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Generalizations of leaky forcing

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Motivated by the inverse eigenvalue problem, vertex leaky forcing was recently introduced as a new variation of zero forcing in order to show how vertex leaks can disrupt the zero forcing process in a graph. An edge leak is an edge that is not allowed to be forced across during the zero forcing process. The ℓ-edge-leaky forcing number of a graph is the size of a smallest zero forcing set that can force the graph blue despite ℓ edge leaks. This paper contains an analysis of the effect of edge leaks on the zero forcing process instead of vertex leaks. Furthermore, specified ℓ-leaky forcing is introduced. The main result is that ℓ-leaky forcing, ℓ-edge-leaky forcing, and specified ℓ-leaky forcing are equivalent. Furthermore, all of these different kinds of leaks can be mixed so that vertex leaks, edge leaks, and specified leaks are used. This mixed ℓ-leaky forcing number is also the same as the (vertex) ℓ-leaky forcing number.

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1. Introduction

Zero forcing was introduced by the AIM Minimum Rank and Special Graphs Work Group in [2], in order to find upper bounds for the maximum nullity for the family of real symmetric matrices whose off-diagonal entries are described by a graph. The zero forcing process uses a set of blue vertices in a graph that color other vertices blue given a color change rule. Given a graph G and a blue vertex $v \in V(G)$, if v has one white neighbor w, then v forces w (v colors w blue). Formally, this process is known as the *zero forcing color-change rule.* A *zero forcing set* for G is an initial set of blue vertices B such that after iteratively and exhaustively applying the zero forcing color-change rule, every vertex in G is blue. The *zero forcing number* of a graph is the size of a minimum zero forcing set, and is denoted $\mathrm{Z}(G)$.

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Zero forcing has shown up as a way to control quantum systems [6, 11]. In fact, it was shown that if a set of vertices is a zero forcing set, then the associated quantum system is controllable [5]. Another system that utilizes the zero forcing process is the electric power system. In [10], Haynes et al. looked into the problem of monitoring an electric power system by placing as few measurement devices as possible. These applications of zero forcing lead to a natural questions: What if something breaks in the system? Is there a way to keep control? These questions were the main focus in [1] and [9]. In [9], Dillman and Kenter introduced *leaky forcing*, which is a variation on zero forcing that focuses on when vertices in a graph are not able to force. Leaky forcing uses the same color-change rule as zero forcing, but certain vertices are not allowed to perform forces.

Given a graph G, a *vertex leak* (also referred to as a leak) is a vertex in G that is not able to perform a force. An *ℓ-leaky forcing set*, is a zero forcing set such that for any set of ℓ vertex leaks in G, exhaustively applying the color-change rule results in every vertex in G becoming blue. The *ℓ-leaky forcing number* for a graph G is the size of a minimum ℓ-leaky forcing set, and we denote this by $\mathrm{Z}_{(\ell)}(G)$. Notice that $\mathrm{Z}(G) = \mathrm{Z}_{(0)}(G)$. Furthermore, the notion of how resilient a graph is to leaks, and which structures need to be circumvented in a graph for a zero forcing set to be an ℓ-leaky forcing set were explored in [1]. The notation used in this paper will follow the notation introduced in [1]. The rest of this section contains results from [1] which are useful for exploring variations of leaky forcing.

In general, let $B \subseteq V(G)$ be an initial set of blue vertices in G. If vertex u colors v blue, then u forces v and we denote this by $u \to v$. The symbol $u \to v$ is called a force. A *set of forces F of B in G* is a set of forces such that there is a chronological ordering of the forces in F where each force is valid and the whole graph turns blue. When the set B is clear form context, F may be referred to as a forcing process of B or a forcing process F (suppressing the reference to B). Intuitively, F represents the instructions for how B can force G blue, or provides a proof that B is a zero forcing set. Implicitly, F gives rise to discrete time steps in which sets of white vertices turn blue. A set B' such that $B \subseteq B' \subseteq V(G)$ is *obtained from B using F* if B can color B' blue using only a subset of forces in a forcing process F. More generally, B' is obtained from B if there is some forcing process F by which B can color B' blue.

Given an initial set of blue vertices B, the set $B^{[\infty]}$ is the set of blue vertices obtained after exhaustively applying the zero forcing color-change rule. Therefore, if B is a zero forcing set, then $B^{[\infty]} = V$. Furthermore, let

$B_L^{[\infty]}$ be the set of blue vertices obtained from B after the zero forcing color-change rule has been exhaustively applied, and a set of leaks L are present in the graph. In particular, $B_L^{[\infty]}$ will be determined after a set of leaks L has been chosen.

Let $\mathcal{F}(B)$ denote the set of all possible forces given a vertex set B. That is, $u \to v \in \mathcal{F}(B)$ if there exists a set of forces F of B in G that contains $u \to v$. Given this notation, B is an ℓ-leaky forcing set if for every $L \subseteq V(G)$ with $|L| = \ell$ there exists a forcing process F such that if $u \to v \in F$, then $u \notin L$.

