Journal of Combinatorics
Volume 15, Number 1, 77–87, 2024

On the size of an r-wise fractional L-intersecting family*

Tapas Kumar Mishra†

Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0, 1)$ is an irreducible fraction. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n]$. We say $\mathcal{F}$ is an *r-wise fractional L-intersecting family* if for every distinct $i_1, i_2, \ldots, i_r \in [m]$, there exists an $\frac{a}{b} \in L$ such that $|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}| \in \{\frac{a}{b}|A_{i_1}|, \frac{a}{b}|A_{i_2}|, \ldots, \frac{a}{b}|A_{i_r}|\}$. In this paper, we introduce and study the notion of r-wise fractional L-intersecting families. This is a generalization of notion of fractional L-intersecting families studied in [Niranjan et.al, Fractional L-intersecting families, The Electronic Journal of Combinatorics, 2019].

Keywords and phrases: Intersecting family, fractional L-intersecting family.

1. Introduction

A family $\mathcal{F}$ of subsets of $[n] = \{1, \ldots, n\}$ is said to be *L-intersecting* if for every $A_i, A_j \in \mathcal{F}$ with $A_i \neq A_j$, we have $|A_i \cap A_j| \in L$. This problem has been studied extensively in literature. One of the earliest results on the problem is by Ray-Chaudhuri and Wilson [1] who proved that $|\mathcal{F}| \leq \binom{n}{s}$ provided $\mathcal{F}$ is t-uniform. Frankl and Wilson [2] proved that $|F| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$ when the uniformity restriction on $\mathcal{F}$ is revoked. Alon, Babai and Suzuki [3] proved the above result using an ingenious linear algebraic argument. In the same paper, the authors generalized the notion of L-intersecting families and obtained the following result.

Theorem 1. *[3] Let $L = \{l_1, \ldots, l_s\}$ be a set of s non negetive integers, and $K = \{k_1, \ldots, k_q\}$ be a set of integers satifying $k_i > s - q$ for each i. Suppose $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n]$ such that $|A_i| \in K$ for each*

arXiv: 1909.13217
*r-wise fractional L-intersecting family.
†ORCID: 0000-0002-9825-3828.

$1 \le i \le m$ and $|A_i \cap A_j| \in L$ for each pair with $i \neq j$. Then,

$$m \le \binom{n}{s} + \ldots + \binom{n}{s-q+1}.$$

This upper bound is tight as given by the family of all subsets of $[n]$ of size between $s-q+1$ and s. Gromuluz and Sudakov [4] extended the results of Frankl-Wilson and Alon-Babai-Suzuki to r-wise L-intersecting families.

Definition 1. Let $r \ge 2$ and $L = \{l_1, \ldots, l_s\}$ be a set of s non-negative integers. If $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n]$ such that $|A_1 \cap \ldots \cap A_r| \in L$ for every collection of r elements in $\mathcal{A}$, then $\mathcal{A}$ is an r-wise L-intersecting family.

Theorem 2. *[4] Let $\mathcal{A}$ be an r-wise L-intersecting family with $L = \{l_1, \ldots, l_s\}$ where $s \ge 1$ and each $l \in L$ are non-negative integers. Then,*

$$|\mathcal{A}| \le (r-1)\left(\binom{n}{s} + \ldots + \binom{n}{0}\right).$$

Moreover, if the sizes of every member of $\mathcal{A}$ lies in $K = \{k_1, \ldots, k_q\}$ where each $k_i > s-q$, then

$$|\mathcal{A}| \le (r-1)\left(\binom{n}{s} + \ldots + \binom{n}{s-q+1}\right).$$

Fűredi and Sudakov [5] improved the above bound and showed that their bound is asymptotically optimal.

Theorem 3. *[5] Let L be a subset of non-negative integers of size s, $r \ge 2$ and $\mathcal{A}$ be an r-wise L-intersecting family of subsets of an n-element set. Then there exists an integer $n_0 = n_0(r, s)$ such that for all $n > n_0$,*

$$|\mathcal{A}| \le \frac{r-s+1}{s+1}\binom{n}{s} + \sum_{i<s}\binom{n}{i}.$$

An improvement to the above bound was provided by Kang et.al. [6] who proved the following theorem.

