Journal of Combinatorics
Volume 14, Number 1, 111–138, 2023

Semi-transitivity of directed split graphs generated by morphisms

Kittitat Iamthong and Sergey Kitaev

A directed graph is semi-transitive if and only if it is acyclic and for any directed path $u_1 \to u_2 \to \cdots \to u_t$, $t \geq 2$, either there is no edge from u_1 to u_t or all edges $u_i \to u_j$ exist for $1 \leq i < j \leq t$.

In this paper, we study semi-transitivity of families of directed split graphs obtained by iterations of morphisms applied to the adjacency matrices and giving in the limit infinite directed split graphs. A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. We fully classify semi-transitive infinite directed split graphs when a morphism in question can involve any $n \times m$ matrices over $\{-1, 0, 1\}$ with a single natural condition.

AMS 2000 subject classifications: Primary 05C62; secondary 68R15.
Keywords and phrases: Word-representable graph, semi-transitive orientation, permutation.

1. Introduction

The notion of a semi-transitive orientation of a graph was introduced by Halldórsson et al. in [4] (also see [5]) as means to completely characterize so-called word-representable graphs [7, 8]: A graph is word-representable if and only if it admits a semi-transitive orientation. Word-representable graphs, and thus semi-transitive graphs (i.e. semi-transitively orientable graphs), generalize several important classes of graphs, e.g. circle graphs, 3-colorable graphs and comparability graphs. Semi-transitive orientations are also interesting in their own right as a generalization of transitive orientations.

Split graphs [3] are graphs in which the vertices can be partitioned into a clique and an independent set. The study of split graphs attracted much attention in the literature (e.g. see [2] and references therein). Related to our context, the study of semi-transitive orientability of split graphs was initiated in [1, 9], where certain subclasses of semi-transitive split graphs were characterized in terms of forbidden subgraphs. Also, split graphs were

arXiv: 2108.05483

instrumental in [1] to solve a 10 year old open problem in the theory of word-representable graphs.

In a recent work [6], the first author of this paper extended the studies in [1, 9] by characterizing semi-transitive split graphs in terms of permutations of columns of the adjacency matrices. Moreover, [6] studies semi-transitivity of split graphs obtained by iterations of morphisms applied to the adjacency matrices, and thus giving yet another link to combinatorics on words [10] (the original link comes from the definition of a word-representable graph). A number of general theorems and a complete classification of semi-transitive orientability in the case of morphisms defined by 2×2 matrices are given in [6].

In this paper, we study families of *directed* split graphs obtained by iterations of morphisms (involving three matrices A, B, C) applied to the adjacency matrices and giving as the limit infinite directed split graphs. For each of such a family we ask the question on whether all graphs in the family are oriented semi-transitively (i.e. are semi-transitive) or a finite iteration k of the morphism produces a non-semi-transitive orientation (which will stay non-semi-transitive for all iterations $> k$). In the former case, we say that the infinite split graph's index of semi-transitivity is ∞ (denoted $\mathrm{IST}(A, B, C) = \infty$; see Definition 20), and in the latter case it is k (assuming k is minimal possible).

The novelty of our paper is in the study of directed graphs in connection to semi-transitive orientations (as opposed to undirected graphs in the long list of relevant research papers cited in [7, 8]), and in that we offer a way to generate interesting (from semi-transitivity point of view) families of directed split graphs using adjacency matrices and iterations of morphisms. Our research will contribute to improving further known algorithms to recognise semi-transitive orientations (on directed split graphs and beyond). It comes somewhat as a surprise that we were able to completely classify infinite directed split graphs with the index of semi-transitivity ∞, where morphisms in question involve almost arbitrary $n \times m$ matrices over $\{-1, 0, 1\}$ as opposed to, say, 2×2 matrices in [6] (in a different context though); the only natural condition, to ensure that our definitions work, is that A has a 0. Our classification is done via several results depending on the structures of matrices A, B, C in question, and it is summarised in the diagram in Figure 1. Following the diagram, one can easily determine whether $\mathrm{IST}(A, B, C) = \infty$ for any given A, B, C.

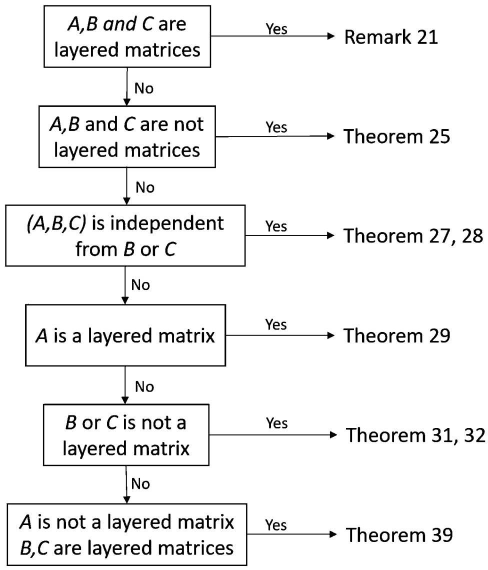

Figure 1: A guide to the classification results where A is assumed to have a 0 (a natural condition to ensure that our definitions work). For example, if none of A, B, C is a layered matrix then Theorem 25 is to be applied; see Definition 14 for the notion of a layered matrix.

2. Preliminaries

2.1. Semi-transitive orientations and split graphs

Graphs in this paper have no loops or multiple edges. Any split graph S_n on n vertices can be partitioned into a *maximal* clique K_m and an independent set E_{n-m}, and we write $S_n = (E_{n-m}, K_m)$.

A directed graph is oriented *semi-transitively* if and only if it is acyclic and for any directed path $u_1 \to u_2 \to \cdots \to u_t$, $t \geq 2$, either there is no edge from u_1 to u_t or all edges $u_i \to u_j$ exist for $1 \leq i < j \leq t$. Graphs admitting semi-transitive orientations are *semi-transitive*.

In this paper, we will need the following results on semi-transitive orientations and split graphs, where a *source* (resp., *sink*) is a vertex of in-degree (resp., out-degree) 0.

Lemma 1 ([9]). *Let K_m be a clique in a graph G. Then any acyclic orientation of G induces a transitive orientation on K_m (where the presence of edges $u \to v$ and $v \to z$ implies the presence of the edge $u \to z$). In particu-*

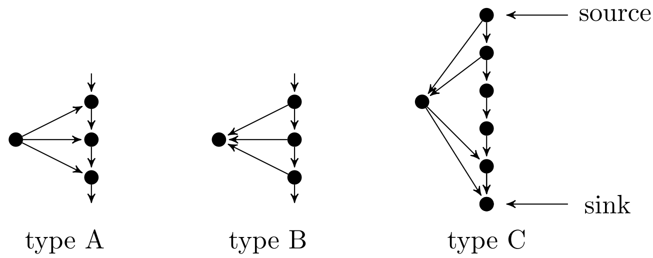

Figure 2: Three types of vertices in E_{n-m} in a semi-transitive orientation of (E_{n-m}, K_m). The vertical oriented paths are a schematic way to show (parts of) $\vec{P}$.

lar, any semi-transitive orientation of G induces a transitive orientation on K_m. In either case, the orientation induced on K_m contains a single source and a single sink.

Theorem 2 ([9])**.** *Any semi-transitive orientation of a split graph $S_n = (E_{n-m}, K_m)$ subdivides the set of all vertices in E_{n-m} into three, possibly empty, groups corresponding to each of the following types (also shown schematically in Figure 2), where $\vec{P} = p_1 \to \cdots \to p_m$ is the longest directed path in K_m:*

- *A vertex in E_{n-m} is of* type A *if it is a source and is connected to all vertices in $\{p_i, p_{i+1}, \ldots, p_j\}$ for some $1 \leq i \leq j \leq m$;*
- *A vertex in E_{n-m} is of* type B *if it is a sink and is connected to all vertices in $\{p_i, p_{i+1}, \ldots, p_j\}$ for some $1 \leq i \leq j \leq m$;*
- *A vertex $v \in E_{n-m}$ is of* type C *if there is an edge $x \to v$ for each $x \in I_v = \{p_1, p_2, \ldots, p_i\}$ and there is an edge $v \to y$ for each $y \in O_v = \{p_j, p_{j+1}, \ldots, p_m\}$ for some $1 \leq i < j \leq m$.*

Theorem 3 ([9])**.** *Let $S_n = (E_{n-m}, K_m)$ be oriented semi-transitively with $\vec{P} = p_1 \to \cdots \to p_m$. For a vertex $x \in E_{n-m}$ of type C, there is no vertex $y \in E_{n-m}$ of type A or B, which is connected to both $p_{|I_x|}$ and $p_{m-|O_x|+1}$. Also, there is no vertex $y \in E_{n-m}$ of type C such that either I_y, or O_y contains both $p_{|I_x|}$ and $p_{m-|O_x|+1}$.*

Theorem 4 ([9])**.** *An orientation of a split graph $S_n = (E_{n-m}, K_m)$ is semi-transitive if and only if*

(i) *K_m is oriented transitively;*
(ii) *each vertex in E_{n-m} is of one of the three types in Theorem 2;*
(iii) *the restrictions in Theorem 3 are satisfied.*

2.2. Directed split graphs

A directed graph is *semi-transitive* if its orientation is semi-transitive. The *adjacency matrix* $A = [a_{ij}]$ of a directed graph on n vertices is a binary matrix such that $a_{ij} = 1$ if $j \to i$ is an edge, and $a_{ij} = 0$ otherwise. Let $L(A) = [\ell_{ij}]$ be the $n \times n$ lower triangular matrix such that, for any $i > j$,

$$\ell_{ij} = \begin{cases} 1 & \text{if } \ a_{ij} = 1, \\ -1 & \text{if } \ a_{ji} = 1, \\ 0 & \text{otherwise} \end{cases}$$

and $\ell_{ij} = 0$ for any $i \leq j$.

