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THE $\boldsymbol{N}$-COPY OF A TOPOLOGICALLY TRIVIAL LEGENDRIAN KNOT.

K.Mishachev

We consider Legendrian knots and links in the standard 3-dimensional contact space. In 1997 Chekanov [Ch] introduced a new invariant for these knots. At the same time, a similar construction was suggested by Eliashberg [E1] within the framework of his joint work with Hofer and Givental on Symplectic Field Theory ([E2],[EGH]). To a knot diagram, they associated a differential algebra A. Its stable isomorphism type is invariant under Legendrian isotopy of the knot.

In this paper, we introduce an additional structure on this algebra in the case of a Legendrian link. For a link of N components, we show that its algebra splits $A = \oplus_{g\in G} A_g$. Here G is a free group on $(N-1)$ variables. The splitting is determined by the order of the knots and is preserved by the differential. It gives a tool to show that some permutations of link components are impossible to produce by Legendrian isotopy.

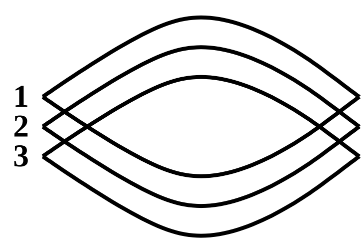

Figure 1: 3-copy of the Legendrian unknot.

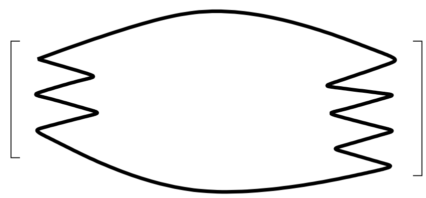

Figure 2: Topologically trivial Legendrian knots.

For example, take three copies of the Legendrian unknot as is shown in Fig. 1. Then the permutation 123 to 231 is possible but 123 to 132 is not. This is the simplest case of the following:

Theorem. *Take a topologically trivial Legendrian knot and shift it N times in the direction transversal to the contact planes. (The shift is small and hence well-defined.) Only cyclic permutations of this link are possible via Legendrian isotopy.*

We prove this theorem in Section 5.

Notes. Topologically trivial Legendrian knots are classified in [EF] and they are shown in Fig. 2. The Legendrian unknot corresponds to $s = r = 1$.

The result about three copies of the Legendrian unknot can be also derived from the results of Traynor [Tr].

1. Preliminaries and the splitting theorem.

1.1. Link diagrams. We work in the standard contact space $(R^3, dz - ydx)$. A Legendrian link is a curve tangent to the contact planes. A Legendrian isotopy is an isotopy via Legendrian links. By Gray's stability theorem, it is equivalent to an ambient isotopy of the contact manifold.

There are two ways to represent a Legendrian link by a 2-dimensional diagram. One can project the knot either to the xy or xz planes. The images are called the plane diagram and the front diagram (or the wavefront) respectively. A wavefront is a smooth curve with cusps and nowhere vertical tangent lines. The knot is completely determined by its wavefront since the y coordinate equals the slope of the tangent line. A plane diagram is a smooth curve of zero area and it determines the knot up to translations along the z-axis. More details can be found in [A][B][EF][FT] and other sources.

For each intersection point on a plane diagram, we include the information of which branch is the overcrossing into the diagram. The result will be a topological link diagram with additional area restrictions. This area information is not essential: two Legendrian links with the same topological diagram are Legendrian isotopic.

In the next two subsections, we briefly recall the construction of the differential algebra from [Ch].

1.2. Admissible disks. The construction of the differential algebra of the Legendrian link is based on the holomorphic disks in the symplectization $R^3 \times R$ (with respect to a special choice of almost complex structure). In [Ch] the disks are visualized as immersions of regular n-gons to R^2. However, in order to describe the link splitting, it is convenient to visualize the disks in R^3.

Consider a Legendrian link L in R^3 with a regular plane diagram. For every point of self-intersection of the diagram, draw a vertical segment in R^3 that joints two points of the link. This segment is called a Reeb chord.

We call a disk $D^2 \to R^3$ admissible if ∂D^2 is mapped to the link with Reeb chords, and the projection of the interior of D^2 onto R^2 is an immersion. The boundary of an admissible disk consists of link arcs and Reeb chords. Near every Reeb chord the image of the projection of the disk forms an angle. We require the angle to be convex ($< \pi$). This is also an admissibility condition. We orient an admissible disk so that it has negative area in R^2. Its boundary is oriented counterclockwise.

Two admissible disks are homotopic if they are homotopic as admissible disks. Let Imm be the corresponding set of homotopy classes.

1.3. Differential algebra. To every Reeb orbit assign a letter a_i. More precisely, assign a letter a_i^+ to an upward-oriented vertical segment and a_i^- to a downward-oriented vertical segment.

To a disk in Imm there corresponds a word in the alphabet $\langle a_i^+, a_i^- \rangle$. It is the product of all letters of the Reeb orbits in ∂D^2. We pick up a_i^+ or a_i^- depending on the orientation of ∂D^2. The word is determined up to cyclic permutation.

Definition 1.3a. A Hamiltonian H is a formal sum of words associated to all disks in Imm.

Note that monomials of H are defined up to cyclic permutations. To be precise, H is an element of the algebra $C\langle a_i^{\pm}\rangle$ of cyclic words generated by letters $a_i^{\pm}$.

Let $A = \mathbb{Z}_2\langle a_i^-\rangle$ be a free associative $\mathbb{Z}_2$-algebra with unit, generated by letters a_i^-. Define the differential ∂ by

$$\partial(a_i^-) = \frac{\partial}{\partial a_i^+} H|_{a_j^+=0\ \forall j}$$

The right part of the formula is an element of A, despite the fact that H is not. By $\frac{\partial}{\partial a_i^+} H|_{a_j^+=0\ \forall j}$ we mean the following. Take words from H with exactly one positive letter a_i^+. Represent them by noncyclic words that begin with a_i^+ and then differentiate, i.e., drop a_i^+. In other words, when we take a derivative of a cyclic word (i.e. drop a letter) the result is not cyclic anymore, since dropped letter marks the beginning of the word.

Extend ∂ to the rest of the algebra by the Leibniz rule.

Definition 1.3b. (A, ∂) is called the differential algebra of a Legendrian link.

Note. To be precise, we should call it the differential algebra of a plane diagram. However, for all links in this paper it is clear which diagram is meant.