Theorem 4. *[6] Let $L = \{l_1, \ldots, l_s\}$ be a set of non-negative integers of size s and $\mathcal{A}$ be an r-wise L-intersecting family of subsets of an n-element* *set. Then, if* $|\cap_{A\in\mathcal{A}} A| < l_1$, $|\mathcal{A}| = o(n^s)$*. Moreover, if* $|\cap_{A\in\mathcal{A}} A| \geq l_1$ *and* n *sufficiently large,*

$$|\mathcal{A}| \leq \frac{r-s+1}{s+1}\binom{n-l_1}{s} + \sum_{i<s}\binom{n-l_1}{i}.$$

Various researchers have worked on many variants of the L-intersecting families, see [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] for detail.

Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0,1)$ is an irreducible fraction. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n]$. We say $\mathcal{F}$ is a *fractional L-intersecting family* if for every distinct $i, j \in [m]$, there exists an $\frac{a}{b} \in L$ such that $|A_i \cap A_j| \in \{\frac{a}{b}|A_i|, \frac{a}{b}|A_j|\}$. Niranjan et.al. [17] introduced the notion of fractional L-intersecting families and proved that $m = \mathcal{O}\left(\binom{n}{s}\left(\frac{\log^2 n}{\log\log n}\right)\right)$. When $L = \{\frac{a}{b}\}$, the bound on m improves to $\mathcal{O}(n \log n)$. In this paper, we generalize the notion of fractional L-intersecting family to r-wise fractional L-intersecting family in the natural way.

Definition 2 (r-wise fractional L-intersecting family)**.** Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0,1)$ is an irreducible fraction. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n]$. We say $\mathcal{F}$ is a *r-wise fractional L-intersecting family* if for every distinct $i_1, i_2, \ldots, i_r \in [m]$, there exists an $\frac{a}{b} \in L$ such that $|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}| \in \{\frac{a}{b}|A_{i_1}|, \frac{a}{b}|A_{i_2}|, \ldots, \frac{a}{b}|A_{i_r}|\}$.

In Section 2, we prove the following theorem.

Theorem 5. *Let n be a positive integer. Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0,1)$ is an irreducible fraction. Let $\mathcal{F}$ be an r-wise fractional L-intersecting family of subsets of $[n]$, where $r \geq 3$. Then, $|\mathcal{F}| \leq 2\frac{\ln^2 n}{\ln\ln n}(r-1)\left(\sum_{l=0}^{s}\binom{n}{l}\right)$. Moreover, the bound improves to $2\frac{\ln^2 n}{\ln\ln n}(r-1)\binom{n}{s}$, if $s \leq n+1-2\ln n$.*

Consider the following examples for an r-wise fractional L-intersecting family.

Example 3. Let $L = \{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$, where s $(= |L|)$ is a constant. The collection of all the s-sized subsets of $[n]$ is an r-wise fractional L-intersecting family of cardinality $\binom{n}{s}$. In this case, the bound given by Theorem 5 is asymptotically tight up to a factor of $(r-1)\frac{\ln^2 n}{\ln\ln n}$. We believe that if $\mathcal{F}$ is an r-wise fractional L-intersecting family of maximum cardinality, where s $(= |L|)$ is a constant, then $|\mathcal{F}| \in \Theta(rn^s)$.

We note that the linear algebraic techniques which are useful to derive the bounds on fractional L-intersecting families are no longer directly applicable in this case due to the requirements. In Section 2, we use a special refinement trick to reduce it into a form such that linear algebraic methods can be used.

Next, we turn our attention to the case when $|L| = s = 1$. In the context of classical L intersecting families, when $|L| = s = 1$, the Fisher's Inequality (see Theorem 7.5 in [18]) yields $|\mathcal{F}| \leq n$, where $\mathcal{F}$ is an L intersecting family. Study of such intersecting families was initiated by Ronald Fisher in 1940 (see [19]) in the context of design theory. Analogously, consider the scenario when $L = \{\frac{a}{b}\}$ is a singleton set. Can we get a tighter bound (compared to Theorem 5) in this case? We show in Theorem 6 that if b is a constant prime we do have a tighter bound.