Clearly, there is a one-to-one correspondence between directed graphs of order n and $n \times n$ lower triangular matrices over $\{-1, 0, 1\}$ with the diagonal elements equal 0. Thus, $L(A)$ can play the role of the adjacency matrix of a directed graph. For $i > j$, the connectivity between vertices i and j is $j \to i$ if $\ell_{ij} = 1$, and is $i \to j$ if $\ell_{ij} = -1$, and there is no edge if $\ell_{ij} = 0$.

Example 5. If $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ is an adjacency matrix of a directed graph G, then $L(A) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$ and the set of edges of G (on 6 vertices) is $\{1 \to 2,\ 2 \to 4,\ 1 \to 6,\ 5 \to 6,\ 3 \to 1,\ 5 \to 1,\ 4 \to 3,\ 6 \to 4\}$.

Our interest is in acyclically (without directed cycles) oriented split graphs since only such graphs have a chance to be semi-transitive. For any acyclically oriented split graph G, by Lemma 1, we know that the induced orientation of the maximal clique in G is transitive, so the following notion can be introduced.

Definition 6. *An acyclically oriented split graph G with a maximal clique of order n is* well-labelled *if the vertex set of G is $V(G) = \{1, 2, \ldots, |V(G)|\}$ and the longest directed path in the maximal clique is $1 \to 2 \to \cdots \to n$.*

Note that the adjacency matrix A of a well-labelled split graph $S = (E_m, K_n)$ (where K_n is maximal) of order $m+n$ satisfies

$$L(A) = \begin{bmatrix} L_n & O_{n,m} \\ M & O_m \end{bmatrix}$$

for some $m \times n$ matrix M with each row having a 0, where $O_{n,m}$ and O_m are $n \times m$ and $m \times m$ zero matrices, respectively, and L_n is the $n \times n$ matrix such that all entries strictly below the main diagonal are 1's, and all other entries are 0's. Hence, every directed split graph with a maximal clique of order n and an independent set of order m can be represented by an $m \times n$ matrix M appearing in $L(A)$ and recording directed edges between K_n and E_m. Thus, generating a matrix M with entries in $\{-1, 0, 1\}$, we generate an acyclically oriented split graph. Note that in the ways we will be generating M in this paper, sometimes we will obtain rows with no 0. In that case, we will apply Lemma 10 or Remark 11 to bring the problem to the case of well-labelled split graphs.

Definition 7. *Let $M = [m_{ij}]$ be an $m \times n$ matrix such that $m_{ij} \in \{-1, 0, 1\}$ for $1 \le i \le m$ and $1 \le j \le n$. Define*

$$S_o(M) = \begin{bmatrix} L_n & O_{n,m} \\ M & O_m \end{bmatrix}$$

where the subscript o stands for "oriented" and S stands for "split". We denote the directed split graph corresponding to $S_o(M)$ by $G_o(M)$.

Example 8. If $M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then

$$S_o(M) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

is the adjacency matrix of the directed graph $G_0(M)$ shown in Figure 3.

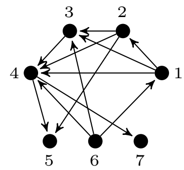

Figure 3: The directed split graph $G_o(M)$ given by $S_o(M)$ in Example 8.

For convenience, we will represent rows of an $m \times n$ matrix M by strings of length n. For example, we will represent the three rows of $\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ by $1(-1)01$, $01(-1)0$ and 0001.

Note that in Definition 7, the maximal clique of $G_o(M)$ is of order $n+1$ if there is a row of the form $11\cdots 1$ or $(-1)(-1)\cdots(-1)$ in M, and the maximal clique is of order n otherwise. In the former case, $G_o(M)$ may not be well-labelled. In the case of $n = 1$, the graph $G_o(M)$ is a tree which is always semi-transitive. Thus, throughout this paper, we can assume that $n \geq 2$.

Remark 9. *If M is a zero matrix, then $G_o(M)$ is semi-transitive as it is a disjoint union of a transitively oriented clique and isolated vertices.*

In what follows, x^r denotes $xx\cdots x$, where $x \in \{-1, 0, 1\}$ is repeated r times.

Lemma 10. *Let $M := [m_{ij}]_{m\times n}$ be an $m \times n$ matrix over $\{-1, 0, 1\}$ such that $m_{p1} = m_{p2} = \cdots = m_{pr} = 1$ and $m_{p(r+1)} = m_{p(r+2)} = \cdots = m_{pn} = -1$ for some $p \in \{1, 2, \ldots, m\}$ and $r \in \{0, 1, \ldots, n\}$. If*

$$N = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1r} & 0 & m_{1(r+1)} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2r} & 0 & m_{2(r+1)} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{(p-1)1} & m_{(p-1)2} & \cdots & m_{(p-1)r} & 0 & m_{(p-1)(r+1)} & \cdots & m_{(p-1)n} \\ m_{(p+1)1} & m_{(p+1)2} & \cdots & m_{(p+1)r} & 0 & m_{(p+1)(r+1)} & \cdots & m_{(p+1)n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{m1} & m_{m2} & \cdots & m_{mr} & 0 & m_{m(r+1)} & \cdots & m_{mn} \end{bmatrix}$$

is an $(m-1) \times (n+1)$ matrix, then $G_o(M)$ is isomorphic to $G_o(N)$.

Proof. The p-th row in M, which is $1^r(-1)^{n-r}$, represents the vertex $n+p$ in the independent set connected to all vertices in $K_n = \{1, 2, \ldots, n\}$. So K_n

is not the maximal clique in $G_o(M)$, but $K_n \cup \{n+p\}$ is the maximal clique. Note that $\ell \to n+p$ for every vertex $\ell \in \{1,2,\ldots,r\}$ and $n+p \to \ell$ for all vertex $\ell \in \{r+1, r+2, \ldots, n\}$. We relabel the vertex $n+p$ to be $r+1$ and relabel a vertex ℓ to be $\ell+1$ for each $\ell \in \{r+1, r+2, \ldots, n+p-1\}$. The relabelling gives the graph that can be represented by the matrix $S_o(N)$. Hence, $G_o(M)$ is isomorphic to $G_o(N)$. □

Remark 11. *Let M be an $m \times n$ matrix over $\{-1,0,1\}$. If $a_1a_2\cdots a_n$ is the p-th row in M such that $a_q = -1$ and $a_r = 1$ for some $1 \leq q < r \leq n$, then $q \to r \to n+p \to q$ forms a cycle in $G_o(M)$. Hence, $G_o(M)$ is not semi-transitive if there is a 1 occurring to the right of a -1 in a row in M. Consequently, if there is a row in M such that it has no 0 and it is not of the form $11\cdots1(-1)(-1)\cdots(-1)$, then $G_o(M)$ is not semi-transitive.*

Let M be an $m \times n$ matrix over $\{-1,0,1\}$. We can see that the maximal clique of $G_o(M)$ is of order n or $n+1$. Moreover, the maximal clique of $G_o(M)$ is the clique of order $n+1$ if there is a row in M containing no 0. In this case, the matrix M does not represent only edges between vertices in the maximal clique and vertices in the independent set, but also a vertex in the maximal clique. By Remark 11, we can assume that M does not contain a row which has no 0 and is not of the form $1^r(-1)^{n-r}$ for some $0 \leq r \leq n$. Hence, if a row of M has no 0, it must be $1^r(-1)^{n-r}$ for some $1 \leq r \leq n$ for graph $G_o(M)$ to have a chance to be semi-transitive. Further, if $1^r(-1)^{n-r}$ is a row of M for some $0 \leq r \leq n$, by Lemma 10, we can consider the $(m-1) \times (n+1)$ matrix N in the statement of the lemma instead of M, and every row of N has a 0.

Theorem 12. *Let M be an $m \times n$ matrix over $\{-1,0,1\}$. The directed split graph $G_o(M)$ is semi-transitive if and only if M satisfies the following conditions:*

(i) *every row of M is of the form $0^r1^s0^t$ or $0^r(-1)^s0^t$ or $1^r0^s(-1)^t$ for $r,s,t \geq 0$;*
(ii) *for each row of M of the form $1^a0^b(-1)^c$ where $a,b,c > 0$, there is no other row having 1's in all positions from a to $a+b+1$;*
(iii) *for each row of M of the form $1^a0^b(-1)^c$ where $a,b,c > 0$, there is no other row having (-1)'s in all positions from a to $a+b+1$.*

Proof. "$\Leftarrow$" Firstly, suppose that every row of M has a 0. Note that the vertices in the independent set will then be of types A, B and C, and taking into account conditions (ii) and (iii), Theorem 4 can be applied to see that $G_o(M)$ is semi-transitive.

For the remaining case, suppose that there is a row p of M of the form $1^r(-1)^{n-r}$ where $1 \le p \le m$ and $0 \le r \le n$. Then, $\{1, 2, \ldots, n, n+p\}$ is the maximal clique in the directed graph $G_o(M)$. By Lemma 10, we have that $G_o(M)$ is isomorphic to $G_o(N)$, where N is the matrix obtained from M by deleting row p and adding a zero-column between columns r and $r+1$ (in the cases of $r = 0$ and $r = n$, the zero-column will be the first column and the last column, respectively). Note that N still satisfies conditions (i), (ii) and (iii) and every row of N has a 0, so the first case can be applied to see that $G_o(N)$ and $G_o(M)$ are semi-transitive.

"$\Rightarrow$" Firstly, suppose that every row of M has a 0. One can see that $G_o(M)$ is well-labelled, so the clique is oriented transitively and its longest path is $1 \to 2 \to \cdots \to n$. Moreover, conditions (ii) and (iii) in Theorem 4 give conditions (i), (ii) and (iii) in this theorem.

