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Inversion sequences avoiding a pair of vincular patterns of type $(2,1)$

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An *inversion sequence* of length n is a sequence of integers $e = e_0 \cdots e_n$ which satisfies $0 \leq e_i \leq i$, for all $i = 0, 1, \ldots, n$. We say that e avoids a pattern ab-c of type $(2,1)$ if does not exist i, j such that $0 \leq i < j-1 \leq n-1$ and the subsequence π_i, π_{i+1}, π_j has the same order isomorphic as abc. For a set of patterns B, let $\mathbf{I}_n(B)$ be the set of inversion sequences of length n that avoid all the patterns from B. We say that two sets of patterns B and C are *I-Wilf equivalent* if $|\mathbf{I}_n(B)| = |\mathbf{I}_n(C)|$, for all $n \geq 0$. In this paper, we show that the number of I-Wilf equivalences among pairs of patterns of type $(2,1)$ is 72. In particular, we present connections to Bell numbers, ascent sequences, and permutations avoiding length-4 vincular pattern.

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1. Introduction

An *inversion sequence of length* n is a word $e = e_0 \cdots e_n$ such that $0 \leq e_i \leq i$, for all $i = 0, 1, \ldots, n$. We denote the set of inversion sequences of length n by $\mathbf{I}_n$. The study of pattern-avoiding inversion sequences initiated in [4, 10] for length-3 patterns. Later, these works extended the notion of pattern avoidance to binary relations, and pairs/triples of length-3 patterns (see [3, 8, 12, 11, 14]).

A *reduction* of a word $\sigma = \sigma_1 \cdots \sigma_k$ is the word obtained by replacing the i-th smallest entry of σ with $i-1$, for all $i = 1, 2, \ldots, k$. Here, we denote the reduction of σ by $red(\sigma)$. For example, the reduction of $\sigma = 22404$ is $red(\sigma) = 11202$. Let abc be any pattern over alphabet $\{0, 1, 2\}$, we say that $e = e_0 e_1 \cdots e_n \in \mathbf{I}_n$ *avoids* a pattern ab-c of type $(2,1)$ if does not exist i, j such that $0 \leq i < j-1 \leq n-1$ and $red(\pi_i, \pi_{i+1}, \pi_j) = abc$. For a set of patterns B, let $\mathbf{I}_n(B)$ be the set of inversion sequences of length n that avoid all the patterns from B. We say that two sets of patterns B and C are *I-Wilf-equivalent*, denoted $B \overset{\mathbf{I}}{\sim} C$, if $|\mathbf{I}_n(B)| = |\mathbf{I}_n(C)|$, for all $n \geq 0$.

As an extension of [4, 10], Lin and Yan [9] and Auli and Elizalde [1] considered the case of patterns of type $(2,1)$. Among other results, they showed that $\{01\text{-}0\} \overset{\mathbf{I}}{\sim} \{01\text{-}1\}$ and $\{10\text{-}0\} \overset{\mathbf{I}}{\sim} \{10\text{-}1\}$. The aim of this paper is to reprove again these results and as well as give all the I-Wilf-equivalences for pairs of patterns of type $(2,1)$. Here, we will use the algorithmic approach based on generating trees developed in [8] to obtain the generating tree for $\mathbf{I}_n(B)$. More precisely, the main result of this paper can be formulated as follows.

Theorem 1.1. *There are exactly* 72 *I-Wilf equivalences among pairs of patterns of type* $(2,1)$.

Note that there are 78 pairs of patterns of type $(2,1)$. Thus, there are only 6 I-Wilf equivalences such that each has only two pairs. Throughout the proof of Theorem 1.1, we enumerate each pair in these 6 I-Wilf equivalences and we show Table 1.

Table 1: Enumeration pairs in the 6 I-Wilf equivalences

B Pair of patterns of type $(2,1)$	$\sum_{n\geq 0} \lvert\mathbf{I}_n(B)\rvert x^{n+1}$	Reference
$\{01\text{-}1, 01\text{-}2\}$,$\{01\text{-}0, 01\text{-}2\}$	$\frac{x}{1-x} + \sum_{j\geq 0} \frac{x^{2j+2}}{(1-x)^{j+2}\prod_{i=0}^{j}((1-x)^i - x)}$	Theorem 3.2
$\{01\text{-}1, 02\text{-}1\}$,$\{01\text{-}0, 02\text{-}1\}$	$\frac{1-2x-\sqrt{1-4x}}{2x}$	Theorem 3.3
$\{01\text{-}0, 12\text{-}0\}$,$\{01\text{-}0, 10\text{-}1\}$	$\sum_{j\geq 1} \frac{x^j}{\prod_{i=1}^{j-1}(1-ix)}$	Theorem 3.4
$\{10\text{-}1, 20\text{-}1\}$,$\{10\text{-}0, 20\text{-}1\}$	$\sum_{n\geq 0} \lvert\mathbf{I}_n(\{101, 201\})\rvert x^{n+1}$	Theorem 3.5
$\{10\text{-}1, 21\text{-}0\}$,$\{10\text{-}0, 21\text{-}0\}$	See [13, Sequence A137538]	Theorem 3.6
$\{00\text{-}0, 10\text{-}0\}$,$\{00\text{-}0, 10\text{-}1\}$	See [13, Sequence A138265]	Theorem 3.8

The paper is organized as follows. In Section 2, we recall the algorithmic approach based on generating trees developed in [8] which leads to the main result of this paper. In Section 3, we reprove $\{01\text{-}0\} \overset{\mathbf{I}}{\sim} \{01\text{-}1\}$ and $\{10\text{-}0\} \overset{\mathbf{I}}{\sim} \{10\text{-}1\}$, and then we prove Theorem 1.1, as described in Table 1.

2. Inversion sequences and generating trees

Following [8], we define the generating tree (see [15]) $\mathcal{T}(B)$ to be a plain tree as follows. Let $\mathbf{I}_B = \cup_{n=0}^{\infty}\mathbf{I}_n(B)$. Clearly, the tree $\mathcal{T}(B)$ is empty whenever

$0 \in B$. Otherwise, the root can always be taken as 0. Starting with this root which stays at level 0, we construct the remainder of the nodes of the tree $\mathcal{T}(B)$ as follows: the children of $e_0e_1\cdots e_n \in \mathbf{I}_n(B)$ are obtained from the set $\{e_0e_1\cdots e_ne_{n+1} \mid e_{n+1} = 0, 1, \ldots, n+1\}$ by obeying the pattern-avoiding restrictions of the patterns in B.

Let $D(B)$ be the set of all nodes of $\mathcal{T}(B)$. We denote the subtree consisting of the inversion sequence e as the root and its descendants in $\mathcal{T}(B)$ by $\mathcal{T}(B; e)$. For any $e, e' \in D(B)$, we say that e is *equivalent* to e' if and only if

$$\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$$

(in the sense of plain trees). Let $\mathcal{T}'(B)$ be the same tree $\mathcal{T}(B)$ where we replace each node e by the first node $e' \in \mathcal{T}(B)$ from top to bottom and from left to right in $\mathcal{T}(B)$ such that $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$. From now, we identify $\mathcal{T}'(B)$ with $\mathcal{T}(B)$.

