* S, *and* $\Gamma(L) \subset \Gamma(E)$ *is an ideal.*

Proof. Let $\mathcal{X} \in \mathfrak{X}^1(Z)$. Cartan's formula yields

$$\{\alpha, \beta\}(\mathcal{X}) = (\varpi^\sharp \circ \beta) \cdot \alpha(\mathcal{X}) - (\varpi^\sharp \circ \alpha) \cdot \beta(\mathcal{X}) + (\mathrm{L}_{\mathcal{X}} \varpi)(\alpha, \beta). \tag{1.6}$$

Since $\varpi^\sharp(E) = TS$, this shows that $\{\alpha, \beta\}|_S$ depends only on $\alpha|_S$ and $\beta|_S$. For $\alpha \in \Gamma(E), \beta \in \Gamma(L)$, it follows from (1.6) with $\mathcal{X}|_S \in \mathfrak{X}^1(S)$ and thus $\mathrm{L}_{\mathcal{X}} \varpi|_S \in \mathfrak{X}^2(S)$ that $\{\alpha, \beta\} \in \Gamma(L)$. The other assertions are clear. □

Corollary 1.9. *The bundle E with the bracket* (1.5) *on* $\Gamma(E)$ *and the anchor map $-\varpi^\sharp$ is a transitive Lie algebroid. Furthermore, L is a natural Lie subalgebroid of E, and E^* and L^* carry natural Poisson structures.*

Definition 1.10. A 1-form $\theta \in \Gamma(T^*S \otimes_S E)$ on S with values in E is called an *E-connection form* iff the associated bundle morphism $\theta^\flat : TS \to E$ is a splitting of (1.4), that is, $-\varpi^\sharp \circ \theta^\flat = Id_{TS}$. The set of E-connection forms is denoted by $\mathfrak{A}$.

As for principal connections, the Leibnitz identity implies:

Proposition 1.11. *Let θ be an E-connection form. There exists a 2-form on S with values in L satisfying*

$$\Phi^\theta(\mathcal{X}_1, \mathcal{X}_2) = \{\theta(\mathcal{X}_1), \theta(\mathcal{X}_2)\} - \theta([\mathcal{X}_1, \mathcal{X}_2]) \qquad \forall \mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}^1(S). \tag{1.7}$$

Definition 1.12. The 2-form Φ^θ is called the *curvature* of θ.

If $\phi : Z \to Z$ is a Poisson automorphism such that $\phi(S) = S$, its tangent map induces a map $\phi^* : \Gamma(\wedge^k T^*S \otimes_S E) \to \Gamma(\wedge^k T^*S \otimes_S E)$ of ϕ given by

$$(\phi^* \alpha)^\flat = (T\phi|_S)^* \circ \alpha^\flat \circ \wedge^k T(\phi|_S) \qquad \forall \alpha \in \Gamma(\wedge^k T^*S \otimes_S E), \tag{1.8}$$

where $\alpha^\flat$ denotes the bundle morphism associated to the form α. It is straightforward to show that $(\phi \circ \psi)^* \alpha = \psi^* \circ \phi^*$ for any two ϕ and ψ preserving S, that the action of ϕ preserves the space $\mathfrak{A}$ and the bracket defined in Lemma 1.5, and thus $\Phi^{\phi^* \theta} = \phi^* \Phi^\theta$. This is analogous to active gauge transformations for usual principal connections. Since Poisson manifolds are locally split, we might call such transformations *splitting transformations*. In deed, one can easily see that every splitting map determines a local E-connection form with vanishing curvature, and that it is possible to write down splitting transformations as transitions between splittings, recovering formulas similar to local gauge transformations ([**18**]). Recall that the fibers of E are all isomorphic to some fixed Lie algebra $\mathfrak{g}$.

Definition 1.13. The bundle $r : R \to S$ whose fiber at $x \in S$ consists of all Lie algebra isomorphisms $f : \mathfrak{g} \to L_x$ is called the *Lie frame bundle* at S.

Proposition 1.14. *R is a principal bundle over S with structure group $Aut(\mathfrak{g})$, where the group action is given by $(f, a) \to f \circ a$ for $(f, a) \in R \times Aut(\mathfrak{g})$. With respect to the diagonal* (*right*) *action of $Aut(\mathfrak{g})$ on $R \times \mathfrak{g}$, we* *have a canonical isomorphism $L \cong \mathfrak{g}(R)$. Any E-connection form θ defines a covariant derivative on L given by*

$$(1.9) \qquad \nabla^\theta : \Gamma(L) \longrightarrow \Gamma(T^*S \otimes_S L) \qquad \eta \longmapsto \{\theta, \eta\}.$$

Here, the brackets denote the bracket of section of E defined in Lemma 1.8. Furthermore, it defines a principal connection form A^θ in the principal bundle R, which takes values in the inner derivations of $\mathfrak{g}$. The curvature form of ∇^θ is given by

$$(1.10) \quad F^\theta_\nabla(\mathcal{X}_1, \mathcal{X}_2) = \nabla^\theta_{\mathcal{X}_1}\nabla^\theta_{\mathcal{X}_2} - \nabla^\theta_{\mathcal{X}_2}\nabla^\theta_{\mathcal{X}_1} - \nabla^\theta_{[\mathcal{X}_1,\mathcal{X}_2]} = (ad \circ \Phi^\theta)(\mathcal{X}_1, \mathcal{X}_2)$$

for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}^1(S)$. Here, ad denotes the adjoint action of the Lie algebra structure of the fibers of L. In particular, the curvature form takes its values in the inner derivations on the fibers of L, and the corresponding curvature 2-form $F^\theta = dA^\theta + \frac{1}{2}[A^\theta, A^\theta]$ on R in the inner derivations of $\mathfrak{g}$, i.e., $ad(\mathfrak{g})$.

Proof. Since all fibers are isomorphic to $\mathfrak{g}$, $L \cong \mathfrak{g}(R)$ by standard constructions (cf [**15**]). Let now θ be an E_X-connection form. Since $\Gamma(L) \subset \Gamma(E)$ is an ideal, (1.9) makes sense, and one can easily check that by (1.3) it defines a covariant derivative corresponding to a linear connection on L, and induces a principal connection in the corresponding frame bundle. Since ∇^θ acts by the Lie bracket of forms, it is a derivation of the Lie algebra structure in the fibers, and thus, parallel transport will act by automorphisms. Consequently, the connection in the principal frame bundle reduces to the Lie frame bundle R. Furthermore, the action of ∇^θ is by inner derivations of the fibers of $\mathfrak{g}(P)$. Thus, the principal connection form takes values in $ad(\mathfrak{g})$. Finally, we obtain (1.10) as an immediate consequence of the Jacobi identity for $\{\cdot,\cdot\}$. Thus, the 2-form F^θ on R takes values in $ad(\mathfrak{g})$, and F^θ_∇ takes values in the inner derivations of the fibers of L. □

2. Sternberg-Weinstein approximation

Here we construct a natural linear approximation to a Hamiltonian system on Z by a Wong system. The construction depends on the choice of an embedded *Lagrangian submanifold* $i_X : X \to S$ of the fixed leaf $S \subset Z$.

2.1. The Poisson structure. Let $(TX)^0$ and $(\underline{TX})^0$ denote the subbundles of $T^*Z|_X$ and $T^*S|_X$ given by the annihilator spaces of the fibers of TX as a subbundle of $TZ|_X$ and $TS|_X$, respectively. Since X is Lagrangian, the restriction $-\varpi^\sharp_X = -\varpi^\sharp|_{(TX)^0}$ yields a map

$$(2.1) \quad -\varpi^\sharp_X = Ti_S \circ (-\omega^\flat_S)^{-1} \circ (Ti_S)^*|_{(TX)^0} : \; (TX)^0 \to (\underline{TX})^0 \to TX.$$

If we define the (*dual*) *Lie algebroid* and (*dual*) *Lie bundle at* X by

$$\begin{aligned} E_X &= (TX)^0 \subset T^*Z|_X = E|_X & E^*_X &= TZ|_X/TX = E^*|_X/TX \\ L_X &= (TS)^0|_X = L|_X & L^*_X &= TZ|_X/TS|_X = L^*|_X, \end{aligned}$$

respectively, E_X, E_X^* and L_X, L_X^* being duals, we obtain an exact sequence

$$0 \longrightarrow L_X \longrightarrow E_X \longrightarrow TX \longrightarrow 0. \tag{2.2}$$

Lemma 2.1. *There is a well-defined bracket on the space of sections of E_X*

$$\{\cdot,\cdot\} : \Gamma(E_X) \times \Gamma(E_X) \to \Gamma(E_X) \qquad \{\alpha|_X, \beta|_X\} = \{\alpha, \beta\}|_X \tag{2.3}$$

for all $\alpha, \beta \in \Omega^1(Z)$ such that $\alpha|_X$ and $\beta|_X$ take values in E_X, where the bracket on the right is the bracket of Proposition 1.7. *Furthermore, $\Gamma(L_X) \subset \Gamma(E_X)$ with the fiberwise bracket is an ideal.*

Proof. Let $\alpha, \beta \in \Omega^1(Z)$ such that $\alpha|_X$ and $\beta|_X$ take values in E_X, and $\alpha^\sharp = \varpi^\sharp \circ \alpha$ etc. From Cartan's formula, we saw that for $\mathcal{X} \in \mathfrak{X}^1(Z)$,

$$\{\alpha, \beta\}(\mathcal{X}) = \mathrm{i}_{\alpha^\sharp}(\mathrm{i}_{\mathcal{X}} d\beta) - \mathrm{i}_{\beta^\sharp}(\mathrm{i}_{\mathcal{X}} d\alpha) + \mathcal{X} \cdot \varpi(\alpha, \beta) \tag{2.4}$$

$$= \beta^\sharp \cdot \alpha(\mathcal{X}) - \alpha^\sharp \cdot \beta(\mathcal{X}) + (\mathrm{L}_{\mathcal{X}} \varpi)(\alpha, \beta). \tag{2.5}$$

Now, (2.1) shows that $\alpha^\sharp|_X$ and $\beta^\sharp|_X$ take their values in TX, and thus, the restriction to X of (2.5) depends only on the values of α and β on X. Furthermore, assume that $\mathcal{X}|_X \in \mathfrak{X}^1(X)$. Then, the last term in (2.4) vanishes on X, as does the first term $\mathrm{i}_{\alpha^\sharp}(\mathrm{i}_{\mathcal{X}} d\beta) = d\beta(\mathcal{X}, \alpha^\sharp) = \alpha^\sharp \cdot \beta(\mathcal{X}) - \mathcal{X} \cdot \beta(\alpha^\sharp) - \beta([\mathcal{X}, \alpha^\sharp])$, and also the second term for similar reasons. This shows that the bracket of E_X-valued forms is E_X-valued. The last assertions follow from Lemma 1.8. □

Corollary 2.2. *The bundle E_X with the bracket* (2.3) *on $\Gamma(E_X)$ and the anchor map $-\varpi_X^\sharp$ is a transitive Lie algebroid with L_X as a Lie subalgebroid, and E_X^* and L_X^* are endowed with natural Poisson structures.*

Definition 2.3. A 1-form $\alpha : X \to T^*X \otimes_X E_X$ on X with values in E_X is called an E_X-*connection form* if the associated bundle morphism over X $\alpha^\flat : TX \to E_X$ is a splitting of (2.2), i.e., satisfies $-\varpi_X^\sharp \circ \alpha^\flat = Id_{TX}$. The set of E_X-connection forms is called $\mathfrak{A}_X$.

Proposition 2.4. *For any $\theta \in \mathfrak{A}$, $i_X^*\theta \in \mathfrak{A}_X$, and $\mathfrak{A}_X = i_X^*\mathfrak{A}$. For any $\alpha = i_X^*\theta \in \mathfrak{A}_X$, there exists a 2-form on X with values in L_X satisfying, for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}^1(X)$, $\Phi^\alpha(\mathcal{X}_1, \mathcal{X}_2) = i_X^*\Phi^\theta = \{\alpha(\mathcal{X}_1), \alpha(\mathcal{X}_2)\} - \alpha([\mathcal{X}_1, \mathcal{X}_2])$. If ϕ is a Poisson isomorphism preserving S and X, it yields an action on $\mathfrak{A}_X$ as in* (1.8) *by setting $\phi^*(i_X^*\theta) = i_X^*(\phi^*\theta)$ for all $\theta \in \mathfrak{A}$, and $\Phi^{\phi^*\alpha} = \phi^*\Phi^\alpha$. Finally, any $\alpha \in \mathfrak{A}_X$ induces a principal connection in the restricted principal bundle $R_X := R|_X$ and the associated bundle $\mathfrak{g}(R_X) = \mathfrak{g}(R)|_X$.*

Definition 2.5. The 2-form Φ^α is called the *curvature* of α.

Now we turn to the Poisson structure on E^*_X and show that it is a natural linear approximation to Z near X by describing its construction explicitly. We denote by $\tau : E^*_X \to X$ the projection induced by $\tau_Z : TZ \to Z$.

Proposition 2.6. *There is a natural extension of the bracket defined in Lemma* 2.1 *on the sections of E_X, seen as vertically linear functions on the dual bundle E^*_X, to an exact Poisson bracket $\{\cdot,\cdot\}'$ on $C^\infty(E^*_X)$.*

Proof. Clearly, it suffices to specify the Poisson bracket on a family of functions whose differentials span $T^*E^*_X$ at each point. Let the subspace $\tau^*C^\infty(X) \oplus \bar{\Gamma}(E_X) \subset C^\infty(E^*_X)$ be the sum of the subspaces of vertically constant and linear functions, respectively, the last corresponding to elements of $\Gamma(E_X)$. One easily verifies that $T^*_t(E^*_X) = d(\tau^*C^\infty(X) \oplus \bar{\Gamma}(E_X))_t$ for all $t \in E^*_X$. We can naturally define the Lie bracket $\{\cdot,\cdot\}'$ to be the trivial Lie bracket on $\tau^*C^\infty(X)$ and induced on $\bar{\Gamma}(E_X)$ by the Lie algebroid bracket on $\Gamma(E_X)$. From the Leibnitz identities for the Lie algebroid brackets and the bracket $\{\cdot,\cdot\}'$, it follows that for $\mathcal{V} \in \Gamma(E)$, denoting the corresponding function as $\bar{\mathcal{V}} \in \bar{\Gamma}(E_X)$, and $f \in C^\infty(X)$, we must have

$$\{\bar{\mathcal{V}}, \tau^* f\}' = -X_{\bar{\mathcal{V}}}\tau^* f = \tau^*((-\varpi^\sharp_X \circ \mathcal{V})f) = ((-\varpi^\sharp_X)^* \circ df)\bar{\mathcal{V}} = X_{\tau^* f}\bar{\mathcal{V}}, \tag{2.6}$$

where the section $(-\varpi^\sharp_X)^* \circ df : X \to T^*X \to E^*_X$ of E^*_X is seen as a constant vertical vector field on E^*_X, which is thus the Hamiltonian vector field of $\tau^* f$. (2.6) gives us a map $\alpha : \bar{\Gamma}(E_X) \to \mathfrak{der}(\tau^*C^\infty(X))$, and a Poisson bracket $\{\cdot,\cdot\}'$ on $\tau^*C^\infty(X) \oplus \bar{\Gamma}(E_X) \subset C^\infty(E^*_X)$ can thus be naturally defined as the semidirect sum $\tau^*C^\infty(X) \rtimes_\alpha \bar{\Gamma}(E_X)$. It remains to verify that the extension of $\{\cdot,\cdot\}'$ to $C^\infty(E^*_X)$ satisfies the Jacobi identity.

We take Darboux coordinates (x^μ, p_μ, r_a), $1 \le \mu \le m = \frac{1}{2}rk(w_{x_0})$ and $1 \le a \le n = \dim Z - 2m$, centered at x_0, such that (with Einstein's convention)

$$\varpi = \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\mu} + \tfrac{1}{2}\varpi_{ab}(r_a))\frac{\partial}{\partial r_a} \wedge \frac{\partial}{\partial r_b}$$

where $\varpi_{ab}(r_a) = \varpi^c_{ab}(r_a)r_c = c^c_{ab}r_c + O(r^2)$ and $c^c_{ab} = \varpi^c_{ab}(0)$ are the structure functions and constants of the transverse structure and Lie algebra $\mathfrak{g}$ at x_0, respectively. Jacobi's identity reads $\sum_{cycl} \varpi^{lj}\frac{\partial \varpi^{ik}}{\partial x^l} = 0$ (sum over ijk).