For the remaining case, suppose that there is a row p of M of the form $1^r(-1)^{n-r}$, where $1 \le p \le m$ and $0 \le r \le n$. Then, $\{1, 2, \ldots, n, n+p\}$ is the maximal clique in the directed graph $G_o(M)$. By Lemma 10, we have that $G_o(M)$ is isomorphic to $G_o(N)$, where N is the matrix obtained from M by deleting row p and adding a zero-column between columns r and $r+1$ (in the cases of $r = 0$ and $r = n$, the zero-column will be the first column and the last column, respectively). Since $G_o(M)$ is word-representable, then $G_o(N)$ is also word-representable. So N satisfies conditions (i), (ii) and (iii) in this theorem as every row of N has a 0. Therefore, every row of M, except for row p, satisfies (i), (ii) and (iii). For row p of M, if there is row q having 1's in r and $r+1$ position, then the row in N obtained from adding a 0 to row q of M does not satisfy the condition (i), which is a contradiction. Similarly, the occurrence of row q having (-1)'s in columns r and $r+1$ implies a contradiction. Hence, M satisfies conditions (i), (ii) and (iii). $\square$

In this paper, Theorem 12 plays an important role to determine if $G_o(M)$ is word-representable for a given matrix M. The next result is a straightforward corollary of Theorem 12.

Corollary 13. *Let M be an $m \times n$ matrix over $\{-1, 0, 1\}$. If every row of M is of the form $0^r1^s0^t$ or $0^r(-1)^s0^t$ for $r, s, t \ge 0$, then the graph $G_o(M)$ is semi-transitive.*

Definition 14. *A matrix M is said to be a* layered matrix *if all entries in the same row of M are identical.*

The next result is a straightforward corollary of Corollary 13.

Corollary 15. *Let M be an $m \times n$ matrix over $\{-1, 0, 1\}$. If M is a layered matrix, then $G_o(M)$ is semi-transitive.*

3. Directed split graphs generated by iterations of morphisms

Definition 16. *Let A, B, C be $m \times n$ matrices over $\{-1, 0, 1\}$. The matrix $M^k(A, B, C)$ is the k^{th}-iteration of the 2-dimensional morphism applied to the 1×1 matrix $[0]$ which maps $[0] \to A$, $[1] \to B$ and $[-1] \to C$. Moreover, we write $S_o^k(A, B, C)$ for the matrix $S_o(M^k(A, B, C))$ and $G_o^k(A, B, C)$ for the graph with the adjacency matrix $S_o^k(A, B, C)$.*

Example 17. *Let $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Then we have $M^0(A, B, C) = \begin{bmatrix} 0 \end{bmatrix}$, $M^1(A, B, C) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and*

$M^2(A, B, C) = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$. Hence, $S_o^2(A, B, C)$ is the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $G_o^2(A, B, C)$ is shown in Figure 4.

Remark 18. *If A is a zero matrix, then $M^k(A, B, C)$ is always a zero matrix for any $m \times n$ matrices B and C and $k \geq 0$. Thus, by Remark 9, $G_o^k(A, B, C)$ is semi-transitive in this case.*

Proposition 19. *If A, B and C are layered matrices over $\{-1, 0, 1\}$, then $G_o^k(A, B, C)$ is semi-transitive for any $k \geq 0$.*

Proof. Let A, B and C be $m \times n$ matrices. Since every row in A, B and C is either 0^n or 1^n or $(-1)^n$, we have that every row in $M^k(A, B, C)$ is either 0^{n^k} or 1^{n^k} or $(-1)^{n^k}$, so by Corollary 15, $G_o^k(A, B, C)$ is semi-transitive. $\square$

If $A = [a_{ij}]_{m \times n}$ contains at least one 0, say $a_{ij} = 0$, then the entry in row i and column j of $M^1(A, B, C)$ is 0. By mapping this 0 to A in the next

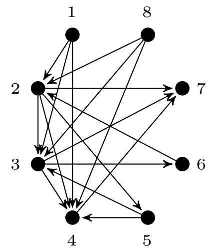

Figure 4: The directed split graph $G_o^2(A,B,C)$ corresponding to the adjacency matrix $S_o^2(A,B,C)$ in Example 17.

iteration of morphism, we obtain $A = M^1(A,B,C)$ as the $m \times n$ submatrix of $M^2(A,B,C)$ given by intersection of rows $(i-1)n+1, (i-1)n+2, \ldots, in$ and columns $(j-1)m+1, (j-1)m+2, \ldots, jm$. More generally, the $m^{k-1} \times n^{k-1}$ submatrix of $M^k(A,B,C)$ given by intersection of rows $(i-1)n^{k-1}+1, (i-1)n^{k-1}+2, \ldots, in^{k-1}$ and columns $(j-1)m^{k-1}+1, (j-1)m^{k-1}+2, \ldots, jm^{k-1}$ is $M^{k-1}(A,B,C)$. So, we can consider the bottommost, then leftmost zero in A as the start of a chain of induced subgraphs generated by the morphism. Thus, the limit $\lim_{k\to\infty} M^k(A,B,C)$, called a *fixed point of the morphism*, is well-defined. So, we have that $G_o^i(A,B,C)$ is an induced subgraph of $G_o^k(A,B,C)$ for $i \leq k$, and the notion of the infinite split graph $G_o(A,B,C)$ is well-defined in the case when A has a 0. Note that this is not a necessary condition for $G_o(A,B,C)$ to be well-defined (for example, A,B,C could be all one matrices). We are interested in the smallest integer ℓ (possibly non-existing) such that $G_o^\ell(A,B,C)$ is not semi-transitive for given A,B and C (then $G_o^i(A,B)$ is not semi-transitive for $i \geq \ell$).

Definition 20. *Let A,B,C be $m \times n$ matrices such that A has a 0 as an entry. The* index of semi-transitivity $\mathrm{IST}(A,B,C)$ *of an infinite directed split graph $G_o(A,B,C)$ is the smallest integer ℓ such that $G_o^\ell(A,B,C)$ is not semi-transitive. If such an ℓ does not exist, that is, if $G_o^\ell(A,B,C)$ is semi-transitive for all ℓ, then $\ell := \infty$.*

Note that since $G_o^0(A,B,C)$ is a graph with one vertex for any A,B,C, we have $\mathrm{IST}(A,B,C) \geq 1$.

Remark 21. *It follows from Proposition 19 that* $\mathrm{IST}(A,B,C) = \infty$ *if A,B and C are layered matrices.*

The following three lemmas give sufficient conditions for A,B and C to have $\mathrm{IST}(A,B,C) = \infty$.

Lemma 22. *Let A, B and C be $m \times n$ matrices over $\{-1, 0, 1\}$ such that A has a* 0 *and* $\mathrm{IST}(A, B, C) = \infty$*. Then,*

- *If A is not a layered matrix, then there is no row in $M^k(A, B, C)$ containing two 0's for any $k \geq 0$.*
- *If B is not a layered matrix, then there is no row in $M^k(A, B, C)$ containing two 1's for any $k \geq 0$.*
- *If C is not a layered matrix, then there is no row in $M^k(A, B, C)$ containing two (-1)'s for any $k \geq 0$.*

Proof. We will prove the first bullet point; the other bullet points can be proved analogously.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and a_{ir}, a_{is} be two entries in row i of A such that $a_{ir} \neq a_{is}$ where $1 \leq r < s \leq n$. Denote $\mu^k(i, j) \in \{-1, 0, 1\}$ the entry of $M^k(A, B, C)$ in row i and column j. Suppose that row a of $M^k(A, B, C)$ contains at least two 0's for some k, say $\mu^k(a, b) = \mu^k(a, c) = 0$ where $b < c$. Consider the intersection of rows $(a-1)m + 1, (a-1)m + 2, \ldots, am$ and columns $(b-1)n + 1, (b-1)n + 2, \ldots, bn$ in $M^{k+1}(A, B, C)$, which is the matrix A because $\mu^k(a, b) = 0$. Similarly, the submatrix of $M^{k+1}(A, B, C)$ formed by rows $(a-1)m + 1, (a-1)m + 2, \ldots, am$ and columns $(c-1)n + 1, (c-1)n + 2, \ldots, cn$ is A. Hence, we have

$$\mu^{k+1}((a-1)m + i, (b-1)n + r) = \mu^{k+1}((a-1)m + i, (c-1)n + r) = a_{ir}$$

and

$$\mu^{k+1}((a-1)mi, (b-1)n + s) = \mu^{k+1}((a-1)m + i, (c-1)n + s) = a_{is}.$$

Thus, the submatrix of $M^{k+1}(A, B, C)$ formed by row $(a-1)m + i$ and columns $(b-1)n+r, (b-1)n+s, (c-1)n+r, (c-1)n+s$ is $\left[a_{ir}, a_{is}, a_{ir}, a_{is}\right]$. That is, row $(a-1)m + i$ of $M^{k+1}(A, B, C)$ cannot be of the form $0^r 1^s 0^t$ or $0^r(-1)^s 0^t$ or $1^r 0^s (-1)^t$. By Theorem 12, $G_o^{k+1}(A, B, C)$ is not semi-transitive, which is a contradiction with $\mathrm{IST}(A, B, C) = \infty$. $\square$

Lemma 23. *Let A, B and C be $m \times n$ matrices over $\{-1, 0, 1\}$ such that A has a* 0 *and* $\mathrm{IST}(A, B, C) = \infty$*. Then,*

- *If A and B are not layered matrices, then every entry of C is (-1).*
- *If A and C are not layered matrices, then every entry of B is 1.*

Proof. Both statements are proved by similar arguments, so we will prove here only the first one. Suppose both A and B are not layered matrices. By Lemma 22, every row of $M^k(A, B, C)$ contains at most one 0 and at most one 1 for any $k \geq 2$. Then, there are at least $n^k - 2$ copies of (-1) in every row of $M^k(A, B, C)$. By Lemma 22, C is a layered matrix.