Suppose $S \subseteq V(G)$ and F is a forcing process. Let

$$F(S) = \{x \to y \in F : y \notin S\}.$$

By extension,

$$F \setminus F(S) = \{x \to y \in F : y \in S\}.$$

The following lemma proves that abandoning process F to follow process F' creates a new forcing process.

Lemma 1.1 ([1])**.** *Let B be a blue set in G with forcing processes F and F'. Then $(F \setminus F(B')) \cup F'(B')$ is a forcing process of B for any B' obtained from B using F.*

The next lemma shows that for any $(\ell - 1)$-leaky forcing set B and set of ℓ vertex leaks L, there exists a time when all ℓ leaks in L are blue. Furthermore, there is also a time when all but one of the ℓ leaks in L are blue.

Lemma 1.2 ([1])**.** *If B is an $(\ell-1)$-leaky forcing set and L is a set of $k \geq \ell$ vertex leaks, then $|L \setminus B_L^{[\infty]}| \leq k - \ell$.*

The previous two lemmas are used to prove Theorem 1.3. The gist of the proof is to use a forcing process that turns all but one of the leaks blue. This is possible by Lemma 1.2. At this point, the forcing process is abandoned for a process that will completely force the graph despite the remaining leak. Switching forcing processes is justified by Lemma 1.1.

Theorem 1.3 ([1])**.** *A set B is an ℓ-leaky forcing set if and only if B is an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ vertex leaks L and $v \in V(G) \setminus B$ there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

A natural generalization of ℓ-leaky forcing is to consider what happens when forces are prohibited from passing over particular edges. An edge xy

is an *edge leak* if neither $x \to y$ nor $y \to x$ are allowed. A set of blue vertices B is an ℓ-edge-leaky forcing set if B can turn the whole graph G blue given any set of ℓ edge leaks. Denote the ℓ-edge-leaky forcing number of a graph G by $\mathrm{Z}'_{(\ell)}(G)$.

Edge-leaky forcing is analysed in Section 2. The main result of this section is that the ℓ-edge-leaky forcing number is the same as the ℓ-leaky forcing. Section 3 introduces specified leaks, and shows that preventing a directional force is also equivalent to vertex leaky forcing. In Section 4, vertex leaks, edge leaks, and specified leaks are mixed in the leak set without changing the underlying behavior of leaky forcing. Furthermore, in Section 5, sets of leaks with a particular underlying structure are explored. In general, analogs of Lemma 1.2 will be used to conclude that the condition in Theorem 1.3 applies for ℓ-edge-leaky forcing and specified ℓ-leaky forcing.

1.1. Motivation for variations of leaky forcing

The Inverse Eigenvalue Problem for a graph (IEPG) G on n vertices is to determine all possible spectra of matrices in

$$\mathcal{S}(G) = \{A \in \mathbb{R}^{n\times n} : a_{i,j} = a_{j,i}, \text{ for } i \neq j,\ a_{i,j} \neq 0 \text{ iff } ij \in E(G)\}.$$

In particular, the IEPG is about the spectra of real symmetric matrices, with free diagonal entries, and off diagonal entries that respect the pattern of a graph's edges.

Zero forcing on G was introduced as a way to bound the maximum nullity of matrices in $\mathcal{S}(G)$. Consider the following system:

$$\begin{aligned}
&(1) \qquad & a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 &= 0\\
&(2) \qquad & a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + 0x_4 &= 0\\
&(3) \qquad & a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + 0x_4 &= 0\\
&(4) \qquad & a_{4,1}x_1 + 0x_2 + 0x_3 + a_{4,4}x_4 &= 0
\end{aligned}$$

or equivalently,

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & 0 & 0 & a_{4,4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Zero forcing studies the minimum number of entries in the x vector that need to be set to 0 before one can conclude that the whole x vector is identically 0

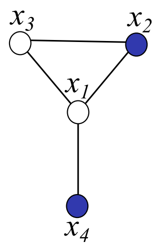

Figure 1: The paw graph with a corresponding zero forcing set.

under the additional assumption that $a_{i,j} = a_{j,i} \neq 0$ for $i \neq j$. In particular, determining that $x_2 = x_4 = 0$ implies $x_1 = x_3 = 0$ is equivalent to seeing that $\{x_2, x_4\}$ is zero forcing set in Figure 1. The bound on the maximum nullity of $\mathcal{S}(G)$ comes from the fact, that if the entries in x indexed by B are zero implies that all entries are zero, then the columns of A indexed by $V(G) \setminus B$ must be linearly independent.

Though vertex-leaky forcing is a natural generalization of zero forcing, its relationship to the linear algebra roots of zero forcing is less clear. We can interpret a vertex leak as a prohibition of using the corresponding equation to force zeros in the x vector. For example, supposing that x_2 is a leak in the paw graph in Figure 1 would prevent us from using equation (2) to solve for $x_3 = 0$ in the zero forcing process. This is an unsatisfying interpretation of vertex-leaky forcing since forgetting an equation out of a linear system is somewhat unnatural.