Theorem 6. *Let n be a positive integer. Let $\mathcal{G}$ be an r-wise fractional L-intersecting families of subsets of $[n]$, where $L = \{\frac{a}{b}\}$, $\frac{a}{b} \in [0, 1)$, and b is a prime. Then, $|\mathcal{G}| \leq (b-1)(r-1)(n+1)\lceil\frac{\ln n}{\ln b}\rceil + r - 1$.*

Assuming $L = \{\frac{1}{2}\}$, Examples 5 in Section 3 give r-wise fractional L-intersecting families on $[n]$ of cardinality $\Omega(n \ln r)$ thereby implying that the bound obtained in Theorem 6 is asymptotically tight up to a factor of $r\frac{\ln n}{\ln r}$ when b is a constant prime. We believe that the cardinality of such families is at most crn, where $c > 0$ is a constant.

The rest of the paper is organized in the following way: in Section 2, we give the proof of Theorem 5 after introducing some necessary lemmas in the beginning. In Section 3, we consider the case when L is a singleton set and give the proof of Theorem 6. Finally, we conclude with some remarks, some open questions, and a conjecture.

Before moving on to the proof of Theorem 5, we state few key lemmas that will be essential in the proof.

Lemma 7 (Lemma 13.11 in [18], Proposition 2.5 in [20])**.** *For $i = 1, \ldots, m$ let $f_i : \Omega \to \mathbb{F}$ be functions and $v_i \in \Omega$ elements such that*

(a) $f_i(v_i) \neq 0$ for all $1 \leq i \leq m$;
(b) $f_i(v_j) = 0$ for all $1 \leq j < i \leq m$.

Then $f_1, \ldots, f_m$ are linearly independent members of the space $\mathbb{F}^\Omega$.

Lemma 8 (Lemma 5.38 in [20])**.** *Let p be a prime; $\Omega = \{0,1\}^n$. Let $f \in \mathbb{F}_p^\Omega$ be defined as $f(x) = \sum_{i=1}^n x_i - k$. For any $A \subseteq [n]$, let $V_A \in \{0,1\}^n$ denote its 0-1 incidence vector and let $x_A = \Pi_{j \in A} x_j$. Assume $0 \leq s, k \leq p-1$ and $s + k \leq n$. Then, the set of functions $\{x_A f \ : \ |A| \leq s-1\}$ is linearly independent in the vector space $\mathbb{F}_p^\Omega$ over $\mathbb{F}_p$.*

2. Proof of Theorem 5

Let $\mathcal{F}$ be an r-wise fractional L-intersecting family of subsets of $[n]$, where $r \geq 3$, L is as defined in the theorem. Let p be a prime. We partition $\mathcal{F}$ into p parts, namely $\mathcal{F}_0, \ldots, \mathcal{F}_{p-1}$, where $\mathcal{F}_j = \{A \in \mathcal{F} \ : \ |A| \equiv j \ (mod\ p)\}$.

Estimating $|\mathcal{F}_j|$, when $j > 0$ If for every pair of sets $A, B \in \mathcal{F}_j$, $|A \cap B| \in \{\frac{a_1}{b_1}|A|, \ldots, \frac{a_s}{b_s}|A|, \frac{a_1}{b_1}|B|, \ldots, \frac{a_s}{b_s}|B|\}$, choose the set A with largest cardinality in $\mathcal{F}_j$, set $X_1 = A$ and $Y_1 = A$, and remove A from $\mathcal{F}_j$. Otherwise, there is a collection of k sets $\{A_1, \ldots, A_k\}$ such that $|\cap_{i=1}^k A_i| \notin \{\frac{a_1}{b_1}|A_1|, \ldots, \frac{a_s}{b_s}|A_1|, \ldots, \frac{a_1}{b_1}|A_k|, \ldots, \frac{a_s}{b_s}|A_k|\}$, and addition of any more set A into $\{A_1, \ldots, A_k\}$ makes $|\cap_{i=1}^k A_i \cap A| \in \{\frac{a_1}{b_1}|A_1|, \ldots, \frac{a_s}{b_s}|A_1|, \ldots, \frac{a_1}{b_1}|A_k|, \ldots,$ $\frac{a_s}{b_s}|A_k|, \frac{a_1}{b_1}|A|, \ldots, \frac{a_s}{b_s}|A|\}$. Set $X_1 = A_1$ and $Y_1 = \cap_{i=1}^k A_i$. Remove $A_1, \ldots, A_k$ from $\mathcal{F}_j$. Repeat the process until no more set is left in $\mathcal{F}_j$. Let X_i, Y_i be sets constructed as above, $1 \leq i \leq m$. Observe that