Suppose that there is no (-1) in A and B. Since every row of $M^1(A, B, C) = A$ has at most one 0 and at most one 1 and no (-1), then $n = 2$ (recall our assumption of $n \geq 2$). Therefore, $M^2(A, B, C)$ has 4 columns with every row having more than one 0 or more than one 1, which is a contradiction.

If (-1) is an entry of A, then $M^1(A, B, C) = A$ has (-1) as an entry. So C is a submatrix of $M^2(A, B, C)$ as (-1) is mapped to C. Since every row of C has the same entries, and there is no more than one 0 and one 1 in each row of $M^2(A, B, C)$, we have that each entry of C must be (-1).

Finally, if there is no (-1) in A, but B contains (-1) as an entry, then $M^1(A, B, C) = A$ contains 1 as an entry. Since 1 maps to B, $M^2(A, B, C)$ contains B as a submatrix. So there is an entry (-1) in $M^2(A, B, C)$, and then C is a submatrix of $M^3(A, B, C)$. Since every row of C has entries equal to each other, and there is no more than one 0 and one 1 in each row of $M^2(A, B, C)$, then each entry of C is (-1). □

Lemma 24. *Let A, B and C be $m \times n$ matrices over $\{-1, 0, 1\}$ such that A has a 0 and $\mathrm{IST}(A, B, C) = \infty$. If B and C are not layered matrices, then all entries of A are 0.*

Proof. Suppose B and C are not layered matrices. By Lemma 22, every row of $M^k(A, B, C)$ contains at most one 1 and at most one (-1) for any $k \geq 2$. Then there are at least $n^k - 2$ zeroes in every row of $M^k(A, B, C)$. By Lemma 22, A is a layered matrix.

Assume that there is a row r in $A := [a_{ij}] = M^1(A, B, C)$ of the form $11 \cdots 1$. Also, suppose that a row s in $B := [b_{ij}]$ has two distinct entries, say $b_{sp} \neq b_{sq}$ for some $1 \leq p < q \leq n$. Note that the intersection of rows $(r-1)m+1, (r-1)m+2, \ldots, rm$ and columns $(\ell-1)n+1, (\ell-1)n+2, \ldots, \ell n$ in $M^2(A, B, C)$ is B for $\ell = 1, 2, \ldots, m$. Then the submatrix of $M^2(A, B, C)$ formed by row $(r-1)m + s$ and columns $p, q, n + p, n + q, 2n + p, 2n + q, \ldots, (m-1)n + p, (m-1)n + q$ is

$$\begin{bmatrix} b_{sp} & b_{sq} & b_{sp} & b_{sq} & \cdots & b_{sp} & b_{sq} \end{bmatrix}.$$

Since every row of $M^k(A, B, C)$ has at most one 1 and at most one (-1) for any k, we have $b_{sp} = b_{sq} = 0$, which is a contradiction. Thus, there is no row in A of the form $11 \cdots 1$. Similarly, we can show that there is no row in A of the form $(-1)(-1) \cdots (-1)$. Hence, A is an all 0 matrix. □

From Lemmas 23 and 24 we have the following theorem.

Theorem 25. *Let A, B and C be $m \times n$ matrices over $\{-1, 0, 1\}$ such that A has a 0. If A, B and C are not layered, then $\mathrm{IST}(A, B, C)$ is finite.*

Definition 26. *Let A, B, C be $m \times n$ matrices over $\{-1, 0, 1\}$. The triple (A, B, C) is said to be independent from B if there are no 1's in A and C. Similarly, the triple (A, B, C) is said to be independent from C if there are no (-1)'s in A and B.*

For convenience, we write $R(M)$ for the set of strings representing rows of M. Moreover, if A, B and C are $m \times n$ matrices over $\{-1, 0, 1\}$, then define $R^k(A, B, C)$ to be the set of strings representing rows of $M^k(A, B, C)$. So, every element of $R^k(A, B, C)$ is a string over $\{-1, 0, 1\}$ of length n^k. Each element of $R^k(A, B, C)$ is called a *row pattern of $M^k(A, B, C)$*.

Theorem 27. *Let A, B and C be $m \times n$ matrices over $\{-1, 0, 1\}$ such that A has a 0 and (A, B, C) is independent from C. Then, $\mathrm{IST}(A, B, C) = \infty$ if and only if A and B satisfy one of the following conditions, where $a_i \in \{0, 1\}$:*

(1) *A and B are layered matrices, or*

(2) $A = \begin{bmatrix} a_1 & 1 & 1 & \cdots & 1 \\ a_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ a_m & 1 & 1 & \cdots & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$, *or*

(3) $A = \begin{bmatrix} 1 & 1 & \cdots & 1 & a_1 \\ 1 & 1 & \cdots & 1 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & a_m \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

Proof. "$\Leftarrow$" There is no (-1) in A and B, and row patterns of $M^k(A, B, C)$ generated by A, B and C in (1), (2) and (3) are in the set

$$\{1^{n^k}, 0^{n^k}, 01^{n^k-1}, 1^{n^k-1}0\}.$$

By Corollary 13, $M^k(A, B, C)$ is semi-transitive for all $k \geq 0$.
"$\Rightarrow$" Since (A, B, C) is independent from C, every entry of $M^k(A, B, C)$ is either 0 or 1. Assume $\mathrm{IST}(A, B, C) = \infty$ and let $R(B) = \{b_1, b_2, \ldots, b_p\}$ where b_i is a binary string of length n. By Theorem 12, we have that every row of $M^k(A, B, C)$ is of the form $0^r 1^s 0^t$. If A is a layered matrix, then $R^1(A, B, C) \subseteq \{0^n, 1^n\}$ and

$$R^2(A, B, C) \subseteq \{0^{n^2}, 1^{n^2}, (b_1)^n, (b_2)^n, \ldots, (b_p)^n\}.$$

So, $R(B) \subseteq \{0^n, 1^n\}$ as otherwise, some strings in $R^2(A,B,C)$ are not of the form $0^r1^s0^t$. Thus, B is a layered matrix. Suppose A is not a layered matrix. By Lemma 22, $R^1(A,B,C) \subseteq \{01^{n-1}, 1^{n-1}0, 1^n\}$. If both 01^{n-1} and $1^{n-1}0$ are rows in A, then $1^{n-1}0(b_i)^{n-1}$ is a row pattern in $R^2(A,B,C)$ for some i. Since every row of $M^k(A,B,C)$ contains at most one 0, b_i must be 1^n, which contradicts $1^{n-1}0(b_i)^{n-1}$ not being of the form $0^r1^s0^t$. So, we have

$$A = \begin{bmatrix} a_1 & 1 & 1 & \cdots & 1 \\ a_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ a_m & 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 1 & \cdots & 1 & a_1 \\ 1 & 1 & \cdots & 1 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & a_m \end{bmatrix}$$

where $a_i \in \{0,1\}$. Note that each row of A is 1^n, 01^{n-1} or $1^{n-1}0$. If row i in A is 1^n, then row $((i-1)m+i)$ in $M^2(A,B,C)$ is x^n, where x is row i in B. Since x^n cannot contain more than one 0, we have $x = 1^n$. If row i in A is 01^{n-1}, then row $((i-1)m+i)$ in $M^2(A,B,C)$ is $01^{n-1}x^{n-1}$, where x is row i in B. So, $x = 1^n$ because $01^{n-1}x^{n-1}$ contains at most one 0. Similarly, if row i in A is $1^{n-1}0$, then row i in B is 1^n. Hence, B is an all 1 matrix. $\square$

Next theorem can be proved similarly to Theorem 27.

Theorem 28. *Let A, B and C be $m \times n$ matrices over $\{-1,0,1\}$ such that A has a 0 and (A,B,C) is independent from B. Then, $\mathrm{IST}(A,B,C) = \infty$ if and only if A and C satisfy one of the following conditions, where $a_i \in \{0,1\}$:*

(1) *A and C are layered matrices, or*

(2) $A = \begin{bmatrix} a_1 & -1 & -1 & \cdots & -1 \\ a_2 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ a_m & -1 & -1 & \cdots & -1 \end{bmatrix}$ *and* $C = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & -1 \end{bmatrix}$, *or*

(3) $A = \begin{bmatrix} -1 & -1 & \cdots & -1 & a_1 \\ -1 & -1 & \cdots & -1 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & \cdots & -1 & a_m \end{bmatrix}$ *and* $C = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & -1 \end{bmatrix}$.

Theorem 29. *Let A, B and C be $m \times n$ matrices over $\{-1,0,1\}$ such that A has a 0 and (A,B,C) is not independent from B and C. Suppose A is a layered matrix. Then, $\mathrm{IST}(A,B,C) = \infty$ if and only if B and C are layered matrices.*

Proof. Suppose $\mathrm{IST}(A,B,C) = \infty$. Note that 1^n or $(-1)^n$ is a row in A because (A,B,C) is not independent from B and C. W.L.O.G., we suppose that 1^n is a row in $A = M^1(A,B,C)$. By Lemma 22, we have B is a layered matrix. If A also contains a row $(-1)^n$, then C is a layered matrix with the same reason. If A does not contain a row $(-1)^n$, then $(-1)^n$ must be a row of B because (A,B,C) is not independent from B and C. Since 1^n is a row of A, we have $BB\cdots B$ are m consecutive rows in $M^2(A,B,C)$. As $(-1)^n$ is a row in B, we have that $(-1)^{n^2}$ is a row in $M^2(A,B,C)$. By Lemma 22, C is a layered matrix.