Definition 1.3c. The stabilization SA of a differential algebra A is a free product $A \coprod S$. Here $S = \mathbb{Z}_2\langle a, b\rangle$ with $\partial a = b,\ \partial b = 0$.

Theorem 1.3d. [Ch] $\partial \circ \partial = 0$. *Differential algebras of the Legendrian isotopic Legendrian links are stable isomorphic, $S^n A = S^m A'$ for some n, m.*

Notes. In fact, Chekanov proves that A and A' are stable *tame* isomorphic. We do not need the notion of *tame* isomorphism in this paper.

The differential algebra is called *stably trivial* if it is stably isomorphic to the algebra with generators $a_0 \dots a_n$ and differential $\partial a_0 = 1$, $\partial a_i = 0$ for $i > 0$.

In [ENS] it is shown how to define the algebra over $\mathbb{Z}[t, t^{-1}]$. It is done by assigning proper signs to the disks.

There is an additional structure on the algebra, the Maslov class grading. We do not consider it here.

1.4. Splitting theorem. For a link L, consider the relative homotopy group $G_N = \pi_1(R^3/L)$. This is a free group on $(N-1)$ variables, where N is the number of components of the link.

Theorem 1.4a. *The differential algebra A splits, $A = \bigoplus_{g\in G_N} A_g$. The differential ∂ preserves this splitting. Algebras of isotopic Legendrian links are componentwise stable isomorphic,*

$$\bigoplus_{g\in G_N} S^n A_g = \bigoplus_{g\in G_N} S^m A'_g$$

In particular, a permutation σ of components of the Legendrian link by Legendrian isotopy induces a componentwise automorphism,

$$\bigoplus_{g\in G_N} S^n A_g \to \bigoplus_{\sigma g\in G_N} S^m A_{\sigma g}$$

Notes. The permutation σ acts naturally on $\pi_1(R^3/L) = G_N$. We will discuss this action in Section 1.5. below.

The stabilization SA is now defined as $A \coprod \mathbb{Z}_2\langle a, b\rangle$ with $g(a) = g(b)$.

We call $g(w)$ a *link degree* of w.

Proof. Each vertical segment represents an element of G_N. Denote it by $g(a_i^{\pm})$. Extend g to the whole algebra by multiplicativity. It gives the splitting $A = \oplus_{g\in G_N} A_g$.

Consider a word $a_i^+ a_j^- \dots a_k^-$ of the Hamiltonian. The corresponding disk in R^3 gives a homotopy to the identity, it implies $g(a_i^+ a_j^- \dots a_k^-) = 1$ or $g(a_i^+)g(a_j^- \dots a_k^-) = 1$. Now $g(a_i^-) = g(a_i^+)^{-1} = g(a_k^- \dots a_l^-)$. We proved that every monomial in ∂a_i^- has the same link degree as a_i^-. Therefore, ∂ preserves the link degree.

To prove existence of componentwise isomorphism it is necessary to repeat step-by-step Chekanov's original proof and check that the splitting is preserved at every step. The most subtle point is to check that all elementary automorphisms used in [Ch] preserve the splitting. $\square$

1.5. The group G_N. Enumerate componentss of a link by $1\dots N$. Denote an arc from component i to component j by $[i \to j]$. Also, set $[i \to i] = 1$. G_N is generated by elements $[i \to j]$ that are subject to relations $[i \to j][j \to k] = [i \to k]$. In particular, $[i \to j][j \to i] = 1$. The permutation σ acts by $\sigma[i \to j] = [\sigma(i) \to \sigma(j)]$. The group G_N is freely generated by $(N-1)$ generators $[1 \to i]$ but this way we lose symmetry.

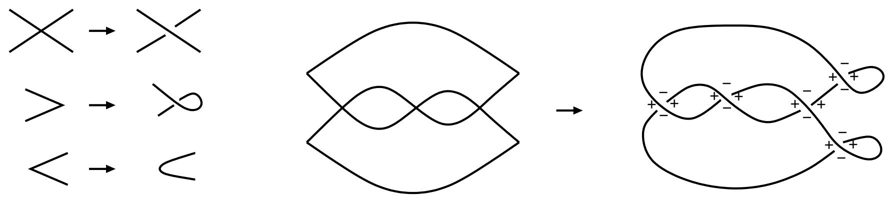

Figure 3: Wavefront resolution.

One can abelianize $G_N = \pi_1(R^3/L)$ to get the relative homology group $H_1(R^3/L) = \mathbb{Z}^{N-1}$. Consider an injective map $m : H^1(R^3/L) \to \mathbb{Z}^N$ defined by $m([i \to j]) = e_i - e_j$ where e_i are basis vectors. Then the action of a permutation group S_N on $H^1(R^3/L)$ is just the standard action of S_N on Z^N.

We established the splitting theorem and now it is convenient to use the original terminology from [Ch]. In the rest of the paper, disks are immersed n-gons in the plane and Reeb orbits are positive and negative disk corners. Also, we write a_i instead of a_i^- and denote by Imm_1 the subset of Imm that consists of disks with one positive corner.

2. Algebras.

2.1. Computing Imm_1. Given a plane diagram, it is a nontrivial problem to find all the disks. It is much easier to do it from a wavefront. This approach was introduced in [Ng].

Here is a way to convert a wavefront to a plane diagram.

Proposition 2.1a. (see [Ng]) *Resolve intersections and cusps of a wavefront as shown in Fig. 3. The result is a plane diagram of a Legendrian link that is Legendrian isotopic to the original.* □

We keep this resolution in mind and draw the disks directly on a wavefront.

A wavefront is called *simple* if all right cusps have the same x-coordinate. Moving all right cusps to the right makes any front simple.

Definition 2.1b. A node of a wavefront is either a self-intersection point *or* a right cusp.

Proposition 2.1c.([Ng]) Imm_1 *of a simple wavefront consists of* ***embedded*** *disks of the following type. Take a left cusp and a node and connect them by two strands. The strands go directly from left to the right, their*

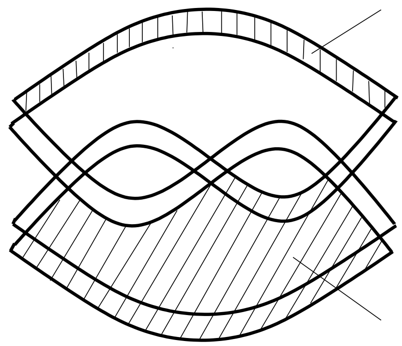

Figure 4: Thick and thin disks.

projections onto x are embedded intervals. The disk is the region between the strands. We require all corners to be convex. Also, for every right cusp there is a 1-vertex disk which is not seen on the wavefront diagram. $\square$

We call a 1-vertex disk a *unit* disk, since it gives rise to a unit term in the differential. Also, we call a unit disk *small* if it corresponds to the resolution of a right cusp, and *big* otherwise. For example, a Legendrian unknot has exactly two unit disks: a small one and a big one.