$$m \geq \frac{|\mathcal{F}_j|}{r-1}. \tag{1}$$

Let $X_i = B_1$, $Y_i = B_1 \cap \ldots \cap B_k$ be a pair of sets constructed as above for some k and i. By construction,

$$|X_i \cap Y_i| = |Y_i| \notin \{\frac{a_1}{b_1}|B_1|, \ldots, \frac{a_s}{b_s}|B_1|, \ldots, \frac{a_1}{b_1}|B_k|, \ldots, \frac{a_s}{b_s}|B_k|\}, \text{ and}$$
$$|X_r \cap Y_i| \in \{\frac{a_1}{b_1}|B_1|, \ldots, \frac{a_s}{b_s}|B_1|, \ldots, \frac{a_1}{b_1}|B_k|, \ldots, \frac{a_s}{b_s}|B_k|, \frac{a_1}{b_1}|X_r|, \ldots, \frac{a_s}{b_s}|X_r|\},$$

for all $r > i$.

With each X_i and Y_i, associate the 0-1 incidence vector x_i and y_i, where $x_i(l) = 1$ if and only if $l \in X_i$. Define m functions f_1 to f_m, where each $f \in \mathbb{F}_p^{\{0,1\}^n}$, in the following way.

$$f_i(x) = \left(\langle x, y_i\rangle - \frac{a_1}{b_1}j\right)\left(\langle x, y_i\rangle - \frac{a_2}{b_2}j\right)\cdots\left(\langle x, y_i\rangle - \frac{a_s}{b_s}j\right). \tag{2}$$

It follows that

$$f_i(x_i) = \left(\langle x_i, y_i\rangle - \frac{a_1}{b_1}j\right)\left(\langle x_i, y_i\rangle - \frac{a_2}{b_2}j\right)\cdots\left(\langle x_i, y_i\rangle - \frac{a_s}{b_s}j\right) \neq 0$$

for $1 \leq i \leq m$, unless $j = 0$. Moreover, $f_i(x_r) = 0$ for $1 \leq i < r \leq m$. Using Lemma 7, it follows that the multilinear polynomials $f_1, \ldots, f_m$ are linearly independent over $\mathbb{F}_p^{\{0,1\}^n}$. The dimension of the space is $\sum_{l=0}^{s}\binom{n}{l}$. Therefore, $\sum_{l=0}^{s}\binom{n}{l} \geq m \geq \frac{|\mathcal{F}_j|}{r-1}$. This implies that $|\mathcal{F}_j| \leq (r-1)\left(\sum_{l=0}^{s}\binom{n}{l}\right)$.

The maximum value of j is $p-1$ and we will show shortly that the maximum value of p needed in the proof is $2\ln n$. So, choosing $s \leq n+1-2\ln n$, the requirements of Lemma 8 are satisfied. We can now improve the upper bound on $|\mathcal{F}_j|$ by using the swallowing trick and Lemma 8 to prove that $\{f_i : 1 \leq i \leq m\} \cup \{x_A f : |A| < s\}$ (where $f(x) = \sum_{i=1}^{n} x_i - j$) is a collection of functions that is linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over $\mathbb{F}_p$. These functions can be obtained as a linear combination of distinct monomials of degree at most s. This implies that $\sum_{l=0}^{s}\binom{n}{l} \geq m + \sum_{l=0}^{s-1}\binom{n}{l}$, that is $m \leq \binom{n}{s}$. This yields $|\mathcal{F}_j| \leq (r-1)\binom{n}{s}$.