Example 2.1. *Let $B = \{00\text{-}0, 01\text{-}2\}$. Clearly, the children of $0 \in \mathcal{T}(B)$ are 00 and 01. The children of 00 are 000, 001 and 002. By obeying the pattern-avoiding restrictions of the patterns in B, we see there are two children 001 and 002. Note that $\pi = 002\pi' \in \mathbf{I}_n(B)$ if and only if $01\pi''$ where π'' obtained from π' by decreasing each letter by 1, so $\mathcal{T}(B; 002) \cong \mathcal{T}(B; 01)$. Thus, the children of 00 in $\mathcal{T}(B)$ are 001 and 01. So, up to now, we have two rules $0 \rightsquigarrow 00, 01$ and $00 \rightsquigarrow 001, 01$. Similarly, we see that $01 \rightsquigarrow 010, 011$; $001 \rightsquigarrow 0011$; $010 \rightsquigarrow 011, 01$ and $011 \rightsquigarrow 001$. Note that there are no children for the node $0011 \in \mathcal{T}(B)$ because all of its children, namely 00110, 00111, 00112, 00113, 00114 does not avoid B.*

Example 2.2. *Let $B = \{00\text{-}1, 02\text{-}1, 12\text{-}0\}$, then we see that $\mathcal{T}(B)$ can be presented by nodes $a_m = (01)^m$, $b_m = (01)^m0$, $c_m = (01)^m02$, $d_m = (01)^m020$, $e = 0110$, $f = 011$, and $g = 00$. More precisely, the rules of $\mathcal{T}(B)$ are given by*

$$\begin{array}{ll} b_0 \rightsquigarrow g, a_1, & g \rightsquigarrow g, \\ f \rightsquigarrow h, f, & h \rightsquigarrow g, f, \\ a_m \rightsquigarrow b_m, f, a_m, c_{m-1}, \ldots, a_2, c_1, a_1, & b_m \rightsquigarrow g, a_{m+1}, c_m, \ldots, a_2, c_1, a_1, \\ c_m \rightsquigarrow d_m, f, b_m, a_m, \ldots, b_1, a_1, & d_m \rightsquigarrow g, c_{m+1}, a_{m+1}, \ldots, c_1, a_1. \end{array}$$

Since the similarity, let us prove 4 of these rules:

- *$b_0 \rightsquigarrow g, a_1$: holds because all the children of $b_0 = 0$ are $00 = g$ and $a_1 = 01$.*

- $g \rightsquigarrow g$*: since the only inversion sequence* $\pi = 00\pi'$ *avoids* B *is* $\pi = 00\cdots0$*, we obtain that the rule holds.*
- $a_1 \rightsquigarrow b_1, f, a_1$*; the children of* a_1 *are* $010 = b_1$*,* $011 = f$*, and* 012*. Note that* $\pi = 012\pi'$ *is an inversion sequence avoids* B *if and only if* $01\pi''$ *is an inversion sequence avoids* B*, where* π'' *obtained from* π' *by subtracting* 1 *from each letter, so* $\mathcal{T}(B; 012) \cong \mathcal{T}(B; a_1)$*. Thus, the rule holds.*
- $d_m \rightsquigarrow g, c_{m+1}, a_{m+1}, \ldots, c_1, a_1$*; By ordering/removing letters of* $\pi = d_m j\pi' \in \mathbf{I}_n(B)$ *with* $j = 0, 2, 3, \ldots, 2m+3$*, we have that* $\mathcal{T}(B; d_m 0) \cong \mathcal{T}(B; g)$*,* $\mathcal{T}(B; d_m(2s)) \cong \mathcal{T}(B; c_{m+2-s})$ *with* $s = 1, 2, \ldots, m+1$*, and* $\mathcal{T}(B; d_m(2s+1)) \cong \mathcal{T}(B; a_{m+2-s})$ *with* $s = 1, 2, \ldots, m+1$*, which implies that the rule holds.*

For given a rule $v \rightsquigarrow v_1, \ldots, v_s$, v is called a *father* and $v_1, \ldots, v_s$ are called *children* of v. In [8] the authors described an algorithm on how to guess and prove the generating tree $\mathcal{T}(B)$ for given a set of pattern B. Briefly, the algorithm is working as follows:

Algorithm KMY:

- Given $D \geq 1$ (Usually, we take D to be small number).
- By computer programming, we can find the generating tree $\mathcal{T}_D(B)$, which is the same tree $\mathcal{T}(B)$ up to level D. Let $R_D(B)$ all the rules of $\mathcal{T}_D(B)$.
- We say that a set of rules $R_1, R_2, \ldots, R_s$ can be written by one index rule $R^{(i)}$, if $R^{(i)} = R_i$ for all $i = 1, 2, \ldots, s$. In this case, we say the set $\{R_1, R_2, \ldots, R_s\}$ is minimize to one index rule $R^{(i)}$. Then, consider any subset R' of $R_D(B)$ and check if R' minimizes to a one index rule ${R'}^{(i)}$. If yes, define $R_D(B)$ to be $(R_D(B)\backslash R') \cup \{{R'}^{(i)}\}$. Otherwise, move to the next step.
- Consider the rules of $R_D(B)$ and try to prove (as done in Example 2.2), by considering children of each father in a rule, that the rules of $R_D(B)$ are exactly the rules of $\mathcal{T}(B)$.

We end this section, by presenting one example for finding the generating tree $\mathcal{T}(\{01\text{-}0, 02\text{-}1\})$. By applying the algorithm for $D = 6$, we obtain the following rules

$$\begin{aligned}
&0 \rightsquigarrow 00, 0,\\
&00 \rightsquigarrow 000, 00, 0,\\
&000 \rightsquigarrow 0000, 000, 00, 0,\\
&0000 \rightsquigarrow 00000, 0000, 000, 00, 0,
\end{aligned}$$

$$00000 \rightsquigarrow 000000000000000000000$$
$$000000 \rightsquigarrow 0000000, 000000, 00000, 0000, 000, 00, 0.$$

So, it is easy to see that this set of rules can be minimized to a one index rule $a_m \rightsquigarrow a_{m+1}, \ldots, a_1$ with $a_m = 0^m$. Here, for a symbol k and an integer d, the constant sequence $k, k, \ldots, k$ of length d is denoted by k^d. To prove the rule, we have to consider only the children of a_m, which are $a_m j$ with $j = 0, \ldots, m$. Note that the inversion sequence $\pi = a_m j \pi'$ avoids B if and only if $a_{m+1-j}\pi'^{(j)}$, where $\pi'^{(j)}$ is a word obtained from π' by subtracting j from each letter of π', which implies that $\mathcal{T}(B; a_m j) \cong \mathcal{T}(B; a_{m+1-j})$ for all $j = 0, 1, \ldots, m$. Hence, the rule $a_m \rightsquigarrow a_{m+1}, \ldots, a_1$ holds, and the generating tree $\mathcal{T}(B)$ satisfies only this rule.

Before we end this section, we state the following observation that is used in Section 3. We define $B \overset{g}{\sim} B'$ whenever $\mathcal{T}(B) = \mathcal{T}(B')$. So, by the definitions, we have the following observation.