2.2. Disks of a general N-copy.

Definition 2.2a. Shift a Legendrian knot N times in the transversal direction. The shift is small. The result is a link of N components. It is called the N-copy of the initial knot.

A wavefront of the N-copy is an original wavefront shifted (a little) N times in the z-direction. To each intersection point of the original wavefront there corresponds a junction of N^2 intersection points of the N-copy. Each left or right cusp gives a junction of $N(N-1)/2$ points.

Let us start with a simple wavefront. Consider a *stick-together* map $s : R^2 \to R^2$ with the following properties. Draw the N-copy of a knot in the first R^2 and the knot itself in the second R^2. The map s takes every copy of the knot to the knot itself, as well as the space between them. It is a diffeomorphism outside.
A disk of the original wavefront gives rise to a family of disks of its N-copy. Namely, for every corner of the original disk, we choose some corner in the corresponding junction.

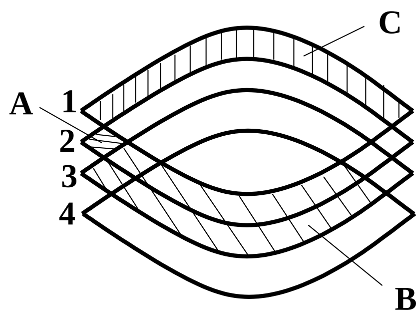

Figure 5: N-copy of the Legendrian unknot.

Definition 2.2b. A disk D of the N-copy is called *thick* if $s(D)$ is a disk of the initial knot.

Here we require $s|_{\partial D}$ to be injective. It means that for every vertex of $s(D)$ there is only one vertex of D in the corresponding junction. Therefore all thick disks come from the original knot, as described above.

For every smooth path connecting a node and a left cusp of the initial knot, there is a family of strip-like disks of the N-copy. Also, there are square-like disks in the left cusp junctions.

Definition 2.2c. A disk D of the N-copy is called *thin* if $s(D)$ lies in the wavefront.

A thin disk always has exactly four corners. One of them is a left cusp. Therefore thin disks are either strip-like or square-like, as described above.

Note. Small unit disks are considered to be neither thick nor thin.

Proposition 2.2d. *Disks of the N-copy of a knot with a simple wavefront are either thick or thin or small unit disks.*

Proof. If a disk has only one consecutive corner per junction it is thick. If there are two consecutive corners at the same junction then it is forced to be thin. $\square$

Note. Proposition 2.2d is true for a wavefront that is not necessarily simple, but has no disks with negative right cusp vertices. However, for generic wavefronts the situation is more complicated.

2.3. N-copy of the Legendrian unknot. Consider a circle with N marked points, $1 \dots N$. Assign a letter a_{ij} to the clockwise oriented arc from i to j. The letter a_{ii} corresponds to an arc of length 2π.

Definition 2.3a. The *circular* algebra O_N is a free algebra of N^2 variables $\mathbb{Z}_2\langle a_{ij}\rangle$ with a differential defined as:

$$\partial a_{ij} = \sum_{\text{k between i and j on the circle}} a_{ik}a_{kj}$$

In other words, the range of k is $i < k < j$ or $k < j \leq i$ or $j \leq i < k$.

Proposition 2.3b. *The differential algebra of the N-copy of the Legendrian unknot is O_N. Link degree is $g(a_{ij}) = [i \to j]$.*

Proof. The Legendrian unknot has only two unit disks: one big and one small. There are $2N$ unit disks in the N-copy. They cancel each other in the differential. The remaining disks are thin. Disks of types A,B,C in Fig. 5 correspond to the ranges $i < k < j$, $k < j \leq i$ and $j \leq i < k$ respectively. □

2.4. Interval algebras. Consider the algebra of a piece of the wavefront in Fig. 6. Rays from i^{th} and j^{th} left cusps intersect at node a_{ij}, where $1 \leq i < j \leq K$ and $j - i \leq N$. The differential ∂ is

$$\partial a_{ij} = \sum_{i<k<j} a_{ik}a_{kj} \qquad \text{for } j - i < N$$

$$\partial a_{ij} = 1 + \sum_{i<k<j} a_{ik}a_{kj} \qquad \text{for } j - i = N$$

Definition 2.4a. This is the *interval* algebra $I_N(K)$.

One can visualize $I_N(K)$ as an interval with K marked points. Variables are subintervals of length $\leq N$. Note that $I_N(N)$ is also well defined and has no units in its differential.

Consider a free product $I_N(K) \coprod I_N(L)$. Identify interval ends of length N. Namely, take the quotient by relations $a_{ij} = a'_{ij}$ and $a_{K+1-i,K+1-j} = a'_{L+1-i,L+1-j}$ for $i, j \leq N$. The differential is still well-defined.

Definition 2.4b. This is the *double interval* algebra $\mathbb{I}_N(K, L)$.

We can visualize it as two intervals with glued ends. See Fig. 7B.

Definition 2.4c. A *circular* algebra $O_N(K)$ is defined to be $\mathbb{I}_N(K + N, N)$.

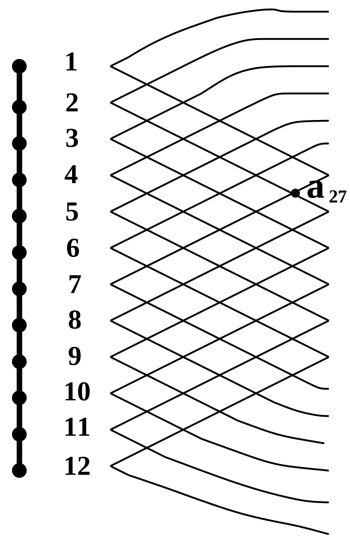

Figure 6: Interval algebra $I_6(12)$.