The main result of this paper is Corollary 2.4 which states that edge-leaky forcing sets and vertex leaky forcing sets are the same. An edge leak at x_4x_1 corresponds to assuming that $a_{1,4}$ and $a_{4,1}$ are zero divisors. That is, if $a_{4,1}$ is a zero divisor and $x_4 = 0$, then $a_{4,1}x_1 + a_{4,4}x_4 = 0$ does not imply that $x_1 = 0$. Equivalently, if x_1x_4 is an edge leak, then x_1 cannot be used to turn x_4 blue. We will return to this interpretation shortly.

A key insight in the study of the IEPG is that $\mathcal{S}(G)$ is a manifold. This allows us to use the Implicit Function Theorem to study graph minor monotone parameters defined on $\mathcal{S}(G)$, by transversely intersecting $\mathcal{S}(G)$ with manifolds corresponding to various spectral properties (the rank manifold, multiplicity list manifold, spectrum manifold, etc.). For references on the use of the Implicit Function Theorem to define strong properties see [7, 8, 3, 4]. Close inspection of $\mathcal{S}(G)$ shows that $\mathcal{S}(G)$ is the disjoint union of cones in $\mathbb{R}^{n\times n}$ were the discontinuities arise from the fact that $a_{i,j}$ cannot equal zero if ij is an edge in the graph. Therefore, a natural thing to wonder is which cones contained in $\mathcal{S}(G)$ can transversely intersect a spectrally defined manifold $\mathcal{M}$, since it may be the case that each cone exhibits its own spectral behavior.

These considerations motivate the definition and study of the Inverse Eigenvalue Problem for Signed Multi-Graphs. For a signed multi-graph G_s with underlying graph G, we say the patter of a matrix A respects G_s, or $A \in \mathcal{S}(G_s)$ if

- $a_{ij} = 0$ whenever the pair $\{i, j\}$ has no edge,
- $a_{ij} > 0$ whenever the pair $\{i, j\}$ has at least one positive edge but no negative edge, and
- $a_{ij} < 0$ whenever the pair $\{i, j\}$ has at least one negative edge but no positive edge.

Notice that pairs $\{i, j\}$ that have both positive and negative edges in G_s can be associated with any real valued entry in a_{ij}. The manifold $\mathcal{S}(G_s)$ for a simple underlying graph G would pick out a particular cone of the manifold corresponding to the unsigned graph underlying G_s. The signed multi-edges allow us to consider the manifolds where we glue together the various cones from the unsigned case, by allowing an entry $a_{i,j}$ to take the boundary value 0 for some edges.

While gluing together cones makes the manifolds in question topologically nicer, it also introduces zero divisors into the zero forcing process. That is, allowing $a_{i,j}$ to possibly take the value zero for some edge $ij \in E(G)$ amounts to placing an edge leak at edge ij. This alone should motivate the study of zero forcing on signed multi-graphs. In this setting, the edge leaks would be given before choosing a zero forcing set since the leaks must appear on particular edges of the signed multi-graph. One trivial bound for a signed multi-graph G_s with underlying graph G would be $\mathrm{M}(G_s) \leq \mathrm{Z}(G_s) \leq \mathrm{Z}(G) + \ell$ where $\mathrm{M}(G_s)$ is the maximum nullity over matrices in $\mathcal{S}(G_s)$ and ℓ is the number of pairs of vertices with both positive and negative edges between them.

In general, zero forcing on a signed multi-graph should be harder than zero forcing on the simple underlying graph, since in the signed multi-graph case there may be some edges that we are not allowed to use. Furthermore, there are many signings and "multi-ing" of a simple graph, which leads to a combinatorial explosion in the amount of work that needs to be done to understand $\mathrm{Z}(G_s)$ for even just small signed multi-graphs.

The ℓ-edge-leaky forcing number of a simple graph G (and by Corollary 2.4, the ℓ-vertex-leaky forcing number) addresses the combinatorial explosion by seeking a zero forcing bound that holds over manifolds obtained by gluing together ℓ adjacent cones in $\mathbb{R}^{n\times n}$ (which are sub-manifolds of $\mathcal{S}(G)$). Symbolically, $\mathrm{Z}(G_s) \leq \mathrm{Z}'_{(\ell)}(G)$ where G_s is a multi-signing of G with at most ℓ edges that are both positive and negative. With this in mind, a reasonable program for further research would be to characterize graphs G such that $\mathrm{Z}(G) = \mathrm{Z}'_{(\ell)}(G)$, where we consider Z as an imperfect proxy for the maximum nullity (there are graphs where Z and M are different). Some work in this direction comes immediately through the equivalence of $\mathrm{Z}_{(\ell)}$ and $\mathrm{Z}'_{(\ell)}$. In fact, the authors of this paper have shown that if $\mathrm{Z}_{(\ell)}(G) = \mathrm{Z}(G)$, then G cannot have small and compact edge cuts (see Theorem 3.4 in [1]).