Observation 2.3. *Let B, B' be any two sets of patterns. If $B \overset{g}{\sim} B'$ then $B \overset{\mathbf{I}}{\sim} B'$.*

3. Patterns of type (2,1) in inversion sequences

As an application of Algorithm KMY, in the next subsections, we present all the I-Wilf equivalences among single patterns of type $(2, 1)$ and among pairs of patterns of type $(2, 1)$.

3.1. Single pattern

As mentioned in the introduction, Lin and Yan [9] and as well as Auli and Elizalde [1] showed that $\{01\text{-}0\} \overset{\mathbf{I}}{\sim} \{01\text{-}1\}$ and $\{10\text{-}0\} \overset{\mathbf{I}}{\sim} \{10\text{-}1\}$. Algorithm KMY gives new proof for these facts.

Theorem 3.1. *We have*

(1) $\{01\text{-}0\} \overset{g}{\sim} \{01\text{-}1\}$. *Moreover, the rules of the generating tree $\mathcal{T}(\{01\text{-}0\})$ are given by*

$$a_m \rightsquigarrow a_{m+1}, a_m, b_{m,2}, \ldots, b_{m,m},$$
$$b_{m,j} \rightsquigarrow a_{m+1}, b_{m+1,2}, \ldots, b_{m+1,j}, b_{m,j}, \ldots, b_{m,m},$$

where $a_m = 0^m$ and $b_{m,j} = 0^m j$ with $2 \leq j \leq m$.

(2) $\{10\text{-}0\} \stackrel{g}{\sim} \{10\text{-}1\}$. *Moreover, the rules of the generating tree* $\mathcal{T}(\{10\text{-}0\})$ *are given by*

$$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m},$$
$$b_{m,j} \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,j-1}, b_{m+1,j}, \ldots, b_{m+1,m+1},$$

where $a_m = 0^m$ *and* $b_{m,j} = 0^m j$ *with* $1 \le j \le m$.

3.2. Pairs of patterns

By finding all the sequences $|\mathbf{I}_n(B)|_{n=0}^{9}$ whenever B is pair of patterns of type $(2,1)$, we present Table 2.

Table 2: Number inversion sequences in $\mathbf{I}_n(B)$, where B is a pair of patterns of type $(2,1)$

Beginning of Table 2					
Class	B	$\lvert\mathbf{I}_n(B)\rvert$	Class	B	$\lvert\mathbf{I}_n(B)\rvert$
1	{00-1, 01-0}	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	2	{00-1, 01-2}	1, 2, 3, 5, 8, 13, 21, 34, 55, 89
3	{00-1, 01-1}	1, 2, 3, 6, 13, 35, 109, 394, 1611, 7387	4	{00-0, 00-1}	1, 2, 3, 9, 33, 158, 919, 6279, 49273, 436517
5	{00-1, 10-1}	1, 2, 4, 10, 29, 102, 422, 2025, 11040, 67324	6	{00-0, 01-0}	1, 2, 4, 10, 29, 98, 378, 1644, 7971, 42692
7	{00-1, 12-0}	1, 2, 4, 10, 30, 109, 468, 2300, 12650, 76508	8	{00-1, 10-0}	1, 2, 4, 10, 32, 124, 571, 3035, 18197, 121147
9	{00-1, 11-0}	1, 2, 4, 10, 34, 154, 874, 5914, 46234, 409114	10	{00-1, 02-1}	1, 2, 4, 11, 36, 137, 586, 2742, 13791, 73538
11	{00-1, 21-0}	1, 2, 4, 11, 38, 160, 789, 4422, 27526, 187216	12	{00-1, 20-1}	1, 2, 4, 11, 38, 161, 797, 4447, 27250, 180065
13	{00-0, 01-2}	1, 2, 4, 6, 7, 8, 8, 8, 8, 8	14	{00-1, 10-2}	1, 2, 4, 9, 22, 58, 163, 485, 1519, 4985
15	{01-0, 01-2} {01-1, 01-2}	1, 2, 4, 8, 17, 39, 96, 251, 691, 1990	16	{00-0, 01-1}	1, 2, 4, 9, 23, 67, 222, 832, 3501, 16412

Continuation of Table 2					
Class	B	$\lvert\mathbf{I}_n(B)\rvert$	Class	B	$\lvert\mathbf{I}_n(B)\rvert$
17	$\{01\text{-}0, 01\text{-}1\}$	1, 2, 4, 9, 24, 75, 267, 1062, 4665, 22437	18	$\{01\text{-}2, 02\text{-}1\}$	1, 2, 5, 12, 27, 58, 121, 248, 503, 1014
19	$\{01\text{-}2, 10\text{-}0\}$	1, 2, 5, 12, 28, 65, 153, 369, 916, 2343	20	$\{01\text{-}2, 10\text{-}1\}$	1, 2, 5, 12, 28, 66, 161, 410, 1089, 3003
21	$\{01\text{-}2, 11\text{-}0\}$	1, 2, 5, 12, 29, 73, 194, 544, 1604, 4957	22	$\{01\text{-}2, 21\text{-}0\}$	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181
23	$\{01\text{-}2, 20\text{-}1\}$	1, 2, 5, 13, 34, 90, 243, 671, 1893, 5442	24	$\{01\text{-}2, 12\text{-}0\}$	1, 2, 5, 13, 35, 98, 284, 847, 2592, 8131
25	$\{01\text{-}1, 10\text{-}2\}$	1, 2, 5, 13, 35, 98, 285, 857, 2652, 8413	26	$\{01\text{-}2, 10\text{-}2\}$	1, 2, 5, 13, 35, 98, 285, 859, 2677, 8604
27	$\{01\text{-}0, 02\text{-}1\}$ $\{01\text{-}1, 02\text{-}1\}$	1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796	28	$\{01\text{-}1, 10\text{-}0\}$	1, 2, 5, 14, 43, 144, 523, 2048, 8597, 38486
29	$\{01\text{-}1, 12\text{-}0\}$	1, 2, 5, 14, 45, 164, 669, 3012, 14789, 78430	30	$\{00\text{-}0, 10\text{-}2\}$	1, 2, 5, 15, 47, 157, 555, 2061, 7997, 32303
31	$\{01\text{-}0, 10\text{-}2\}$	1, 2, 5, 15, 51, 187, 721, 2889, 11954, 50869	32	$\{01\text{-}1, 21\text{-}0\}$	1, 2, 5, 15, 52, 202, 860, 3951, 19372, 100543
33	$\{01\text{-}1, 20\text{-}1\}$	1, 2, 5, 15, 52, 203, 876, 4118, 20838, 112389	34	$\{01\text{-}0, 11\text{-}0\}$	1, 2, 5, 15, 52, 203, 879, 4184, 21765, 123193
35	$\{01\text{-}0, 10\text{-}1\}$ $\{01\text{-}0, 12\text{-}0\}$	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975	36	$\{01\text{-}0, 10\text{-}0\}$	1, 2, 5, 15, 52, 205, 908, 4473, 24283, 144076
37	$\{01\text{-}0, 21\text{-}0\}$	1, 2, 5, 15, 53, 216, 992, 5024, 27570, 161773	38	$\{01\text{-}0, 20\text{-}1\}$	1, 2, 5, 15, 53, 216, 993, 5047, 27898, 165556
39	$\{01\text{-}1, 10\text{-}1\}$	1, 2, 5, 15, 53, 216, 997, 5134, 29139, 180514	40	$\{01\text{-}1, 11\text{-}0\}$	1, 2, 5, 15, 53, 216, 997, 5136, 29195, 181508
41	$\{00\text{-}0, 12\text{-}0\}$	1, 2, 5, 16, 60, 261, 1281, 6987, 41791, 271261	42	$\{00\text{-}0, 02\text{-}1\}$	1, 2, 5, 16, 61, 265, 1274, 6628, 36756, 214812