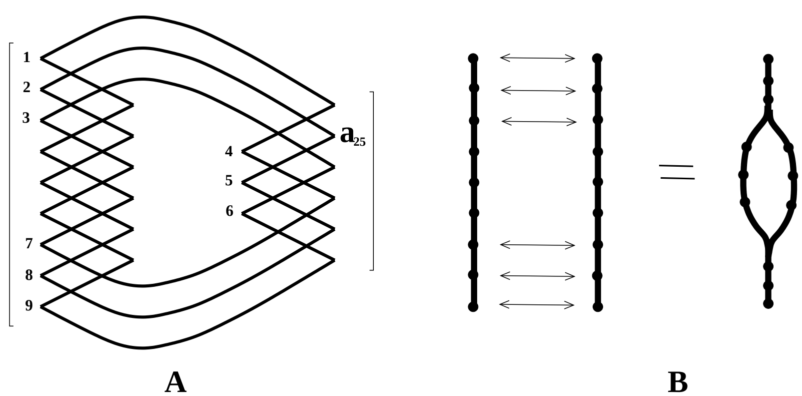

Figure 7: Double interval algebra $I_3(9,9)$.

This is a generalization of a definition from the previous section, $O_N(N) = O_N$. We can visualize $O_N(K)$ as an algebra of arcs of length $\leq N$ in a circle with K marked points. See Fig. 8.

2.5. N-copies of topologically trivial Legendrian knots. These are shown in Fig. 2.

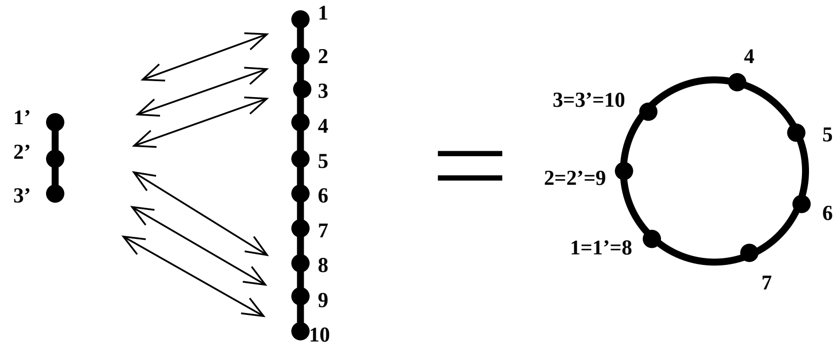

Figure 8: Circular algebra.

Proposition 2.5a. *The differential algebra of the N-copy of a topological unknot with r left cusps and s right cusps is $\mathbb{I}_N(rN, sN+N)$ as soon as $s > 1$. Link degree is $g(a_{ij}) = [i \operatorname{mod} \mathrm{N} \to \mathrm{j} \operatorname{mod} \mathrm{N}]$.*

Proof. When $s > 1$ a wavefront has no disks except small unit disks. The N-copy has only thin disks. The diagram of the N-copy is given by Fig. 7A with $p = rN$, $q = sN$. The algebra $I_N(rN)$ corresponds to the left side of Fig. 7A. To see how the right side of the picture corresponds to $I_N(sN+N)$ we mark the left cusps by $1 \dots (sN+N)$ as it is shown. Now a'_{ij} is a point of intersection of rays from i^{th} and j^{th} left cusps. One can check that all the disks fit together to give a correct differential. $\square$

2.6. N-copies of negative torus knots. Legendrian torus knots are classified in [EH]. They are divided into two categories: positive and negative. All negative torus knots are shown in Fig. 7A. with $(N, p+q) = 1$. When $(N, p+q) \neq 1$ Fig. 7A gives $(N, p+q)$-copy of a negative torus knot.

Proposition 2.6a. The algebra of a link shown in Fig. 7A is $\mathbb{I}_N(p, q+N)$ as soon as $q \geq 2N$.

We do not consider torus links in this paper and omit the proof of the proposition. However, it is likely that our methods can be used to prove the following:

Conjecture 2.6b. Only cyclic permutations are possible for the N-copy of a negative torus knot.

3. Algebraic tools.

We need some computable invariants of differential algebras that survive stabilization. The homology ring $H = \operatorname{Ker}\partial/\operatorname{Im}\partial$ is difficult to compute. (A, ∂) can be viewed as a vector field on a non-commutative affine space. Thus, the invariants of (A, ∂) are subject to algebraic geometry language rather than homology groups language. Zeroes of a vector field are called augmentations and we study their local invariants. Augmentations and corresponding linear homology groups were introduced in the original paper [Ch] and studied further in [F].

3.1. Local homology groups.

Definition 3.1a. An augmentation ϵ of A is a ring homomorphism $\epsilon : A \to \mathbb{Z}_2$ such that $\epsilon \circ \partial = 0$ and $\epsilon(1) = 1$.

Alternatively, an augmentation is a zero of the vector field ∂ or a maximal differential ideal $A^1 = \operatorname{Ker}\epsilon$ so that $A = 1 \oplus A^1$ and $\partial A^1 \subset A^1$.

The homomorphism ϵ is defined as soon as we know all $\epsilon(a_i)$. Those are the coordinates of a point. Change the coordinate system via $a_i := a_i + \epsilon(a_i)$. Now ϵ is an augmentation if in the new coordinate system ∂ has no constant terms (units) i.e. the vector field vanishes at the origin.

For the rest of the section, we assume that A has some fixed augmentation with kernel A^1. Also, we substitute $a_i := a_i + \epsilon(a_i)$ and assume $\epsilon(a_i) = 0$ for all generators of A.

Power ideals $A^N = (A^1)^N$ are preserved by ∂ as well. The augmentation gives rise to the power filtration $A = A^0 \supset A^1 \supset A^2 \ldots \supset A^N \supset \ldots$. It is the usual filtration of A by word length.

Consider $\mathbb{T} = A^1/A^2$. It is a $\mathbb{Z}_2$-vector space spanned by a_i. The differential ∂ induces the differential $\partial_1 : \mathbb{T} \to \mathbb{T}$. It is exactly the linear part of ∂.

Definition 3.1b. The linear homology group H^1 is $\operatorname{Ker}\partial_1/\operatorname{Im}\partial_1$.

For any $i < j$ there is a space A^i/A^j with the induced differential and homology group $H^{i,j}$. In particular, $H^{1,2} = H^1$ and $H^{0,\infty} = H$.

There is also a system of inclusion/surjection maps $A^i/A^j \to A^{i'}/A^{j'}$ for $i \geq i', j \geq j'$. They are compositions of inclusions $A^i/A^j \to A^{i'}/A^j$ and surjections $A^i/A^j \to A^i/A^{j'}$. The induced maps $i_* : H^{i,j} \to H^{i',j'}$ are also well defined.