For the converse direction, it is clear from Proposition 19 that if A, B and C are layered matrices, then $\mathrm{IST}(A,B,C) = \infty$. $\square$

Definition 30. *Let A, B, C be $m \times n$ matrices over $\{-1,0,1\}$. The triple (A,B,C) is said to be*

- *an* all-but-leftmost-negative triple *if $R(A), R(B) \subseteq \{0(-1)^{n-1}, 1(-1)^{n-1}\}$ and C is an all (-1) matrix,*
- *an* all-but-rightmost-negative triple *if $R(A), R(B) \subseteq \{(-1)^{n-1}0, (-1)^{n-1}1\}$ and C is an all (-1) matrix,*
- *an* all-but-leftmost-positive triple *if $R(A), R(B) \subseteq \{01^{n-1}, (-1)1^{n-1}\}$ and C is an all 1 matrix,*
- *an* all-but-rightmost-positive triple *if $R(A), R(B) \subseteq \{1^{n-1}0, 1^{n-1}(-1)\}$ and C is an all 1 matrix.*

From Definition 30, we can easily see that

- If (A,B,C) is all-but-leftmost-negative, then
$$R^k(A,B,C) \subseteq \{0(-1)^{n^k-1}, 1(-1)^{n^k-1}\},$$
- If (A,B,C) is all-but-rightmost-negative, then
$$R^k(A,B,C) \subseteq \{(-1)^{n^k-1}0, (-1)^{n^k-1}1\},$$
- If (A,B,C) is all-but-leftmost-positive, then
$$R^k(A,B,C) \subseteq \{01^{n^k-1}, (-1)1^{n^k-1}\},$$
- If (A,B,C) is all-but-rightmost-positive, then
$$R^k(A,B,C) \subseteq \{1^{n^k-1}0, 1^{n^k-1}(-1)\}.$$

With this observation, we can prove the following theorem.

Theorem 31. *Let A, B, C be $m \times n$ matrices over $\{-1,0,1\}$ such that A has a 0 and (A,B,C) is not independent from B and C. Suppose A and B are not layered matrices and C is a layered matrix. Then, $\mathrm{IST}(A,B,C) = \infty$ if and only if (A,B,C) is an all-but-leftmost-negative triple.*

Proof. "$\Leftarrow$" Let (A,B,C) be all-but-leftmost-negative. Then, for any $k \geq 1$,

$$M^k(A,B,C) = \begin{bmatrix} x_1 & -1 & -1 & \cdots & -1 \\ x_2 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ x_{m^k} & -1 & -1 & \cdots & -1 \end{bmatrix}$$

where $x_i \in \{0,1\}$. So $M^k(A,B,C)$ satisfies both conditions in Theorem 12, and hence $\mathrm{IST}(A,B,C) = \infty$.

"$\Rightarrow$" Suppose $\mathrm{IST}(A,B,C) = \infty$. From Lemma 23, we have that C is an all (-1) matrix. By Lemma 22, every row of $M^k(A,B,C)$ does not contain more than one 0 and more than one 1. Note that every row of A must be of the form $0^r1^s0^t$, $0^r(-1)^s0^t$ or $1^r0^s(-1)^t$, where $r,s,t \geq 0$. So, all possible row patterns of A are in

$$\{01, 10, 0(-1)^{n-1}, (-1)^{n-1}0, (-1)^n, 1(-1)^{n-1}, 10(-1)^{n-2}\}.$$

Suppose that $n = 2$ and row i in A is 01. Then, the submatrix of $M^2(A,B,C)$ formed by rows $(i-1)m+1, (i-1)m+2, \ldots, im$ and columns $1,2,3,4$ is AB. So, row $(i-1)m+i$ in $M^2(A,B,C)$ is $01x$, where x is row i in B. Note that $01x$ must be of the form $0^r1^s0^t$, where $r,s,t \geq 0$. Therefore, x is 11 because $M^2(A,B,C)$ contains at most one 0. So, $01x$ contains more than one 1, which contradicts Lemma 22. Hence, 01 cannot be a row in A. Similarly, we obtain that 10 is also not a row in A. Hence, we have that 01 and 10 cannot be a row in A.

Suppose row i in A is $10(-1)^{n-2}$. Then there is m consecutive rows in $M^2(A,B,C)$ built by $BACC\cdots C$. Note that row i in $BACC\cdots C$ is $y10(-1)^{n-2}zz\cdots z$, where y and z are rows i in B and C, respectively. Since $\mathrm{IST}(A,B,C) = \infty$, $y10(-1)^{n-2}zz\cdots z$ must be of the form $1^r0^s(-1)^t$, where $r,s,t \geq 0$. Thus, $y = 1^n$ and $z = (-1)^n$. This contradicts to the fact that any row in $M^2(A,B,C)$ has at most one 1. Hence, $10(-1)^{n-2}$ cannot be a row in A.

Now, all possible row patterns of A are in

$$\{0(-1)^{n-1}, (-1)^{n-1}0, (-1)^n, 1(-1)^{n-1}\}.$$

If $1(-1)^{n-1}$ is not a row in A, then (A,B,C) is independent from B. Then $1(-1)^{n-1}$ must be a row in A. If $(-1)^{n-1}0$ or $(-1)^n$ is a row in A, then condition (ii) of Theorem 12 is not satisfied. So, $G^1_o(A,B,C)$ is not semi-transitive. Therefore $(-1)^{n-1}0$ and $(-1)^n$ are not rows in A and we have

$$A = \begin{bmatrix} a_1 & -1 & -1 & \cdots & -1 \\ a_2 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ a_m & -1 & -1 & \cdots & -1 \end{bmatrix} \quad \text{where} \quad a_i \in \{0,1\}.$$

Since both $0(-1)^{n-1}$ and $1(-1)^{n-1}$ are rows in A, there are m consecutive rows of $M^2(A,B,C)$ built by $ACC\cdots C$ and $BCC\cdots C$. Then $1(-1)^{n^2-1}$ is a row in $M^2(A,B,C)$. Note that row i in $BCC\cdots C$ is $b_{i1}b_{i2}\cdots b_{in}(-1)^{n^2-n}$ where $b_{i1}b_{i2}\cdots b_{in}$ is row i in B. Since $M^2(A,B,C)$ is semi-transitive and $b_{i1}b_{i2}\cdots b_{in}(-1)^{n^2-n}$ is a row in $M^2(A,B,C)$, we have $b_{i1}b_{i2}\cdots b_{in}$ is $0^r(-1)^{n-r}$ or $10^s(-1)^{n-s-1}$ for some $0 \le r \le n$ and $0 \le s \le n-1$. As $M^2(A,B,C)$ contains at most one 0, we obtain that $b_{i1}b_{i2}\cdots b_{in}$ must be $1(-1)^{n-1}$ or $0(-1)^{n-1}$ or $10(-1)^{n-2}$ for any $1 \le i \le m$. If $10(-1)^{n-2}$ is a row of B, then there are m consecutive rows in $M^3(A,B,C)$ such that $ABCC\cdots C$ is its prefix. So, $x10(-1)^{n-2}yy\cdots y$ is a row in $M^3(A,B,C)$ where x is a row in A and y is a row in C. That is, $x = 1^n$, which is a contradiction. So, $10(-1)^{n-2}$ cannot be a row in B. Hence,

$$B = \begin{bmatrix} b_1 & -1 & -1 & \cdots & -1 \\ b_2 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ b_m & -1 & -1 & \cdots & -1 \end{bmatrix} \quad \text{where} \quad b_i \in \{0,1\}. \qquad \square$$

Using similar arguments, we can prove the following theorem.

Theorem 32. *Let A, B, C be $m \times n$ matrices over $\{-1,0,1\}$ such that A has a 0 and (A,B,C) is not independent from B and C. Suppose A and C are not layered matrices and B is a layered matrix. Then,* $\mathrm{IST}(A,B,C) = \infty$ *if and only if (A,B,C) is all-but-rightmost-positive.*

By now, we already have a classification for triples (A,B,C) with the index of semi-transitivity infinity except for the case when A is not a layered matrix and B and C are layered matrices and (A,B,C) is not independent from B and C. To solve the remaining cases, we begin with a definition of a type of a triple (A,B,C).

Definition 33. *Let A,B,C be $m \times n$ matrices over $\{-1,0,1\}$. A triple (A,B,C) is* left-0-invariant *if A, B, C satisfy the following properties:*

- *every row in A is in $\{01^{n-1}, 1^n, 0(-1)^{n-1}, (-1)^n\}$;*

- *every row in B and C is in $\{1^n, (-1)^n\}$;*
- *if 01^{n-1} appears as a row in A, then*
  - *row i in A is 01^{n-1} implies row i in B is 1^n;*
  - *row i in A is 1^n implies row i in B is 1^n;*
  - *row i in A is $0(-1)^{n-1}$ implies row i in B is $(-1)^n$;*
  - *row i in A is $(-1)^n$ implies row i in B is $(-1)^n$;*
- *if $0(-1)^{n-1}$ appears as a row in A, then*
  - *row i in A is 01^{n-1} implies row i in C is 1^n;*
  - *row i in A is 1^n implies row i in C is 1^n;*
  - *row i in A is $0(-1)^{n-1}$ implies row i in C is $(-1)^n$;*
  - *row i in A is $(-1)^n$ implies row i in C is $(-1)^n$.*

Definition 34. *Let A, B, C be $m \times n$ matrices over $\{-1, 0, 1\}$. A triple (A, B, C) is* right-0-invariant *if A, B, C satisfy the following properties:*

- *every row in A is in $\{1^{n-1}0, 1^n, (-1)^{n-1}0, (-1)^n\}$;*
- *every row of B and C is in $\{1^n, (-1)^n\}$;*
- *if $1^{n-1}0$ appears as a row in A, then*
  - *row i in A is $1^{n-1}0$ implies row i in B is 1^n;*
  - *row i in A is 1^n implies row i in B is 1^n;*
  - *row i in A is $(-1)^{n-1}0$ implies row $i$in B is $(-1)^n$;*
  - *row i in A is $(-1)^n$ implies row i in B is $(-1)^n$;*
- *if $(-1)^{n-1}0$ appears as a row in A, then*
  - *row i in A is $1^{n-1}0$ implies row i in C is 1^n;*
  - *row i in A is 1^n implies row i in C is 1^n;*
  - *row i in A is $(-1)^{n-1}0$ implies row i in C is $(-1)^n$;*
  - *row i in A is $(-1)^n$ implies row i in C is $(-1)^n$.*

To classify the triples (A, B, C) with the index of semi-transitivity infinity, where A is not a layered matrix and B and C are layered matrices and (A, B, C) is not independent from B, we need the following four lemmas.