Continuation of Table 2

Class	B	$\lvert\mathbf{I}_n(B)\rvert$	Class	B	$\lvert\mathbf{I}_n(B)\rvert$
43	{00-0, 10-0} {00-0, 10-1}	1, 2, 5, 16, 61, 271, 1372, 7795, 49093, 339386	44	{00-0, 11-0}	1, 2, 5, 16, 63, 300, 1696, 11186, 84687, 725406
45	{00-0, 20-1}	1, 2, 5, 17, 71, 350, 1960, 12156, 81936, 591811	46	{00-0, 21-0}	1, 2, 5, 17, 71, 350, 1962, 12219, 83168, 611437
47	{10-2, 12-0}	1, 2, 6, 21, 77, 287, 1079, 4082, 15522, 59280	48	{02-1, 10-2}	1, 2, 6, 21, 79, 312, 1280, 5418, 23539, 104529
49	{10-2, 11-0}	1, 2, 6, 21, 80, 320, 1327, 5669, 24867, 111791	50	{10-0, 10-2}	1, 2, 6, 21, 81, 335, 1470, 6788, 32793, 164990
51	{10-1, 10-2}	1, 2, 6, 21, 81, 337, 1492, 6965, 34055, 173503	52	{10-2, 20-1}	1, 2, 6, 22, 88, 368, 1584, 6968, 31192, 141656
53	{10-2, 21-0}	1, 2, 6, 22, 88, 370, 1619, 7349, 34534, 167637	54	{02-1, 12-0}	1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098
55	{02-1, 10-0}	1, 2, 6, 22, 91, 412, 2003, 10312, 55653, 312487	56	{02-1, 11-0}	1, 2, 6, 22, 92, 424, 2105, 11092, 61382, 353938
57	{10-0, 12-0}	1, 2, 6, 22, 92, 424, 2113, 11238, 63204, 373381	58	{10-1, 12-0}	1, 2, 6, 22, 92, 425, 2127, 11383, 64545, 385155
59	{02-1, 10-1}	1, 2, 6, 22, 93, 437, 2229, 12140, 69762, 419206	60	{10-1, 11-0}	1, 2, 6, 22, 94, 454, 2438, 14398, 92790, 648702
61	{10-0, 10-1}	1, 2, 6, 22, 94, 456, 2466, 14670, 95026, 664838	62	{10-0, 11-0}	1, 2, 6, 22, 94, 456, 2470, 14780, 96930, 692276
63	{11-0, 12-0}	1, 2, 6, 22, 95, 464, 2516, 14924, 95836, 660908	64	{02-1, 21-0}	1, 2, 6, 23, 103, 511, 2722, 15275, 89206, 537666
65	{02-1, 20-1}	1, 2, 6, 23, 103, 514, 2779, 15984, 96582, 607562	66	{12-0, 20-1}	1, 2, 6, 23, 104, 528, 2918, 17205, 106744, 690006
67	{10-0, 20-1} {10-1, 20-1}	1, 2, 6, 23, 104, 530, 2958, 17734, 112657, 750726	68	{12-0, 21-0}	1, 2, 6, 23, 104, 531, 2980, 18059, 116715, 797204

Continuation of Table 2					
Class	B	$\lvert\mathbf{I}_n(B)\rvert$	Class	B	$\lvert\mathbf{I}_n(B)\rvert$
69	$\{10\text{-}0, 21\text{-}0\}$ $\{10\text{-}1, 21\text{-}0\}$	1, 2, 6, 23, 104, 532, 3004, 18426, 121393, 851810	70	$\{11\text{-}0, 20\text{-}1\}$	1, 2, 6, 23, 105, 547, 3161, 19863, 133751, 954492
71	$\{11\text{-}0, 21\text{-}0\}$	1, 2, 6, 23, 105, 549, 3207, 20577, 143239, 1071704	72	$\{20\text{-}1, 21\text{-}0\}$	1, 2, 6, 24, 116, 632, 3720, 23072, 148528, 983072
End of Table 2					

Table 2 suggests there are exactly 6 I-Wilf equivalences. So, the aim of this section is to prove that there the only 6 I-Wilf equivalences among pairs of patterns of type $(2, 1)$, namely, we show the following equivalences:

$$\{01\text{-}0, 01\text{-}2\} \overset{\mathbf{I}}{\sim} \{01\text{-}1, 01\text{-}2\} \text{ (Thm. 3.2)},$$
$$\{01\text{-}0, 02\text{-}1\} \overset{\mathbf{I}}{\sim} \{01\text{-}1, 02\text{-}1\} \text{ (Thm. 3.3)},$$
$$\{01\text{-}0, 10\text{-}1\} \overset{\mathbf{I}}{\sim} \{01\text{-}0, 12\text{-}0\} \text{ (Thm. 3.4)},$$
$$\{10\text{-}0, 20\text{-}1\} \overset{\mathbf{I}}{\sim} \{10\text{-}1, 20\text{-}1\} \text{ (Thm. 3.5)},$$
$$\{10\text{-}0, 21\text{-}0\} \overset{\mathbf{I}}{\sim} \{10\text{-}1, 21\text{-}0\} \text{ (Thm. 3.6)},$$
$$\{00\text{-}0, 10\text{-}0\} \overset{\mathbf{I}}{\sim} \{00\text{-}0, 10\text{-}1\} \text{ (Thm. 3.8)}.$$

In order to prove these 6 I-Wilf equivalences, we use Algorithm KMY to guess the generating tree for each pair, and then we prove the rules as explained in Section 2. Since it is routine procedure to prove the rules, we omit the proofs.

Theorem 3.2. *We have* $\{01\text{-}1, 01\text{-}2\} \overset{\mathbf{I}}{\sim} \{01\text{-}0, 01\text{-}2\}$. *Moreover, the rules of the generating tree* $\mathcal{T}(\{01\text{-}1, 01\text{-}2\})$ *are given by*

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_1, \ldots, b_m, \\ b_m &\rightsquigarrow c_{m,0}, \ldots, c_{m,m-2}, b_m, \\ c_{m,j} &\rightsquigarrow c_{m,0}, \ldots, c_{m,j}, b_{j+1}, \ldots, b_{m-1}, \end{aligned}$$

where $a_m = 0^m$, $b_m = 0^m m$, *and* $c_{m,j} = 0^m mj$ *with* $0 \le j \le m-2$. *The rules of the generating tree* $\mathcal{T}(\{01\text{-}0, 01\text{-}2\})$ *are given by*

$$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m,$$

$$\begin{aligned} b_m &\rightsquigarrow d_{m,1},\ldots,d_{m,m-1},b_m,\\ d_{m,j} &\rightsquigarrow d_{m,1},\ldots,d_{m,j},b_j,\ldots,b_{m-1}, \end{aligned}$$

where $a_m = 0^m$, $b_m = 0^m m$, *and* $d_{m,j} = 0^m mj$ *with* $1 \le j \le m-1$.