Definition 3.1c. We call the groups $H^{i,j}$ together with homomorphisms i_* *a system of local homology groups* that corresponds to the augmentation ϵ.

For an augmentation ϵ of A there is an augmentation $S\epsilon$ of $SA = A\coprod \mathbb{Z}_2\langle a, b\rangle$ defined by $\epsilon(a) = \epsilon(b) = 0$.

Proposition 3.1d. *The system of local homology groups survive stabilization, i.e.,* $H^{i,j}(A) = H^{i,j}(SA)$ *and these isomorphisms commute with the maps* i_*.

Proof. In [Ch] the invariance of H is proved by means of the chain homotopy operator h (see Lemma 2.2. in [Ch]). The same technique works here. One has to check that h commutes with inclusion/surjection maps and preserves the power filtration. □

Notes. We will use only the map $i_* : H \to H^1$.

If we think about an augmentation ϵ as a zero (or a singular point) of the vector field, then the system of local homology groups is an invariant of the stable geometric type of this zero.

The Maslov class grading makes $\mathbb{T}$ a graded vectorspace. Then instead of the dimension of H^1 there is a Poincaré polynomial.

The invariance of H^1 can be also proved as follows. Stabilization SA raises the dimension of $\mathbb{T}$ by 2. The differential is $\partial_1 a = b,\ \partial_1 b = 0$, so the homology is preserved.

3.2. Augmentation classes. Unfortunately, for every augmentation ϵ of A there are two augmentations for SA. One is $S\epsilon$ the other is $\epsilon(a) = 1,\ \epsilon(b) = 0$. They are equivalent but it causes some trouble in definitions.

Definition 3.2a. Two augmentations ϵ and ϵ' are called (stably) equivalent if there is a differential preserving automorphism $\alpha : S^nA \to S^nA$ with $S^n\epsilon' = S^n\epsilon \circ \alpha$. A set of equivalent augmentations forms *an augmentation class*.

An augmentation class is the geometrical type of a zero up to stabilization. All zeros of SA come from zeros of A, but nonequivalent zeroes of A can become equivalent in SA. In fact, the only way to check that two zeroes are not stably equivalent is to compute their local invariants, i.e. homology groups.

Proposition 3.2b. *If A and A' are stably isomorphic then there is a one-to-one correspondence between their augmentation classes and isomorphisms of corresponding systems of local homology groups. In particular, the number of augmentation classes is an invariant.* $\square$

3.3. Splitting and proper augmentations. If the knot under consideration is a link, then everything splits: $H = \oplus_g H_g$, $\mathbb{T} = \oplus_g \mathbb{T}_g$, $H^1 = \oplus_g H_g^1$ etc. For $g = [i \to j]$ we will use shorter notations like $H_{[i \to j]} = H_{ij}$, $\mathbb{T}_{[i \to j]} = \mathbb{T}_{ij}$ etc.

The algebra of a link has another class of differential ideals besides the augmentation ideals. Namely, denote by $\bar{A}_g$ the two-sided differential ideal generated by A_g. $\bar{A}_g$ is a proper subset for $g \neq 1$.

Let L be a link formed by knots K_i.

Proposition 3.3a. *There is a natural homomorphism $A(L) \to \coprod_i A(K_i)$.*

Proof. This is exactly the quotient map $A(L) \to A(L)/\oplus_{g \neq 1} \bar{A}_g(L)$. $\square$

Definition 3.3b. An augmentation ϵ is called proper if $\epsilon(A_g) = 0$ for $g \neq 1$.

Proposition 3.3c. *Proper augmentations of the link are in one-to-one correspondence with augmentations of its knot components.*

Proof. It follows from the previous proposition. $\square$

3.4. Characteristic algebra. There is an alternative approach to the problem of distinction of differential algebras.

Definition 3.4a. [Ng] A characteristic algebra of A is $C(A) = A/\mathrm{Im}\,\partial$.

The characteristic algebra has relations. The stabilization $SC(A)$ is defined as $SC(A) := C(SA)$. It is $C(A)$ with one new generator added and no new relations. There are some algebraic properties of $C(A)$ that survive stabilization. For example,

Proposition 3.4b. *Both $C(A)$ and $SC(A)$ have divisors of zero or neither have.* $\square$

More examples and applications of characteristic algebras can be found in [Ng]. In particular, the Legendrian mirror problem of [FT] is solved there.

3.5. Reduced algebra. Consider a differential algebra A with a fixed proper augmentation. Denote the intersection of the augmentation ideal A^1 with A_1 by A_0. Thus, $A = \oplus_{g\neq 1} A_g \oplus A_0 \oplus 1$. Also, let $\bar{A}_0$ be the corresponding differential ideal.

Definition 3.5a. A reduced algebra $\tilde{A}$ is $A/\bar{A}_0$.

The reduced algebra $\tilde{A}$ is generated by the letters of A with the link degree $\neq 1$. It is not free. The products of the generators that have unit link degree are equal to 0. These are all the relations. Here is an alternative description of the linear homology and $\mathbb{T}_g$ for $g = [i \to j]$.

Proposition 3.5b. $1 \oplus \mathbb{T}_{ij} = \tilde{A} / \bigoplus\limits_{g\neq[i\to j],\neq 1} \bar{A}_g$.

Proof. All generators of link degree $\neq [i \to j]$ disappear. If $g(a) = g(b) = [i \to j]$ then $g(ab) = [i \to j]^2 \neq [i \to j]$ hence the quadratic terms also disappear. $\square$

3.6. ijk**-localization.** The reduced algebra has a filtration by powers of A_g. For our purposes it is enough to exploit only the simplest nonlinear term of this filtration.

Definition 3.6a. Let $\gamma = \{[i \to j], [j \to k], [i \to k]\}$. The ijk-localization for A is the space $\mathbb{T}_{ijk}$ defined by $1 \oplus \mathbb{T}_{ijk} = \tilde{A} / \bigoplus\limits_{g\notin\gamma,\neq 1} \bar{A}_g$.

Proposition 3.6b. *$\mathbb{T}_{ijk}$ is generated by generators of A with link degrees from γ. The only quadratic elements are products of generators of degree $[i \to j]$ and $[j \to k]$. As a $\mathbb{Z}_2$-vector space, $\mathbb{T}_{ijk}$ is isomorphic to $\mathbb{T}_{ij} \oplus \mathbb{T}_{jk} \oplus \mathbb{T}_{ik} \oplus (\mathbb{T}_{ij} \otimes \mathbb{T}_{jk})$. The differential ∂_{ijk} is obtained from ∂ by erasing all nonlinear terms except products ab with $g(a) = [i \to j]$, $g(b) = [j \to k]$.* $\square$

Note. We can now define $H_{ijk} = \operatorname{Ker} \partial_{\mathrm{ijk}} / \operatorname{Im} \partial_{\mathrm{ijk}}$. These groups, and their generalizations, are well-defined invariants of the stable type, but they are beyond the scope of this paper. We use a more computable characteristic algebra approach.