On its face, the motivation for vertex-leaky forcing comes from a desire to understand zero forcing as a graph coloring process. Vertex leaks obstruct the zero forcing process in the sense that vertex leaks must be turned blue, will not force other vertices, and can stop their neighbors from forcing other vertices as long as the leaks are white. This kind of obstruction is stronger than vertex deletion, since deleting vertex v is equivalent to coloring v blue (for free) and making it a leak. Admittedly, this is not particularly motivating if the primary interest in zero forcing comes from its applications to the inverse eigenvalue problem. Furthermore, the obstruction caused by a vertex leak may or may not actually correspond to the physical disruption of a network. Edge leaks, on the other hand, do have a linear algebraic interpretation. The edge-leaky forcing number is essentially a measurement of how much worse the zero forcing bound on the maximum nullity can get across some glued together cones of $\mathcal{S}(G)$.

To round out this section, we will spare a few words for the motivation of specified leaks, mixed-leaky forcing, and sets of independent leaks. While undefined at this point in the paper, each of these variants of leaky forcing takes a finer grained approach to obstructing the zero forcing process. This is motivated by the somewhat surprising result that $\mathrm{Z}_{(\ell)}$ and $\mathrm{Z}'_{(\ell)}$ are the same parameter despite the fact that edge leaks seem more permissive than vertex leaks. By considering finer grained versions of leaky forcing, we hope to understand the nature of the equivalence of vertex and edge leaks.

2. On edge-leaky forcing

Recall that an edge leak is an edge xy where we prohibit $x \to y$ and $y \to x$ in the zero forcing process. On the other hand, a vertex leak v is a vertex that is not allowed to perform a force. Setting both x and y as vertex leaks is a strictly stronger constraint on the zero forcing process than setting xy as an edge leak. However, setting x as a vertex leak is not obviously as strong as setting xy as an edge leak, since setting x as a vertex leak still allows $y \to x$. This makes the following result somewhat surprising.

Proposition 2.1. *A set B is a* 1*-edge-leaky forcing set if and only if for all $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}(B)$ with $y \neq x$.*

Proof. Assume that B is a 1-edge-leaky forcing set. This implies that B is a zero forcing set with forcing process F. Let $v \in V(G) \setminus B$ and $x \to v \in F$. Since B is a 1-edge-leaky forcing set, there exists a forcing process F' by which B turns G blue despite setting xv as an edge leak. Therefore, F' must contain a force $y \to v$ where $y \neq x$. Thus, $x \to v, y \to v \in \mathcal{F}(B)$, proving the forward direction.

Assume that for all $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}(B)$ with $y \neq x$. Clearly, this implies that B is a zero forcing set of G with forcing process F. Let xv be an arbitrary edge leak. If neither $x \to v$ nor $v \to x$ are in F, then there is nothing to show. Therefore, without loss of generality, assume that $x \to v \in F$. Let B' be a set of blue vertices obtained from B using F such that $x \to v$ is valid given F (were it not for xv being an edge leak), and $v \notin B'$. By assumption, there exists $y \to v \in \mathcal{F}(B)$ where $y \neq x$. This implies that there exists a set of forces F' of B in G with $y \to v$. Since $x \in B'$, it follows that $v \to x \notin F'(B')$. Therefore, $(F \setminus F(B')) \cup F'(B')$ is a forcing process of B by Lemma 1.1, that does not use xv. □

With Proposition 2.1, it's not as surprising that the ℓ-edge-leaky forcing number is equivalent to the ℓ-leaky forcing number for all $\ell \geq 0$. The next lemma finds an appropriate time to switch forcing processes and controls how edge leaks and forcing sets interact. Given a set of edges L and $S \subseteq V(G)$, let $L - S$ denote the edges in L that do not have vertices in S. Explicitly,

$$L - S = \{xy \in L : x, y \notin S\}.$$

By extension,

$$L \setminus (L - S) = \{xy \in L : x \in S \text{ or } y \in S\}.$$

Lemma 2.2. *If B is an $(\ell - 1)$-edge-leaky forcing set and L is a set of $k \geq \ell$ edge leaks, then $|L - B_L^{[\infty]}| \leq k - \ell$.*

Proof. Let L be a set of $k \geq \ell$ edge leaks, and assume $|L - B_L^{[\infty]}| \geq k - \ell + 1$. Furthermore, let $L' = L \setminus (L - B_L^{[\infty]})$. Since $|L'| = |L| - |L - B_L^{[\infty]}|$, it follows that $|L'| \leq k - k + \ell - 1 = \ell - 1$. Notice that edge leaks $uv \in L - B_L^{[\infty]}$ did not change the zero forcing behavior of B. In particular, these edge leaks never played a role in stopping B from propagating because their endpoints never were forced. Therefore, L' is a set of at most $\ell - 1$ edge leaks which shows that B is not an $(\ell - 1)$-edge-leaky forcing set. □

As in the vertex leaky setting, Lemma 2.2 says that if B is an $(\ell-1)$-edge-leaky forcing set and L is a set of ℓ edge leaks, then B forces at least one vertex in every edge leak.

Theorem 2.3. *A set B is an ℓ-edge-leaky forcing set if and only if B is an $(\ell-1)$-leaky forcing set such that for every set of $\ell-1$ vertex leaks L and $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

Proof. Proceed by induction on ℓ. Notice Proposition 2.1 is the base case when $\ell = 1$. Assume that the claim holds for all $r < \ell$.