Lemma 35. *Let A, B, C be $m \times n$ matrices over $\{-1, 0, 1\}$ such that A has a 0 and (A, B, C) is not independent from B and C. Then,*

(1) *if both 01^{n-1} and $1^{n-1}0$ are rows in A, then $\mathrm{IST}(A, B, C)$ is finite;*

(2) *if both* $0(-1)^{n-1}$ *and* $(-1)^{n-1}0$ *are rows in* A, *then* $\mathrm{IST}(A,B,C)$ *is finite;*
(3) *if both* 01^{n-1} *and* $(-1)^{n-1}0$ *are rows in* A, *then* $\mathrm{IST}(A,B,C)$ *is finite;*
(4) *if both* $0(-1)^{n-1}$ *and* $1^{n-1}0$ *are rows in* A, *then* $\mathrm{IST}(A,B,C)$ *is finite;*
(5) *if both* $1^p0(-1)^{n-p-1}$ *and* $1^q0(-1)^{n-q-1}$ *are rows in* A, *where* $0 \le p < q \le n-1$, *then* $\mathrm{IST}(A,B,C)$ *is finite.*

Proof.

(1) Suppose that $\mathrm{IST}(A,B,C) = \infty$ and row i and row j in A are 01^{n-1} and $1^{n-1}0$, respectively. Note that $B^{n-1}A$ gives m consecutive rows in $M^2(A,B,C)$ obtained by applying the morphism to $1^{n-1}0$. Row i in $B^{n-1}A$ is $x^{n-1}01^{n-1}$, where x is row i in B. Since A is not a layered matrix, by Lemma 22, there is no 0 in x. So $x^{n-1}01^{n-1}$ cannot be of the form $0^r1^s0^t$, $0^r(-1)^s0^t$ or $1^r0^s(-1)^t$. This contradicts to Theorem 12.
(2) Suppose that $\mathrm{IST}(A,B,C) = \infty$ and row i and row j in A are $0(-1)^{n-1}$ and $(-1)^{n-1}0$, respectively. Note that AC^{n-1} gives m consecutive rows in $M^2(A,B,C)$ obtained by applying the morphism to $0(-1)^{n-1}$. Row j in AC^{n-1} is $(-1)^{n-1}0x^{n-1}$, where x is row j in B. Since A is not a layered matrix, by Lemma 22, there is no 0 in x. So $(-1)^{n-1}0x^{n-1}$ cannot be of the form $0^r1^s0^t$, $0^r(-1)^s0^t$ or $1^r0^s(-1)^t$. This contradicts to Theorem 12.
(3) Suppose that $\mathrm{IST}(A,B,C) = \infty$ and row i and row j in A are 01^{n-1} and $(-1)^{n-1}0$, respectively. Note that AB^{n-1} gives m consecutive rows in $M^2(A,B,C)$ obtained by applying the morphism to 01^{n-1}. Row j in AB^{n-1} is $(-1)^{n-1}0x^{n-1}$, where x is row j in B. Note that $(-1)^{n-1}0x^{n-1}$ must be of the form $0^r(-1)^s0^t$, and so $x = 0^n$. Thus, $(-1)^{n-1}0x^{n-1} = (-1)^{n-1}0^{(n^2-n-1)}$ is a row in $M^2(A,B,C)$ having more than one 0, which contradicts to Lemma 22.
(4) Suppose that $\mathrm{IST}(A,B,C) = \infty$ and row i and row j in A are $0(-1)^{n-1}$ and $1^{n-1}0$, respectively. Note that AC^{n-1} gives m consecutive rows in $M^2(A,B,C)$ obtained by applying the morphism to $0(-1)^{n-1}$. Row j in AC^{n-1} is $1^{n-1}0x^{n-1}$, where x is row j in C. Since A is not a layered matrix, by Lemma 22, there is no 0 in x. Therefore, $1^{n-1}0x^{n-1}$ is of the form $1^r0^s(-1)^t$. So $x = (-1)^n$ and $1^{n-1}0x^{n-1} = 1^{n-1}0(-1)^{n^2-n}$ is a row in $M^2(A,B,C)$. Note that $B^{n-1}A$ gives m consecutive rows in $M^2(A,B,C)$ obtained by application of the morphism to $1^{n-1}0$. Row j in $B^{n-1}A$ is $y^{n-1}1^{n-1}0$, where y is row j in B. Since A is not a layered matrix, by Lemma 22, there is no 0 in y. Therefore, $y^{n-1}1^{n-1}0$ is of the form $0^r1^s0^t$. So $y = 1^n$ and $y^{n-1}1^{n-1}0 = 1^{n^2-1}0$ is a row in $M^2(A,B,C)$. Note that $1^{n-1}0(-1)^{n^2-n-1}$ and $1^{n^2-1}0$ break the second

condition of Theorem 12. Hence, $G_o^2(A,B,C)$ is not semi-transitive and this leads to a contradiction.

(5) Suppose that $\mathrm{IST}(A,B,C) = \infty$ and row i and row j in A are $1^p0(-1)^{n-p-1}$ and $1^q0(-1)^{n-q-1}$, respectively, where $0 \le p < q \le n-1$. Note that B^pAC^{n-p-1} gives m consecutive rows in $M^2(A,B,C)$ obtained by applying the morphism to $1^p0(-1)^{n-p-1}$. Row i in B^pAC^{n-p-1} is $x^p1^p0(-1)^{n-p-1}y^{n-p-1}$ where x is row i in B and y is row i in C. Since A is not a layered matrix, by Lemma 22, there is no more than one 0 in any row of $M^2(A,B,C)$. By Theorem 12, we obtain $x^p1^p0(-1)^{n-p-1}y^{n-p-1}$ equals $1^{np+p}0(-1)^{n^2-np-p-1}$. Note that B^qAC^{n-q-1} gives m consecutive rows in $M^2(A,B,C)$ obtained by application of the morphism to $1^q0(-1)^{n-q-1}$, and $x^q1^q0(-1)^{n-q-1}y^{n-q-1}$ is its row i. Similarly to the above, we have $x^q1^q0(-1)^{n-q-1}y^{n-q-1} = 1^{nq+q}0(-1)^{n^2-nq-q-1}$. That is, both $1^{np+p}0(-1)^{n^2-np-p-1}$ and $1^{nq+q}0(-1)^{n^2-nq-q-1}$ are rows of $M^2(A,B,C)$. If $p=0$ and $q=n-1$, then $0(-1)^{n-1}$ and $1^{n-1}0$ are rows in A which is a contradiction by (4). So, one of $1^{np+p}0(-1)^{n^2-np-p-1}$ and $1^{nq+q}0(-1)^{n^2-nq-q-1}$ is of the form $1^r0^s(-1)^t$ for some $r,s,t>0$.
Note that $G_o^2(A,B,C)$ is not semi-transitive because the second condition of Theorem 12 is not satisfied, and this is a contradiction. □

Lemma 36. *Let A, B, C be $m \times n$ matrices over $\{-1,0,1\}$ such that A has a 0 and (A,B,C) is not independent from B and C. Then,*

(1) *if $1^p0(-1)^{n-p-1}$ and 01^{n-1} are rows in A, where $1 \le p \le n-2$, then $\mathrm{IST}(A,B,C)$ is finite;*
(2) *if $1^p0(-1)^{n-p-1}$ and $0(-1)^{n-1}$ are rows in A, where $1 \le p \le n-2$, then $\mathrm{IST}(A,B,C)$ is finite;*
(3) *if $1^p0(-1)^{n-p-1}$ and $1^{n-1}0$ are rows in A, where $1 \le p \le n-2$, then $\mathrm{IST}(A,B,C)$ is finite;*
(4) *if $1^p0(-1)^{n-p-1}$ and $(-1)^{n-1}0$ are rows in A, where $1 \le p \le n-2$, then $\mathrm{IST}(A,B,C)$ is finite.*

Proof.

(1) Suppose that $1^p0(-1)^{n-p-1}$ and 01^{n-1} are rows i and j in A, respectively, and $\mathrm{IST}(A,B,C) = \infty$. Note that B^pAC^{n-p-1} gives m consecutive rows in $M^2(A,B,C)$ obtained by applying the morphism to $1^p0(-1)^{n-p-1}$ in $M^1(A,B,C)$. Row j in B^pAC^{n-p-1} is $b^p01^{n-1}c^{n-p-1}$, where b and c are row j in B and C, respectively. So $b^p01^{n-1}c^{n-p-1}$ must be $0^r1^s0^t$ for some $r,s,t \ge 0$. Hence, $b = 0^n$ and $c = 1^n$. As A is not a layered matrix, every row in $M^2(A,B,C)$ contains at most one 0, which is a contradiction. Therefore, $\mathrm{IST}(A,B,C) < \infty$.