Moreover, the generating function $\sum_{n\ge0}|\mathbf{I}_n(\{01\text{-}1,01\text{-}2\})|x^{n+1}$ *is given by*

$$\frac{x}{1-x}+\sum_{j\ge0}\frac{x^{2j+2}}{(1-x)^{j+2}\prod_{i=0}^{j}((1-x)^i-x)}.$$

Proof. By Algorithm KMY, we obtain the rules as stated in the current theorem. Note that, by mapping the label $d_{m,j}$ to the label $c_{m,j-1}$, we obtain that the two generating trees $\mathcal{T}(\{01\text{-}1,01\text{-}2\})$ and $\mathcal{T}(\{01\text{-}0,01\text{-}2\})$ are isomorphic as plain trees. Thus, $\{01\text{-}1,01\text{-}2\}\overset{\mathbf{I}}{\sim}\{01\text{-}0,01\text{-}2\}$.

Now let us focus on the generating tree $\mathcal{T}(B)$, where $B=\{01\text{-}1,01\text{-}2\}$. Define $A_m(x)$ (respectively, $B_m(x)$, $C_{m,j}(x)$) to be the generating function for the number of nodes at level $n\ge0$ for the subtree of $\mathcal{T}(B;a_m)$ (respectively, $\mathcal{T}(B;b_m)$, $\mathcal{T}(B;c_{m,j})$), where its root stays at level 0. Then, these rules lead to

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + xB_1(x) + \cdots + xB_m(x),\\ B_m(x) &= x + xC_{m,0}(x) + \cdots + xC_{m,m-2}(x) + xB_m(x),\\ C_{m,j}(x) &= x + xC_{m,0}(x) + \cdots + xC_{m,j}(x) + xB_{j+1}(x) + \cdots + xB_{m-1}(x). \end{aligned}$$

Define $F(v)=\sum_{m\ge1}F_m(x)v^{m-1}$ with $F\in\{A,B\}$ and

$$C(v,u)=\sum_{m\ge1}\sum_{j=0}^{m-2}C_{m,j}(x)u^jv^{m-2}.$$

Then, the recurrence can be written as

$$\begin{aligned} A(v) &= \frac{x}{1-v}+\frac{x}{v}(A(v)-A(0))+\frac{x}{1-v}B(v),\\ B(v) &= \frac{x}{1-v}+xvC(v,1)+xB(v),\\ C(v,u) &= \frac{x}{(1-v)(1-uv)}+\frac{x}{1-u}(C(v,u)-uC(uv,1))\\ &\quad+\frac{x}{(1-u)(1-v)}(B(v)-uB(uv)). \end{aligned}$$

By equation of $C(v, u)$, we have

$$C(v/u, u) = \frac{x}{(1-v/u)(1-v)} + \frac{x}{1-u}(C(v/u,u) - uC(v,1)) + \frac{x}{(1-u)(1-v/u)}(B(v/u) - uB(v)).$$

Thus, by taking $u = 1 - x$, we obtain

$$C(v, 1) = \frac{x}{(1-x-v)(1-v)} + \frac{1}{1-x-v}(B(v/(1-x)) - (1-x)B(v)).$$

Hence, the equation of $B(v)$ gives

$$B(v) = \frac{x}{(1-x)(1-v)} + \frac{xv}{(1-x)^2(1-v)}B(v/(1-x)).$$

By iterating this equation with assuming $|x| < 1$, we obtain

$$\begin{aligned} B(v) &= \frac{x}{(1-x)(1-v)} + \frac{x^2 v}{(1-x)^2(1-v)((1-x)-v)} \\ &\quad + \frac{x^2v^2}{(1-x)^3(1-v)(1-x-v)}B(\frac{v}{(1-x)^2}) \\ &= \sum_{j=0}^{2} \frac{x^{j+1}v^j}{(1-x)^{j+1}\prod_{i=0}^{j}((1-x)^i - v)} \\ &\quad + \frac{x^3v^3}{(1-x)^3(1-v)(1-x-v)((1-x)^2-v)}B(\frac{v}{(1-x)^3}) \\ &= \cdots, \end{aligned}$$

which implies

$$B(v) = \sum_{j \geq 0} \frac{x^{j+1}v^j}{(1-x)^{j+1}\prod_{i=0}^{j}((1-x)^i - v)}.$$

Equation of $A(v)$ with $v = x$ gives

$$A(0) = \frac{x}{1-x} + \frac{x}{1-x}B(x),$$

which completes the proof. $\square$

Theorem 3.3. *We have* $\{01\text{-}1, 02\text{-}1\} \overset{g}{\sim} \{01\text{-}0, 02\text{-}1\}$. *Moreover, the rules of the generating tree* $\mathcal{T}(\{01\text{-}1, 02\text{-}1\})$ *are given by*

$$a_m \rightsquigarrow a_{m+1}, a_m, \ldots, a_1,$$

where $a_m = 0^m$. *Moreover, for all* $n \geq 0$,

$$|\mathbf{I}_n(\{01\text{-}1, 02\text{-}1\})| = \frac{1}{n+2}\binom{2n+2}{n+1}.$$

Proof. By Algorithm KMY, the rules of the generating tree $\mathcal{T}(\{01\text{-}1, 02\text{-}1\})$ are the same as the rules of the generating tree $\mathcal{T}(\{01\text{-}0, 02\text{-}1\})$ (see end of Section 2) and they are given by $a_m \rightsquigarrow a_{m+1}, a_m, \ldots, a_1$, where $a_m = 0^m$. Thus, $\{01\text{-}1, 02\text{-}1\} \overset{g}{\sim} \{01\text{-}0, 02\text{-}1\}$.

Let $B = \{01\text{-}0, 02\text{-}1\}$. Define $A_m(x)$ to be the generating function for the number of nodes at level $n \geq 0$ for the subtree of $\mathcal{T}(B; a_m)$, where its root stays at level 0. Then, these rules lead to

$$A_m(x) = x + x\sum_{i=1}^{m+1} A_i(x).$$

Define $A(x, v) = \sum_{m\geq 1} A_m(x)v^{m-1}$. Thus,

$$A(x, v) = \frac{x}{1-v} + \frac{x}{v}(A(x, v) - A(x, 0)) + \frac{x}{1-v}A(x, v).$$