3.7. Characteristic ijk**-algebra.** We call $\mathbb{T}_{ijk}$ a *free* ijk-algebra. It has no relations except those dictated by the link degree. The stabilization operation SA induces an obvious stabilization operation $S\mathbb{T}_{ijk}$ of free ijk-algebras. We need some invariants that survive stabilization.

Definition 3.7a. The characteristic ijk-algebra CH_{ijk} is $\mathbb{T}_{ijk} / \operatorname{Im} \partial_{\mathrm{ijk}}$.

The characteristic algebra is not necessarily free. The stabilization operation $S(CH_{ijk})$ is the addition of one new generator of any appropriate link degree.

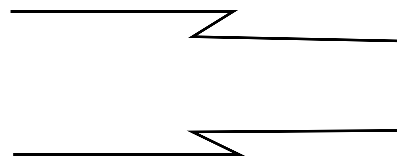

Figure 9: Kinks.

Definition 3.7b. An ijk-algebra has no divisors of zero if $ab = 0$ only by degree reasons. In other words, $ab \neq 0$ as soon as $g(a) = [i \to j],\ g(b) = [j \to k]$.

Proposition 3.7c. *Both CH_{ijk} and $S(CH_{ijk})$ have divisors of zero or neither have.* □

4. N-copies.

4.1. N-copy alternative.

Proposition 4.1a. *For any link L either*

1) *All permutations of the N-copy of L are possible by Legendrian isotopy, and that is when all permutations of the 3-copy are possible, or*
2) *Only cyclic permutations are possible, and that is when 12 to 21 is possible and 123 to 132 is not, or*
3) *No permutations are possible.*

Proof. If we can exchange knots of the 2-copy, we can exchange a knot and its $(N-1)$-copy. It gives a cyclic permutation of the N-copy. If there is a non-cyclic permutation of the N-copy we can drop all the knots but three of them to get a non-cyclic permutation of the 3-copy. Given a permutation 123 to 132 one can generate any permutation of the N-copy. □

Notes. The proposition is true in any contact manifold. The main theorem of this paper states that topologically trivial knots are of type 2. The result of [Tr] can be viewed as an example of type 3 knot in $J^1(S^1)$. Recently, L. Ng produced an example of type 3 knot in R^3.

4.2. Kinked knots and cyclic permutations. One can add a positive or negative zigzag to the Legendrian knot K (see Fig. 9.) The result is a kinked knot $Z^{\pm}(K)$.

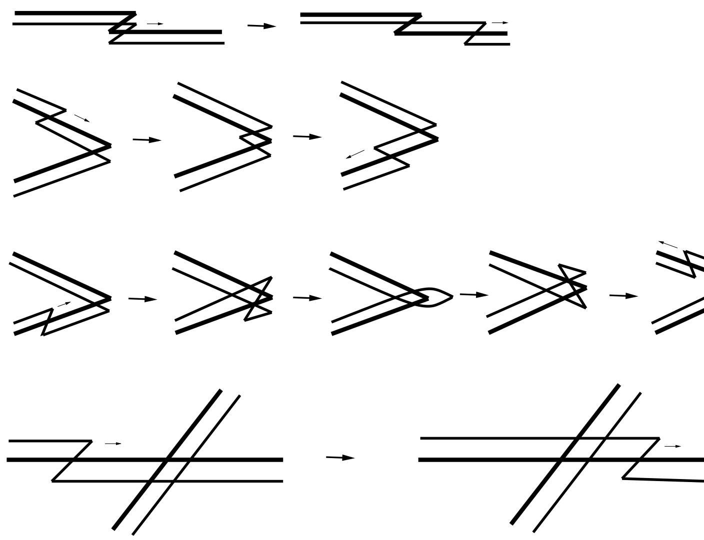

Figure 10: Zigzag journey.

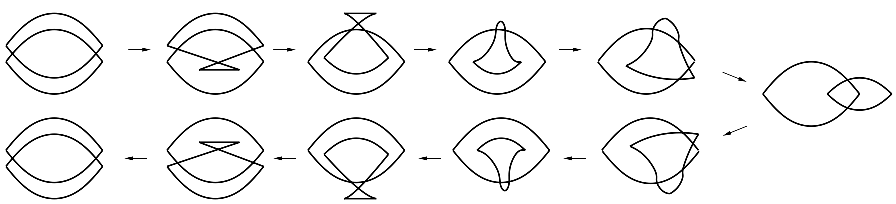

Figure 11: Permutation of the 2-copy of the Legendrian unknot.

Proposition 4.2a. *Cyclic permutations of the N-copy of the kinked knot $Z^{\pm}(K)$ are possible.*

Proof. Fig. 10. shows how the zigzag travels through the 2-copy of the knot to exchange the knots. It is shown how the zigzag goes through the cusps and the intersection points. □

Proposition 4.2b. *Cyclic permutations of the N-copy of the Legendrian unknot are possible.*

Proof. The permutation 12 to 21 is shown in Fig. 11. Alternatively, consider the torus $|z_1|^2 + |z_2|^2 = 1,\ \mathrm{Im}\, z_1/z_2 = 0$ in the contact sphere $S^3 \subset \mathbb{C}^2$. It is transversal to the contact structure and is foliated by Legendrian unknots. □

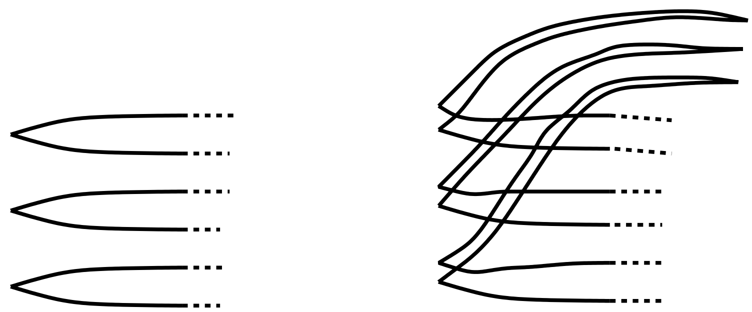

Figure 12: Getting rid of big unit disks.