Since our claim is local, we can assume that our Darboux coordinates on Z cover all of X. Then they induce coordinates $(x^\mu, \dot{x}^\mu, \dot{p}_\mu, \dot{r}_a)$ on $TZ|_X$ with respect the vector fields $(\partial/\partial x^\mu, \partial/\partial p_\mu, \partial/\partial r_a)$ whose values at x span T_xZ for all $x \in X$, and coordinates $(x^\mu, [\dot{p}_\mu], [\dot{r}_a])$ on $E^*_X = TZ|_X/TX$ with respect to the sections $([\partial/\partial p_\mu], [\partial/\partial r_a])$ of E^*_X whose values at x span $(E^*_X)_x$ for all $x \in X$. On the other hand, the values at x of the sections $(dp_\mu|_X, dr_a|_X)$ of T^*Z_X span $E_X = (TX)^0$ at each $x \in X$. By definition,

$x^\mu \in \tau^* C^\infty(X)$, while $[\dot p_\mu] = \bar{dp_\mu}|_X, [\dot r_a] = \bar{dr_a}|_X \in \bar\Gamma(E_X)$. Thus, we have

$$\begin{aligned}
\{x^\mu, x^\nu\}' &= 0\\
\{[\dot p_\mu], [\dot p_\nu]\}' &= \overline{[dp_\mu|_X, dp_\nu|_X]_{E_X}} = \overline{(d\{p_\mu, p_\nu\}|_X)} = 0\\
\{x^\mu, [\dot p_\nu]\}' &= ((\varpi_X^\sharp \circ dp_\nu|_X) \cdot x^\mu = \partial x^\mu/\partial x^\nu = \delta^\mu_\nu\\
\{x^\mu, [\dot r_b]\}' &= ((\varpi_X^\sharp \circ dr_b|_X) \cdot x^\mu = 0\\
\{[\dot p_\mu], [\dot r_b]\}' &= \overline{[dp_\mu|_X, dr_b|_X]_{E_X}} = \overline{(d\{p_\mu, r_b\}|_X)} = 0\\
\{[\dot r_a], [\dot r_b]\}' &= \overline{[dr_a|_X, dr_b|_X]_{E_X}} = \overline{(d\{r_a, r_b\}|_X)} = c^c_{ab}\bar{dr_c}|_X = c^c_{ab}[\dot r_c]
\end{aligned}$$

where we used the definitions above. Hence, with respect to the coordinates $(x^\mu, [\dot p_\mu], [\dot r_a])$, the bivector ϖ' defined by the bracket $\{\cdot,\cdot\}'$ on $C^\infty(E_X^*)$ reads

$$\varpi' = \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial[\dot p_\mu]} + \tfrac12 c^c_{ab}[\dot r_c]\frac{\partial}{\partial[\dot r_a]} \wedge \frac{\partial}{\partial[\dot r_b]}. \tag{2.7}$$

Since the c^c_{ab} are structure constants of the Lie algebra $\mathfrak{g}$, it follows that ϖ' satisfies the Jacobi identity $[\varpi', \varpi'] = 0$ as required. If $\mathcal{I} \in \mathfrak{X}(E_X^*)$ is the linear vertical vector field on E_X^* corresponding to the identity bundle morphism $Id_{E_X^*} \in \Gamma(E_X^* \otimes_X E_X)$, then $\mathcal{I} = [\dot p_\mu]\partial/\partial[\dot p_\mu] + [\dot r_a]\partial/\partial[\dot r_a]$, and one easily verifies that $\varpi' = -[\varpi', \mathcal{I}]$, that is, ϖ' is even exact. In addition, we note that $(x^\mu, [\dot p_\mu], [\dot r_a])$ are Darboux coordinates for ϖ' on E_X^* at x_0. □

Corollary 2.7. *Let $\bar\Gamma(S^\bullet E_X) = \overline{\Gamma(S^\bullet E_X)}$ be the subalgebra of fiber-polynomial functions on E_X^*. If $\mathfrak{C}_X = \{f \in C^\infty(Z) | f|_X = 0\}$, there is a natural map $\mathfrak{C}_X \to \bar\Gamma(S^1 E_X)$ given by $f \mapsto \bar{df}|_X$. For any $f \in \mathfrak{C}_X$, the adjoined action of $\bar{df}|_X$ on $\bar\Gamma(S^\bullet E_X)$ is given by*

$$\{\bar P, \bar{df}|_X\}' = X_{\bar{df}|_X} \cdot \bar P = \overline{(\mathrm{L}_{X_f}\hat P)|_X} \qquad \forall P \in \Gamma(S^\bullet E_X), \tag{2.8}$$

where $\hat P \in \Gamma(S^\bullet(T^*Z))$ is any extension of P to a neighborhood of X.

Proof. The identifications are obviously canonical. Thus, since $\{\cdot, X_{\bar{df}|_X}\}'$ and L_{X_f} are both derivatives of the respective associative algebra structures, it suffices to verify (2.8) for elements of $\Gamma(S^0 E_X) = C^\infty(X)$ and $\Gamma(S^1 E_X) = \Gamma(E_X)$. For $P \in C^\infty(X)$, we have $\bar P = \tau^* P$, and thus

$$\{\bar P, \bar{df}|_X\}' = -\overline{((-\varpi_X^\sharp \circ df|_X) \cdot P)} = \overline{(X_f|_X \cdot P)} = \overline{(\mathrm{L}_{X_f}\hat P)|_X}$$

for any extension $\hat P$ of P since $X_f|_X$ is tangent to X for $f \in \mathfrak{C}_X$. For $P \in \Gamma(E_X)$ and any extension $\hat P$, we have further

$$\begin{aligned}
\{\bar P, \bar{df}|_X\}' &= \overline{\{P, df|_X\}} = \overline{(\mathrm{i}_{\varpi^\sharp\circ df} d\hat P - \mathrm{i}_{\varpi^\sharp\circ\hat P} d(df) + d(\varpi(\hat P, df)))|_X}\\
&= \overline{(\mathrm{i}_{X_f} \circ d\hat P + d \circ \mathrm{i}_{X_f}\hat P)|_X} = \overline{(\mathrm{L}_{X_f}\hat P)|_X}.
\end{aligned}$$

□

Corollary 2.8. *There is a canonical injection* $\Gamma(\wedge^n T^*X \otimes_X E_X) \to \Omega^n(E_X^*)$*, defined by the tensor product of the pull-back by* τ *with the identification of corollary* 2.7*, which we denote again by a bar. Any* $\alpha \in \mathfrak{A}_X$ *defines a projection* $h^\alpha : \mathfrak{X}^1(E_X^*) \to \mathfrak{X}^1(E_X^*)$ *with* $\tau_* \circ h^\alpha = \tau_*$ *given by*

$$h^\alpha = \overline{ad}(\alpha) \circ \tau_*, \qquad \textit{where} \quad \overline{ad}(\alpha)(\mathcal{X}) = -X_{\overline{\alpha(\mathcal{X})}} \qquad \forall \mathcal{X} \in \mathfrak{X}(X).$$

The following structure equation and Bianchi identity hold good:

$$d\bar{\alpha} \circ h^\alpha = \bar{\Phi}^\alpha + \tfrac{1}{2}\{\bar{\alpha}, \bar{\alpha}\}' \qquad\qquad d\bar{\Phi}^\alpha \circ h^\alpha = \tfrac{1}{2}\{\bar{\alpha}, \bar{\Phi}^\alpha\}',$$

where $\{\bar{\alpha}, \bar{\alpha}\}'(\mathcal{Y}_1, \mathcal{Y}_2) = \{\bar{\alpha}(\mathcal{Y}_1), \bar{\alpha}(\mathcal{Y}_2)\}' - \{\bar{\alpha}(\mathcal{Y}_2), \bar{\alpha}(\mathcal{Y}_1)\}'$ *etc.*

Proof. Because of (2.6), we have in deed $\tau_* \circ h^\alpha(\mathcal{Y}) = -\varpi^\sharp \circ \alpha^\flat \circ \tau_*(\mathcal{Y}) = \tau_*(\mathcal{Y})$. Let $\mathcal{Y}_i \in \mathfrak{X}^1(E_X^*)$, $\mathcal{X}_i = \tau_*\mathcal{Y}_i, i = 0, 1, 2$. Then, we obtain

$$\begin{aligned} h^\alpha(\mathcal{Y}_i) \cdot \bar{\alpha}(h^\alpha(\mathcal{Y}_j)) &= -X_{\overline{\alpha^\flat \circ \tau_*(\mathcal{Y}_i)}} \cdot \overline{\alpha(\tau_* \circ h^\alpha(\mathcal{Y}_j))} \\ &= \{\overline{\alpha(\mathcal{X}_i)}, \overline{\alpha(\mathcal{X}_j)}\}', \end{aligned}$$

$$\begin{aligned} &\bar{\Phi}^\alpha(\mathcal{Y}_1, \mathcal{Y}_2) \\ &= \overline{\{\alpha(\mathcal{X}_1), \alpha(\mathcal{X}_2)\} - \alpha([\mathcal{X}_1, \mathcal{X}_2])} \\ &= \{\overline{\alpha(\mathcal{X}_1)}, \overline{\alpha(\mathcal{X}_2)}\}' - \{\overline{\alpha(\mathcal{X}_2)}, \overline{\alpha(\mathcal{X}_1)}\}' - \overline{\alpha([\mathcal{X}_1, \mathcal{X}_2])} - \{\overline{\alpha(\mathcal{X}_1)}, \overline{\alpha(\mathcal{X}_2)}\}' \\ &= h^\alpha(\mathcal{Y}_1) \cdot \bar{\alpha}(h^\alpha(\mathcal{Y}_2)) - h^\alpha(\mathcal{Y}_2) \cdot \bar{\alpha}(h^\alpha(\mathcal{Y}_1)) - \bar{\alpha}([h^\alpha(\mathcal{Y}_1), h^\alpha(\mathcal{Y}_2)]) \\ &\quad - \{\bar{\alpha}(\mathcal{Y}_1), \bar{\alpha}(\mathcal{Y}_2)\}' = (d\bar{\alpha} \circ h^\alpha - \tfrac{1}{2}\{\bar{\alpha}, \bar{\alpha}\}')(\mathcal{Y}_1, \mathcal{Y}_2) \end{aligned}$$

which implies the structure equation. For the Bianchi identity, we compute

$$\begin{aligned} &\bar{\Phi}^\alpha \circ h^\alpha(\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2) \\ &= d(d\bar{\alpha} - \tfrac{1}{2}\{\bar{\alpha}, \bar{\alpha}\}') \circ h^\alpha(\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2) \\ &= -\tfrac{1}{2} d(\{\bar{\alpha}, \bar{\alpha}\}')(h^\alpha(\mathcal{Y}_0), h^\alpha(\mathcal{Y}_1), h^\alpha(\mathcal{Y}_2)) \\ &= -\sum_{\text{cycl}} \left\{\overline{\alpha(\mathcal{X}_0)}, \left\{\overline{\alpha(\mathcal{X}_1)}, \overline{\alpha(\mathcal{X}_2)}\right\}'\right\}' + \sum_{\text{cycl}} \left\{\overline{\alpha([\mathcal{X}_0, \mathcal{X}_1])}, \overline{\alpha(\mathcal{X}_2)}\right\}' \\ &= \sum_{\text{cycl}} \left\{\overline{\alpha(\mathcal{X}_0)}, \left\{\overline{\alpha(\mathcal{X}_1)}, \overline{\alpha(\mathcal{X}_2)}\right\}'\right\}' - \sum_{\text{cycl}} \left\{\overline{\alpha(\mathcal{X}_0)}, \overline{\alpha([\mathcal{X}_1, \mathcal{X}_2])}\right\}' \\ &= \sum_{\text{cycl}} \left\{\overline{\alpha(\mathcal{X}_0)}, \overline{\{\alpha(\mathcal{X}_1), \alpha(\mathcal{X}_2)\} + \alpha([\mathcal{X}_1, \mathcal{X}_2])}\right\}' \\ &= \tfrac{1}{2}\{\bar{\alpha}, \bar{\Phi}^\alpha\}'(\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2), \end{aligned}$$

where the summation runs over all cyclic permutations of the indices $0, 1, 2$, and we used the Jacobi identity for the fourth equality. □

Definition 2.9. For any Poisson manifold (Z, ϖ) and any Lagrangian submanifold $X \subset S$ of a symplectic leaf S, the *Sternberg-Weinstein approximation* at X is given by the Poisson manifold (Z', ϖ'), where $Z' = E_X^* =$

$TZ|_X/TX$, and ϖ' is the Poisson tensor defined by the Poisson bracket $\{\cdot,\cdot\}'$ on $C^\infty(Z')$ in lemma 2.6, given in Darboux coordinates by (2.7).

Corollary 2.10. *The subbundle $S' = TS_{|X}/TX \subset Z'$ is a symplectic leaf, which contains the zero section, identified with X, as a Lagrangian submanifold. Setting $\underline{E}_X = (\underline{TX})^0$ and $\omega_X^\flat = \omega_S^\flat|_{TX}$, we have bundle isomorphisms*

$$-\omega_X^{\flat -1} : \underline{E}_X \longrightarrow TX \qquad\qquad (-\omega_X^\flat)^* : S' = \underline{E}_X^* \longrightarrow T^*X$$

which are the anchor map for the Lie algebroid structure on $\underline{E}_X$ (set $Z = S$) and a symplectomorphism with respect to the canonical symplectic structure on T^*X, respectively. The subspace $\mathfrak{C}_{S'} = \{f \in C^\infty(Z')|f|_{S'} = 0\}$ is an ideal of $C^\infty(Z')$, and the subspace $\bar{\Gamma}(L_X)$ of the functions on E_X^* defined by sections of L_X is an ideal of $\bar{\Gamma}(E_X)$. The canonical projection

$$l : Z' \longrightarrow L_X^* \cong \mathfrak{g}^*(R_X), \tag{2.9}$$

is a Poisson morphism for the canonical Poisson structure on L_X^*. That is,

$$\{l^*f, l^*g\}(l^{-1}(n)) = \{f|_{(L_X^*)_x}, g|_{(L_X^*)_x}\}_x(n), \tag{2.10}$$

for all $f, g \in C^\infty(L_X^*)$ and $n \in (L_X^*)_x$, where $\{\cdot,\cdot\}_x$ denotes the Poisson bracket on $(L_X^*)_x \cong \mathfrak{g}^*$. Finally, the Sternberg-Weinstein approximation of Z' at $X \subset S'$ is canonically isomorphic to Z'.

Proof. The assertions are easily checked in our Darboux coordinates. For (2.10), $L_X = \ker(\varpi_X^\sharp)$ implies $\{\tau^* C^\infty(X), \bar{\Gamma}(L_X)\} = 0$ and by lemma 2.1, $\bar{\Gamma}(L_X) \subset \bar{\Gamma}(E_X)$ is an ideal. For the last fact, we use that for every vector bundle $M \to N$, we have vector bundle isomorphisms $TM|_{o(N)}/T(o(N)) \cong V(M)|_{o(N)} \cong M$, where $o(N) \cong N$ is the image of the zero section. □

Remark 2.11. We can see here that Z' is locally equivalent to $T^*X \times \mathfrak{g}^*$, and in particular, transversally linear at $S' = T^*X$. Since Z is locally equivalent to $V \times N$, and there is always a local symplectic isomorphism from V to T^*X, we see that Z is locally Poisson equivalent to Z' iff it is linearizable at V. In corollary 4.4 we will see that the Sternberg-Weinstein phase space is naturally isomorphic to its Sternberg-Weinstein approximation. ◊

2.2. The Wong system.

We will now approximate a given Hamiltonian system on Z by a Wong system on Z'.