From the discussion above, it is clear that

$$|\mathcal{F}_j| \leq \begin{cases} (r-1)\binom{n}{s}, & \text{if } s \leq n+1-2\ln n \\ (r-1)\left(\sum_{l=0}^{s}\binom{n}{l}\right), & \text{otherwise} \end{cases} \quad \text{for } j > 0. \tag{3}$$

Estimating $|\mathcal{F}_0|$ In order to estimate $|\mathcal{F}_0|$, we choose a collection $p_1 < p_2 < \ldots < p_t$ of t smallest primes such that $p_1 p_2 \ldots p_t > n$. This implies that every set F in $\mathcal{F}$ has a prime p such that $p \nmid |F|$ – that is, F will be counted in the estimation of $|\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{p-1}|$. So,

$$|\mathcal{F}| \leq \begin{cases} t * (p_t - 1)(r-1)\binom{n}{s}, & \text{if } s \leq n+1-2\ln n, \\ t * (p_t - 1)(r-1)\left(\sum_{l=0}^{s}\binom{n}{l}\right), & \text{otherwise.} \end{cases} \tag{4}$$

Now, the only thing that remains is to estimate t and p_t. The product of the first t primes is the *primorial function* $p_t\#$ and it is known that $p_t\# = e^{(1+o(1))t\ln t}$. Setting $p_t\# = e^{(1+o(1))t\ln t} > n$, we get $t \leq \frac{\ln n}{\ln\ln n}$. Moreover, using the Prime Number Theorem (see Section 5.1 of [21]), the tth largest prime is at most $2t\ln t$. Using these facts and Inequality 4, Theorem 5 follows. □

3. L is a singleton set

As explained in Section 1, Fisher's Inequality is a special case of the classical L-intersecting families, where $|L| = 1$. In this section, we study r-wise fractional L-intersecting families with $|L| = 1$; a fractional variant of the Fisher's inequality.

3.1. Proof of Theorem 6

Statement of Theorem 6: Let n be a positive integer. Let $\mathcal{G}$ be an r-wise fractional L-intersecting families of subsets of $[n]$, where $L = \{\frac{a}{b}\}$, $\frac{a}{b} \in [0,1)$, and b is a prime. Then, $|\mathcal{G}| \leq (b-1)(r-1)(n+1)\lceil\frac{\ln n}{\ln b}\rceil + r - 1$.

Proof. It is easy to see that if $a = 0$, then $|\mathcal{G}| \leq n$ with the set of all singleton subsets of $[n]$ forming a tight example to this bound. So assume $a \neq 0$. Let $\mathcal{F} = \mathcal{G} \setminus \mathcal{H}$, where $\mathcal{H} = \{A \in \mathcal{G} \;:\; b \nmid |A|\}$. From the definition of an r-wise fractional $\frac{a}{b}$-intersecting family it is clear that $|\mathcal{H}| \leq r-1$. The rest of the proof is to show that $|\mathcal{F}| \leq (b-1)(r-1)(n+1)\lceil\frac{\ln n}{\ln b}\rceil$. We do this by partitioning $\mathcal{F}$ into $(b-1)\lceil\log_b n\rceil$ parts and then showing that each part is of size at most $(r-1)(n+1)$. We define F_i^j as

$$\mathcal{F}_i^j = \{A \in \mathcal{F} \,|\, |A| \equiv j (\text{mod } i)\}.$$

Since b divides $|A|$, for every $A \in \mathcal{F}$, under this definition $\mathcal{F}$ can be partitioned into families $\mathcal{F}_{b^k}^{ib^{k-1}}$, where $2 \leq k \leq \lceil\log_b n\rceil$ and $1 \leq i \leq b-1$. We show that, for every $i \in [b-1]$ and for every $2 \leq k \leq \lceil\log_b n\rceil$, $|\mathcal{F}_{b^k}^{ib^{k-1}}| \leq (r-1)(n+1)$.

