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Small domination-type invariants in random graphs

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For $c \in \mathbb{R}^+ \cup \{\infty\}$ and a graph G, a function $f : V(G) \to \{0, 1, c\}$ is called a c-self dominating function of G if for every vertex $u \in V(G)$, $f(u) \geq c$ or $\max\{f(v) : v \in N_G(u)\} \geq 1$, where $N_G(u)$ is the neighborhood of u in G. The minimum weight $w(f) = \sum_{u \in V(G)} f(u)$ of a c-self dominating function f of G is called the c-self domination number of G. The c-self domination concept is a common generalization of three domination-type invariants; (original) domination, total domination and Roman domination. In this paper, we investigate a behavior of the c-self domination number in random graphs for small c.

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1. Introduction

Throughout this paper, we let $\mathbb{R}^+$ and $\mathbb{Z}^+$ denote the sets of positive numbers and positive integers, respectively. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For a vertex $u \in V(G)$, we let $N_G(u)$ denote the *neighborhood* of u in G; thus, $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. A set $S \subseteq V(G)$ is a *dominating set* (resp. a *total dominating set*) of G if each vertex in $V(G) \setminus S$ (resp. each vertex in $V(G)$) is adjacent to a vertex in S. The minimum size of a dominating set (resp. a total dominating set) of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is called the *domination number* (resp. the *total domination number*) of G. Because a graph G with isolated vertices has no total dominating set, the total domination number has been typically defined only for graphs without isolated vertices. However, for convenience, we define $\gamma_t(G)$ as $\gamma_t(G) = \infty$ if G has an isolated vertex. Domination and total domination are important invariants in graph theory because they have many applications in both theoretical and applied problems [5, 6, 7].

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The first author [4] recently defined a novel domination-type concept as follows: Let G be a graph. For a function $f : V(G) \to \mathbb{R}^+ \cup \{0, \infty\}$, the *weight* $w(f)$ of f is defined by $w(f) = \sum_{u \in V(G)} f(u)$. Let $c \in \mathbb{R}^+ \cup \{\infty\}$. A function $f : V(G) \to \mathbb{R}^+ \cup \{0, \infty\}$ is a *c-self dominating function* (or *c-SDF*) of G if for each $u \in V(G)$, $f(u) \geq c$ or $\max\{f(v) : v \in N_G(u)\} \geq 1$. Then the following proposition holds.

Proposition 1.1 (Furuya [4]). *Let $c \in \mathbb{R}^+ \cup \{\infty\}$, and let G be a graph. If f is a c-SDF of G, then there exists a c-SDF g of G such that $w(g) \leq w(f)$ and $g(u) \in \{0, 1, c\}$ for all $u \in V(G)$.*

Based on Proposition 1.1, the minimum weight of a c-SDF of G is well-defined. The minimum weight of a c-SDF of G, denoted by $\gamma^c(G)$, is called the *c-self domination number* of G. Note that $\gamma^1(G) = \gamma(G)$ and $\gamma^\infty(G) = \gamma_t(G)$ for all graphs G (see [4]). Furthermore, the $\frac{1}{2}$-self domination number is equal to half of the Roman domination number defined in Subsection 1.1. Hence the self domination concept is a common generalization of three well-studied invariants.

In this paper, our primary objective is to analyze the behavior of the c-self domination number in Erdős–Rényi model random graphs $G(n, p)$ on $[n] := \{1, 2, \ldots, n\}$. For $p \in (0, 1)$ and $n \in \mathbb{Z}^+ \setminus \{1\}$, let $a_p(n) = \log_{1/(1-p)} \frac{n}{\log_{1/(1-p)} n \ln n}$. Then the following are elucidated.

Theorem A (Wieland and Godbole [9]). *For any $p \in (0, 1)$ a fixed constant, $\gamma(G(n, p)) \in \{\lfloor a_p(n) \rfloor + 1, \lfloor a_p(n) \rfloor + 2\}$ with a probability that tends to 1 as $n \to \infty$.*

Theorem B (Bonato and Wang [2]). *For any $p \in (0, 1)$ a fixed constant, $\gamma_t(G(n, p)) \in \{\lfloor a_p(n) \rfloor + 1, \lfloor a_p(n) \rfloor + 2\}$ with a probability that tends to 1 as $n \to \infty$.*

Remark 1. Recall that our definition of total domination is not conventional because we define $\gamma_t(G) = \infty$ for graphs G with an isolated vertex. Hence the total domination in Theorem B strictly differs from the one presented in this paper. However, Bonato and Wang [2] proved that $G(n, p)$ has a total dominating set of size $\lfloor a_p(n) \rfloor + 2$ with a probability that tends to 1 as $n \to \infty$. Furthermore, since $\gamma(G) \leq \gamma_t(G)$ for all graphs G, it follows from Theorem A that $G(n, p)$ has no total dominating set of size $\lfloor a_p(n) \rfloor$ with a probability that tends to 1 as $n \to \infty$. Hence Theorem B holds under our definition.