Theorem 2.12. *Let $X \subset S \overset{i_S}{\hookrightarrow} Z$ be a Lagrangian submanifold of the symplectic leaf (S, ω_S) of the Poisson manifold (Z, ϖ). Every Hamiltonian $H \in C^\infty(Z)$ with $dH|_X = 0$ naturally defines a bundle morphism $t_X^2 H : \underline{E}_X^* \to E_X$, in the notation of 2.1. If the composition $t_X^2 H_0 = (Ti_S)^* \circ t_X^2 H : \underline{E}_X^* \to \underline{E}_X$ is invertible, then the bundle morphisms*

$$\begin{aligned} \gamma_H^\flat &= (-\omega_X^\flat)^* \circ (t_X^2 H_0)^{-1} \circ (-\omega_X^\flat) : \; TX \longrightarrow T^*X \\ \alpha_H^\flat &= t_X^2 H \circ (t_X^2 H_0)^{-1} \circ (-\omega_X^\flat) : \quad TX \longrightarrow E_X \end{aligned}$$

define a metric γ_H and an E_X-connection form α_H on X, as well as a principal connection form $A_H = A^{\alpha_H}$ on R_X.

Proof. The tangent map of $dH|_S : S \to E$ yields bundle morphisms

$$T(dH)|_{TS|_X} :\ TS|_X \longrightarrow (dH|_X)^*T(E) = TS|_X \oplus_X E|_X$$

$$pr_2 \circ T(dH)|_{TS|_X} :\ TS|_X \longrightarrow E|_X \tag{2.11}$$

$$(Ti_S)^* \circ pr_2 \circ T(dH)|_{TS|_X} :\ TS|_X \longrightarrow T^*S|_X, \tag{2.12}$$

the decomposition being canonical over the zero section. Since $dH|_X = 0$, the later maps vanish on TX. On the other hand, since $d(i_S^* dH) = d(d(H|_S)) = 0$, (2.12) defines symmetric bilinear forms on the fibers of $TS|_X$. Thus, it induces symmetric bilinear forms on the fibers of the quotient bundle $\underline{E}_X^*$ and an associated bundle morphism $\underline{E}_X^* \to \underline{E}_X$. Hence, (2.11) must take values in E_X, and as claimed, we obtain the bundle morphisms

$$t_X^2 H :\ \underline{E}_X^* \longrightarrow E_X \qquad t_X^2 H_0 = (Ti_S)^* \circ t_X^2 H :\ \underline{E}_X^* \longrightarrow \underline{E}_X. \tag{2.13}$$

If $t_X^2 H_0$ is invertible, then γ_H is obviously a metric on X. On the other hand, $-\varpi_X^\sharp \circ \alpha_H^\flat = Ti_S \circ (-\omega_S^\flat)^{-1} \circ (Ti_S)^* \circ t_X^2 H \circ (t_X^2 H_0)^{-1} \circ (-\omega_X^\flat) = Id_{TX}$, showing that α_H is an E_X-connection form on X, inducing a principal connection form on R_X. In summary, we have the diagram

$$\begin{array}{ccccccc}
E|_X & \longrightarrow & E_X & \longrightarrow & \underline{E}_X & \xleftarrow{-\omega_X^\flat} & TX \\
\uparrow{\scriptstyle pr_2\circ T(dH)|_{TS|_X}} & & & {\scriptstyle t_X^2 H}\nwarrow & {\scriptstyle t_X^2 H_0}\uparrow\ \downarrow{\scriptstyle t_X^2 H_0^{-1}} & & \downarrow{\scriptstyle \gamma_H} \\
TS|_X & \longrightarrow & & & \underline{E}_X^* & \xrightarrow{-(\omega_X^\flat)^*} & T^*X
\end{array}$$

where $\qquad \alpha_H^\flat = t_X^2 H \circ (t_X^2 H_0)^{-1} \circ (-\omega_X^\flat) :\ TX \longrightarrow E_X.$ □

Corollary 2.13. *Any section $h \in \Gamma(E)$ with $h|_X = 0$ naturally defines a bundle morphism $t_X h : TS|_S \to E_X$. If the composition $t_X h_0 = (Ti_S)^* \circ t_X h$ is invertible and $d(i_S^* h)|_X = 0$, then $t_X h$ uniquely defines a metric, an E_X-connection form and an associated principal connection form on R_X. Given in addition a linear connection in the bundle E, which provides a splitting $TE = TS \oplus_S VE$ over S, any $h \in \Gamma(E)$ with $d(i_S^* h) = 0$ defines a metric and an E-connection form on the leaf S.*

By Theorem 2.12, a Hamiltonian H on Z with $dH|_X = 0$ defines a map $(\alpha_H^\flat)^* : Z' \to T^*X$ and, via the metric γ_H, a Hamiltonian H_0' on T^*X. Thus, we can define a Hamiltonian system on Z' by

$$H_1' = H_0' \circ (\alpha_H^\flat)^*, \qquad \text{where} \quad H_0'(p) = \tfrac{1}{2} p(\gamma_H^{\flat -1}(p)) \quad \forall p \in T^*X. \tag{2.14}$$

Notice that while the definition of H_1' requires the invertibility of $t_X^2 H_0$, that of H_0' does not. We now define a bundle morphism over X by

$$\psi_H = ((\alpha_H^\flat)^*, l): \quad Z' \longrightarrow L_X^\pi = T^*X \oplus_X L_X^*,$$

which is obviously a diffeomorphism. It can be used to induce an H-depending Poisson structure $\varpi_H' = (\psi_H)_* \varpi'$ on L_X^π. Then, H_0' pulls back to $H_{\hat{0}}' \in C^\infty(L_X^\pi)$, and by construction, the system $(L_X^\pi, \varpi_H', H_{\hat{0}}')$ is equivalent to the system on Z'. But L_X^π is fibred over T^*X and X and admits natural physical coordinates for the equations of motion, if we can calculate ϖ_H.

The affine bundle morphism $(Tpr_1 \oplus_{TX} Tpr_2) : T(L_X^\pi) \xrightarrow{\sim} T(T^*X) \times_{TX} T(L_X^*)$ induces a graded algebra map $hor_H^\pi : \mathfrak{X}(T^*X) \to \mathfrak{X}(L_X^\pi)$ by

$$\begin{aligned} \mathcal{Z} = hor_H^\pi(\mathcal{Y}) \quad &\Longleftrightarrow \quad pr_{1*}(\mathcal{Z}) = \mathcal{Y}, \quad pr_{2*}(\mathcal{Z}) = hor_H(\pi_{X*}(\mathcal{Y})) \\ hor_H(\mathcal{X}) \quad &= \quad (l_* \circ \overline{ad})(\alpha_H)(\mathcal{X}) = -l_* X_{\overline{\alpha_H(\mathcal{X})}} \qquad \forall \mathcal{X} \in \mathfrak{X}^1(X), \end{aligned}$$

where the Hamiltonian vector field is defined by ϖ' and projectable by l since $\bar{\Gamma}(L_X) \subset \bar{\Gamma}(E_X)$ is an ideal. Since by Corollary 2.8 $(\tau_{L_X^*})_* \circ hor_H = Id_{TX}$, where $\tau_{L_X^*} : L_X^* \to X$, the map is well defined. Furthermore, the vertical restriction $V(L_X^\pi) \cong V(T^*X) \oplus_X V(L_X^*)$ induces injections

$$\begin{aligned} \Gamma(\wedge^n T^*X \otimes_X L_X) \ni \varphi \quad &\longmapsto \quad \overrightarrow{\varphi} \in \Gamma(\wedge^n V(L_X^\pi)) \hookrightarrow \mathfrak{X}^n(L_X^\pi) \\ \Gamma(\wedge^n V(L_X^*)) \ni w \quad &\longmapsto \quad \overleftarrow{w} \in \Gamma(\wedge^n V(L_X^\pi)) \hookrightarrow \mathfrak{X}^n(L_X^\pi), \end{aligned}$$

where in the first line, we used also the natural injections $\wedge^n \Gamma(T^*X) \to \wedge^n V(T^*X)$ and $\Gamma(L_X) \to C^\infty(L_X^\pi)$ as constant vertical multi-vector fields and as vertically linear functions, respectively.

Theorem 2.14. *The Poisson structure ϖ_H on L_X^π is given by*

$$\varpi_H = hor_H^\pi \circ w_{T^*X} + \overrightarrow{\Phi}^{\alpha_H} + \overleftarrow{w}_{L_X^*}, \tag{2.15}$$

where w_{T^*X} is the Poisson tensor on T^*X, Φ^{α_H} is the curvature of α_H, and $w_{L_X^*}$ is the Poisson tensor on L_X^*.

Proof. We determine the Poisson structure ϖ_H' on L_X^π on the characteristic set of functions on L_X^π given by the functions on T^*X induced by functions and vector fields on X, and functions induced by sections of L_X, denoted by a double bar. The dual of ψ_H decomposes into a direct sum as

$$\psi_H^* = (\alpha_H^\flat, l^*) : \ (L_X^\pi)^* = TX \oplus_X L_X \quad \xrightarrow{\sim} \quad \alpha_H^\flat(TX) \oplus_X L_X \cong E_X.$$

It follows that for $f_i \in C^\infty(X) = \mathfrak{X}^0(X)$, $\mathcal{X}_i \in \mathfrak{X}^1(X)$, and $\mathcal{V}_i \in \Gamma(L_X)$,

$$\psi_H^* \bar{\bar{f}}_i = \tau^* f_i \qquad \psi_H^* \bar{\bar{\mathcal{X}}}_i = \overline{\alpha_H(\mathcal{X}_i)} \qquad \psi_H^* \bar{\bar{\mathcal{V}}}_i = \bar{\mathcal{V}}_i \qquad i = 1, 2.$$

Denoting by $\{\cdot,\cdot\}'_H$ the Poisson bracket defined by ϖ'_H, this implies that

$$
\begin{aligned}
\{\bar{\bar{f}}_1, \bar{\bar{f}}_2\}'_H &= (\psi_H^{-1})^*\{\tau^* f_1, \tau^* f_2\}' = 0 = \{\bar{\bar{f}}_1, \bar{\bar{f}}_2\}_{T^*X}\\
\{\bar{\bar{\mathcal{X}}}_1, \bar{\bar{f}}_2\}'_H &= (\psi_H^{-1})^*\{\overline{\alpha_H(\mathcal{X}_1)}, \tau^* f_2\}' = (\psi_H^{-1})^*\tau^*((-\varpi_X^\sharp \circ \alpha_H(\mathcal{X}_1)) \cdot f_2)\\
&= (\psi_H^{-1})^*\tau^*(\mathcal{X}_1 \cdot f_2) = \overline{\overline{\mathcal{X}_1 \cdot f_2}} = \{\bar{\bar{\mathcal{X}}}_1, \bar{\bar{f}}_2\}_{T^*X}\\
\{\bar{\bar{f}}_1, \bar{\bar{\mathcal{V}}}_2\}'_H &= (\psi_H^{-1})^*\{\tau^* f_1, \bar{\mathcal{V}}_2\}' = -(\psi_H^{-1})^*\tau^*((-\varpi_X^\sharp \circ \mathcal{V}_2) \cdot f_1) = 0\\
\{\bar{\bar{\mathcal{X}}}_1, \bar{\bar{\mathcal{V}}}_2\}'_H &= (\psi_H^{-1})^*\{\overline{\alpha_H(\mathcal{X}_1)}, \bar{\mathcal{V}}_2\}' = (\psi_H^{-1})^*\overline{\{\alpha_H(\mathcal{X}_1), \mathcal{V}_2\}} = \overline{\overline{\{\alpha_H(\mathcal{X}_1), \mathcal{V}_2\}}}\\
\{\bar{\bar{\mathcal{X}}}_1, \bar{\bar{\mathcal{X}}}_2\}'_H &= (\psi_H^{-1})^*\{\overline{\alpha_H(\mathcal{X}_1)}, \overline{\alpha_H(\mathcal{X}_2)}\}'\\
&= (\psi_H^{-1})^*(\overline{\Phi^{\alpha_H}(\mathcal{X}_1, \mathcal{X}_2)} + \overline{\alpha_H([\mathcal{X}_1, \mathcal{X}_2])})\\
&= \overline{\overline{\Phi^{\alpha_H}(\mathcal{X}_1, \mathcal{X}_2)}} + \overline{\overline{[\mathcal{X}_1, \mathcal{X}_2]}} = \overline{\overline{\Phi^{\alpha_H}(\mathcal{X}_1, \mathcal{X}_2)}} + \{\bar{\bar{\mathcal{X}}}_1, \bar{\bar{\mathcal{X}}}_2\}_{T^*X}\\
\{\bar{\bar{\mathcal{V}}}_1, \bar{\bar{\mathcal{V}}}_2\}'_H &= (\psi_H^{-1})^*\{\bar{\mathcal{V}}_1, \bar{\mathcal{V}}_2\}' = \overline{\overline{\{\mathcal{V}_1, \mathcal{V}_2\}}} = \{\bar{\bar{\mathcal{V}}}_1, \bar{\bar{\mathcal{V}}}_2\}_{L_X^*}
\end{aligned}
$$

where we also wrote $\{\cdot,\cdot\}_{T^*X}$ and $\{\cdot,\cdot\}_{L_X^*}$ for the Poisson bracket on T^*X and L_X^*, respectively, omitting obvious restrictions. By testing on the characteristic set of functions, we verify expression (2.15) for ϖ_H. For example,

$$
\begin{aligned}
&\varpi_H(d\bar{\bar{\mathcal{X}}}_1, d\bar{\bar{\mathcal{X}}}_2)\\
&= (pr_{1*} \circ hor_H^\pi \circ w_{T^*X})(d\bar{\bar{\mathcal{X}}}_1|_{T^*X}, d\bar{\bar{\mathcal{X}}}_2|_{T^*X}) + \overrightarrow{\Phi}^{\alpha_H}(d\bar{\bar{\mathcal{X}}}_1, d\bar{\bar{\mathcal{X}}}_2)\\
&= \{\bar{\bar{\mathcal{X}}}_1, \bar{\bar{\mathcal{X}}}_2\}_{T^*X} + \overline{\overline{\Phi^{\alpha_H}(\mathcal{X}_1, \mathcal{X}_2)}}\\
&= \{\bar{\bar{\mathcal{X}}}_1, \bar{\bar{\mathcal{X}}}_2\}'_H,
\end{aligned}
$$

where we used that by construction, $\overrightarrow{\varphi}(d\bar{\bar{\mathcal{X}}}) = d\bar{\bar{\mathcal{X}}}(\overrightarrow{\varphi}) = \overline{\overline{\varphi(\mathcal{X})}}$ for all $\mathcal{X} \in \mathfrak{X}(X)$ and $\varphi \in \Gamma(T^*X \otimes_X L_X)$. □

Definition 2.15. The equivalent systems (Z', ϖ', H'_1) and $(L_X^\pi, \varpi'_H, H'_{\hat{0}})$ will be called the *Wong system* and the *gauged Wong system* associated to the Hamiltonian system (Z, ϖ, H) at X, respectively.

Wong's equations ([**29**]) can now be easily written down. [**20**] provides a detailed calculation and discussion of the Poisson structure (2.15) for the original Sternberg-Weinstein phase space and the (left) gauged Wong system. Note that the original Wong system is associated to itself as wanted.

2.3. The Einstein-Mayer system. If $t_X^2 H_0$ in Theorem 2.12 in degenerate, it is still possible to define an approximated system on the Sternberg-Weinstein approximation of the underlying Poisson manifold.

Theorem 2.16. *Let $X \subset S \overset{i_S}{\hookrightarrow} Z$ be a Lagrangian submanifold of the symplectic leaf (S, ω_S) of the Poisson manifold (Z, ϖ). Every Hamiltonian $H \in C^\infty(Z)$ such that $dH|_X = 0$ naturally defines a section $d_X^2 H : X \to$*

S^2E_X of symmetric bilinear forms on the fibers of E_X^*, in the notation of 2.1. If $d_X^2H|_{\underline{E}_X^*}$ is nondegenerate, then d_X^2H determines a metric γ_H and an E_X-connection form α_H on X, a principal connection form A_H on R_X, and fiber-quadratic function H'_{20} on L_X^*. Furthermore, if the corresponding field of quadratic forms is nondegenerate, then d_X^2H is nondegenerate and defines a field η of scalar products on the fibers of E_X, which determines, and is determined by, the triple $(\gamma_H, \alpha_H, \chi_H)$ of fields on X, where χ_H is the field of scalar products on $L_X = \mathfrak{g}(R_X)$ defining H'_{20}.