Let B be an $(\ell-1)$-vertex-leaky forcing set such that for every set of $\ell-1$ vertex leaks T and $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_T(B)$. Clearly, B is an $(\ell-2)$-vertex-leaky forcing set such that for every set of $\ell-2$ vertex leaks T and $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_T(B)$. Therefore, by the induction hypothesis, B is an $(\ell-1)$-edge-leaky forcing set.

Let L be a set of ℓ edge leaks. By Lemma 2.2, it is possible to apply forces one by one until every edge in L contains a blue vertex. Let B' be the resulting set of blue vertices. Notice that B' is an $(\ell-1)$-edge-leaky set since $B \subseteq B'$. Therefore, if B' contains an edge in L, then there is nothing left to show. Thus, assume that every edge in L contains exactly one blue vertex in B'.

Let $A \subseteq \{x \in B' : xy \in L\}$ such that $|L - A| \leq 1$ and $|A| \leq \ell - 1$. Observe that because every edge in L contains at most one blue vertex in B', no vertex of A can ever perform a force. Therefore, assume that vertices in A never perform a force; otherwise, there is nothing left to show.

Let $G^* = G - A$. Since B, and hence B', is an $(\ell-1)$-vertex-leaky forcing set, it follows that $B^* = B' \setminus A$ is a zero forcing set of G^*. At this point there is at most one edge leak from L in G^*. Let $v \in V(G^*) \setminus B^*$. By assumption, there exists $x \to v, y \to v \in \mathcal{F}_A(B)$ in G with $y \neq x$. Notice that $x, y \notin A$ since v is a white neighbor of both x and y, and both x and y are adjacent to a white vertex on an edge leak. Therefore, $x, y \in V(G^*)$ and $x \to v, y \to v \in \mathcal{F}_A(B^*)$. By Proposition 2.1, B^* is a 1-edge-leaky forcing set of G^*. Thus, B^* can color G^* blue, demonstrating that B is an ℓ-edge-leaky forcing set.

To prove the contrapositive of the forward direction, assume that B is an $(\ell-1)$-vertex-leaky forcing set, $L = \{x_1, \ldots, x_{\ell-1}\}$ is a set of vertex leaks, and $v \in V(G) \setminus B$ such that if $x \to v, y \to v \in \mathcal{F}_L(B)$, then $x = y$. Since B is an $(\ell-1)$-vertex-leaky forcing set, there exist $x_0 \to v \in \mathcal{F}_L(B)$. Let $L' = L \cup \{x_0\}$ be a set of ℓ vertex leaks and exhaustively apply the zero forcing rule so that $B_{L'}^{[\infty]}$ is blue. Notice that x_0 is the only vertex that can force v by assumption; therefore, L' is a set of ℓ leaks that prevents B from forcing all the vertices in G. Therefore, $B_{L'}^{[\infty]} \subset V(G)$ and B is not an ℓ-vertex-leaky forcing set.

To complete the proof, the set of vertex leaks L' will be converted into a set of edge leaks. By Lemma 1.2, $L' \subseteq B_{L'}^{[\infty]}$. It follows that every vertex $x_i \in L'$ has exactly one white neighbor y_i when the blue set of G is $B_{L'}^{[\infty]}$; otherwise, it is possible to remove a vertex from L' to conclude that B is not an $(\ell-1)$-vertex-leaky forcing set. Let $L^* = \{x_i y_i : 0 \leq i \leq \ell - 1\}$. Now L^* demonstrates that B is not an ℓ-edge-leaky forcing set. □

By Theorems 1.3 and 2.3, we immediately get the following.

Corollary 2.4. *For any graph G and $\ell \geq 0$,*

$$\mathrm{Z}_{(\ell)}(G) = \mathrm{Z}'_{(\ell)}(G).$$

In particular, B is an ℓ-leaky forcing set if and only if B is an ℓ-edge-leaky forcing set.

The combination of Theorems 1.3 and 2.3 provide insight into how leaks interact with the zero forcing rule. Furthermore, vertex leaks are nicer than edge leaks. Once a vertex leak turns blue, it can safely be deleted from the graph and disregarded for the rest of the process. Edge leaks do not afford us the same luxury. Even if an endpoint of an edge leak turns blue, the vertex cannot be deleted without further care, since it might perform a force later.

3. On specified-leaky forcing

Throughout this section, $v \to u$ is a *specified leak* if v is prohibited from forcing u. In this sense, setting a vertex v as a leak represents the set of specified leaks $\{v \to u : u \in N(v)\}$, and setting an edge uv as a leak represents the set of specified leaks $\{v \to u, u \to v\}$.

It seems as though prohibiting $v \to x$ and $v \to y$ is not more restrictive than prohibiting just $v \to x$ or $v \to y$, but not both. This is more intuitive after considering the fact that in any particular forcing process F, only setting $v \to x$ or $v \to y$ as a specified leak poses a problem since $v \to x$ and $v \to y$ are not both in F. Furthermore, the strength of leaks being picked after the initial blue sets makes a single leak $v \to x$ as devastating as two leaks $v \to x, v \to y$.

To formalize this intuition a little more, consider the following definitions. A set B is a *specified ℓ-leaky forcing set of G* if B can color G blue when any set of ℓ forces are prohibited. Let $\mathrm{Z}^{s}_{(\ell)}(G)$ be the minimum size of a specified ℓ-leaky forcing set of G.