(2) This is given by (5) in Lemma 35.
(3) This is given by (5) in Lemma 35.
(4) Suppose that $1^p0(-1)^{n-p-1}$ and $(-1)^{n-1}0$ are rows i and j in A, respectively, and $\mathrm{IST}(A,B,C) = \infty$. Note that B^pAC^{n-p-1} gives m consecutive rows in $M^2(A,B,C)$ obtained by applying the morphism to $1^p0(-1)^{n-p-1}$ in $M^1(A,B,C)$. Row j in B^pAC^{n-p-1} is $b^p(-1)^{n-1}0c^{n-p-1}$, where b and c are row j in B and C, respectively. So, $b^p(-1)^{n-1}0c^{n-p-1}$ must be $0^r(-1)^s0^t$ for some $r,s,t \geq 0$. Hence, $b = (-1)^n$ and $c = 0^n$. As A is not a layered matrix, every row in $M^2(A,B,C)$ contains at most one 0, which is a contradiction. Therefore, $\mathrm{IST}(A,B,C) < \infty$. □

Lemma 37. *Let A, B, C be $m \times n$ matrices over $\{-1,0,1\}$ such that A has a 0. Then,*

(1) *If (A,B,C) is left-0-invariant and $01^{n-1} \notin R(A)$, then $01^{n^k-1} \notin R^k(A,B,C)$ for any $k \geq 0$.*
(2) *If (A,B,C) is left-0-invariant and $0(-1)^{n-1} \notin R(A)$, then $0(-1)^{n^k-1} \notin R^k(A,B,C)$ for any $k > 0$.*
(3) *If (A,B,C) is right-0-invariant and $1^{n-1}0 \notin R(A)$, then $1^{n^k-1}0 \notin R^k(A,B,C)$ for any $k > 0$.*
(4) *If (A,B,C) is right-0-invariant and $(-1)^{n-1}0 \notin R(A)$, then $(-1)^{n^k-1}0 \notin R^k(A,B,C)$ for any $k > 0$.*

Proof. As all of the statements are proved in similar ways, we will only prove (1). Assume (A,B,C) is left-0-invariant and $01^{n-1} \notin R(A)$. For $k = 1$, it is obvious that $M^1(A,B,C) = A$ and then $01^{n-1} \notin R^1(A,B,C)$. Suppose $k \geq 2$ and $01^{n^k-1} \in R^k(A,B,C)$. Let $0x_1x_2\cdots x_{n^{k-1}-1}$ be a row in $M^{k-1}(A,B,C)$ such that applying to it the morphism creates row 01^{n^k-1}. That is, 01^{n^k-1} is a row in the matrix $AX_1X_2\cdots X_{n^{k-1}-1}$, where $X_i \in \{A,B,C\}$, obtained from $0x_1x_2\cdots x_{n^{k-1}-1}$ by application of the morphism. This is a contradiction because $01^{n-1} \notin R(A)$. Hence, $01^{n^k-1} \notin R^k(A,B,C)$. □

Lemma 38. *Let A, B, C be $m \times n$ matrices over $\{-1,0,1\}$ such that A has a 0. If (A,B,C) is left-0-invariant (resp., right-0-invariant), then $\mathrm{IST}(A,B,C) = \infty$.*

Proof. Suppose that (A,B,C) is left-0-invariant. We will prove that for any $k > 0$, $R^k(A,B,C) \subseteq \{01^{n^k-1}, 1^{n^k}, 0(-1)^{n^k-1}, (-1)^{n^k}\}$ by induction on k. From the definition of a left-0-invariant triple, we have that $R^1(A,B,C) = R(A) \subseteq \{01^{n-1}, 1^n, 0(-1)^{n-1}, (-1)^n\}$. Suppose $R^k(A,B,C) \subseteq \{01^{n^k-1}, 1^{n^k},$

$0(-1)^{n^k-1}, (-1)^{n^k}\}$ for some k. If $01^{n-1} \notin R(A)$, then $0(-1)^{n-1} \in R(A)$ and, by Lemma 37, $01^{n^k-1} \notin R^k(A,B,C)$. So, every row in $M^k(A,B,C)$ is 1^{n^k}, $0(-1)^{n^k-1}$ or $(-1)^{n^k}$. As every row in $M^{k+1}(A,B,C)$ is a row in an $m \times n^{k+1}$ matrix obtained by applying the morphism to a row in $M^k(A,B,C)$, we have that

$$R^{k+1}(A,B,C) = R(B^{n^k}) \cup R(AC^{n^k-1}) \cup R(C^{n^k}).$$

We can see that $R(B^{n^k})$ and $R(C^{n^k})$ are subset of $\{1^{n^{k+1}}, (-1)^{n^{k+1}}\}$. Row i in AC^{n^k-1} is $1^{n^{k+1}}$, $(-1)^{n^{k+1}}$ and $0(-1)^{n^{k+1}-1}$ if row i in A is 1^n, $(-1)^n$ and $0(-1)^n$, respectively. Hence, $R^{k+1}(A,B,C) \subseteq \{1^{n^{k+1}}, 0(-1)^{n^{k+1}-1}, (-1)^{n^{k+1}}\}$ in the case of $01^{n-1} \notin R(A)$. For the case of $0(-1)^{n-1} \notin R(A)$, we can follow similar arguments to see that $R^{k+1}(A,B,C) \subseteq \{01^{n^{k+1}-1}, 1^{n^{k+1}}, (-1)^{n^{k+1}}\}$. Assume both 01^{n-1} and $0(-1)^{n-1}$ are in $R(A)$. So, every row in $M^k(A,B,C)$ is 01^{n^k-1}, 1^{n^k}, $0(-1)^{n^k-1}$ or $(-1)^{n^k}$ and

$$R^{k+1}(A,B,C) = R(AB^{n^k-1}) \cup R(B^{n^k}) \cup R(AC^{n^k-1}) \cup R(C^{n^k}).$$

Note that $R(B^{n^k}), R(C^{n^k}) \subseteq \{1^{n^{k+1}}, (-1)^{n^{k+1}}\}$. Row i in AC^{n^k-1} is $1^{n^{k+1}}$, $(-1)^{n^{k+1}}$, $01^{n^{k+1}-1}$ and $0(-1)^{n^{k+1}-1}$ if row i in A is 1^n, $(-1)^n$, 01^n and $0(-1)^n$, respectively. Row i in AB^{n^k-1} is $1^{n^{k+1}}$, $(-1)^{n^{k+1}}$, $01^{n^{k+1}-1}$ and $0(-1)^{n^{k+1}-1}$ if row i in A is 1^n, $(-1)^n$, 01^n and $0(-1)^n$, respectively. Hence, $R^{k+1}(A,B,C) \subseteq \{1^{n^{k+1}}, 0(-1)^{n^{k+1}-1}, (-1)^{n^{k+1}}\}$. Thus, we have shown that, for any $k > 0$,

$$R^k(A,B,C) \subseteq \{01^{n^k-1}, 1^{n^k}, 0(-1)^{n^k-1}, (-1)^{n^k}\}.$$

By Corollary 13, $G_o^k(A,B,C)$ is semi-transitive for any $k > 0$, which means that $\mathrm{IST}(A,B,C) = \infty$. □

Theorem 39. *Let A, B, C be $m \times n$ matrices over $\{-1,0,1\}$ such that A has a 0 and (A,B,C) is not independent from B and C. Suppose A is not a layered matrix but B and C are layered matrices. Then,* $\mathrm{IST}(A,B,C) = \infty$ *if and only if one of the following conditions holds:*

- *(A,B,C) is left-0-invariant.*
- *(A,B,C) is right-0-invariant.*
- *$R(A) = \{1^p 0 1^{n-p-1}\}$ for some $p \in \{1,2,\dots,n-2\}$, and B and C are all 1 and (-1) matrices, respectively.*

Proof. Assume $\mathrm{IST}(A,B,C) = \infty$. Since A is not a layered matrix, by Lemma 22, every row of $M^1(A,B,C) = A$ contains at most one 0. Then, by Theorem 12, every row in A is 01^{n-1}, $1^{n-1}0$, 1^n, $0(-1)^{n-1}$, $(-1)^{n-1}0$,

$(-1)^n$, $1^p0(-1)^{n-p-1}$ or $1^q(-1)^{n-q}$ for some $p \in \{1,2,\dots,n-2\}$ and $q \in \{1,2,\dots,n-1\}$. Since (A,B,C) is not independent from B and C, and B and C are layered matrices, we have every row of B and C must be 1^n or $(-1)^n$, otherwise there is a row in $M^k(A,B,C)$ having more than one 0 for some k.

If $1^q(-1)^{n-q}$ is row i in A for $q \in \{1,2,\dots,n-1\}$, then every row in A, except for row i, is $1^q(-1)^{n-q}$, $1^{q-1}0(-1)^{n-q}$ or $1^q0(-1)^{n-q-1}$. By (5) in Lemma 35, we have that A cannot contain both $1^{q-1}0(-1)^{n-q}$ and $1^q0(-1)^{n-q-1}$ as its rows.

If $1^{q-1}0(-1)^{n-q} \notin R(A)$, then

$$A = \begin{bmatrix} 1^q & a_1 & (-1)^{n-q-1} \\ 1^q & a_2 & (-1)^{n-q-1} \\ \vdots & \vdots & \vdots \\ 1^q & a_m & (-1)^{n-q-1} \end{bmatrix}$$

for $a_i \in \{0,1\}$. Since A has a 0, there is row j in A of the form $1^q0(-1)^{n-q-1}$. Let b and c be row j in B and C, respectively. Note that B^qAC^{n-q-1} is m consecutive rows of M^2 obtained by applying the morphism to $1^q0(-1)^{n-q-1}$. Then, $b^q1^q0(-1)^{n-q-1}c^{n-q-1}$ is row j in $M^2(A,B,C)$ and it must be of the form $1^r0^s(-1)^t$ for some $r,s,t \geq 0$. So, we obtain $b = 1^n$ and $c^n = (-1)^n$ and $1^{nq+q}0(-1)^{n^2-nq-q-1}$ is a row of $M^2(A,B,C)$. Note that B^qC^{n-q} is m consecutive rows of M^2 obtained by applying the morphism to $1^q(-1)^{n-q}$, and $1^{nq}(-1)^{n(n-q)}$ is row j in B^qC^{n-q}. Since $1^{nq+q}0(-1)^{n^2-nq-q-1}$ and $1^{nq}(-1)^{n(n-q)}$ are rows in $M^2(A,B,C)$, the conditions of Theorem 12 are not satisfied for $M^2(A,B,C)$. So, $M^2(A,B,C)$ is not semi-transitive, which is a contradiction. By the same argument, we also obtain a contradiction in the case of $1^{q-1}0(-1)^{n-q} \notin R(A)$. Hence $1^q(-1)^{n-q}$ cannot be a row in A.