Proof. The first jet of dH with $dH|_X = 0$ yields maps

$$J^1dH_{|X} : X \to T^*Z_{|X} \otimes_X (dH_{|X})^*T(T^*Z) = T^*Z_{|X} \otimes_X (TZ_{|X} \oplus_X T^*Z_{|X})$$

$$pr_2 \circ J^1(dH)|_X : X \longrightarrow S^2(T^*Z_{|X}), \tag{2.16}$$

where the field of (Hessian) bilinear forms on the fibers of $TZ|_X$ defined in the second line is symmetric since dH is closed. Since in addition, the corresponding bundle morphism vanishes on TX because of $dH|_X = 0$, we get an induced field of symmetric bilinear forms on the fibers of E_X^*

$$d_X^2H : X \longrightarrow S^2E_X, \qquad \text{with} \qquad d_X^2H^\sharp : E_X^* \longrightarrow E_X.$$

Restriction to the subbundle $\underline{E}_X^* \subset E_X^*$ yields a field $d_X^2H_0 : X \to S^2\underline{E}_X$ of scalar products on the fibers of $\underline{E}_X^*$ and a bundle morphism t_X^2H as in (2.13), with $t_X^2H_0 = d_X^2H_0^\sharp$. If $d_X^2H_0$ is nondegenerate, we obtain a metric and an E_X-connection form as in theorem 2.12 given by

$$\begin{aligned} \gamma_H^\flat &= (-\omega_X^\flat)^* \circ d_X^2H_0^{\sharp-1} \circ (-\omega_X^\flat) : \quad TX \longrightarrow T^*X \\ \alpha_H^\flat &= d_X^2H^\flat \circ Ti_S \circ d_X^2H_0^{\sharp-1} \circ (-\omega_X^\flat) : \ TX \longrightarrow E_X. \end{aligned}$$

In addition, α_H yields a principal connection form A_H in R_X, and the bundle identification $\psi_H : E_X^* \xrightarrow{\sim} T^*X \oplus_X L_X^*$. Let $l^{\alpha_H} : L_X^* \to \ker(\alpha_H^\flat)^* \subset E_X^*$ be the induced inclusion, whose image is by construction orthogonal to $T^*X \cong \underline{E}_X^* \subset E_X^*$ with respect to d_X^2H. Then, if l is the projection (2.9),

$$d_X^2H = (\alpha_H^\flat \circ (-\omega_X^{\flat-1}))^* d_X^2H_0 + l^*((l^{\alpha_H})^* d_X^2H), \tag{2.17}$$

and H'_{20} is defined as the function given by the field $(l^{\alpha_H})^* d_X^2H \in S^2(L_X)$. If this field is nondegenerate, then d_X^2H is nondegenerate, too. The remaining assertions follow by dualizing (2.17). Summarizing, we obtain the diagram:

$$\begin{array}{ccccccc} L_X & \longrightarrow & E_X & \xrightarrow{(Ti_S)^*} & \underline{E}_X & \xleftarrow{-\omega_X^\flat} & TX \\ \downarrow{\scriptstyle \chi_H^\flat} & & {\scriptstyle \eta^\flat}\downarrow\uparrow{\scriptstyle d_X^2H^\sharp} & & {\scriptstyle d_X^2H_0^\sharp}\uparrow\downarrow{\scriptstyle d_X^2H_0^{\sharp-1}} & & \downarrow{\scriptstyle \gamma_H^\flat} \\ L_X^* & \longleftarrow & E_X^* & \xleftarrow{Ti_S} & \underline{E}_X^* & \xrightarrow{(-\omega_X^\flat)^*} & T^*X \end{array} \tag{2.18}$$

where $$\alpha_H^\flat = d_X^2H^\sharp \circ Ti_S \circ d_X^2H_0^{\sharp-1} \circ (-\omega_X^\flat).$$ □

Corollary 2.17. *Let $X \subset S \overset{i_S}{\hookrightarrow} Z$ be a Lagrangian submanifold of the symplectic leaf S of the Poisson manifold Z. Any 1-form $h \in \Omega^1(Z)$ with $h|_X = 0$ naturally defines a field $d_X h : X \to S^2 E_X$ of bilinear forms on the fibers of E_X^*. If $d_X h$ is nondegenerate and $dh|_X = 0$, then it defines a field η of scalar products on the fibers of E_X and, if $d_X h|_{\underline{E}_X^*}$ and $\eta|_{L_X}$ are nondegenerate, a triple $(\gamma_h, \alpha_h, \chi_h)$ of fields on X, where γ_h is a metric, α_h is an E_X-connection form, determining a principal connection in R_X, and χ_h is a field of scalar products on the associated vector bundle $L_X = \mathfrak{g}(R_X)$.*

From (2.17), we deduce that the Hamiltonian defined by $d_X^2 H$ on E_X^*, i.e., the Sternberg-Weinstein approximation (Z', ϖ') to (Z, ϖ) at X, is given by

$$H_2' = \tfrac{1}{2}\overline{d_X^2 H} = H_1' + l^* H_{20}' \tag{2.19}$$

if H_1' is the Hamiltonian (2.14) of the Wong system for nondegenerate $d_X^2 H_0$.

Definition 2.18. The equivalent systems (Z', ϖ', H_2') and (if it is defined) $(L_X^\pi, \varpi_H', H_{\hat{0}}' + pr_2^* H_{20}')$ are called the *Einstein-Mayer system* and the *gauged Einstein-Mayer system* associated to (Z, ϖ, H) at X, respectively.

Remark 2.19. In 1932, Einstein and Mayer considered a unified theory of gravitation and electricity inspired by Kaluza-Klein theory (cf. section 4.1) which was based on an alternative tangent bundle to the four-dimensional space-time manifold ([**7**]). This work could be regarded as a precursor of Lie algebroid ideas in physics before their apparition in mathematics. The Wong system can be regarded a special case of the Einstein-Mayer system (cf. Definition 4.8), or as an approximation to it exploiting only part of the Hessian of H in (2.19) and requiring the nondegeneracy of $d_X^2 H_0$.

3. Dimensionally reduced approximation

Poisson reduction theory ([**25**]) allows to obtain new Poisson manifolds from constraints in a given one. Under Sternberg-Weinstein approximation, this process becomes related to the dimensional reduction procedure for symmetric gauge fields ([**4**]). For the original Sternberg-Weinstein construction, this relation was pointed out in [**22**]. Let us fix an embedded *coisotropic* submanifold $i_Q : Q \hookrightarrow (Z, \varpi)$, i.e., $\varpi|_{(TQ)^0} \equiv 0$, or, equivalently, $\varpi^\sharp((TQ)^0) \subset TQ$. A standard result of reduction theory is the following.

Proposition 3.1. *If the subcharacteristic distribution $\mathsf{N}(Q) = \varpi^\sharp((TQ)^0)$ has constant dimension, it is differentiable and integrable to a subcharacteristic foliation $\mathcal{C}$. If $\mathcal{C}$ is transversal to the symplectic leaves of Z and given by the fibers of a submersion $q : Q \to \tilde{Z} = Q/\mathcal{C}$, there is a reduced Poisson structure on $\tilde{Z}$ whose symplectic foliation is given by the symplectically reduced symplectic leaves of Z with respect to their intersections with Q.*

In addition, let us suppose that Q is *locally closed*. Let $\mathfrak{P} = C^\infty(Z)$, and $\mathfrak{C} = \{f \in \mathfrak{P} | f|_Q = 0\}$ be the subspace of *constraints*. Then Q is coisotropic iff $\forall f \in \mathfrak{C}$, $X_f|_Q \in \mathfrak{X}^1(Q)$ or equivalently, $\{\mathfrak{C}, \mathfrak{C}\} \subset \mathfrak{C}$. Let $\mathfrak{N} = \mathfrak{N}_{\mathfrak{P}}(\mathfrak{C})$ be the idealizer subalgebra of *admissible functions*. They are precisely those whose Hamiltonian vector fields are tangent to Q, and which are constant along the leaves of $\mathcal{C}$, that is, $C^\infty(\tilde{Z}) \cong q^* C^\infty(\tilde{Z}) = i_Q^* \mathfrak{N} \cong \mathfrak{N}/\mathfrak{C}$. We see that if the conditions of proposition 3.1 are not satisfied, a reduced Poisson bracket can still be defined as a quotient algebra on the subspace

$$\tilde{\mathfrak{P}} = i_Q^* \mathfrak{N} \cong \mathfrak{N}/\mathfrak{C} \tag{3.1}$$

of $C^\infty(Q)$, which we will call the *reduced Poisson algebra* of $\mathfrak{P}$.

3.1. Reduced Sternberg-Weinstein approximation. Let us suppose $S \subset Q$ and apply the reduction procedure sketched above to Z'. We set

$$A_X = (TQ)^0_{|X} \qquad \tilde{E}_X = (TX)^0 \subset T^*Q_{|X} \qquad \tilde{L}_X = (TS)^0_{|X} \subset T^*Q_{|X}$$
$$Q' = \tilde{E}^*_X = TQ_{|X}/TX \qquad \tilde{L}^*_X = TQ_{|X}/TS_{|X},$$

with the canonical bundle isomorphisms $\tilde{E}_X \cong E_X/A_X$ and $\tilde{L}_X \cong L_X/A_X$.

Lemma 3.2. *The submanifold $i_{Q'} : Q' \to Z'$ is given as the common zero level set of the functions in $\bar{\Gamma}(A_X)$ defined by differentials $(d\mathfrak{C})|_X \in \Gamma(A_X)$ as in Corollary 2.7, and the fibers of $(TQ')^0 \subset T^*Z'$ are spanned at each point by the differentials of these functions. In particular, Q' is coisotropic.*

Proof. It suffices to notice that the subspaces $(A_X)_x \subset (L_X)_x$ are spanned by the differentials $(d\mathfrak{C})_x$ at each point $x \in X$. In particular, $\{df, dg\}_x = (d\{f, g\})_x$ shows that Q' is coisotropic as soon as Q is. □

Definition 3.3. Let $\mathfrak{P}' = C^\infty(Z')$. The subalgebras $\mathfrak{C}' = \{f \in \mathfrak{P}' | f|_{Q'} = 0\}$ and $\mathfrak{N}' = \mathfrak{N}_{\mathfrak{P}'}(\mathfrak{C}')$ are called the *Sternberg-Weinstein constraint algebra* and *admissible function algebra*, respectively. Setting $\tilde{\mathfrak{P}}' = i_{Q'}^* \mathfrak{N}' \cong \mathfrak{N}'/\mathfrak{C}'$, we call $(Q', \tilde{\mathfrak{P}}')$ the *(Q-)reduced Sternberg-Weinstein approximation* of Z at X.

Let $\tilde{l} : Q' \to \tilde{L}^*_X$ denote the natural projection. Clearly, $\tilde{l}^* C^\infty(\tilde{L}^*_X) = i_{Q'}^*(l^* C^\infty(L^*_X))$. The fact that the Sternberg-Weinstein bracket of functions on Z' which are pull-backs of functions in L^*_X only depends on their restrictions to the fibers allows us to characterize, by means of two assumptions, those elements of $\tilde{l}^* C^\infty(\tilde{L}^*_X) \cap \tilde{\mathfrak{P}}'$ which are vertically polynomial.

From the proof of lemma 3.2 we know that for all $x \in X$, $(A_X)_x$ is a Lie subalgebra of $(L_X)_x \cong \mathfrak{g}$. We will make the following assumption:

(3.2) *The fibers of A_X are all isomorphic to some subalgebra $\mathfrak{c} \subset \mathfrak{g}$.*

We will further make the stronger assumption that for every fiber there exists an isomorphism which is the restriction of an isomorphism $a \in (R_X)_x$. If

$N_{Aut(\mathfrak{g})}(\mathfrak{c})$ denotes the stabilizer subgroup of $\mathfrak{c}$ in $Aut(\mathfrak{g})$, this means that:

(3.3) *The Lie-frame bundle R_X can be reduced to $N = N_{Aut(\mathfrak{g})}(\mathfrak{c})$.*

Definition 3.4. If (3.3) is valid, the reduced subbundle of R_X given by

$$R_X^{\mathfrak{c}} \to X \qquad (R_X^{\mathfrak{c}})_x = \{a \in R_x | a : \mathfrak{c} \to (TQ)_x^0\} \qquad \forall x \in X,$$

with structure group N, is called the *constraint Lie frame bundle*. We write

$$S^i(\tilde{L}_X) = (R_X^{\mathfrak{c}} \times S^i(\mathfrak{g}/\mathfrak{c}))/N \qquad S^i(\tilde{L}_X^*) = (R_X^{\mathfrak{c}} \times S^i(\mathfrak{c}^0))/N,$$

where $\mathfrak{c}^0 \subset \mathfrak{g}^*$ is the annihilator subspace, for the natural identification of the i-th symmetric powers with associated vector bundles to $R_X^{\mathfrak{c}}$ for the naturally induced action of N on $\mathfrak{g}/\mathfrak{c}$ and $\mathfrak{c}^0$, respectively.

Definition 3.5. Let C be the analytic subgroup defined by $\mathfrak{c} \subset \mathfrak{g}$ in some Lie group with Lie algebra $\mathfrak{g}$, and let $S^i(\mathfrak{g}/\mathfrak{c})^C$ and $S^i(\mathfrak{c}^0)^C$ denote the subspaces of invariant elements under the (well-defined) induced (co)adjoint C-action. If $k_i : N \to Gl(S^i(\mathfrak{g}/\mathfrak{c})^C)$ are the naturally induced group homomorphisms, we set $\tilde{R}_i = R_X^{\mathfrak{c}}/K_i$ and $\tilde{N}_i = N/K_i$, where $K_i = \ker k_i$. We define subbundles of $S^i(\tilde{L}_X)$ and $S^i(\tilde{L}_X^*)$, respectively, by setting

$$(3.4) \qquad \tilde{L}_i = (\tilde{R}_i \times S^i(\mathfrak{g}/\mathfrak{c})^C)/\tilde{N}_i \qquad \tilde{L}_i^* = (\tilde{R}_i \times S^i(\mathfrak{c}^0)^C)/N.$$

In particular, $\tilde{R}_X = \tilde{R}_1$, $\tilde{N} = \tilde{N}_1$, and $\tilde{L}_1 = (\tilde{R}_X \times (\mathfrak{g}/\mathfrak{c})^C)/\tilde{N}$ are called the *reduced Lie frame bundle*, *reduced structure group*, and *reduced Lie algebra bundle*, respectively.