Consider the following definitions before proceeding with the proof of Theorem 3.1: If $v \to u$ is a specified leak, then v is called *the tail of the leak* $v \to u$ and u is the *head of the leak* $v \to u$. Let $T(L) = \{x : x \to y \in L\}$ be the set of tails of L and $H(L) = \{y : x \to y \in L\}$ be the set of heads of L.

Theorem 3.1. *A set B is a specified ℓ-leaky forcing set if and only if B is an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ vertex leaks L and $v \in V(G) \setminus B$, there exist $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

Proof. Let L be a set of ℓ specified leaks that shows that B is not a specified ℓ-leaky forcing set. Notice that $|T(L)| \leq \ell$. Therefore, $T(L)$ demonstrates that B is not an ℓ-leaky forcing set. Thus, by Theorem 1.3, there exist $v \in V(G) \setminus B$ such that if $x \to v, y \to v \in \mathcal{F}_{T(L)}(B)$ then $y = x$.

Suppose that B is an $(\ell-1)$-leaky forcing set, $L = \{x_1, \ldots, x_{\ell-1}\}$ is a set of $\ell - 1$ vertex leaks, and $v_0 \in V(G) \setminus B$ is such that $x \to v_0, y \to v_0 \in \mathcal{F}(L)$ implies $y = x$. Since B is an $(\ell - 1)$-leaky forcing set, there exists $x_0 \to v_0 \in \mathcal{F}_L(B)$. Let $L' = L \cup \{x_0\}$. By Lemma 1.2,

$$|L' \setminus B^{[\infty]}_{L'}| = 0.$$

Notice that if there exists $y \in L'$ with at least two white neighbors, then $L' \setminus \{y\}$ would show that B is not an $(\ell - 1)$-leaky forcing set. Therefore, each vertex $x_i \in L'$ has exactly one white neighbor $v_i \in V(G) \setminus B^{[\infty]}_{L'}$. The set of specified leaks

$$\{x_i \to v_i : 0 \leq i \leq \ell - 1\}$$

shows that B is not a specified ℓ-leaky forcing set. ☐

By Theorems 1.3 and 3.1, we immediately get the following.

Corollary 3.2. *For any graph G and $\ell \geq 0$,*

$$\mathrm{Z}^{s}_{(\ell)}(G) = \mathrm{Z}_{(\ell)}(G).$$

In particular, B is an ℓ-leaky forcing set if and only if B is a specified ℓ-leaky forcing set.

4. On mixed-leaky forcing

This section investigates what happens when a system has various types of leaks preventing the zero forcing process from finishing. A set $B \subseteq V(G)$ is a *mixed ℓ-leaky forcing set* of a graph G if B can color G blue despite any set of ℓ vertex leaks, edge leaks, or specified leaks (refer to these collectively as *leaks*). Let $\mathrm{Z}^m_{(\ell)}(G)$ be the minimum size of a mixed ℓ-leaky forcing set.

Lemma 4.1. *Let $L = L_1 \cup L_2 \cup L_3$ be a set of $k \geq \ell$ leaks where L_1 is the set of vertex leaks, L_2 is the set of edge leaks, and L_3 is the set of specified leaks. If B is a mixed $(\ell-1)$-leaky forcing set, then $|L_1 \setminus B_L^{[\infty]}| + |L_2 - B_L^{[\infty]}| + |L_3 - B_L^{[\infty]}| \leq k - \ell$.*

Proof. To prove the contrapositive, assume that $|L_1 \setminus B_L^{[\infty]}| + |L_2 - B_L^{[\infty]}| + |L_3 - B_L^{[\infty]}| \geq k - \ell + 1$. Every vertex in $B_L^{[\infty]}$ has either 0, 1, or at least 2 white neighbors. If $v \in B_L^{[\infty]}$ such that v has exactly one white neighbor u, then either $v \in L_1$, $vu \in L_2$, or $v \to u \in L_3$. Let $L' = [L_1 \setminus (L_1 \setminus B_L^{[\infty]})] \cup [L_2 \setminus (L_2 - B_L^{[\infty]})] \cup [L_3 \setminus (L_3 - B_L^{[\infty]})]$. Since this is a disjoint union,

$$\begin{aligned} |L'| &= |L_1 \setminus (L_1 \setminus B_L^{[\infty]})| + |L_2 \setminus (L_2 - B_L^{[\infty]})| + |L_3 \setminus (L_3 - B_L^{[\infty]})| \\ &\leq \ell - 1. \end{aligned}$$

Notice that any leak in either $L_1 \setminus B_L^{[\infty]}$, $L_2 - B_L^{[\infty]}$, or $L_3 - B_L^{[\infty]}$ did not change the zero forcing behavior of B. In particular, these leaks never played a role in stopping B from propagating because the vertex leaks were never forced blue, the tails of the specified leaks were never forced blue, and the endpoints of the edge leaks were never forced blue. Therefore, L' is a set of at most $\ell - 1$ leaks which shows B is not a mixed $(\ell-1)$-leaky forcing set. □

Notice that if L_2 and L_3 are empty, then Lemma 1.2 is recovered. With this more general formulation of leaky forcing, the next theorem can be proven.