In order to estimate $|\mathcal{F}_{b^k}^{ib^{k-1}}|$, for each $A \in \mathcal{F}_{b^k}^{ib^{k-1}}$, create a vector X_A as follows:

$$X_A(j) = \begin{cases} \frac{1}{\sqrt{b^{k-2}}}, \text{ if } j \in A; \\ 0, \text{ otherwise.} \end{cases}$$

Definition 4. Let $x^1, \ldots, x^r \in \mathbb{F}^n$ for some field $\mathbb{F}$, where $x^i = (x_1^i, \ldots, x_n^i)$. The r-wise dot product, denoted as $\langle x^1, \ldots, x^r \rangle$ is defined as $\langle x^1, \ldots, x^r \rangle = \sum_{i=1}^n x_i^1 x_i^2 \ldots x_i^r$.

Note that, for distinct sets $A_1, \ldots, A_r \in \mathcal{F}_{b^k}^{ib^{k-1}}$

$$\begin{aligned} &\langle X_{A_j}, X_{A_j} \rangle \equiv b (\text{mod } b^2), \\ &\langle X_{A_1}, \ldots, X_{A_r} \rangle \equiv ai (\text{mod } b). \end{aligned} \tag{5}$$

Estimating $|\mathcal{F}_{b^k}^{ib^{k-1}}|$ If for every pair of sets $A, B \in \mathcal{F}_{b^k}^{ib^{k-1}}$, $|A \cap B| \equiv ai (\text{mod } b)$, choose the set A with largest cardinality in $\mathcal{F}_{b^k}^{ib^{k-1}}$, set $C_1 = A$ and $D_1 = A$, and remove A from $\mathcal{F}_{b^k}^{ib^{k-1}}$. Otherwise, there is a collection of k sets $\{A_1, \ldots, A_k\}$ such that $|\cap_{j=1}^k A_j| \not\equiv ai (\text{mod } b)$, and addition of any more set A into $\{A_1, \ldots, A_k\}$ makes $|\cap_{j=1}^k A_j \cap A| \equiv ai (\text{mod } b)$. Set $C_1 = A_1$ and $D_1 = \cap_{j=1}^k A_j$. Remove $A_1, \ldots, A_k$ from $\mathcal{F}_{b^k}^{ib^{k-1}}$. Repeat the process until no more set is left in $\mathcal{F}_{b^k}^{ib^{k-1}}$. Let C_j, D_j be sets constructed as above, $1 \le j \le m$. Observe that

$$m \ge \frac{|\mathcal{F}_{b^k}^{ib^{k-1}}|}{r-1}. \tag{6}$$

Let $C_j = B_1$, $D_j = B_1 \cap \ldots \cap B_k$ be a pair of sets constructed as above for some k and j. By construction, $|C_j \cap D_j| = |D_j| \notin \{\frac{a}{b}|B_1|, \ldots, \frac{a}{b}|B_k|\}$, and $|C_r \cap D_j| \in \{\frac{a}{b}|B_1|, \ldots, \frac{a}{b}|B_k|, \frac{a}{b}|C_r|\}$, for all $r > j$. From the definition of $\mathcal{F}_{b^k}^{ib^{k-1}}$, Equation 5, and construction above, it follows that for any $1 \le j, l \le m$,

$$\langle X_{C_j}, X_{D_l} \rangle \begin{cases} \not\equiv ai(\text{mod } b), & \text{if } j = l, \\ \equiv ai(\text{mod } b), & \text{if } j > l, \end{cases}$$

Define m functions f_1 to f_m, where each $f_j \in \mathbb{F}_b^{\mathbb{R}^n}$, in the following way.

$$f_j(x) = (\langle x, X_{D_j} \rangle - ai).$$

It follows that

$$f_j(X_{C_r}) \begin{cases} \neq 0, & \text{if } j = r, \\ = 0, & \text{if } r > j, \end{cases}$$

So, f_j's are linearly independent in the vector space ${\mathbb{F}_b}^{\mathbb{R}^n}$ over $\mathbb{F}_b$ (by Lemma 7). Each f_j is thus an appropriate linear combination of distinct monomials of degree at most one. Therefore, $m \le \sum_{j=0}^{1} \binom{n}{j} = n + 1$. Thus, using Equation 6, $|\mathcal{F}_{b^k}^{ib^{k-1}}| \le (r-1)(n+1)$. This concludes the proof of the Theorem.