Lemma 3.6. *The vertically polynomial functions in $\tilde{l}^* C^\infty(\tilde{L}_X^*) \cap \tilde{\mathfrak{P}}'$ are those defined on Q' by the sections of $\oplus_{i\geq 0}\tilde{L}_i$. In particular,*

$$(3.5) \quad \bar{\Gamma}(\tilde{L}_X) \cap \tilde{\mathfrak{P}}' = \bar{\Gamma}(\tilde{L}_1), \quad \textit{and} \quad \tilde{L}_1 = (\tilde{R}_X \times \tilde{\mathfrak{g}})/\tilde{N} \quad \textit{with} \quad \tilde{\mathfrak{g}} = \mathfrak{N}_{\mathfrak{g}}(\mathfrak{c})/\mathfrak{c}.$$

Proof. Let $\bar{s} \in \tilde{l}^* C^\infty(\tilde{L}_X^*)$ be a vertically polynomial function defined by $s \in \Gamma(S^\bullet \tilde{L}_X)$, and let $\bar{s} = i_{Q'}^* \hat{\bar{s}}$, where $\hat{\bar{s}} \in l^* C^\infty(L_X^*)$ is defined by $\hat{s} \in \Gamma(S^\bullet L_X)$. Let $f \in \mathfrak{C}$ define $df|_X \in \Gamma(L_X)$ and $\bar{df}|_X \in \bar{\Gamma}(L_X)$. We can interpret $\hat{s}$, s and $df|_X$ as equivariant maps

$$\underline{\hat{s}} : \quad R_X^{\mathfrak{c}} \to S^\bullet \mathfrak{g} \qquad \underline{s} = \underline{c} \circ \underline{\hat{s}} : R_X^{\mathfrak{c}} \to S^\bullet(\mathfrak{g}/\mathfrak{c}) \qquad \underline{df}|_X : R_X^{\mathfrak{c}} \to \mathfrak{c}$$

where the restriction $C^\infty(\mathfrak{g}^*) \supset S^\bullet \mathfrak{g} \xrightarrow{\underline{c}} S^\bullet(\mathfrak{g}/\mathfrak{c}) \subset C^\infty(\mathfrak{c}^0)$ is given by the natural projection. Let now $m \in (E_X^*)_x$ and $n = l(m) = [p, d] \in \mathfrak{g}^*(R_X^{\mathfrak{c}})$, $p \in (R_X^{\mathfrak{c}})_x, d \in \mathfrak{g}^*$. Thanks to the equivariance, for any choice of p and d,

$$\begin{aligned} \{\hat{\bar{s}}, \bar{df}|_X\}'(m) &= \{\hat{\bar{s}}|_{(L_X^*)_x}, \bar{df}|_X|_{(L_X^*)_x}\}_x(l(m)) = \{\underline{\hat{s}}(p), \underline{df}|_X(p)\}_{\mathfrak{g}^*}(d) \\ &= X_{\underline{df}|_X(p)} \cdot (\underline{\hat{s}}(p))(d) = d/dt|_{t=0}(Ad^*(-t\,\underline{df}|_X(p)))^* \underline{\hat{s}}(p)(d) \end{aligned}$$

where $\{\cdot,\cdot\}_x$ and $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ denote the Poisson brackets on $(L_X^*)_x$ and $\mathfrak{g}^*$, respectively, Ad^* the coadjoint action, and we used that for all $D \in \mathfrak{g}_{\mathsf{L}} \subset$

$C^\infty(\mathfrak{g}_L^*), l \in \mathfrak{g}_L^*$, $X_{-D}(l) = d/dt|_0 Ad^*(\exp(tD))(l)$, where $\mathfrak{g}_L$ is the anti-isomorphic Lie algebra. Since at all points, the differentials df_x span $(A_X)_x \cong \mathfrak{c}$, and those of the functions $\bar{d}f|_X$ span $(TQ')^0$, we have

$$\begin{aligned}
\bar{s} \in \tilde{\mathfrak{P}}' &\iff \hat{\hat{s}} \in \mathfrak{N}' \iff \{\hat{\hat{s}}, \mathfrak{C}'\}' \subset \mathfrak{C}' \iff i_{Q'}^*\{\hat{\hat{s}}, \bar{d}f|_X\}' = 0 \qquad \forall f \in \mathfrak{C} \\
&\iff d/dt|_{t=0}(Ad^*(-t\,\underline{df}|_X(p)))^*\underline{\hat{s}}(p)(d) = 0 \qquad \forall p \in R_X^{\mathfrak{c}}, d \in \mathfrak{c}^0 \\
&\iff Ad_*(C)((\underline{c} \circ \underline{\hat{s}})(p)) = (\underline{c} \circ \underline{\hat{s}})(p) \qquad \forall p \in R_X^{\mathfrak{c}} \\
&\iff Ad_*(C) \circ \underline{s} = \underline{s} \iff s \in \bar{\Gamma}(\oplus_{i \geq 0} \tilde{L}_i)
\end{aligned}$$

where $Ad_*(C)$ denotes the induced action on $S^\bullet(\mathfrak{g}/\mathfrak{c})$. For $i = 1$, we have

$$(3.6) \qquad [n] \in \tilde{\mathfrak{g}} \iff ad(\mathfrak{c})[n] = 0 \iff Ad_*(C)([n]) = [n] \iff [n] \in (\mathfrak{g}/\mathfrak{c})^C,$$

completing the proof. In particular, $\tilde{L}_1$ is a bundle of Lie algebras. □

3.2. The reduced Einstein-Mayer and Wong systems. Let $H \in \mathfrak{P}$ such that $dH|_X = 0$. We will show that it defines an admissible Einstein-Mayer Hamiltonian on Q' if H is admissible, i.e., $\tilde{H} = i_Q^* H \in \tilde{\mathfrak{P}}$.

Theorem 3.7. *Let $X \subset S \overset{i_S}{\hookrightarrow} Z$ be a Lagrangian submanifold of the symplectic leaf (S, ω_S) of the Poisson manifold (Z, ϖ), and $Q \supset S$ a locally closed coisotropic submanifold. Every $\tilde{H} \in C^\infty(Q)$ such that $d\tilde{H}|_X = 0$ naturally defines a field $d_X^2\tilde{H} : X \to S^2\tilde{E}_X$ of symmetric bilinear forms on the fibers of $\tilde{E}_X^*$, in the notation of* 3.1 *and thereabove . If $\tilde{H} = i_Q^* H$ for $H \in \mathfrak{P}$, then $\tilde{H}_2' = i_{Q'}^* H_2'$, where H_2' is given by* (2.19)*, and $\tilde{H}_2' = 1/2\,\overline{d_X^2\tilde{H}} \in C^\infty(Q')$. Furthermore, $\tilde{H} \in \tilde{\mathfrak{P}}$ implies $\tilde{H}_2' \in \tilde{\mathfrak{P}}'$.*

Proof. The transversal Hessian of $\tilde{H}$ yields $d_X^2\tilde{H}$ as in theorem 2.16 with (Z, H) replaced by $(Q, \tilde{H})$. If $\tilde{H} = i_Q^* H$ for $H \in \mathfrak{P}$ with $dH|_X = 0$, then by construction $d_X^2\tilde{H} = (Ti_Q)^* d_X^2 H$, i.e., $\tilde{H}_2' = i_{Q'}^* H_2'$. Let us show that $\tilde{H} \in \tilde{\mathfrak{P}}$ implies $\tilde{H}_2' \in \tilde{\mathfrak{P}}'$ or, equivalently, $H \in \mathfrak{N}$ implies $H_2' \in \mathfrak{N}'$.

Since Q is coisotropic, X_f is tangent to Q for all $f \in \mathfrak{C}$ and its flow defines a local family of diffeomorphisms $\exp(tX_f) : Q \to Q, t \in \mathbb{R}$ (omitting obvious restrictions). The first jet prolongation of $(T\exp(tX_f))^*$ can be restricted to the image of X under the zero section of T^*Q, and under the identification (2.16) of jets with tensor fields over the zero section, it becomes identified with $\otimes^2(T\exp(tX_f)|_X)^*$ acting on $\otimes^2 T^*Q|_X$. Since because of $df|_S \in (TQ)^0|_S \subset (TS)^0$, $X_f|_S = 0$ and $\exp(tX_f)|_S$ is the identity, this induces bundle morphisms of $S^2\tilde{E}_X$. Finally, for any section $d\tilde{H} : Q \to T^*Q$

$$J^1(T^*\exp(tX_f)) \circ J^1(d\tilde{H}) \circ \exp^{-1}(tX_f) = J^1(d(\exp(-tX_f)^*\tilde{H})),$$

which reads with our identification (since $X_f|_X = 0$)

$$
\begin{array}{llrcll}
(3.7) & & \otimes^2 T^* \exp(tX_f)|_X \circ d_X^2 \tilde{H} & = & d_X^2(\exp(-tX_f)^* \tilde{H}) & \forall f \in \mathfrak{C}, \\
(3.8) & \iff & \mathrm{L}_{X_f}(d_X^2 \tilde{H}) & = & d_X^2(-X_f \cdot \tilde{H}) & \\
 & \iff & i_{Q'}^*(\overline{\mathrm{L}_{X_f}(d_X^2 H)}) & = & i_{Q'}^*(\overline{d_X^2(-X_f \cdot H)}) & \\
(3.9) & \iff & 2\, i_{Q'}^*\{H_2', \bar{d}f|_X\}' & = & \overline{d_X^2(i_Q^*\{f, H\})} & \forall f \in \mathfrak{C},
\end{array}
$$

where we used Corollary 2.7. By Lemma 3.2, the differentials of the functions $df|_X$ span $(TQ')^0 \subset T^*Z'|_{Q'}$ at each point of Q'. Hence, it follows from (3.9) that $H \in \mathfrak{N}$ implies $H_2' \in \mathfrak{N}'$ and thus, $\tilde{H}_2' \in \tilde{\mathfrak{P}}'$. □

We expect that generically, $d_X^2 \tilde{H}$ should define corresponding reduced fields $\tilde{\eta}$, $\alpha_{\tilde{H}}$ $\chi_{\tilde{H}}$, $d_X^2 \tilde{H}_0 = d_X^2 H_0$, and $\gamma_{\tilde{H}} = \gamma_H$, as shown in the diagram

$$
(3.10) \qquad
\begin{array}{ccccccc}
\tilde{L}_X & \longrightarrow & \tilde{E}_X & \longrightarrow & \underline{E}_X & \xleftarrow{-\omega_X^\flat} & TX \\
\Big\downarrow \chi_{\tilde{H}}^\flat & & \tilde{\eta}^\flat \Big\downarrow \Big\uparrow d_X^2 \tilde{H}^\sharp & & d_X^2 \tilde{H}_0^\sharp \Big\uparrow \Big\downarrow d_X^2 \tilde{H}_0^{\sharp -1} & & \Big\downarrow \gamma_{\tilde{H}}^\flat \\
\tilde{L}_X^* & \longleftarrow & \tilde{E}_X^* & \xleftarrow{Ti_S} & \underline{E}_X^* & \xrightarrow{(-\omega_X^\flat)^*} & T^*X
\end{array}
$$

$$
(3.11) \qquad \text{where} \quad \alpha_{\tilde{H}}^\flat \; = \; d_X^2 \tilde{H}^\sharp \circ Ti_S \circ d_X^2 \tilde{H}_0^{\sharp -1} \circ (-\omega_X^\flat) \; = \; (Ti_Q)^* \circ \alpha_H^\flat
$$

is called the $\tilde{E}_X$*-connection form* defined by $\tilde{H}$. There is the following result.

Theorem 3.8. *If the field $d_X^2 \tilde{H}$ defined by $\tilde{H} = i_Q^* H$ in Theorem* 3.7 *is nondegenerate, it naturally defines fields of scalar products $\tilde{\eta}$ on $\tilde{E}_X$ and $\chi_{\tilde{H}}$ on $\tilde{L}_X$, a metric $\gamma_{\tilde{H}}$ and an $\tilde{E}_X$-connection form $\alpha_{\tilde{H}}$ on X as in* (3.10) *and* (3.11), *provided that $d_X^2 \tilde{H}|_{\underline{E}_X^*}$ and $\tilde{\eta}|_{\tilde{L}_X}$ are also nondegenerate. $\tilde{\eta}$ and the triple $(\gamma_{\tilde{H}}, \alpha_{\tilde{H}}, \chi_{\tilde{H}})$ determine each other. Furthermore, suppose that $\tilde{H} \in \tilde{\mathfrak{P}}$, and that the assumptions* (3.2, 3.3) *are valid. Then, for every $\mathcal{X} \in \mathfrak{X}(X)$, $\overline{\mathrm{i}_{\mathcal{X}} \alpha_{\tilde{H}}} \in \bar{\Gamma}(\tilde{E}_X) \cap \tilde{\mathfrak{P}}'$, and $\alpha_{\tilde{H}}$ defines a principal connection form $A_{\tilde{H}}$ on $\tilde{R}_X$ taking values in $ad(\tilde{\mathfrak{g}})$. Finally, $\chi_{\tilde{H}} \in \Gamma(\tilde{L}_2^*)$, with $\tilde{L}_2^*$ defined in* (3.4).

Proof. The first two assertions follow from Theorem 2.16 with (Z, H) replaced by $(Q, \tilde{H})$. Let now $\tilde{H} \in \tilde{\mathfrak{P}}$ and $\tilde{H} = i_Q^* H$ with $H \in \mathfrak{N}$. Since $\exp(-tX_f)|_S$ is the identity, (3.7) implies $(T\exp(tX_f)|_X)^* \circ \alpha_{\tilde{H}}^\flat = \alpha_{\tilde{H}}^\flat$ for all $f \in \mathfrak{C}$. Hence, using Corollary 2.7,

$$
\begin{array}{ll}
 & \overline{\mathrm{L}_{X_f}(\mathrm{i}_{\mathcal{X}} \alpha_{\tilde{H}})} = i_{Q'}^*(\overline{\mathrm{L}_{X_f}(\mathrm{i}_{\mathcal{X}} \alpha_H)}) = i_{Q'}^*\{\overline{\mathrm{i}_{\mathcal{X}} \alpha_H}, \bar{d}f|_X\}' = 0 \qquad \forall f \in \mathfrak{C} \\
(3.12) & \iff \overline{\mathrm{i}_{\mathcal{X}} \alpha_H} \in \bar{\Gamma}(E_X) \cap \mathfrak{N}' \iff \overline{\mathrm{i}_{\mathcal{X}} \alpha_{\tilde{H}}} = i_{Q'}^*(\overline{\mathrm{i}_{\mathcal{X}} \alpha_H}) \in \bar{\Gamma}(\tilde{E}_X) \cap \tilde{\mathfrak{P}}'
\end{array}
$$

for all $\mathcal{X} \in \mathfrak{X}^1(X)$. Since $\bar{\Gamma}(\tilde{L}_X) \cap \tilde{\mathfrak{P}}' = \bar{\Gamma}(\tilde{L}_1)$ by Lemma 3.6, this means that $\alpha_{\tilde{H}}$ defines an adjoined action on $\bar{\Gamma}(\tilde{L}_1)$ by the reduced bracket on $\tilde{\mathfrak{P}}'$

and thus, a covariant derivative on $\Gamma(\tilde{L}_1)$ and, since $\tilde{N}$ acts effectively on the standard fiber of $\tilde{L}_1$, a principal connection form on $\tilde{R}_X$. This is a form

$$(3.13)\qquad A_{\tilde{H}} : \tilde{R}_X \longrightarrow T^*\tilde{R}_X \otimes ad(\tilde{\mathfrak{g}}) \qquad ad(\tilde{\mathfrak{g}}) \subset \tilde{\mathfrak{n}} = Lie(\tilde{N}) = \mathfrak{n}/\mathfrak{k}_1,$$

where $\mathfrak{n} = Lie(N)$ and $\mathfrak{k}_1 = Lie(K_1)$. In deed, we deduce from (3.12) and Lemma 3.6 that the form $A_H = A^{\alpha_H}$ on R_X takes values in $ad(\mathfrak{g}) \cap \mathfrak{n} = ad(\mathfrak{N}_{\mathfrak{g}}\mathfrak{c})$, and, consequently, restricts to $R_X^{\mathfrak{c}}$. Its equivariance implies that $Lie(k_1) \circ A_H|_{R_X^{\mathfrak{c}}}$, where $Lie(k_1)|_{ad(\mathfrak{N}_{\mathfrak{g}}\mathfrak{c})} : ad(\mathfrak{N}_{\mathfrak{g}}\mathfrak{c}) \to ad(\tilde{\mathfrak{g}})$ is the induced map, is constant on the fibers of the projection $pr : R_X^{\mathfrak{c}} \to \tilde{R}_X$, and we have $Lie(k_1) \circ A_H|_{R_X^{\mathfrak{c}}} = pr^* A_{\tilde{H}}$, showing (3.13).

From (3.8), we deduce that for $H \in \mathfrak{N}$, $\mathrm{L}_{X_f}(d_X^2\tilde{H}) = 0 = \mathrm{L}_{X_f}\tilde{\eta}$ for all $f \in \mathfrak{C}$. Now, the function on $\tilde{E}_X$ defined by $\tilde{\eta} \in \Gamma(S^2\tilde{E}_X^*)$ can be restricted to the function on $\tilde{L}_X$ defined by $\chi_{\tilde{H}} \in \Gamma(S^2\tilde{L}_X^*)$, and since S is preserved,

$$\mathrm{L}_{X_f}\chi_{\tilde{H}} = 0\ \forall f \in \mathfrak{C} \iff \tilde{l}^*\tilde{H}'_{20} \in \bar{\Gamma}(S^2\tilde{L}_X) \cap \tilde{\mathfrak{P}}' = \bar{\Gamma}(\tilde{L}_2) \iff \chi_{\tilde{H}} \in \Gamma(\tilde{L}_2^*),$$

where $\tilde{H}'_{20}$ is the function defined by $\chi_{\tilde{H}}^{\flat -1}$ on $\tilde{L}_X^*$, and we used Corollary 2.7 and Lemma 3.6. This completes the proof of the theorem. □

Corollary 3.9. *There is a well-defined curvature* $\Phi_{\tilde{H}} = \Phi^{\alpha_{\tilde{H}}}$ *of* $\alpha_{\tilde{H}}$

$$\Phi_{\tilde{H}}(\mathcal{X}_1, \mathcal{X}_2) = \{\alpha_{\tilde{H}}(\mathcal{X}_1), \alpha_{\tilde{H}}(\mathcal{X}_2)\} - \alpha_{\tilde{H}}([\mathcal{X}_1, \mathcal{X}_2]) \qquad \forall \mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}^1(X),$$

taking values in $\tilde{L}_1$. *The curvature form of* $A_{\tilde{H}}$ *takes values in* $ad(\tilde{\mathfrak{g}})$.