By taking $v = \frac{1-\sqrt{1-4x}}{2}$, we obtain $A(x, 0) = \frac{1-\sqrt{1-4x}}{2x} - 1$, the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ with $n \geq 1$. Hence,

$$|\mathbf{I}_n(\{01\text{-}1, 02\text{-}1\})| = \frac{1}{n+2}\binom{2n+2}{n+1},$$

for all $n \geq 0$. □

Theorem 3.4. *We have* $\{01\text{-}0, 12\text{-}0\} \overset{\mathbf{I}}{\sim} \{01\text{-}0, 10\text{-}1\}$. *Moreover, the rules of the generating tree* $\mathcal{T}(\{01\text{-}0, 12\text{-}0\})$ *are given by*

$$a_m \rightsquigarrow a_{m+1}, a_m, b_{m,2}, \ldots, b_{m,m},$$
$$b_{m,j} \rightsquigarrow a_{m+1}, a_{m+1-j}, b_{m+1,2}, \ldots, b_{m+1,j}, b_{m+1-j,2}, \ldots, b_{m1+-j,m+1-j},$$

where $a_m = 0^m$ and $b_{m,j} = 0^m j$ with $2 \le j \le m$. The rules of the generating tree $\mathcal{T}(\{01\text{-}0, 10\text{-}1\})$ are given by

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, a_m, b_{m,2}, \ldots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow a_m, b_{m,2}, \ldots, b_{m,j-1}, b_{m+1,j}, b_{m,j}, \ldots, b_{m,m}, \end{aligned}$$

where $a_m = 0^m$ and $b_{m,j} = 0^m j$ with $2 \le j \le m$.

Moreover, the number of inversion sequences in $\mathbf{I}_n(\{01\text{-}0, 10\text{-}1\})$ is given by the n-th Bell number.

Proof. We proceed with the proof by showing that the number of inversion sequences in $\mathbf{I}_n(\{01\text{-}0, 12\text{-}0\})$ ($\mathbf{I}_n(\{01\text{-}0, 10\text{-}1\})$) is given by the n-th Bell number, that is, the generating function for the number of such inversion sequences is given by $G(x) = \sum_{j\ge1} \frac{x^j}{\prod_{i=1}^{j}(1-ix)}$.

First, we consider the case $B = \{01\text{-}0, 12\text{-}0\}$. By Algorithm KMY, we derive the rules of the generating tree $\mathcal{T}(B)$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level $n \ge 0$ for the subtree of $\mathcal{T}(B; a_m)$ (respectively, $\mathcal{T}(B; b_{m,j})$, where its root stays at level 0. Then, these rules lead to

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + xA_m(x) + x\sum_{i=2}^{m} B_{m,i}(x), \\ B_{m,j}(x) &= x + xA_{m+1}(x) + xA_{m+1-j}(x) + x\sum_{i=2}^{j} B_{m+1,i}(x) \\ &\quad + x\sum_{i=2}^{m+1-j} B_{m+1-j,i}(x). \end{aligned}$$

Define $A(v) = \sum_{m\ge1} A_m(x)v^{m-1}$ and

$$B(v,u) = \sum_{m\ge1}\sum_{j=0}^{m-2} B_{m,j}(x)u^{m-j}v^{m-2}.$$

Then, the recurrence can be written as

(1)

$$\begin{aligned} A(v) &= \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xA(v) + xvB(v,1), \\ B(v,u) &= \frac{x}{(1-v)(1-uv)} + \frac{x}{v^2(1-u)}\left(A(v) - A(0) - \frac{1}{u}(A(uv) - A(0))\right) \end{aligned}$$

$$+\frac{x}{1-v}A(uv)+\frac{x}{v(1-u)}(B(v,1)-B(v,u))+\frac{xuv}{1-v}B(uv,1). \tag{2}$$

Based on the first terms of the generating functions $A(v)$ and $B(v,u)$, we assume

$$A(v)+vB(v,1)=\frac{G(x/(1-v))}{1-v}. \tag{3}$$

Note that from (3), we see that $A(0)=G(x)$. Also, by (1)-(2), we have

$$A(v)=\frac{xG(x)}{2vx-v+x}+\frac{vx(1+G(x/(1-v)))}{(1-v)(2vx-v+x)}$$

and

$$B(v,1)=\frac{x(1-v)G(x)-(vx-v+x)G(x/(1-v))+vx}{v(1-v)(2vx-v+x)}.$$

Hence, by (2), we have an explicit formula for $B(v,u)$:

$$\begin{aligned}
&B(v,u)\\
&=\frac{(-v^2(1-v)(1-u)+v(2uv^2-5uv-2v^2+u+4v)x)xG(x)}{v(uv-v-x)(2uvx-uv+x)(1-v)(2vx-v+x)}\\
&+\frac{(8uv^2-2uv-6v^2+v-1)x^3G(x)}{v(uv-v-x)(2uvx-uv+x)(1-v)(2vx-v+x)}\\
&-\frac{x(v-x)G(x/(1-v))}{v(uv-v-x)(1-v)(2vx-v+x)}\\
&+\frac{(u^2v^2-uv^2-uvx+x)xG(x/(1-uv))}{(uv-v-x)(2uvx-uv+x)(1-uv)(1-v)}\\
&+\frac{(uv^2(1-u)+(2u^2v^2-u^2v-2uv^2+2u-1)x)vx}{(1-v)(2vx-v+x)(1-uv)(2uvx-uv+x)(uv-v-x)}\\
&-\frac{(4uv-2v+1)x^3}{(1-v)(2vx-v+x)(2uvx-uv+x)(uv-v-x)}.
\end{aligned}$$

By using expressions of $B(v,u)$ and $A(v)$, we see that (1)-(3) hold. Hence, $A(0)=G(x)$, that is, the number of inversion sequences in $\mathbf{I}_n(\{01\text{-}0,12\text{-}0\})$ is given by the n-th Bell number.

Second, we consider the case $B=\{01\text{-}0,10\text{-}1\}$. By Algorithm KMY, we derive the rules of the generating tree $\mathcal{T}(B)$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level

$n \geq 0$ for the subtree of $\mathcal{T}(B; a_m)$ (respectively, $\mathcal{T}(B; b_{m,j})$, where its root stays at level 0. Then, these rules lead to

$$\begin{aligned}A_m(x) &= x + xA_{m+1}(x) + xA_m(x) + xB_{m,2}(x) + \cdots + xB_{m,m}(x),\\ B_{m,j}(x) &= x + xA_m(x) + xB_{m,2}(x) + \cdots + xB_{m,j-1}(x) + xB_{m+1,j}(x)\\ &\quad + xB_{m,j}(x) + \cdots + xB_{m,m}(x).\end{aligned}$$

Define $A(v) = \sum_{m\geq 2} A_m(x)v^{m-2}$ and

$$B(v,u) = \sum_{m\geq 1}\sum_{j=0}^{m-2} B_{m,j}(x)u^{m-j}v^{m-2}.$$

Then, the recurrence can be written as

$$A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xA(v) + xB(v,1), \tag{4}$$

$$\begin{aligned}B(v,u) &= \frac{x}{(1-v)(1-uv)} + \frac{x}{1-u}(A(v) - uA(uv))\\ &\quad + \frac{x}{1-u}(B(v,1) - B(v,u))\\ &\quad + \frac{x}{uv}(B(v,u) - B(v,0)) + \frac{x}{1-u}(B(v,u) - uB(uv,1)).\end{aligned} \tag{5}$$