4.3. N-copy trick. If two Legendrian knots are Legendrian isotopic, then so are their N-copies (and vice versa). We can consider invariants of the N-copy instead of invariants of the initial knot. Call it *the N-copy trick*. The N-copy trick is not completely new: the Thurston-Bennequin invariant can be defined as a linking number of a 2-copy.

The differential algebra of a Legendrian knot has two flaws. First, it gives no invariants for kinked knots since such a knot has a stably trivial algebra. This is because $\partial(a) = 1$, where a is a letter of the right cusp of a zigzag. One can prove that it implies stable triviality. Second, it can have no augmentations at all and then it is difficult to deal with (see however [Ng]).

However, the differential algebra of the N-copy of a Legendrian knot is not necessarily stably trivial. The next proposition shows that 2-copies (and hence all $2N$-copies) always have augmentations.

Proposition 4.3a. *A 2-copy of any Legendrian link has at least one augmentation.*

Proof. Consider the initial link. All units in the differential come from unit disks, which are either big or small. A Legendrian isotopy does not alter the existence of the augmentation. Move all right cusps to the right and all left cusps to the left. Then perform the operation shown in Fig. 12. Now there are no big unit disks. Indeed, such a disk is formed by an intersection point of two branches of the link from one left cusp point. But these branches do not intersect at all in Fig. 12.

Consider a 2-copy of the altered link. It has no big unit disks either. Every cusp (left or right) gives a single node b_i of degree $[1 \to 2]$ or $[2 \to 1]$. Let $\epsilon(b_i) = 1$ for those nodes and $\epsilon(a_i) = 0$ for all other nodes.

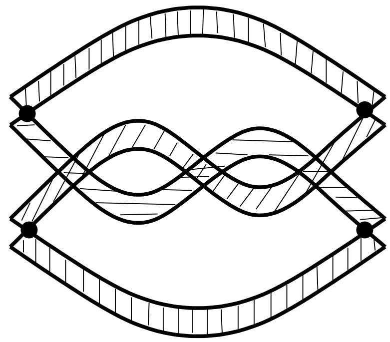

Figure 13: Ruling of the 2-copy.

In Fig. 13. nodes b_i are marked. Now ϵ is an augmentation. Indeed, for every small unit disk there is a thin disk with only b_i vertices, as it is shown in Fig. 13. Hence all units in the differential cancel out. $\square$

Note that thin disks in the Fig. 13. form *a ruling* [F]. We can also use the result of [F] ("ruling implies augmentation") to prove the proposition. Then the preliminary step in Fig. 12. is not unnecessary.

Notes. We saw that the flaws of the differential algebra of the Legendrian knot may be resolved by applying the N-copy trick. The disadvantage of this approach is that the new algebra is much bigger. The advantage is that the link degree g gives a lot of additional structure. Now we have the differential ideals $\bar{A}_g$, filtration by powers of A_g and groups like H_{ijk}, CH_{ijk} to work with.

To produce even more invariants of Legendrian knots, a generalization of the N-copy trick was proposed by the author [M]. It was called the *satellite construction* in [Ng2].

It is likely that these techniques may be used to find practical invariants of stabilized Legendrian knots, though this still seems to be a delicate and difficult problem.

5. Proof of the theorem.

5.1. Legendrian unknot.

Theorem 5.1a. *Only cyclic permutations of the N-copy of the Legendrian unknot are possible.*

Note that they are indeed possible; see Section 4.2. We give two proofs of this theorem. One uses homology groups. The other is a model proof for the general case.

Proof. The algebra is O_N; see Section 2.3. There is a unique proper augmentation class since the Legendrian unknot has one augmentation class. It is given by $\epsilon(a_{ij}) = 0$. The differential has trivial linear part, $\partial_1 = 0$. The homology groups are:

$$H^1 = \mathbb{Z}_2^{N^2}$$
$$H_1^1 = \mathbb{Z}_2^N$$
$$H_{ij}^1 = \mathbb{Z}_2$$

Lemma 5.1b. *The homomorphism $i : H_{ij} \to H_{ij}^1$ is not trivial when $j = i + 1$ or $i = N, j = 1$. It is trivial otherwise.*

Proof. In the first case, $\partial a_{ij} = 0$. Hence a_{ij} represents an element of H_{ij} and $i(a_{ij}) \neq 0$.

In the second case, let s be a representative of a homology class of H_{ij} with $i(s) \neq 0$. Then $s = a_{ij} + w$ where $w \in A^2$. Since $\partial_1 = 0$ we have $\partial(A^2) \subset A^3$ and $\partial w \in A^3$. Now $\partial s = \partial a_{ij} + \partial w = 0$ hence $\partial a_{ij} \in A^3$. This is a contradiction. $\Box$

It follows that Legendrian isotopy should preserve the cycle $1 \to 2 \to \ldots \to N \to 1$ and hence the isotopy may only cyclically permute the components of the N-copy. $\Box$

Another proof of 5.1a. Consider O_3. It is enough to show that the permutation 123 to 132 is impossible. Take $\mathbb{T}_{123}$ and $\mathbb{T}_{132}$. Both have three generators, one in each degree. But $\partial_{123}(a_{13}) = a_{12}a_{23}$ while $\partial_{132} \equiv 0$. Hence characteristic algebra CH_{123} has divisors of zero while CH_{132} has none. $\Box$

5.2. Topologically trivial Legendrian knots.

Theorem 5.2a. *Only cyclic permutations are possible for the N-copy of a topologically trivial Legendrian knot.*

Proof. All these knots are shown in Fig. 2. The case $s = r = 1$ was considered in the previous section. Cyclic permutations are possible since they are kinked knots; see Section 4.2.

The z-mirror of the knot is its reflection along z axis. It is enough to prove the theorem for the z-mirror of the knot. Therefore, we can assume $s > 1$.

We want to prove that the permutation 123 to 132 is impossible for the 3-copy. The algebra of the 3-copy is $\mathbb{I}_3(3r, 3s+3)$; see Section 2.5. This algebra has no proper augmentations since the initial knot has none. We use the 2-copy trick, as recommended in Section 4.3., to obtain the augmentation.

Proposition 5.2b. $\mathbb{I}_2(2r, 2s+2)$ *has a unique (not proper) augmentation class. It is given by* $\epsilon(a_{i,i+1}) = \epsilon(a'_{i,i+1}) = 1$ *and* $\epsilon(a_{i,i+2}) = \epsilon(a'_{i,i+2}) = 0$.