Corollary 3.10. *Let* $\tilde{H}'_1 = \tilde{H}'_0 \circ (\alpha_{\tilde{H}}^{\flat})^*$, *where* $\alpha_{\tilde{H}}^{\flat} : TX \to \tilde{E}_X$, *and* $\tilde{H}'_0$ *denotes the Hamiltonian defined by* $1/2\,\gamma_{\tilde{H}}^{\flat -1}$ *on* T^*X. *Then,* $\tilde{H}'_1 \in \tilde{\mathfrak{P}}'$.

The restriction of ψ_H to Q' is given by the bundle isomorphism

$$\psi_{\tilde{H}} = \psi_H|_{\tilde{E}_X^*} = ((\alpha_{\tilde{H}}^{\flat})^*, \tilde{l}) : \tilde{E}_X^* \longrightarrow T^*X \oplus_X \tilde{L}_X^* =: \tilde{L}_X^{\pi} \subset L_X^{\pi}.$$

Thus, $\tilde{L}_X^{\pi}$ is a coisotropic submanifold of (L_X^{π}, ϖ_H). Let $\tilde{\mathfrak{P}}'_{\tilde{H}} = (\psi_{\tilde{H}}^{-1})^*(\tilde{\mathfrak{P}}')$.

Definition 3.11. The triples $(Q', \tilde{\mathfrak{P}}', \tilde{H}'_1)$ and $(\tilde{L}_X^{\pi}, \tilde{\mathfrak{P}}'_{\tilde{H}}, pr_1^*\tilde{H}'_0)$ are called the *reduced* and *gauged reduced Wong systems*, and the triples $(Q', \tilde{\mathfrak{P}}', \tilde{H}'_2)$ and $(\tilde{L}_X^{\pi}, \tilde{\mathfrak{P}}'_{\tilde{H}}, pr_1^*\tilde{H}'_0 + pr_2^*\tilde{H}'_{20})$ the *reduced* and *gauged reduced Einstein-Mayer systems* associated to $(Z, \varpi, H; Q)$ at X, respectively.

Remark 3.12. The study of the elements of $(S^2\mathfrak{c}^0)^C$ and its decomposition into irreducible subspaces under the action of the reduced structure group was carried out systematically; cf [**18**], [**6**] for the example of Manton [**17**], which yields in particular a field of hermitian forms. Higgs fields of this form appear in interaction terms between spinor fields ([**24**]).

Remark 3.13. As in classical mechanics, the presence of the constraint Q can be related to $d_X^2\tilde{H}$ being precisely the nondegenerate part of a $d_X^2 H$.

4. Local Kaluza-Klein realization

The existence of a locally minimal symplectic realization of the Poisson manifold Z and the associated dual pair ([**27**, **25**]) is related via Sternberg-Weinstein approximation to classical Yang-Mills and Kaluza-Klein theory. In the presence of coisotropic constraints, we obtain the reduction theorem for bundles with homogeneous fibers ([**4**]).

Let us recall that for a fixed Poisson manifold (Z,ϖ) and any $z_0 \in Z$, there exists an open neighborhood $U \ni z_0$ such that (U,ϖ) (omitting the restriction) is realizable by a surjective Poisson submersion $\rho : (W,\omega) \to (U,\varpi)$, where (W,ω) is a symplectic manifold of (locally minimal) dimension $d_{min} = 2(\dim Z - (1/2)\mathrm{rk}(\varpi_z))$. In addition, this realization is essentially unique ([**27**, **25**]). The fibers of ρ define the foliation $\mathcal{F}$ whose leaves we will presume connected. If the dual (i.e., symplectically orthogonal) foliation $\mathcal{F}^\perp$ has connected leaves, and is such that the quotient map $\lambda : W \to W/\mathcal{F}^\perp =: \Upsilon$ is a submersion onto a manifold Υ with coinduced Poisson structure υ,

$$(\Upsilon,\upsilon) \overset{\lambda}{\longleftarrow} (W,\omega) \overset{\rho}{\longrightarrow} (U,\varpi) \tag{4.1}$$

will be a dual pair. Referring to this case, we will simply say that (4.1) exits.

Let $\mathfrak{F} = \rho^* C^\infty(U)$ and $\mathfrak{F}^\perp = \lambda^* C^\infty(\Upsilon)$ be the polar function groups, $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{F})$ denote the center (Casimir functions), and $\mathfrak{Ph}$ be the subalgebra in $C^\infty(W)$ of functions whose Hamiltonian vector field is projectable by ρ_* into the subspace $\mathfrak{Ham}(U)$ of Hamiltonian vector fields on U.

Lemma 4.1. *We have $\mathfrak{F} = \mathfrak{Z}_{C^\infty(W)}(\mathfrak{F}^\perp)$, $\mathfrak{F}^\perp = \mathfrak{Z}_{C^\infty(W)}(\mathfrak{F})$, where $\mathfrak{Z}_{C^\infty(W)}$ denotes the centralizer in $C^\infty(W)$, and $\mathfrak{Z}(C^\infty(U)) \cong \mathfrak{Z} = \mathfrak{F} \cap \mathfrak{F}^\perp = \mathfrak{Z}(\mathfrak{F}^\perp) \cong \mathfrak{Z}(C^\infty(\Upsilon))$. Furthermore, let $K \in \mathfrak{Ph}$ be such that $\rho_* X_K = X_H$ for some $H \in C^\infty(U)$. Then, for some $J \in C^\infty(\Upsilon)$, there are decompositions*

$$K = \rho^* H + \lambda^* J \qquad X_K = X_{\rho^* H} + X_{\lambda^* J}, \tag{4.2}$$

unique for X_K and unique up an element of $\mathfrak{Z}$ for K. Thus $\mathfrak{Ph} = \mathfrak{F} + \mathfrak{F}^\perp$, and the sequence $0 \to \mathfrak{F}^\perp \to \mathfrak{Ph} \to \mathfrak{F}/\mathfrak{Z} \cong \mathfrak{Ham}(U) \to 0$ is exact. In particular, $\mathfrak{Ham}(U) \cong \mathfrak{Ph}/\mathfrak{F}^\perp$.

Proof. For (4.2), we define $\hat{J} \in C^\infty(W)$ by $\hat{J} = K - \rho^* H$. By assumption, $\rho_* X_{\hat{J}} = 0$, and thus, $\mathfrak{Z}_{C^\infty(W)}(\mathfrak{F}) = \mathfrak{F}^\perp \ni \hat{J} = \lambda^* J$ for some $J \in C^\infty(\Upsilon)$. The other assertions are well-known or easily verified. □

Let S denote the symplectic leaf through z_0, and let $V := S \cap U$, $\hat{V} := \rho^{-1}(V)$ and $F := \rho^{-1}(z_0)$. We assume that these manifolds are *connected.*

Lemma 4.2. *The manifold $\hat{V}$ is a coisotropic submanifold of W, and the characteristic foliation is given by the fibers of the submersion $\rho|_{\hat{V}}$. In particular, the fibers are isotropic submanifolds of W, and $F = \rho^{-1}(z_0)$ is a Lagrangian submanifold of $\hat{N} = \rho^{-1}(N)$ for any transverse submanifold N*

to V representing the transverse structure at z_0. Furthermore, U can be chosen such that $\rho|_{\hat{V}} : \hat{V} \to V$ is a fiber bundle over V with standard fiber F. On the other hand, we have $\lambda(w_1) = \lambda(w_2) =: l_0 \in \Upsilon$ for all $w_1, w_2 \in \hat{V}$, and $\mathrm{rk}_{l_0} \upsilon = 0$, i.e., l_0 is a symplectic leaf, and Υ is its own transverse structure at l_0, with linear approximation given by $T_{l_0}\Upsilon \cong \mathfrak{g}_{\mathrm{L}}^*$, with $\mathfrak{g}_{\mathrm{L}}$ denoting the Lie algebra anti-isomorphic to the transverse Lie algebra $\mathfrak{g}$ to V at z_0.

Proof. The first assertions are again well-known ([**27**]). Let us suppose that U is split by a splitting map $sp : U \to V \times N$, where N is an arbitrary transverse submanifold through z_0 to V representing the transverse structure. Since the sp is Poisson, we know that $sp^{-1} \circ (Id_V, \rho|_{\hat{N}}) : V \times \hat{N} \to U$ is a realization of minimal dimension. After restricting U and W, the essential uniqueness of the minimal realization implies that there is a symplectomorphism $\widehat{sp} : W \to V \times \hat{N}$ such that $\rho = sp^{-1} \circ (Id_V, \rho|_{\hat{N}}) \circ \widehat{sp}$. The restriction of $\widehat{sp}$ to $\hat{V}$ yields a trivialization $\widehat{sp}|_{\hat{V}} : \hat{V} \to V \times F$, of the bundle $\hat{V}$ with standard fiber F. By counting dimensions, we find $\dim \Upsilon = \dim N$, which implies the remaining assertions. □

4.1. Kaluza-Klein realization and dynamics. Let $\rho : (W, \omega) \to (U, \varpi)$, V, F as above, and $X \subset V$ an embedded Lagrangian submanifold. Suppose $z_0 \in X$ and that $\rho|_{\hat{V}} : \hat{V} = \rho^{-1}(V) \to V$ *is a fiber bundle with standard fiber* F. Then, we also have a fiber bundle $Y \subset \hat{V}$ over X given by

$$Y = \rho^{-1}(X) \xrightarrow{p} X \qquad p = \rho|_Y. \tag{4.3}$$

The total space is an embedded Lagrangian submanifold $i_Y : Y \to W$, since it is coisotropic as the preimage of a coisotropic submanifold, and lemma 4.2 implies $2\dim Y = 2\dim X + 2\dim F = \dim V + \dim \hat{N} = \dim W$. Thus, the Sternberg-Weinstein approximation to W at Y is given by

$$W' = TW_{|Y}/TY \overset{(-\omega_Y^\flat)^*}{\longrightarrow} T^*Y, \qquad \omega_Y^\flat = \omega^\flat|_{TY} : TY \to (TY)^0,$$

where $(-\omega_Y^\flat)^*$ is a symplectomorphism for the canonical structure on T^*Y.

Proposition 4.3. *The map $\rho' : W' \to U'$ induced by $T\rho|_Y$ is a symplectic realization of U'. If the dual pair* (4.1) *exists, there is an induced dual pair*

$$\mathfrak{g}_{\mathrm{L}}^* \cong \Upsilon' \overset{\lambda'}{\longleftarrow} W' \overset{\rho'}{\longrightarrow} U', \tag{4.4}$$

where U' and Υ' are the Sternberg-Weinstein approximation of U at $X \cap U$ and Υ at l_0, respectively, and λ' is the map induced by $T\lambda|_Y$.

Proof. Since ρ is a surjective submersion, the same is true for ρ'. For Darboux coordinates (x^μ, p_μ, r_a) on U, inducing coordinates $(x^\mu, [\dot{p}_\mu], [\dot{r}_a])$ on U' as in Proposition 2.6, it is always possible to choose Darboux coordinates $(\hat{x}^\mu = \rho^* x^\mu, \hat{p}_\mu = \rho^* p_\mu, \hat{y}^a, \hat{p}_a)$ on W, inducing Darboux coordinates $(\hat{x}^\mu, [\dot{\hat{p}}_\mu] = (\rho')^*[\dot{p}_\mu], \hat{y}^a, [\dot{\hat{p}}_a])$ on $Z' \cong T^*Y$ which are holonomic with respect to coordinates $(\hat{x}^\mu|_Y, \hat{y}^a|_Y)$ on Y. In these coordinates, one easily shows that ρ' induces the Sternberg-Weinstein Poisson structure. If (4.1) exists, we see as before that (4.4) forms a dual pair. By Lemma 4.2, $\Upsilon' = T_{l_0}\Upsilon \cong \mathfrak{g}_{\mathrm{L}}^*$. $\square$

Corollary 4.4. *The Sternberg-Weinstein approximation of the Sternberg-Weinstein phase space T^*P/G associated to a principal fiber bundle $P \to B$ with structure group G at $B \subset T^*B$ is naturally isomorphic to T^*P/G.*

Proof. Obviously, T^*P is a global symplectic realization of T^*P/G of dimension $d_{\min}$. Furthermore, the isomorphism $(T^*P)' = T(T^*P)|_P/TP \cong T^*P$ is natural, symplectic and compatible with the projection maps $\rho : T^*P \to T^*P/G$ and $\rho' : T(T^*P)|_P/TP \to T(T^*P/G)|_B/TB$ since ρ is defined by a fiberwise linear action of G. Thus, it induces a natural Poisson equivalence of the quotient manifolds. $\square$

Definition 4.5. The Sternberg-Weinstein approximation $W' \cong T^*Y$ will be called a *Kaluza-Klein realization* of Z at z_0. We denote by $\pi : W' \to Y$ the natural projection.

Lemma 4.6. *Suppose that the dual pair* (4.4) *exists. Then, the fibers of $\rho'|_Y$ are given by the orbits of a local right action of the connected and simply connected Lie group G_1 with Lie algebra $\mathfrak{g}_{\mathrm{L}}$ on Y, and the fibers of ρ' are given by the orbits of the canonical Hamiltonian lift to $W' \cong T^*Y$ of this local action, which in addition is locally free. Furthermore, there is a natural bundle morphism $a : Y \to R_X$ over Id_X which is locally equivariant with respect to the canonical homomorphism $Ad : G_1 \to Aut(\mathfrak{g})$.*

Proof. It is well-known ([**2**]) that the fibers of ρ' are given by the orbits of the local action of a connected and simply connected Lie group G_1 on W' induced by λ'. The fundamental vector fields of $D \in \mathfrak{g}_{\mathrm{L}} \subset C^\infty(\mathfrak{g}_{\mathrm{L}}^*)$ is the Hamiltonian vector field of $-(\lambda')^*D$. Since λ' is surjective, it follows that these Hamiltonian vector fields span TW' at each point and thus, the action is locally free. For $y \in Y$, $\lambda'|_y : T_yW/T_yY \cong T_y^*Y \to \mathfrak{g}_{\mathrm{L}}^*$ is a linear map, an thus, $-(\lambda')^*D|_{T^*yY}$ is a linear function, that is, an element of T_yY. Hence, $-(\lambda')^*D$ is a vertically linear function on $W' \cong T^*Y$ corresponding to a vector field $D_Y \in \mathfrak{X}(Y)$. On the other hand, it is well-known that the Hamiltonian vector field of a function defined on T^*Y with its canonical symplectic structure by a vector field on Y is precisely the unique Hamiltonian lift of this vector field. In particular, $D_Y = X_{-(\lambda')^*D}|_Y$. Thus, the local action of G_1 on T^*Y is the unique Hamiltonian lift of the restricted local action on Y. Consequently, we can define the map

$$a : Y \to R_X \qquad a(y)(D) = (pr_2 \circ (\omega^{\flat -1} \circ \hat{\rho}^*)^{-1} \circ D_Y)(y) \qquad \forall D \in \mathfrak{g}$$

where $\hat{\rho}^*$ is the bundle morphism to be defined in (4.8). Note that $(\omega^{\flat -1} \circ \hat{\rho}^*)(Y \times_X (TV)^0_{|X}) = VY$ so that we have in deed $a(y) : \mathfrak{g} \to L_{p(y)}$, and since $\omega^\flat$ and $D \to D_Y$ and are Lie algebra anti-automorphisms (since $\mathfrak{g}$ is anti-isomorphic to $\mathfrak{g}_{\mathrm{L}}$), and ρ is Poisson a morphism, this is a Lie algebra isomorphism. On the other hand, a is equivariant since

$$\begin{aligned} a(yg)(D) &= (pr_2 \circ (\hat{\rho}^*)^{-1} \circ \omega^\flat \circ D_Y)(yg) \\ &= (pr_2 \circ (\hat{\rho}^*)^{-1} \circ \omega^\flat \circ TR_g \circ (Ad_*(g)(D))_Y)(y) \\ &= (pr_2 \circ (\hat{\rho}^*)^{-1} \circ (\rho'_{yg,g})^* \circ \omega^\flat \circ (Ad_*(g)(D))_Y)(y) \\ (a(y) \circ Ad_*(g))(D) &= (pr_2 \circ (\hat{\rho}^*)^{-1} \circ \omega^\flat \circ ((Ad_*(g)(D))_Y)(y) \end{aligned}$$

where $R_g : Y \to Y$ denotes the action of $g \in G_1$ on Y whenever it is defined, and $\rho'_{yg,g} = \rho'|^{-1}_{T_yW/T_yY} \circ \rho'|_{T_{yg}W/T_{yg}Y}$. Here we used $(TR_g)^* \circ (-\omega^\flat_Y)^* = (-\omega^\flat_Y)^* \circ \rho'_{yg,g})$ and $\hat{\rho}^*|_{Y \times X(TX)^0} = (\pi, \rho')^*$ (cf lemma 4.12). □