Theorem 4.2. *A set B is a mixed ℓ-leaky forcing set if and only if B is an $(\ell-1)$-leaky forcing set such that for every $\ell-1$ vertex leaks L and $v \in V(G) \setminus B$, there exist $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

Proof. Proceed by induction on ℓ. Notice either Theorem 1.3, Proposition 2.1, or Theorem 3.1 handles the base case when $\ell = 1$. Assume the claim holds for all $r < \ell$.

Let B be an $(\ell-1)$-leaky forcing set such that for every set of $\ell-1$ vertex leaks L and $v \in V(G)$, there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$. Clearly, B is an $(\ell-2)$-leaky forcing set such that for every set of $\ell-2$ vertex leaks L and $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$. Therefore, by the induction hypothesis, B is a mixed $(\ell-1)$-leaky forcing set.

Let $L = L_1 \cup L_2 \cup L_3$ be a set of ℓ leaks where L_1 is the set of vertex leaks, L_2 is the set of edge leaks, and L_3 is the set of specified leaks. By Lemma 4.1, it is possible to apply forces one by one until every vertex leak in L_1 is blue, every edge leak in L_2 contains a blue vertex, and the tails of specified leaks in L_3 are blue. Let B' be the resulting set of blue vertices. Notice that B' is a mixed $(\ell-1)$-leaky forcing set since $B \subseteq B'$. If B' contains an edge from either L_2 or L_3, then there is nothing left to show. Therefore, assume that the edges in L_2 are incident to one blue vertex, and only the tails of forces in L_3 are blue.

Let $L' \subset L$ be a set of $\ell-1$ leaks. Notice that blue vertices in $L_1 \cap L'$, blue vertices incident to edges in $L_2 \cap L'$, and the tails of specified leaks in $L_3 \cap L'$ can be removed since they are blue. Let A be the set of these vertices.

Consider $G^* = G - A$. Since B is an $(\ell-1)$-leaky forcing set, it follows that $B^* = B' \setminus A$ is a zero forcing set for G^*. At this point there is at most one leak from L in G^*. Let $v \in V(G^*) \setminus B^*$. By assumption there exists $x \to v, y \to v \in \mathcal{F}_A(B)$ in G with $y \neq x$. Since $x, y \notin A$, it follows that $x, y \in V(G^*)$ and $x \to v, y \to v \in \mathcal{F}(B^*)$. By Theorems 1.3, 2.3, and 3.1, B^* is a 1-edge-leaky forcing set, a 1-leaky forcing set, and a specified 1-leaky forcing set. Thus, B^* can color G^* blue, demonstrating that B is a mixed ℓ-leaky forcing set.

To prove the contrapositive of the forward direction, assume B is an $(\ell-1)$-leaky forcing set, $L = \{x_1, \ldots, x_{\ell-1}\}$ is a set of vertex leaks and $v \in V(G) \setminus B$ such that if $x \to v, y \to v \in \mathcal{F}_L(B)$, then $x = y$. Since B is an $(\ell-1)$-leaky forcing set, there exists $x_0 \to v \in \mathcal{F}_L(B)$. Let $L' = L \cup \{x_0\}$ be a set of ℓ vertex leaks. Notice that L' demonstrates that B is not an ℓ-leaky forcing set. Thus, B is also not a mixed ℓ-leaky forcing set. □

By Theorems 1.3 and 4.2, we immediately get the following.

Corollary 4.3. *For any graph G and $\ell \geq 0$,*

$$\mathrm{Z}_{(\ell)}(G) = \mathrm{Z}^m_{(\ell)}(G).$$

In particular, B is an ℓ-leaky forcing set if and only if B is a mixed ℓ-leaky forcing set.

5. Independent sets of specified leaks

Corollaries 2.4 and 3.2 suggest that the strength of a set of specified leaks is somewhat independent of the number of leaks or their structure as a subgraph. In particular, a set of ℓ vertex leaks or ℓ edge leaks is at most as strong as a set of ℓ specified leaks even though ℓ vertex or edge leaks corresponds to more than ℓ specified leaks. Thus, arranging specified leaks into sets of out-stars or 2-cycles is in some sense inefficient. The goal of this section is to formally develop what it means for out-stars and 2-cycles in a set of specified leaks to be inefficient.

Let L be a set of specified leaks on $V(G)$. Notice that a specified leak $v \to u$ can be thought of as a directed edge from v to u. Therefore, L naturally corresponds to the edge set of a directed graph on the vertex set $V(G)$. This gives rise to a notion of isomorphic sets of specified leaks. Let L_1 and L_2 be sets of specified leaks on $V(G)$. A set of specified leaks L_1 is *isomorphic* to L_2 if there exists a bijection $\phi : V(G) \to V(G)$ such that $x \to y \in L_1$ if and only if $\phi(x) \to \phi(y) \in L_2$.