Proof. $\partial a_{i,i+2} = 1 + a_{i,i+1}a_{i+1,i+2}$ implies $\epsilon(a_{i,i+1}) = 1$. The value of $\epsilon(a_{i,i+2})$ is not important since there are automorphisms $a_{i,i+2} := 1 + a_{i,i+2}$. $\square$

We can subdivide a link into components that are links themselves. The splitting theorem and all related considerations still work. Consider the 3-copy of the 2-copy of the knot. In other words, consider its 6-copy subdivided into three 2-copy parts. The algebra is $\mathbb{I}_6(6r, 6s+6)$ with the link degree given by $g(a_{ij}) = [i \bmod 3 \to \mathrm{j} \bmod 3]$. We want to show that permutation 123 to 132 is impossible for this algebra.

Note that the algebra $\mathbb{I}_6(6r, 6s+6)$ (viewed as a 3-copy) has a unique proper augmentation. It follows from Proposition 5.2b ($\mathbb{I}_2$ has a unique augmentation) and Proposition 3.3c (proper augmentations of a link are induced by augmentations of its components). Note that the augmentation of 5.2b becomes proper when we view the 2-copy as one component. The following proposition finalizes the proof of the theorem.

Proposition 5.2c. $CH_{123}(\mathbb{I}_6(6r, 6s+6))$ *has divisors of zero, while* $CH_{132}(\mathbb{I}_6(6r, 6s+6))$ *has none.*

Proof. We will evaluate CH_{123} and CH_{132} graphically. The sample picture of the interval algebra $I_6(12)$ is shown in Fig.14. A node at the intersection of the rays from two left cusps marked i and j has degree $[i \to j]$. $\mathbb{T}_{ijk}$ is generated by the nodes grouped into triangles.

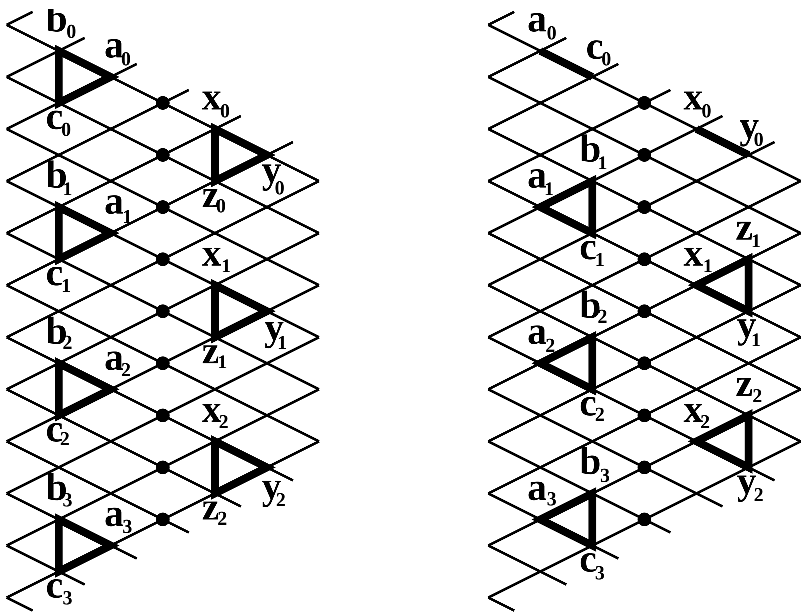

Figure 14: $I_6(12)$

Factoring by the augmentation ideal $\bar{A}_0$ means that circled nodes are equal to 1 since ϵ is equal to 1 there. All other nodes are equal to zero.

Every term of the differential ∂ is represented by a parallelogram. Nonzero terms of ∂_{ijk} come from two types of parallelograms. The first type is a parallelogram with 3 vertices at the triangles. It gives a quadratic term. The second type is a parallelogram with 2 vertices at the triangles and one circled vertex. It gives a linear term.

These calculations show that $\mathbb{T}_{123}(I_6(12))$ is generated by a_0, b_0, c_0, $a_i, b_i, c_i,\ x_i, y_i, z_i,\ i = 1 \ldots 3$. The differential is given by the formulae:

$$\partial b_i = \partial c_i = 0$$
$$\partial a_i = b_i c_i$$
$$\partial x_i = b_i + b_{i+1}$$
$$\partial z_i = c_i + c_{i+1}$$
$$\partial y_i = a_i + a_{i+1} + x_i c_{i+1} + b_i z_i$$

That means $b_0 = b_1 = b_2 = b_3 = b$ and $c_0 = c_1 = c_2 = c_3 = c$ in the CH_{123} and the relation $bc = 0$ gives divisors of zero. Note that $b, c \neq 0$ because *all* relations in CH_{123} are given by the right parts of the displayed formulae.

$\mathbb{T}_{132}(I_6(12))$ is generated by a_0, c_0, a_i, b_i, c_i, $i = 1 \dots 3$ and x_0, y_0, x_i, y_i, z_i, $i = 1, 2$. The differential is given by:

$$\begin{aligned}
&\partial a_i = \partial b_i = \partial c_i = 0 \\
&\partial y_i = c_i + c_{i+1} \\
&\partial z_i = b_i + b_{i+1} \\
&\partial x_i = a_i + a_{i+1} + c_i b_{i+1}
\end{aligned}$$

For the characteristic algebra CH_{132}, this means that $b_1 = b_2 = b_3 = b$ and $c_0 = c_1 = c_2 = c_3 = c$. Also, $cb = a_0 + a_1 = a_1 + a_2 = a_2 + a_3$ which implies $a_0 = a_2 = a$ and $a_1 = a_3 = a'$. The algebra CH_{132} is generated by $a, a', b, c, x_i, y_i, z_i$ with only one relation $cb = a + a'$. It has no divisors of zero. Indeed, let $pq = 0$, with $g(p) = [1 \to 3]$, $g(q) = [3 \to 2]$. Then p and q are sums of linear terms. The product consists of quadratic terms with at most one term cb. This product cannot be zero.

The calculations for the general case $\mathbb{I}_6(6r, 6s + 6)$ are similar to those performed above. $CH_{ijk}(\mathbb{I}_6(6r, 6s + 6))$ is obtained from $CH_{ijk}(I_6(6r))$ and $CH_{ijk}(I_6(6s + 6))$ by variable identification. Still, CH_{123} has divisors of zero and CH_{132} has none. $\square$

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