Suppose that $1^p0(-1)^{n-p-1}$ is row i in A. By Theorem 12, we have that 1^n and $(-1)^n$ are not rows in $M^1(A,B,C)$. By Lemma 36, we have that 01^{n-1}, $1^{n-1}0$, $0(-1)^{n-1}$ and $(-1)^{n-1}0$ are not rows in $M^1(A,B,C)$. If there is a row in A of the form $1^u0(-1)^{n-u-1}$, where $1 \leq u \leq n-2$, by (5) in Lemma 35, we have $p = u$. Hence, we obtain

$$A = \begin{bmatrix} 1^p & 0 & (-1)^{n-p-1} \\ 1^p & 0 & (-1)^{n-p-1} \\ \vdots & \vdots & \vdots \\ 1^p & 0 & (-1)^{n-p-1} \end{bmatrix} \quad \text{where} \quad 1 \leq p < n-2.$$

Let b and c be row j in B and C, respectively, for any $1 \leq j \leq m$. Note that B^pAC^{n-p-1} is m consecutive rows of M^2 obtained by applying the morphism to $1^p0(-1)^{n-p-1}$. Then, $b^p1^p0(-1)^{n-p-1}c^{n-p-1}$ is row j in $M^2(A,B,C)$ and it must be of the form $1^r0^s(-1)^t$ for some $r,s,t \geq 0$. So, we obtain $b = 1^n$ and $c = (-1)^n$. Hence, we see that B and C are all 1 matrix and all (-1) matrix, respectively.

Assume that $1^p0(-1)^{n-p-1}$ is not a row in A for any $1 \leq p \leq n-2$. That is, every row in A is 01^{n-1}, $1^{n-1}0$, 1^n, $0(-1)^{n-1}$, $(-1)^{n-1}0$ or $(-1)^n$. By Lemma 35, we need to consider the following two cases.

Case1: $01^{n-1}, 0(-1)^{n-1} \in R(A)$ and $1^{n-1}0, (-1)^{n-1}0 \notin R(A)$. That is, every row in A is 01^{n-1}, 1^n, $0(-1)^{n-1}$ or $(-1)^n$. Suppose that 01^{n-1} is a row in A. Then, AB^{n-1} is m consecutive rows in $M^2(A,B,C)$. Let row i in B be b. Consider the following subcases:

- If row i in A is 01^{n-1}, then $01^{n-1}b^{n-1}$ is a row in $M^2(A,B,C)$. Since $b \neq 0^n$, we have $b = 1^n$.
- If row i in A is 1^n, then 1^nb^{n-1} is a row in $M^2(A,B,C)$. Since $b \neq 0^n$, we have $b = 1^n$.
- If row i in A is $0(-1)^{n-1}$, then $0(-1)^{n-1}b^{n-1}$ is a row in $M^2(A,B,C)$. Since $b \neq 0^n$, we have $b = (-1)^n$.
- If row i in A is $(-1)^n$, then $(-1)^nb^{n-1}$ is a row in $M^2(A,B,C)$. Since $b \neq 0^n$, we have $b = (-1)^n$.

Suppose that $0(-1)^{n-1}$ is a row in A. Then, AC^{n-1} is m consecutive rows in $M^2(A,B,C)$. Let row i in C be c. Consider the following subcases:

- If row i in A is 01^{n-1}, then $0(-1)^{n-1}c^{n-1}$ is a row in $M^2(A,B,C)$. Since $c \neq 0^n$, we have $c = 1^n$.
- If row i in A is 1^n, then 1^nc^{n-1} is a row in $M^2(A,B,C)$. Since $c \neq 0^n$, we have $c = 1^n$.
- If row i in A is $0(-1)^{n-1}$, then $0(-1)^{n-1}c^{n-1}$ is a row in $M^2(A,B,C)$. Since $c \neq 0^n$, we have $c = (-1)^n$.
- If row i in A is $(-1)^n$, then $(-1)^nc^{n-1}$ is a row in $M^2(A,B,C)$. Since $c \neq 0^n$, we have $c = (-1)^n$.

Thus, we see that (A,B,C) is left-0-invariant.

Case2: $1^{n-1}0, (-1)^{n-1}0 \in R(A)$ and $01^{n-1}, 0(-1)^{n-1} \notin R(A)$. With the same way of the case 1, we can prove that (A,B,C) is right-0-invariant.

Thus, "⇒" has been proved. Lemma 38 gives us the converse. □

4. Direction of further research

In this paper, we fully classified semi-transitivity of infinite families of directed split graphs generated by iterations of morphisms in the cases when the matrix A has a 0. This research is a first step towards a classification of semi-transitive directed graphs in terms of positions of 0s and 1s (and (-1)s in the lower-triangular case) in the adjacency matrices. An application of such a classification could be in finding more efficient algorithms to recognize semi-transitivity of a directed graph, which is a problem solvable in polynomial time [8]. More importantly, a classification of semi-transitive directed graphs via adjacency matrices may lead to a better understanding of which (undirected) graphs admit semi-transitive orientations; this is an NP-complete problem [7, 8]. Should the general problem resist attempts to solve it, one could shift their attention to classification of semi-transitivity of naturally defined (infinite) families of directed graphs. Such a shift should allow discovering new methods to deal with semi-transitivity of oriented graphs, and hence bring us closer to solving the general problem.

For yet another direction of research, note that Definition 20 of the index of semi-transitivity $\mathrm{IST}(A, B, C)$ makes sense in many situations when A has no 0's. For example, if A, B and C contain only 1's, we still can apply Definition 20 to see that $\mathrm{IST}(A, B, C) = \infty$. On the other hand, Definition 20 does not work, for example, in the case when A is any matrix without 0's while B and C contain only 0's, as the infinite graph $G_o(A, B, C)$ is then not well-defined. Indeed, in the later case we see that $G_o^i(A, B, C)$ is not an induced subgraph of $G_o^{i+1}(A, B, C)$ while $G_o^i(A, B, C)$ is an induced subgraph of $G_o^{i+2}(A, B, C)$ for any $i \geq 0$, so that we have two infinite chains of induced subgraphs leading to two different infinite graphs as the limits (one of which is with no edges between the clique and the independent set). For another example, letting A be an all one matrix, B be an all (-1) matrix, and C be an all zero matrix, we witness the situation of three infinite chains of induced subgraphs with three infinite graphs as the limits.

In any case, the problem we solved in this paper can be extended to the case of matrices A with no 0's in the situations when the limiting infinite graph is uniquely defined, and the goal then is to classify such triples (A, B, C) with $\mathrm{IST}(A, B, C) = \infty$. Of course, extra care should be taken about Definition 20 as it still may not work. For example, A without 0's can easily be chosen so that $G_0^1(A, B, C)$ has directed cycles and thus is not semi-transitive, while then choosing B and C be all one matrices, we see that $G_0^k(A, B, C)$ is semi-transitive for $k > 1$, so that the limiting graph is also semi-transitive and it is natural to assume that $\mathrm{IST}(A, B, C) = \infty$, while by

Definition 20, $\mathrm{IST}(A,B,C)=1$. However, natural adjustments to Definition 20 could be introduced. For example, we can define $\mathrm{IST}(A,B,C):=\infty$ if there exists a natural number k such that $G_o^i(A,B,C)$ is semi-transitive for every $i\geq k$.

References

[1] H. Z.Q. Chen, S. Kitaev, A. Saito. Representing split graphs by words, *Discussiones Mathematicae Graph Theory*, to appear 2021.

[2] P. Choudhary, P. Jain, R. Krithika, V. Sahlot. Vertex deletion on split graphs: beyond 4-hitting set. *Theoret. Comput. Sci.* **845** (2020), 21–37. MR4165653

[3] S. Foldes, P. L. Hammer. Split graphs. In Proceedings of the *Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing* (Louisiana State Univ., Baton Rouge, 1977), pages 311–315. Congressus Numerantium, No. XIX. MR0505860

[4] M. Halldórsson, S. Kitaev, A. Pyatkin. Alternation graphs, Lecture Notes in Computer Science **6986** (2011) 191–202. Proceedings of the 37th International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2011, Tepla Monastery, Czech Republic, June 21–24, 2011. MR2914710

[5] M. Halldórsson, S. Kitaev, A. Pyatkin. Semi-transitive orientations and word-representable graphs. *Discr. Appl. Math.* **201** (2016) 164–171. MR3453927

[6] K. Iamthong. Word-representability of split graphs generated by morphisms. arXiv:2104.14872, 2021.

[7] S. Kitaev. A Comprehensive Introduction to the Theory of Word-Representable Graphs, In: Developments in Language Theory: 21st International Conference, DLT, Liege, Aug 7–11, 2017. *Lecture Notes in Comp. Sci.* **10396** (2017) 36–67. MR3691068

[8] S. Kitaev, V. Lozin. Words and Graphs, *Springer*, 2015. MR3408128

[9] S. Kitaev, Y. Long, J. Ma, H. Wu. Word-representability of split graphs, *Journal of Combinatorics*, to appear. MR4376083

[10] M. Lothaire. *Combinatorics on words*, Encyclopedia of Mathematics and its Applications, 17, Addison-Wesley Publishing Co., Reading, Mass, 1983. MR0675953

Kittitat Iamthong
Department of Mathematics and Statistics
University of Strathclyde
26 Richmond Street Glasgow G1 1XH
United Kingdom
E-mail address: kittitat.pele@gmail.com

Sergey Kitaev
Department of Mathematics and Statistics
University of Strathclyde
26 Richmond Street Glasgow G1 1XH
United Kingdom
E-mail address: sergey.kitaev@strath.ac.uk

Received May 28, 2021