In order to solve this system, we guess that $B(v,0) = A(v)$ (based on the first values of the generating functions). By solving (4) for $B(v,1)$, then (5) gives

$$B(v,u) = \frac{A(v) - uA(uv)}{1-u},$$

which implies $B(v,1) = A(v) + v\frac{\partial}{\partial v}A(v)$. Hence, (5) gives

$$A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + 2xA(v) + xv\frac{\partial}{\partial v}A(v).$$

By finding the coefficient of v^{m-2}, we obtain

$$A_m(x) = x + xA_{m+1}(x) + 2xA_m(x) + (m-2)xA_m(x),$$

for all $m \geq 2$. Hence, $A_m(x) = \frac{x}{1-mx} + \frac{x}{1-mx}A_{m+1}(x)$. By induction on m, we see that

$$A_m(x) = \sum_{j\geq 1} \frac{x^j}{\prod_{i=m}^{m+j-1}(1-ix)},$$

which implies

$$A(v) = \sum_{m\geq 2}\sum_{j\geq 1}\frac{x^j v^{m-2}}{\prod_{i=m}^{m+j-1}(1-ix)}$$

and

$$B(v,u) = \sum_{m\geq 2}\sum_{j\geq 1}\frac{x^j(1-u^{m-1})v^{m-2}}{(1-u)\prod_{i=m}^{m+j-1}(1-ix)}.$$

This satisfies (4), (5), and $B(v,0)=A(v)$. Note that $A_1(x)=x+xA_1(x)+xA_2(x)$, that is, $A_1(x)=\frac{x}{1-x}+\frac{x}{1-x}A_2(x)$, so $A_1(x)=G(x)$. Hence, the number of inversion sequences in $\mathbf{I}_n(\{01\text{-}0,10\text{-}1\})$ is given by the n-th Bell number. □

Theorem 3.5. *We have* $\{10\text{-}1,20\text{-}1\}\overset{g}{\sim}\{10\text{-}0,20\text{-}1\}$. *Moreover, the rules of the generating tree* $\mathcal{T}(\{10\text{-}1,20\text{-}1\})$ *are given by*

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1},\ldots,b_{m,m},\\ b_{m,j} &\rightsquigarrow a_{m+2-j}, b_{m+1,j},\ldots,b_{m+1,m+1}, b_{m+2-j,1},\ldots,b_{m+2-j,m+2-j}, \end{aligned}$$

where $a_m=0^m$ *and* $b_{m,j}=0^m j$ *with* $1\leq j\leq m$. *Moreover, the generating function* $\sum_{n\geq 0}|\mathbf{I}_n(\{10\text{-}1,20\text{-}1\})|x^{n+1}$ *is given by [13, Sequence A117106].*

Proof. Algorithm KMY derives that the generating trees $\mathcal{T}(\{10\text{-}1,20\text{-}1\})$ and $\mathcal{T}(\{10\text{-}0,20\text{-}1\})$ with the given rules in the statement. So it remains to show that the generating function $\sum_{n\geq 0}|\mathbf{I}_n(\{10\text{-}1,20\text{-}1\})|x^{n+1}$ is given by [13, Sequence A117106]. To do so, we show that $\mathbf{I}_n(\{10\text{-}1,20\text{-}1\}) = \mathbf{I}_n(\{101,201\})$, where in the right-side we avoid 101 and 201 as subsequences (that is, for any $e\in\mathbf{I}_n(\{101,201\})$, there are no i,j,k such that $0\leq i<j<k\leq n$ and $red(e_ie_je_k)\in\{101,201\}$). Clearly, $\mathbf{I}_n(\{101,201\})\subseteq\mathbf{I}_n(\{10\text{-}1,20\text{-}1\})$. Now, let us show $\mathbf{I}_n(\{10\text{-}1,20\text{-}1\})\subseteq\mathbf{I}_n(\{101,201\})$.

Let $\pi\in\mathbf{I}_n(\{10\text{-}1,20\text{-}1\})$. Assume that π contains 101 as $\pi_a\pi_b\pi_c$ with $0\leq a<b<c\leq n$, a minimal, $a+b$ minimal, and $a+b+c$ minimal, that is, leftmost occurrence of 101 in π. Since π avoids 10-1, so $b>a+1$. Since we select left-most occurrence of 101, we have $\pi_{a+1},\ldots,\pi_{b-1}\geq\pi_a$. So, the reduction of $\pi_{b-1}\pi_b\pi_c$ is either 101 or 201, a contradiction.

Now, assume that π contains 201 as $\pi_a\pi_b\pi_c$ with $0\leq a<b<c\leq n$, a minimal, $a+b$ minimal, and $a+b+c$ minimal, that is, leftmost occurrence of 201 in π. Since π avoids 20-1, we have that $b>a+1$. Since minimality of $a+b$, we see that $\pi_{a+1},\ldots,\pi_{b-1}\geq\pi_c$. Thus, the reduction of $\pi_{b-1}\pi_b\pi_c$ is either 101 or 201, a contradiction.

Hence, $\pi\in\mathbf{I}_n\{101,201\}$, which completes the proof. □

Theorem 3.6. *We have* $\{10\text{-}1, 21\text{-}0\} \overset{g}{\sim} \{10\text{-}0, 21\text{-}0\}$. *Moreover, the rules of the generating tree* $\mathcal{T}(\{10\text{-}1, 21\text{-}0\})$ *are given by*

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}, \\ b_{m,j} &\rightsquigarrow a_{m+1}, \ldots, a_{m+2-j}, b_{m+1,j}, \ldots, b_{m+1,m+1}, \end{aligned}$$

where $a_m = 0^m$ *and* $b_{m,j} = 0^m j$ *with* $1 \le j \le m$. *Moreover,*

$$|\mathbf{I}_n(\{10\text{-}0, 21\text{-}0\})| = |S_{n+1}(1\text{-}24\text{-}3)|,$$

where in the right-side we meant the number of permutations $\pi = \pi_1 \cdots \pi_{n+1}$ *of length* $n+1$ *such that there are no* i, j, k *such that* $0 \le i < j < k-1 \le n-1$ *and* $red(\pi_i \pi_j \pi_{j+1} \pi_k) = 1243$*, See [13, Sequence A137538].*

Proof. Algorithm KMY derives that the generating trees $\mathcal{T}(\{10\text{-}1, 21\text{-}0\})$ and $\mathcal{T}(\{10\text{-}0, 21\text{-}0\})$ with the given rules in the statement. So remains, to show that $|\mathbf{I}_n(\{10\text{-}0, 21\text{-}0\})| = |S_{n+1}(1\text{-}24\text{-}3)|$, for all $n \ge 0$. Here we use the coding of a permutation in $\pi = \pi_1 \cdots \pi_n \in S_{n+1}$ by inversion sequences in $e = e_0 \cdots e_n \in \mathbf{I}_n$: $e_i = |\{j | \pi_{n+1-i} > \pi_{n+1-j},\, i > j \ge 0\}|$. In this case, we write $e = e(\pi) = e_0 \cdots e_n$.