Theorem 4.7. *Suppose that the dual pair* (4.1) *exists. Let $H \in C^\infty(U)$ be a Hamiltonian such that $dH|_X = 0$, and $K \in C^\infty(W)$ such that $dK|_Y = 0$ and $\rho_* X_K = X_H$. Then, the Einstein-Mayer systems (W', ω', K'_2) and (U', ϖ', H'_2) are ρ'_*-related. Writing $\underline{\kappa} = K'_2$, $\underline{\eta} = H'_2$ and $\underline{\iota} = J'_2$, the Einstein-Mayer approximation of J at l_0, we have a decomposition*

$$\underline{\kappa} = (\rho')^* \underline{\eta} + (\lambda')^* \underline{\iota}. \tag{4.5}$$

If $d^2_Y K$, $d^2_X H$, $d^2_{l_0} J$ are nondegenerate, then $\underline{\kappa}$, $\underline{\eta}$, $\underline{\iota}$ are functions defined by a metric κ on Y, η, and some scalar product ι on $\mathfrak{g}_{\mathrm{L}}$, respectively. Furthermore, $(\rho')^* \underline{\eta}$ corresponds to a (*locally*) *G_1-invariant metric on Y.*

Proof. If λ exists, lemma 4.1 yields a decomposition $K = \rho^* H + \lambda^* J$ for some $J \in C^\infty(\Upsilon)$. Thus $dK = \rho^* dH + \lambda^* dJ$, and similarly as in the proof of theorem 3.7, this implies $K'_2 = (\rho')^* H'_2 + (\lambda')^* J'_2$, i.e., (4.5), and $\rho'_* X_{K'_2} = X_{H'_2}$. Under the nondegeneracy assumptions, κ corresponds to a fiber-quadratic function on $(W')^* \cong TY$, i.e., metric on Y. According to lemma 4.6, the term $(\rho')^* \underline{\eta}$ is a (locally) G_1-invariant fiber quadratic function on W', and thus, a G_1-invariant metric on Y . □

Definition 4.8. The triple $(W', \omega', \underline{\kappa})$ is called a *Kaluza-Klein dynamics* corresponding to (Z, ϖ, H) at $z_o \in X$. By a given κ, the fields α_H and γ_H defined by a corresponding $\underline{\eta}$ are well-determined, while $\underline{\chi}_H = l^* H'_{20}$ and $\underline{\iota}$ are determined up to a Casimir function. If $\underline{\chi}_H(\underline{\iota})$ is itself a Casimir, we will call κ of *Sternberg-Weinstein* (*invariant*) type, if both are (not), of *Casimir* (*mixed*) type. Notice that only $d^2_X H_0$ needs to be nondegenerate here.

Darboux's theorem implies that *locally*, we have a symplectic inclusion $i_W : W \hookrightarrow T^*Y \cong W'$ identifying W with an open neighborhood of the zero section in T^*Y. Consequently, it becomes simultaneously a symplectic realization of U and its Sternberg-Weinstein approximation U'. The inclusion i_W allows us to assume that $K \in C^\infty(W)$ with $\rho_* X_K = X_H$ for a $H \in C^\infty(U)$ is well-defined by a metric on Y, that is, $K = K'_2$. Then, theorem 4.7 implies that we also have $\rho'_* X_K = X_{H'_2}$. In summary:

$$(U', \varpi', X_{H'_0}) \xleftarrow{(\rho', \rho'_*)} (W, \omega, X_K) \xrightarrow{(\rho, \rho_*)} (U, \varpi, X_H)$$

Hence, we can consider the Poisson structure ϖ as a nonlinear deformation of the Poisson structure ϖ', obtained by a nonlinear deformation in W of the foliation $\mathcal{F}'$ defined by ρ' to the foliation $\mathcal{F}$ defined by ρ. Equally, the Hamiltonian system defined by H appears as a nonlinear deformation of the Sternberg-Weinstein system defined by H'_2. The fact that both systems are projections of a system defined by the same metric κ restricts the possible deformations to the natural subclass. In particular, it can be shown that the equations of motion obtained in this way are natural non-linear analogues to the Wong equations, involving a non-linear Yang-Mills potential and field strength. An interesting immediate consequence is the following result.

Theorem 4.9. *Let a metric κ on Y be called* projectable *if its induced Hamiltonian vector field projects to a Hamiltonian vector field X_H on U. If Z is linearizable at the leaf $S \supset V$, then, in linear local coordinates, any Hamiltonian H obtained from a projectable metric defines equations of motion which coincide with those of its Einstein-Mayer approximated system.*

Remark 4.10. As the Lie algebras common in physics are semi-simple of compact type, or semidirect products with $\mathbb{R}$, which are C^∞-nondegenerate (cf [**3**, **2**]), we conclude that *Wong's equations are the generic Hamiltonian equations of motion obtained as a projection of geodesic equations.*

4.2. Local $\mathfrak{g}_\mathrm{L}$-principal connection forms. Let us see how E_X-connection forms are related to local $\mathfrak{g}_\mathrm{L}$-principal connection forms.

Lemma 4.11. *Suppose that* (4.4) *exists and that λ' is complete. Then, the bundle Y is a fiber bundle with homogeneous standard fiber $G_{y_0}\backslash G_1$, where G_{y_0} is the discrete stabilizer subgroup of a point $y_0 \in F$, and the G_1-action corresponds to the right action of G_1 on $G_{y_0}\backslash G_1$. In particular, if G_{y_0} is normal, then Y is a principal bundle with structure group $G = G_{y_0}\backslash G_1$. Furthermore, we have natural isomorphisms $L_X \cong \mathfrak{g}(Y)$ and $L_X^\pi \cong T^*X \times_X \mathfrak{g}^*(Y)$, with respect to the* (*co*)*adjoint action of G.*

Proof. The definition of Y implies that there are local trivializations $Y_i \cong X_i \times F$, where $\{X_i, i \in I\}$ is an open covering of X, $Y_i = \rho^{-1}(X_i)$, and $F = \rho^{-1}(z_0)$. The local G_1-action constructed in Lemma 4.6 extends to a global action if λ' is complete. Since we assumed that F was connected, this yields a locally free transitive action on F, and thus, via the choice of $y_0 \in F$, a diffeomorphism $F \cong G_{y_0}\backslash G_1$ as claimed. Note this map will be G_1-equivariant since it is induced by the restriction of a Poisson morphism. If G_{y_0} is normal, then Y becomes a principal fiber bundle with structure group G, and the bundle morphism a of Lemma 4.6 induces isomorphisms of associated bundles defined by $\mathfrak{g}(Y) \ni [y, D] \mapsto [a(y), D] \in L_X$ etc. $\square$

Lemma 4.12. *Let $\theta : V \to T^*V \otimes_V E_{|V}$ and $\alpha : X \to T^*X \otimes_X E_X$ be an E-connection form on V and an E_X-connection form on X, respectively. There are associated connections on the bundles $\rho|_{\hat{V}} : \hat{V} \to V$ and $p : Y \to X$* (4.3)*, respectively, given by the bundle morphisms*

$$\hat{\theta} = -\omega^{\flat -1} \circ \hat{\rho}^* \circ (Id_{\hat{V}}, \theta^\flat) : \quad \hat{V} \times_V TV \to T\hat{V}, \tag{4.6}$$

$$\hat{\alpha} = -\omega^{\flat -1} \circ \hat{\rho}^* \circ (Id_Y, \alpha^\flat) : \quad Y \times_X TX \to TY, \tag{4.7}$$

$$\text{where} \qquad \hat{\rho}^* : W \times_U T^*U \to (T\mathcal{F})^0 \subset T^*W \tag{4.8}$$

is the canonical bijective morphism of fibred manifolds over U induced by the surjective submersion $\rho : W \to U$, whose restriction to $Y \times_X (TX)^0$ is precisely the dual to $\hat{\rho} = (\pi, \rho')$.

Proof. By Lemma 4.2, $\mathcal{F} \cap \hat{V}$, i.e., the fibers of $\rho|_{\hat{V}} : \hat{V} \to V$, form the characteristic foliation of $\hat{V} \subset W$. Thus $\omega^{\flat -1}(T\mathcal{F})^0|_{\hat{V}}) = T\hat{V}$, and $\hat{\theta}$ is well defined. In order to show that it defines a connection in $\rho|_{\hat{V}} : \hat{V} \to V$, it remains to show that $(\tau_{\hat{V}}, T\rho) \circ \hat{\theta} = Id_{\hat{V} \times_V TV}$. Since both sides are bundle morphisms over $Id_{\hat{V}}$, this follows from $(\varpi^\sharp)_{\rho(w)} = T_w\rho \circ (\omega^{\flat -1})_w \circ (T_w\rho)^*$. Proposition 2.4 states that $\alpha = i_X^*\theta$ for some $\theta \in \mathfrak{A}$, and we have $\hat{\alpha} = \hat{\theta} \circ (i_Y, Ti_X)$. Since $(\tau_{\hat{V}}, T\rho) \circ \hat{\theta} = Id_{\hat{V} \times_V TV}$, it follows that $(\tau_Y, Tp) \circ \hat{\alpha} = Id_{Y \times_X TX}$, and this shows that $\hat{\alpha}$ takes its values in TY. Thus, $\hat{\alpha}$ defines a connection on Y. By definition, the restriction $\hat{\rho}^* : Y \times_X (TX)^0 \to (TY)^0$ is dual to $\hat{\rho} = (\pi, \rho') : TW|_Y/TY \to Y \times_X TZ|_X/TX$. □

Proposition 4.13. *Suppose that* (4.4) *exists, λ' is complete, and G_{y_0} is a normal subgroup. If $\alpha \in \mathfrak{A}_X$, then the connection on Y induced by $\hat{\alpha}$ defined in* (4.7) *is given by a principal connection form $\hat{A}$ on Y with*

$$\hat{A}^\flat = -pr_2 \circ (\hat{\lambda}^*)^{-1} \circ \omega^\flat \circ (Id_{TY} - \hat{\alpha} \circ (\tau_Y, Tp)) : \ TY \to T^*_{l_0}\Upsilon = \mathfrak{g}_{\mathrm{L}},$$

$$\text{where} \qquad \hat{\lambda}^* : W \times_\Upsilon T^*\Upsilon \to (T\mathcal{F}^\perp)^0 \subset T^*W$$

is the canonical bijective morphism of fibred manifolds over Υ induced by the surjective submersion $\lambda : W \to \Upsilon$. Furthermore, $ad \circ \hat{A}^\flat = (a^*A^\alpha)^\flat$. The $\mathfrak{g}_{\mathrm{L}}$-valued curvature 2-form $\hat{F} = d\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}]$ of $\hat{A}$ corresponds to

$$\hat{F}^\flat = -pr_2 \circ (\hat{\lambda}^*)^{-1} \circ \hat{\rho}^* \circ (\tau_Y^2, (\Phi^\alpha)^\flat \circ \wedge^2 Tp) : \ \wedge^2 TY \to T^*_{l_0}\Upsilon = \mathfrak{g}_{\mathrm{L}},$$

where $\tau_Y^2 : \wedge^2 TY \to Y$ is the natural map, and $(\Phi^\alpha)^\flat : \wedge^2 TX \to L_X$ is the bundle morphism defined by the curvature of α. Furthermore, $ad(\hat{F}) = a^*F^\alpha$, where F^α denotes the curvature 2-form of A^α.

Proof. In the proof of Lemma 4.6, we saw that the fundamental vector field induced by $D \in \mathfrak{g}_{\mathrm{L}} \subset C^\infty(\Upsilon')$ at $y \in Y$ was given by

$$D_Y(y) = X_{-\lambda'^*D}(y) = -\omega^{\flat -1} \circ \hat{\lambda}^*(y, D) \tag{4.9}$$

where $-\omega^{\flat-1} \circ \hat{\lambda}^* : Y \times T^*_{l_0}\Upsilon \to (T\hat{V})^0_{|Y} \to VY$ is a bundle isomorphism since $\hat{V} = \lambda^{-1}(l_0)$ (Lemma 4.2) is coisotropic with a characteristic foliation given by the fibers of $p = \rho|_Y$ (Lemma 4.2). But (4.9) implies that $\hat{A}(I)_Y = (Id_{TY} - \hat{\alpha} \circ (\tau_Y, Tp))(I)$ for all $I \in TY$. Thus, the vertical projection corresponding to $\hat{\alpha}$ is given by $I \to \hat{A}(I)_Y$, and $\hat{A}$ is a principal connection form corresponding to $\hat{\alpha}$ iff $\hat{\alpha}$ is equivariant, i.e., $TR_g \circ \hat{\alpha} = \hat{\alpha} \circ (R_g, Id_{TX})$ for all $g \in G$. This follows form a calculation as at the end of the proof of Lemma 4.6. The fibers of the Ad_*-equivariant bundle morphism $a : Y \to R_X$ are precisely the orbits of the center $Z = \ker Ad_* \subset G$, and thus $ad \circ \hat{A}^\flat$ is constant on these fibers. It is easy to see that $ad \circ \hat{A}$ and A^α induce the same covariant derivative on $L_X \cong \mathfrak{g}(Y)$, which implies $ad \circ \hat{A}^\flat = (a^* A^\alpha)^\flat$.