By the definition of self domination, if $c, c' \in \mathbb{R}^+ \cup \{\infty\}$ satisfy $c \leq c'$, then $\gamma^c(G) \leq \gamma^{c'}(G)$ for all graphs G. Here, we note that for $c \in (1, \infty)$, the value $\gamma^c(G)$ may be a non-integer if c is a non-integer. Therefore, the following result is obtained as a corollary of Theorems A and B.

Corollary 1.2. *For $c \in [1, \infty)$ and any $p \in (0,1)$ a fixed constant, $\gamma^c(G(n,p)) \in [\lfloor a_p(n) \rfloor + 1, \lfloor a_p(n) \rfloor + 2]$ with a probability that tends to 1 as $n \to \infty$.*

In this paper, we focus on c-self domination in the remaining case, that is, the case where $c \in (0,1)$. To state our main result, we extend the floor $\lfloor * \rfloor$. For $t \in \mathbb{Z}^+$ and $a \in \mathbb{R}$, let $\lfloor a \rfloor_t$ be the largest number in $\{m_1 + \frac{m_2}{t} : m_1, m_2 \in \mathbb{Z},\ m_1 + \frac{m_2}{t} \le a\}$. Recall that $a_p(n) = \log_{1/(1-p)} \frac{n}{\log_{1/(1-p)} n \ln n}$. For $p \in (0,1)$, $t \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+ \setminus \{1\}$, let $b_{p,t}(n) = \lfloor \lfloor a_p(n) \rfloor_t + \frac{1}{t} \rfloor + 1$. Note that if $\lfloor a_p(n) \rfloor_t + \frac{1}{t}$ is a non-integer, then $b_{p,t}(n)$ is the smallest integer more than $a_p(n)$; if $\lfloor a_p(n) \rfloor_t + \frac{1}{t}$ is an integer, then $b_{p,t}(n)$ is the second smallest integer more than $a_p(n)$. Our main result is the following:

Theorem 1.3. *Let s and t be integers with $2 \le s \le t-1$. Then for any $p \in (0,1)$ a fixed constant,*

$$\gamma^{\frac{s}{t}}(G(n,p)) \in \left[\lfloor a_p(n) \rfloor_t + \frac{1}{t},\ b_{p,t}(n) \right] \setminus \left\{ b_{p,t}(n) - \frac{i}{t} : t-s+1 \le i \le t-1 \right\}$$

with a probability that tends to 1 as $n \to \infty$.

Our approach is similar to that of previous problem. In particular, we determine a random variable corresponding to c-SDFs and calculate its expected value in Section 3. Then we obtain a weaker result than Theorem 1.3:

$$\Pr\left(\gamma^{\frac{s}{t}}(G(n,p)) \in \left[\lfloor a_p(n) \rfloor_t + \frac{1}{t},\ b_{p,t}(n) \right] \right) \to 1 \quad (n \to \infty)$$

(Theorem 3.1). The highlight of this paper is presented in Section 4. Although several known results for domination-type invariants in random graphs are proven by simply bounding a random variable, we can refine the above weak result to Theorem 1.3 using an additional graph-theoretic approach. Note that $b_{p,t}(n) \le \lfloor a_p(n) \rfloor_t + \frac{t+1}{t}$ and $\gamma^{\frac{s}{t}}(G) \in \{m_1 + \frac{m_2}{t} : m_1, m_2 \in \mathbb{Z}^+ \cup \{0\}\}$ for all graphs G. Thus Theorem 3.1 claims that $\gamma^{\frac{s}{t}}(G(n,p))$ takes at most $t+1$ values with a high probability, and Theorem 1.3 improves "at most $t+1$" to "at most $t-s+2$". In Subsection 1.1, we focus on the Roman domination number and its related topic.

Remark 2. Using a similar strategy as in Sections 3 and 4, we can estimate $\gamma^c(G(n,p))$ even if $c \in (0,1)$ is an irrational number. However, it seems difficult to describe an optimal formula. However, we can provide the following estimated formula (based on Theorem 3.1): Let $c \in (0,1)$ be an irrational number. Then for any $p \in (0,1)$ a fixed constant and $\varepsilon \in \mathbb{R}^+$, $\Pr(\gamma^c(G(n,p)) \in (a_p(n), a_p(n)+1+\varepsilon]) \to 1$ $(n \to \infty)$.