Given a set of specified leaks L, we say that a set B is an L-leaky forcing set if B can turn G blue despite any set of L_1 leaks that is isomorphic to L_2 where $L_2 \subseteq L$. Correspondingly, the L-leaky forcing number of G, denoted $\mathrm{Z}_{(L)}(G)$, is the size of the smallest L-leaky forcing set.

A set of specified leaks L is a *set of independent leaks* if for all $x \to y, v \to u \in L$, it follows that $x \neq v$ and $y \neq v$. Equivalently, L is independent if $|T(L)| = |L|$ and $T(L) \cap H(L) = \varnothing$. Let $I(L)$ denote the size of the largest set of independent leaks contained by L.

These definitions let us abstract away general structure of a set of specified leaks L and focus on the parameter of L that seems to matter. In particular, a set of specified leaks L is no stronger than a maximum set of independent leaks contained in L.

Theorem 5.1. *Let L be a set of specified leaks on $V(G)$ and let $\ell = I(L)$. If B is a specified ℓ-leaky forcing set then B is an L-leaky forcing set. That is,*

$$\mathrm{Z}_{(L)}(G) \leq \mathrm{Z}^s_{(\ell)}(G).$$

To prove Theorem 5.1, consider *active leaks*. The set of active leaks given a blue set B and a set of specified leaks L is the set of leaks in L that actively prevents B from performing a force. Formally, the set of active leaks is given by

$$A(B, L) = \{x \to y \in L : x \in B, \{y\} = N(x) \setminus B\}.$$

Proof. Consider the contrapositive of the desired result, and suppose that B is not an L-leaky forcing set. Therefore, there exists a set of specified leaks L' which is isomorphic to a subset of L that prevents B from coloring all of G blue. Let $B_{L'}^{[\infty]}$ be the set of blue vertices obtained from B given L' by exhaustively applying forces. Notice that $B_{L'}^{[\infty]} \neq V(G)$, and let $A = A(B_{L'}^{[\infty]}, L')$. Since A is a set of independent leaks, it follows that

$$|A| \leq I(L') \leq I(L).$$

Furthermore, A demonstrates that B is not a specified ℓ-leaky forcing set. □

The converse of Theorem 5.1 holds when $I(L) = 1$.

Proposition 5.2. *Let L be a set of specified leaks on $V(G)$ such that $1 = I(L)$. A set B is an L-leaky forcing set if and only if B is a specified 1-leaky forcing set.*

Proof. The backward direction Proposition 5.2 is covered by Theorem 5.1. Therefore, assume that B is not a specified 1-leaky forcing set. This implies that there exists $L' = \{x \to y\}$ that stops B from turning G blue. By assumption, L has a set of independent leaks of size 1. Therefore, L' is isomorphic to a subset of L. Therefore, B is not an L-leaky forcing set. □

The proof of Proposition 5.2 relies on the fact that, up to isomorphism, there is only one set of independent leaks. Proving the converse of Theorem 5.1 fails since an arbitrary set of ℓ independent leaks cannot always be injected into L when $I(L) = \ell$. To illustrate this point, consider the following example. Let $G = K_{\ell+1} \square K_2$, $\ell \geq 2$ with vertex set $V(G) = \{x_1, \ldots, x_{\ell+1}, y_1, \ldots, y_{\ell+1}\}$ where the sets $\{x_i : 1 \leq i \leq \ell+1\}, \{y_i : 1 \leq i \leq \ell+1\}$ induce cliques, and $\{x_i y_i : 1 \leq i \leq \ell+1\}$ induces a matching. Let $L_1 = \{x_i \to y_i : 1 \leq i \leq \ell+1\}$, and $L_2 = \{x_i \to x_{\ell+1} : 1 \leq i \leq \ell\}$. Suppose that $B = \{x_i : 1 \leq i \leq \ell+1\}$. First, notice that B is not a specified 2-leaky forcing set, since $L = \{x_1 \to y_1, x_2 \to y_2\}$ prevents B from turning y_1, y_2 blue. This also shows that B is not an L_2-leaky forcing set. However, B is an L_1 leaky forcing set. Since $I(L_1) = I(L_2) = \ell$, this example shows that the converse of Theorem 5.1 is false for $\ell \geq 2$. Furthermore, this example shows that in general it is possible that $\mathrm{Z}_L(G) < \mathrm{Z}_{I(L)}^s(G)$.

6. Closing remarks

A vertex leak at a vertex v can be thought of as a set of specified leaks $L = \{v \to u : u \in N(v)\}$. An edge leak at vy can be thought of as the set of specified leaks $L' = \{v \to y, y \to v\}$. Due to the nature of zero forcing, at most one of the specified leaks in $L \cup L'$ can be operative at a time. In particular, if v is in a position to force another vertex, then all but one of its neighbors are already blue. In this case, only one specified leak in L matters. Similarly, both leaks in L' cannot be operative at the same time since v will try to force y or y will try to force v, but not both. This gives some intuition for the equivalence of specified leaky forcing to both vertex- and edge-leaky forcing. Since this intuition relied on the fact that the standard zero forcing rule only allows blue vertices to force unique white neighbors, we would be surprised if edge and vertex leaks are equivalent for other variants of zero forcing (which are generally more permissive than standard zero forcing).

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