Assume that $\pi \in S_{n+1}$ contain 1-24-3 and, then there exist $1 \le a < b < c-1 \le n$ such that $red(\pi_a \pi_b \pi_{b+1} \pi_c) = 1243$. We choose the left-most occurrence of 1-24-3, that is, $a, a+b, a+b+c$ minimal. Let $a \le i \le b-1$ such that $\pi_i < \pi_a$ and i maximal. So, since we selected leftmost occurrence of 1-24-3, we see all the letters $\pi_{i+1}, \ldots, \pi_{b-1}$ are greater than π_c. Hence, π contains 1-24-3 as leftmost occurrence $\pi_a \pi_b \pi_{b+1} \pi_c$ if and only if $e_{n+1-b} > e_{n+1-b} \ge e_{n+1-i}$, that is, if and only if e contains either 10-0 or 21-0. □

In order to present our last I-Wilf equivalence, we need the following definition. A sequence $\pi_1 \cdots \pi_n$ of nonnegative integers is an *ascent sequence* of length n if $\pi_1 = 0$ and for all $i \ge 2$, π_i is at most 1 plus the number of ascents in $\pi_1 \cdots \pi_{i-1}$, that is, $\pi_i \le 1 + |\{j | \pi_j < \pi_{j+1},\, 1 \le j \le i-2\}|$. Note that the area of combinatorics of ascent sequences have been received a lot of attention, see, for example, [2, 5, 6, 7]. Here, we interested on [13, Sequence A138265], the sequence of the number of ascent sequences $\pi_1 \cdots \pi_n$ of length n without flat steps (that is, $\pi_i \ne \pi_{i+1}$ for all $i = 1, 2, \ldots, n$). We denote the set of all ascent sequences of length n without flat steps by AS_n.

Lemma 3.7. *The generating tree* $\mathcal{A}$ *for all the ascent sequences in* $\cup_{n \ge 2} AS_n$ *is given by root* $(1,1)$ *and the following rule*

$$s_{i,j} \rightsquigarrow s_{i,0}, \ldots, s_{i,j-1}, s_{i+1,j+1}, \ldots, s_{i+1,i+1}$$

with $0 \leq j \leq i$ *and* $i \geq 1$*, where* $s_{i,j}$ *is the label for an ascent sequence with* i *ascents and right most letter* j*.*

Proof. Let π be any ascent sequence with i ascents and the last (rightmost) letter j. Then, the children of π are πk where $k = 0, 1, \ldots, j, j+1, \ldots, i+1$. Note that πk has i ascents and last letter is k whenever $k = 0, 1, \ldots, j-1$ and it has $i+1$ ascents and last letter k whenever $k = j+1, j+2, \ldots, i+$. This completes the proof. □

Theorem 3.8. *We have* $\{00\text{-}0, 10\text{-}0\} \overset{\mathbf{I}}{\sim} \{00\text{-}0, 10\text{-}1\}$*. Moreover, the number of inversion sequences in* $\mathbf{I}_n(00\text{-}0, 10\text{-}0)$ *is the same as the number of ascents sequences in* AS_{n+2}*.*

Proof. By Algorithm KMY, we obtain the rules of the generating trees $\mathcal{T}(\{00\text{-}0, 10\text{-}0\})$ and $\mathcal{T}(\{00\text{-}0, 10\text{-}1\})$. More precisely, we have that

(1) the rules of the generating tree $\mathcal{T}(\{00\text{-}0, 10\text{-}0\})$ are given by

$$\begin{aligned} a_m &\rightsquigarrow b_{m,0}, \ldots, b_{0,m}, a_{m+1}, \\ b_{m,j} &\rightsquigarrow b_{m+j,0}, \ldots, b_{m+1,j-1}, b_{m,j+1}, \ldots, b_{0,m+j+1}, a_{m+j+1}, \end{aligned}$$

where $a_m = 01 \cdots m$ and $b_{m,j} = 01 \cdots m01 \cdots j$.

(2) the rules of the generating tree $\mathcal{T}(\{00\text{-}0, 10\text{-}1\})$ are given by

$$\begin{aligned} a'_m &\rightsquigarrow b'_{m,1}, \ldots, b'_{1,m}, c'_m, a'_{m+1}, \\ c'_m &\rightsquigarrow b'_{m+1,1}, \ldots, b'_{1,m+1}, a'_{m+1}, \\ b'_{m,j} &\rightsquigarrow b'_{m+j-1,1}, \ldots, b'_{m+1,j-1}, c'_{m+j-1}, b'_{m,j+1}, \ldots, b'_{1,m+j}, a'_{m+j}, \end{aligned}$$

where $a'_m = 01 \cdots m$, $c'_m = 01 \cdots mm$, and $b'_{m,j} = 01 \cdots mms_j$, where s_j is a sequence of j consecutive letters starting from 0 and does not contain the letter m.

To show that $\{00\text{-}0, 10\text{-}0\} \overset{\mathbf{I}}{\sim} \{00\text{-}0, 10\text{-}1\}$, we describe a simple bijection f between labels of the generating tree $\{00\text{-}0, 10\text{-}0\}$ and labels of the generating tree $\{00\text{-}0, 10\text{-}1\}$ as follows: $f(a_m) = a'_m$, $f(b_{m,0}) = c'_m$, and $f(b_{m,j}) = b'_{j,m+1}$ with $j \geq 1$ and $m \geq 0$. Clearly, we see that each rule in the generating tree $\{00\text{-}0, 10\text{-}0\}$ maps by f to a rule in the generating tree $\{00\text{-}0, 10\text{-}1\}$.

Thus, it is remains to prove that

$$|\mathbf{I}_n\{00\text{-}0, 10\text{-}0\})| = |AS_{n+2}|,$$

for all $n \geq 2$. To do so, we describe a bijection between the generating tree $\mathcal{T}(\{00\text{-}0, 10\text{-}0\})$ and the generating tree of $\mathcal{A}$ (see Lemma 3.7). By mapping

a_m to $(m+1, m+1)$ and $b_{m,j}$ to $(m+j+1, j)$, we see that the root of $\mathcal{T}(\{00\text{-}0, 10\text{-}0\})$ maps to $(1,1)$ and the rules map to

$$\begin{aligned}(m+1,m+1) &\rightsquigarrow (m+1,0),\ldots,(m+1,m),(m+2,m+2),\\ (m+j+1,j) &\rightsquigarrow (m+j+1,0),\ldots,(m+j+1,j-1),\\ &\qquad (m+j+2,j+1),\ldots,(m+j+2,m+j+1),\\ &\qquad (m+j+2,m+j+2),\end{aligned}$$

that is, we can map the root of $\mathcal{T}(\{00\text{-}0, 10\text{-}0\})$ to $(1,1)$ and its rules maps to

$$s_{m,j} \rightsquigarrow s_{m,0},\ldots,s_{m,j-1},s_{m+1,j+1},\ldots,s_{m+1,m+1}.$$

Thus, by Lemma 3.7, we see that the number of nodes at level n (the root is stay at level 0) in $\mathcal{T}(\{00\text{-}0, 10\text{-}0\})$ equals the number of nodes at level n in $\mathcal{A}$. Note that the root $0 \in \mathbf{I}_0$ has length 0 and $01 \in AS_2$ has length 2. Hence, $|\mathbf{I}_n\{00\text{-}0, 10\text{-}0\})| = |AS_{n+2}|$, for all $n \geq 1$, as claimed. □

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