1.1. Roman domination and differential

A function $f : V(G) \to \{0,1,2\}$ is a *Roman dominating function* of G if each vertex $u \in V(G)$ with $f(u) = 0$ is adjacent to a vertex $v \in V(G)$ with $f(v) = 2$. The minimum weight of a Roman dominating function of G, denoted by $\gamma_R(G)$, is called the *Roman domination number* of G. Roman domination was introduced by Stewart [8], and was further studied by Cockayne et al. [3]. Since $\gamma_R(G) = 2\gamma^{\frac{1}{2}}(G)$ for all graphs G, we obtain the following result as a corollary of Theorem 1.3.

Corollary 1.4. *For any $p \in (0,1)$ a fixed constant, $\gamma_R(G(n,p)) \in \{2\lfloor a_p(n)\rfloor_2 + i : 1 \le i \le 3\}$ with a probability that tends to 1 as $n \to \infty$.*

Roman domination is closely related to another important invariant. The *differential* of a graph G, denoted by $\partial(G)$, is defined as $\partial(G) = \max\{|(\bigcup_{u\in X} N_G(u)) - X| - |X| : X \subseteq V(G)\}$. The differential has been widely studied because it was inspired by information diffusion in social networks. Recently, Bermudo et al. [1] validated a very useful result that every graph G satisfies $\gamma_R(G) + \partial(G) = |V(G)|$. Hence Corollary 1.4 gives the following.

Corollary 1.5. *For any $p \in (0,1)$ a fixed constant, $\partial(G(n,p)) \in \{n - 2\lfloor a_p(n)\rfloor_2 - i : 1 \le i \le 3\}$ with a probability that tends to 1 as $n \to \infty$.*

2. Lemmas

In this section, we prepare a few lemmas that will be used in our argument. We start with two fundamental lemmas related to the c-self domination concept.

Lemma 2.1. *Let $a \in \mathbb{R}^+$ and $c \in (0,1)$, and let G be a graph of order at least a. Then $\gamma^c(G) \le a$ if and only if there exists a c-SDF $f : V(G) \to \{0,1,c\}$ of G such that $a - 1 < w(f) \le a$.*

Proof. The “if” part is trivial. Thus it suffices to prove the “only if” part. Suppose that $\gamma^c(G) \le a$. Then by Proposition 1.1, there exists a c-SDF f of G such that $w(f) \le a$ and $f(u) \in \{0,1,c\}$ for all $u \in V(G)$. Choose f so that $w(f)$ is as large as possible under these constraints. If $w(f) = |V(G)|$, then $w(f) = a$ because $w(f) \leq a \leq |V(G)| = w(f)$, as desired. Thus we may assume that $w(f) < |V(G)|$. Since $c \in (0,1)$, there exists a vertex $u_0 \in V(G)$ such that $f(u_0) \in \{0, c\}$. Then the function $g : V(G) \to \{0, 1, c\}$ with

$$g(u) = \begin{cases} 1 & (u = u_0) \\ f(u) & (u \neq u_0) \end{cases}$$

is a c-SDF of G and $w(g) > w(f)$. This together with the maximality of $w(f)$ implies that $a < w(g) \leq w(f) + 1$, and so $a - 1 < w(f) \leq a$. ☐

Lemma 2.2. *Let s and t be integers with $2 \leq s \leq t-1$. Let G be a graph, and suppose that $\gamma^{\frac{s}{t}}(G)$ is a non-integer and $\gamma^{\frac{s}{t}}(G) \leq \lfloor \gamma^{\frac{s}{t}}(G) \rfloor + \frac{s-1}{t}$. Then $\gamma^{\frac{1}{t}}(G) < \lfloor \gamma^{\frac{s}{t}}(G) \rfloor$.*

Proof. Let $f : V(G) \to \{0, 1, \frac{s}{t}\}$ be an $\frac{s}{t}$-SDF of G with $w(f) = \gamma^{\frac{s}{t}}(G)$, and let $U = \{u \in V(G) : f(u) = \frac{s}{t}\}$. Since $\gamma^{\frac{s}{t}}(G)$ is a non-integer, we have $U \neq \emptyset$. If $|U| = 1$, then $\gamma^{\frac{s}{t}}(G) = \lfloor \gamma^{\frac{s}{t}}(G) \rfloor + \frac{s}{t}$, which contradicts the second assumption of the lemma. Thus $|U| \geq 2$.