□

We shall call $\mathcal{F}$ a *r-wise bisection closed family* if $\mathcal{F}$ is a fractional L-intersecting family where $L = \{\frac{1}{2}\}$. We have the following construction that yields an r-wise bisection closed family of cardinality at least $n\{1 + \frac{1}{2} + \ldots + \frac{1}{r}\} - 2r$ on $[n]$.

Example 5. Let n be an even positive integer. Let $\mathcal{B}_1$ denote the collection of 2-sized sets that contain only 1 as a common element in any two sets, i.e. $\{1,2\}, \{1,3\}, \ldots, \{1,n\}$; and let $\mathcal{B}_2$ denote collection of 4-sized sets that contain only $\{1,2\}$ as common elements, i.e. $\{1,2,3,4\}$, $\{1,2,5,6\}, \ldots, \{1,2,n-1,n\}$. Similarly, let $\mathcal{B}_i$ denote collection of $2i$-sized sets that contain only $\{1, 2, \ldots, i\}$ as common elements, i.e. $\{1, 2, \ldots, i, i+1, \ldots, 2i\}$, $\{1, 2, \ldots, i, 2i+1, \ldots, 3i\}, \ldots, \{1, 2, \ldots, i, n-i+1, \ldots, n\}$, for $1 \leq i \leq r$ (possibly excluding the last set in the family if it is not of size $2i$). It is not hard to see that $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_r$ is indeed r-wise bisection closed.

4. Discussion

In this paper, we introduce and study the notion of r-wise fractional L-intersecting families, which is a generalization of notion of fractional L-intersecting families studied in [17]. If $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$, Theorem 5 gives an upper bound of $\mathcal{O}\left(\frac{\ln^2 n}{\ln \ln n} r \binom{n}{s}\right)$ on the size of such families. When L is a singleton set, this translates to an upper bound of $\mathcal{O}\left(rn \frac{\ln^2 n}{\ln \ln n}\right)$ on the size of such families. If $L = \{\frac{a}{b}\}$, $\frac{a}{b} \in [0, 1)$, and b is a prime, Theorem 6 gives an upper bound of $\mathcal{O}\left(rn \ln n\right)$ We believe that in this case, the upper bound should be linear which we pose as an open problem.

Conjecture 9. *Let $\mathcal{F}$ be an r-wise fractional L-intersecting family, where $L = \{a/b\}$. Then, $|\mathcal{F}| = \mathcal{O}(rn)$.*

Let r be a fixed constant and $L = \{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$, where s is a constant. The collection of all the s-sized subsets of $[n]$ is an r-wise fractional L-intersecting family of cardinality $\binom{n}{s}$. In this case, the bound given by Theorem 5 is asymptotically tight up to a factor of $\frac{\ln^2 n}{\ln \ln n}$. We believe that in this case, $|\mathcal{F}| \in \Theta(n^s)$ and improving the bound in Theorem 5 remains open.

In Theorem 6 and Theorem 8 of [17], the authors have shown linear upper bound for fractional L-intersecting families for large sized sets and sets of size nearly $\frac{n}{2}$, respectively. Obtaining similar bounds in the case of r-wise fractional L-intersecting families remains open.

Conflict of Interest

The author declares that there is no conflict of interest.