Now, the map $\hat{\alpha}^\flat \circ (\tau_Y, Tp)$ is the horizontal projection corresponding to $\hat{A}$. For $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}^1(Y)$, writing $Tp(\mathcal{X}_1)$ for $Tp \circ \mathcal{X}_1$ etc., a straightforward calculation using that $-\omega^{\flat-1} \circ \hat{\rho}^*$ is a Lie algebra isomorphism yields

$$\begin{aligned}
&(Id_{TY} - \hat{\alpha} \circ (\tau_Y, Tp))([\hat{\alpha} \circ (\tau_Y, Tp)(\mathcal{X}_1), \hat{\alpha} \circ (\tau_Y, Tp)(\mathcal{X}_2)]) \\
&\quad = -\omega^{\flat-1} \circ \hat{\rho}^* \circ (\tau_Y^2(\mathcal{X}_1 \wedge \mathcal{X}_2), (\Phi^\alpha)^\flat (Tp(\mathcal{X}_1) \wedge Tp(\mathcal{X}_2))), \\
\hat{F}^\flat(\mathcal{X}_1, \mathcal{X}_2) &= d\hat{A}(\hat{\alpha} \circ (\tau_Y, Tp)(\mathcal{X}_1), \hat{\alpha} \circ (\tau_Y, Tp)(\mathcal{X}_2)) \\
&= -\hat{A}([\hat{\alpha} \circ (\tau_Y, Tp)(\mathcal{X}_1), \hat{\alpha} \circ (\tau_Y, Tp)(\mathcal{X}_2)]) \\
&= -pr_2 \circ (\hat{\lambda}^*)^{-1} \circ \hat{\rho}^* \circ (\tau_Y^2(\mathcal{X}_1 \wedge \mathcal{X}_2), (\Phi^\alpha)^\flat (Tp(\mathcal{X}_1) \wedge Tp(\mathcal{X}_2)))
\end{aligned}$$

This last assertion follows easily. □

Theorem 4.14. *In the situation of Theorem* 4.7*, suppose that* $d^2_Y K$ *and* $d^2_X H_0$ *are nondegenerate, and that* κ *is of Sternberg-Weinstein type. With the assumptions of the proposition, let* $\hat{A}_H$ *be the principal connection form determined by* α_H*. Then* $\kappa = p^*\gamma_H + \hat{A}^*_H \iota_H$*, where* ι_H *is the scalar product on* $\mathfrak{g}_{\mathrm{L}}$ *corresponding to* $\underline{\iota} + \underline{\chi}_H$ *seen as a quadratic function on* $\mathfrak{g}^*_{\mathrm{L}}$*, provided that the last is nondegenerate. That is,* κ *is precisely the Kaluza-Klein metric* (1.2) *determined by* γ_H*,* $\hat{A}_H$*, and* ι_H*. In Theorem* 1.6*, the system* $(T^*P, \omega, K) \cong ((T^*P)', \omega', K_2')$ *is a Kaluza-Klein dynamics* (*of Sternberg-Weinstein type*) *corresponding to* $(T^*P/G, \varpi, H) \cong ((T^*P/G)', \varpi', H_1')$.

Proof. By checking the definitions (recall $\mathfrak{Z} = \mathfrak{F} \cap \mathfrak{F}^\perp$ by Lemma 4.1). □

Remark 4.15. For $\mathfrak{der}(\mathfrak{g}) = ad(\mathfrak{g})$, T^*R_X with its $Aut(\mathfrak{g})$-action and the Hamiltonian defined by the Kaluza-Klein metric determined by γ_H, A_H and some metric on $ad(\mathfrak{g})$ is a global symplectic realization of the reduced system defined by H on $Q' \subset Z'$, where Q' is the subbundle of E^*_X annihilating the subbundle of L_X whose fiber at $x \in X$ is the center of $(L_X)_x$. Since E_X-connection forms are related globally to $ad(\mathfrak{g})$- but locally to $\mathfrak{g}_{\mathrm{L}}$-valued principal connection forms, we could say that localization allows the passage from the adjoined structure group to the group itself. A similar transition appears in the duality of quantized gauge theories [**9**].

4.3. Reduced Kaluza-Klein realization. Let $i_Q : Q \to U$ be a locally closed coisotropic submanifold with $V \subset Q$. Then $i_{\hat{Q}} : \hat{Q} = \rho^{-1}(Q) \hookrightarrow W$ is obviously a coisotropic submanifold of W. Let $\hat{\mathcal{C}}$ denote the characteristic foliation of $\hat{Q}$, and suppose that there is a reduction of W via $\hat{Q}$ given by $\hat{q} : \hat{Q} \longrightarrow \hat{Q}/\hat{\mathcal{C}} =: \tilde{W}$, such that $(\tilde{W}, \tilde{\omega})$ is a symplectic manifold with reduced symplectic form $\tilde{\omega}$. Since $\hat{Q}$ is a union of leaves of the foliation $\mathcal{F}$ defined by ρ, $\hat{\mathcal{C}}$ is a subfoliation of the orthogonal foliation $\mathcal{F}^{\perp}$, for which we suppose that (4.1) exists. Consequently, there is a unique map $\tilde{\lambda} : \tilde{W} \to \Upsilon$ such that $\lambda \circ i_{\hat{Q}} = \tilde{\lambda} \circ \hat{q}$. We denote the constraints and admissible functions by $\hat{\mathfrak{C}} = \{f \in C^{\infty}(W) | f_{|\hat{Q}} = 0\}$ and $\hat{\mathfrak{N}} = \mathfrak{N}_{C^{\infty}(W)}(\hat{\mathfrak{C}})$, respectively, such that $\hat{q}^* C^{\infty}(\tilde{W}) = i^*_{\hat{Q}} \hat{\mathfrak{N}} \cong \hat{\mathfrak{N}}/\hat{\mathfrak{C}}$. Then $\mathfrak{F}^{\perp} \subset \hat{\mathfrak{N}}$, every $\lambda^* f \in \mathfrak{F}^{\perp}$ reduces to $\tilde{\lambda}^* f \in C^{\infty}(\tilde{W})$, and $\tilde{\lambda}^*$ is an isomorphism of Poisson algebras. Finally, the orbits of the flow of the Hamiltonian vector fields of $\tilde{\mathfrak{F}}^{\perp} = \tilde{\lambda}^* C^{\infty}(\Upsilon)$ are precisely given by $\tilde{\mathcal{F}} := \hat{q}(\mathcal{F} \cap \hat{Q})$. Since $\tilde{\lambda}$ is not surjective, this will be a nonregular foliation of $\tilde{W}$ in general, and the image of $\tilde{\rho} : \tilde{W} \longrightarrow \tilde{U} := \tilde{W}/\tilde{\mathcal{F}}$ might not be a manifold.

However, we can define $C^{\infty}(\tilde{U}) = (\tilde{\rho}^*)^{-1} C^{\infty}(\tilde{W})$, and an isomorphic subalgebra of $C^{\infty}(\tilde{W})$ by $\tilde{\mathfrak{F}} = \mathfrak{Z}_{C^{\infty}(\tilde{W})}(\tilde{\mathfrak{F}}^{\perp}) = \tilde{\rho}^* C^{\infty}(\tilde{U})$. On the other hand, $\hat{Q} = \rho^{-1}(Q)$ also implies that $\mathfrak{C} \cong \hat{\mathfrak{C}} \cap \mathfrak{F}$ and $\mathfrak{N} \cong \mathfrak{N}_{C^{\infty}(W)}(\hat{\mathfrak{C}} \cap \mathfrak{F}) \cap \mathfrak{F} = \hat{\mathfrak{N}} \cap \mathfrak{F}$, and thus, $\tilde{U}$ can be identified with the image of the projection $q : Q \to \tilde{Z} = Q/\mathcal{C}$. With (3.1) (for $\tilde{U}$ instead of $\tilde{Z}$), we get the commutative diagrams:

$$\begin{array}{ccccc} \Upsilon & \xleftarrow{\lambda} & \hat{Q} & \xrightarrow{\rho} & Q \\ \tilde{\lambda} \nwarrow & & \big\downarrow \hat{q} & & \big\downarrow q \\ & & \tilde{W} & \xrightarrow{\tilde{\rho}} & \tilde{U} \end{array} \qquad \begin{array}{ccc} i^*_{\hat{Q}}(\hat{\mathfrak{N}} \cap \mathfrak{F}) & \xleftarrow{\sim} & \tilde{\mathfrak{P}} \\ \big\uparrow \wr & & \big\uparrow \wr \\ \tilde{\mathfrak{F}} & \xleftarrow{\sim} & C^{\infty}(\tilde{U}) \end{array} \tag{4.10}$$

Lemma 4.16. *Let $K \in C^{\infty}(W)$ be a Hamiltonian such that $\rho_* X_K = X_H$ for some Hamiltonian $H \in C^{\infty}(U)$. Then, K is admissible for $\hat{Q}$ iff H is admissible for Q. In this case, the reduced Hamiltonian $\tilde{K} \in C^{\infty}(\tilde{W})$ is given as $\tilde{K} = \tilde{\rho}^* \tilde{H}_{\tilde{U}} + \tilde{\lambda}^* J$, where $\tilde{H} = q^* \tilde{H}_{\tilde{U}}$, and $\tilde{\rho}^* \tilde{H}_{\tilde{U}}$ is precisely the reduction of $\rho^* H$ by $\hat{Q}$.*

Proof. Lemma 4.1 gives us the decomposition $K = \rho^* H + \lambda^* J$, where $J \in C^{\infty}(\Upsilon)$. $\mathfrak{N} \cong \hat{\mathfrak{N}} \cap \mathfrak{F}$ implies that $\rho^* H$ is admissible for $\hat{Q}$ iff H is admissible for Q. On the other hand, $\mathfrak{F}^{\perp} \subset \hat{\mathfrak{N}}$ shows that every $\lambda^* J \in \mathfrak{F}^{\perp}$ is admissible for $\hat{Q}$. This implies the remaining assertions. ☐

Hence, we may think of the dynamics induced by $\tilde{K}$ on $\tilde{\mathfrak{F}}$ as a "symplectic realization" of the reduced dynamics induced by H on $\tilde{\mathfrak{P}}$.

Proposition 4.17. *Let $\hat{Q} = \rho^{-1}(Q) \subset W$ and $(\tilde{W}, \tilde{\omega})$ be as above. Then $i_{\hat{Q}'} : \hat{Q}' = T\hat{Q}|_Y/TY \to W'$ is a coisotropic submanifold. There is a reduced manifold of W' by $\hat{Q}'$ given by the Sternberg-Weinstein approximation of $\tilde{W}$ at $q(Y)$, that is, by $\tilde{W}' = T\tilde{W}_{|\tilde{Y}}/T\tilde{Y} \cong T^*\tilde{Y}$, where $\tilde{Y} := \hat{q}(Y) \subset \tilde{W}$ is a Lagrangian submanifold, and the identification with $T^*\tilde{Y}$ is given by $(\tilde{\omega}^\flat|_{T\tilde{Y}})^*$. The projection defined by the characteristic foliation $\hat{\mathcal{C}}'$ of $\hat{Q}'$ is given by the map $\hat{q}' : \hat{Q}' \to \tilde{W}'$ naturally induced by $T\hat{q}|_Y$. Furthermore, the restriction of local G_1-action of lemma* 4.6 *to $\hat{Q}'$ projects to a local G_1-action on $\tilde{W}'$ which is the Hamiltonian lift of the restricted projected action on $\tilde{Y}$. There is a natural bijection $\tilde{\rho}' : \tilde{U}' \xrightarrow{\sim} Q'/\mathcal{C}'$ of the orbit space $\tilde{U}'$ with the set of leaves of the subcharacteristic distribution $\mathcal{C}'$ of Q'.*

Proof. Since $\hat{Q}' = (\rho')^{-1}(Q')$, $\hat{Q}'$ is coisotropic. Furthermore, in local Darboux coordinates providing an identification $W \cong W' \cong T^*Y$, it is easy to see that $\ker \omega'|_{\hat{Q}'} = T\hat{\mathcal{C}}|_Y/TY = \ker \hat{q}'$, and obviously, $\hat{q}'(\hat{Q}') = T\tilde{W}|_{\tilde{Y}}/T\tilde{Y} = \tilde{W}'$, where $\tilde{Y} = \hat{q}(Y)$ is a Lagrangian submanifold as the reduced image of a Lagrangian submanifold of W. If $\lambda' : W' \to \mathfrak{g}_{\mathrm{L}}^*$ exists, we see that there is an induced map $\tilde{\lambda}' : \tilde{W}' \to \mathfrak{g}_{\mathrm{L}}^*$ such that $\lambda' \circ i_{\hat{Q}'} = \tilde{\lambda}' \circ \hat{q}'$. Thus, every $(\lambda')^*D, D \in \mathfrak{g}_{\mathrm{L}}$ yields a reduced Hamiltonian $(\tilde{\lambda}')^*D$ on $\tilde{W}'$, and as in the proof of Lemma 4.6, this yields a local G_1-action which is given by the unique Hamiltonian lift of the action on $\tilde{Y}$, and at the same time the projection of the restricted G_1-action on $\hat{Q}'$. However, this action will not be locally free if $\tilde{\lambda}'$ is not surjective. The last assertion follows from the commutativity of (4.10); in fact, it yields a Sternberg-Weinstein approximated analogue of these diagrams. □

Proposition 4.18. *The orbits of the restricted projected G_1-action on $\tilde{Y}$ define a regular foliation of $\tilde{Y}$. Under the hypotheses of Lemma* 4.11 *and Proposition* 4.13*, $\tilde{Y}$ is a fiber bundle over X with homogeneous fibers diffeomorphic to $C_{\mathrm{L}}\backslash G$, where C_{L} is the analytic subgroup of G defined by $\mathfrak{c} \subset \mathfrak{g}$. The projected G-action in given by the natural right action of G on $C_{\mathrm{L}}\backslash G$.*

Proof. Lemma 3.2 implies that $T\mathcal{C}$ and $T\hat{\mathcal{C}}'$ are spanned at each point by the Hamiltonian vector fields of the functions $\tilde{s}$ and $(\rho')^*\tilde{s}$ for all $s \in \Gamma(A_X)$, respectively. Since $V' = S' \cap U' \subset Q'$ and thus $(TQ')^0_{|V'} \subset (TV')^0$, the Hamiltonian vector field of $\tilde{s}$ vanishes on X. Thus, that of $(\rho')^*\tilde{s}$ must be tangent the fibers of ρ' on Y. On the other hand, and as in the proof of Lemma 4.6, we see that the restriction to Y_x depends only on $s(x) \in (A_X)_x$ for all $x \in X$ and corresponds to an infinitesimal Lie algebra action of $(A_X)_x$ on Y_x. Fixing a point $y \in (A_X)_x$, we get (local) isomorphisms $G \cong Y_x$ and $a(y) : \mathfrak{c} \cong (A_X)_x$. Under these identifications, the fiber of $\hat{\mathcal{C}}'$ through y is locally given by the orbit of the local C_{L}-action obtained by integrating the infinitesimal $\mathfrak{c}$-action on G, that is, the natural left action, since $\mathfrak{c} \subset \mathfrak{g}$

corresponds to *right* invariant vector fields on G. Thus, the G_1-orbits in $\tilde{Y} = Y/(\hat{\mathcal{C}}' \cap Y)$ are locally isomorphic to $C_{\mathrm{L}}\backslash G$, and thus, the foliation is regular. Under the additional hypotheses, $\tilde{Y}$ is a fiber bundle over X with homogeneous fibers isomorphic to $C_{\mathrm{L}}\backslash G$. □

Theorem 4.19. *Let $H \in \mathfrak{N} \subset C^\infty(U)$ such that $dH|_X = 0$ be admissible for Q, and $K \in C^\infty(W)$ such that $dK|_Y = 0$ and $\rho_* X_K = X_H$. Then, the reduction of the Hamiltonian defining the Einstein-Mayer system of K on W' is precisely the Hamiltonian $\tilde{K_2'}$ defining the Einstein-Mayer system of the reduced Hamiltonian $\tilde{K}$ of K on $\tilde{W}'$. With our nondegeneracy assumptions,*

$$\tilde{K_2'} = \underline{\tilde{\kappa}} = (\tilde{\rho}')^* \underline{\tilde{\eta}} + (\tilde{\lambda}')^* \underline{\iota}, \tag{4.11}$$

where $\underline{\tilde{\kappa}}$ and $(\tilde{\rho}')^* \underline{\tilde{\eta}} \in \tilde{\mathfrak{F}}'$ correspond to a metric and a G_1-invariant metric for the (local) G_1-action of Lemma 4.6 on $\tilde{Y}$, and $\underline{\iota}$ to a scalar product on $\mathfrak{g}_{\mathrm{L}}$. Here, $\underline{\tilde{\eta}} \in \tilde{\mathfrak{P}}'$ is seen as a function on $\tilde{U}'$, analogous to diagram (4.10).

Proof. Theorem 4.7 yields a decomposition $\underline{\kappa} = (\rho')^* \underline{\eta} + (\lambda')^* \underline{\iota}$. Applying Lemma 4.16 in the Sternberg-Weinstein approximated situation, we obtain the decomposition (4.11) if the reduction of $\underline{\kappa} = K_2'$ is given by $\underline{\tilde{\kappa}} = \tilde{K_2'}$, which follows easily from $\hat{q}^* d\tilde{K} = i^*_{\hat{Q}} dK$. The remaining claims follows from the nondegeneracy assumption and Proposition 4.17, respectively. □

Remark 4.20. Theorems 3.8 and 4.19 yield the reduction theorem for invariant metrics on bundles with homogeneous fibers ([**4**]).

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