References

[1] Dijen K. Ray-Chaudhuri and Richard M. Wilson. On t-designs. *Osaka Journal of Mathematics*, 12(3):737–744, 1975. MR0392624

[2] Peter Frankl and Richard M. Wilson. Intersection theorems with geometric consequences. *Combinatorica*, 1(4):357–368, 1981. MR0647986

[3] Noga Alon, László Babai, and Hiroshi Suzuki. Multilinear polynomials and Frankl-Ray-Chaudhuri-Wilson type intersection theorems. *Journal of Combinatorial Theory, Series A*, 58(2):165–180, 1991. MR1129114

[4] Vince Grolmusz and Benny Sudakov. On k-wise set-intersections and k-wise hamming-distances. *Journal of Combinatorial Theory, Series A*, 99(1):180–190, 2002. MR1911465

[5] Zoltán Füredi and Benny Sudakov. Extremal set systems with restricted k-wise intersections. *Journal of Combinatorial Theory, Series A*, 105, pp. 143–159, 2004. MR2030145

[6] Qingde Kang, Jiuqiang Liu, Larry X.W.Wang. Restricted t-wise $\mathcal{L}$-intersecting families on set systems. *European Journal of Combinatorics*, 32(3), pp. 464–471, 2011. MR2764809

[7] Hunter S. Snevily. A sharp bound for the number of sets that pairwise intersect at k positive values. *Combinatorica*, 23(3):527–533, 2003. MR2021164

[8] Hunter S. Snevily. On generalizations of the deBruijn–Erdös theorem. *Journal of Combinatorial Theory, Series A*, 68(1), pp. 232–238, 1994. MR1295795

[9] William Y.C. Chen, Jiuqiang Liu. Set systems with $\mathcal{L}$-intersections modulo a prime number. *Journal of Combinatorial Theory, Series A*, 116(1), pp. 120–131, 2009. MR2469252

[10] Jian Liu, Jiuqiang Liu. Set systems with cross $\mathcal{L}$-intersection and k-wise $\mathcal{L}$-intersecting families. *Discrete Mathematics*, 309(20), pp. 5920–5925, 2009. MR2552623

[11] Jiuqiang Liu, Wenbo Yang. Set systems with restricted k-wise $\mathcal{L}$-intersections modulo a prime number. *European Journal of Combinatorics*, 36(1), pp. 707–719, 2014. MR3131922

[12] Jiuqiang Liu, Shenggui Zhang, Shuchao Li, Huihui Zhang. Set systems with k-wise $\mathcal{L}$-intersections and codes with restricted Hamming distances *European Journal of Combinatorics*, 58(1), pp. 166–180, 2016. MR3530627

[13] Shuchao Li, Huihui Zhang. Set Systems with L-Intersections and k-Wise L-Intersecting Families. *Journal of Combinatorial Designs*, 24(11), pp. 514–529, 2016. MR3549368

[14] Jiuqiang Liu, Xiaodong Liu. Set systems with positive intersection sizes. *Discrete Mathematics*, 340(10), pp. 2333–2340, 2017. MR3674131

[15] Jimeng Xiao, Jiuqiang Liu, Shenggui Zhang. Families of vector spaces with k-wise $\mathcal{L}$-intersections. *European Journal of Combinatorics*, 58(1), pp. 166–180, 2016. MR3764352

[16] Xin Wang, Hengjia Wei, Gennian Ge. A strengthened inequality of Alon–Babai–Suzuki's conjecture on set systems with restricted intersections modulo p. *Discrete Mathematics*, 341(1), pp. 109–118, 2018. MR3713389

[17] Niranjan Balachandran, Rogers Mathew, and Tapas Kumar Mishra. Fractional L-intersecting families. *The Electronic Journal of Combinatorics*, 26(2):2.40, 2019. MR3982269

[18] Stasys Jukna. *Extremal combinatorics: with applications in computer science*. Springer Science & Business Media, 2011. MR2865719

[19] Ronald Aylmer Fisher. An examination of the different possible solutions of a problem in incomplete blocks. *Annals of Human Genetics*, 10(1):52–75, 1940. MR0002086

[20] László Babai and Péter Frankl. *Linear Algebra Methods in Combinatorics: With Applications to Geometry and Computer Science*. Department of Computer Science, Univ. of Chicago, 1992.

[21] Victor Shoup. *A computational introduction to number theory and algebra*. Cambridge University Press, 2009. MR2488898

Tapas Kumar Mishra
National Institute of Technology Rourkela
Odisha
India
769008
E-mail address: mishrat@nitrkl.ac.in

Received December 1, 2021