Let $g : V(G) \to \{0, 1, \frac{1}{t}\}$ be the function with

$$g(u) = \begin{cases} \frac{1}{t} & (u \in U) \\ f(u) & (u \notin U). \end{cases}$$

Then g is a $\frac{1}{t}$-SDF of G, and hence

$$\begin{aligned} \gamma^{\frac{1}{t}}(G) &\leq w(g) \\ &= w(f) - \frac{|U|(s-1)}{t} \\ &\leq \gamma^{\frac{s}{t}}(G) - \frac{2(s-1)}{t} \\ &\leq \lfloor \gamma^{\frac{s}{t}}(G) \rfloor - \frac{s-1}{t} \\ &< \lfloor \gamma^{\frac{s}{t}}(G) \rfloor, \end{aligned}$$

as desired. ☐

The following lemmas are well-known (or proved by easy argument) in mathematics.

Lemma 2.3 (Stirling's formula)**.** *For $n \in \mathbb{Z}^+$, $n! \geq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.*

Lemma 2.4. *For $x \geq 0$, $1 - x \leq e^{-x}$.*

3. Crude estimation

In this section, we prove the following theorem, which is weaker than Theorem 1.3.

Theorem 3.1. *Let s and t be integers with $1 \leq s \leq t-1$. Then for any $p \in (0,1)$ a fixed constant,*

$$\gamma^{\frac{s}{t}}(G(n,p)) \in \left[\lfloor a_p(n) \rfloor_t + \frac{1}{t},\ b_{p,t}(n)\right]$$

with a probability that tends to 1 as $n \to \infty$.

In [9], Wieland and Godbole implicitly proved the following lemma.

Lemma 3.2 (Wieland and Godbole [9])**.** *Let $\varepsilon \in \mathbb{R}^+$. Then for any $p \in (0,1)$ a fixed constant, $\gamma(G(n,p)) \leq \lceil a_p(n) + \varepsilon \rceil$ with a probability that tends to 1 as $n \to \infty$.*

Lemma 3.3. *For $p \in (0,1)$, $t \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+ \setminus \{1\}$, we have $\lceil a_p(n) + \frac{1}{2t} \rceil \leq b_{p,t}(n)$.*

Proof. There exist non-negative integers m_1 and m_2 such that $m_1 + \frac{m_2}{t} \leq a_p(n) < m_1 + \frac{m_2+1}{t}$ and $0 \leq m_2 \leq t-1$. Suppose $m_2 = t-1$. Since $\lfloor a_p(n) \rfloor_t + \frac{1}{t} = m_1 + \frac{t-1}{t} + \frac{1}{t} = m_1 + 1$ ($\in \mathbb{Z}^+$), we have $b_{p,t}(n) = \lfloor \lfloor a_p(n) \rfloor_t + \frac{1}{t} \rfloor + 1 = m_1 + 2$. On the other hand, $a_p(n) + \frac{1}{2t} < m_1 + 1 + \frac{1}{2t}$, and so $\lceil a_p(n) + \frac{1}{2t} \rceil \leq m_1 + 2 = b_{p,t}(n)$, as desired. Thus we may assume that $0 \leq m_2 \leq t-2$.

Since $\lfloor a_p(n) \rfloor_t + \frac{1}{t} = m_1 + \frac{m_2+1}{t} \leq m_1 + \frac{t-1}{t}$, we have $b_{p,t}(n) = \lfloor \lfloor a_p(n) \rfloor_t + \frac{1}{t} \rfloor + 1 = m_1 + 1$. In contrast, $a_p(n) + \frac{1}{2t} < m_1 + \frac{t-1}{t} + \frac{1}{2t} = m_1 + \frac{2t-1}{2t} < m_1 + 1$, and so $\lceil a_p(n) + \frac{1}{2t} \rceil \leq m_1 + 1 = b_{p,t}(n)$, as desired. □

Proof of Theorem 3.1. Note that $\gamma^{\frac{s}{t}}(G) \leq \gamma^1(G) = \gamma(G)$ for all graphs G. Hence, by Lemma 3.2 with $\varepsilon = \frac{1}{2t}$ and Lemma 3.3,

$$\begin{aligned} \Pr(\gamma^{\frac{s}{t}}(G(n,p)) \leq b_{p,t}(n)) &\geq \Pr(\gamma(G(n,p)) \leq b_{p,t}(n)) \\ &\geq \Pr\left(\gamma(G(n,p)) \leq \left\lceil a_p(n) + \frac{1}{2t} \right\rceil\right) \\ &\to 1 \quad (n \to \infty). \end{aligned}$$

Consequently, we obtain the upper bound of the theorem.

Subsequently, we prove the lower bound of the theorem. Let $\mathbb{M} = \{m_1 + \frac{m_2}{t} : m_1, m_2 \in \mathbb{Z}^+ \cup \{0\}\}$, and for $a \in \mathbb{R}^+$, let $\mathcal{M}(a) = \{(m_1, m_2) : m_1 + \frac{m_2}{t} = a\}$. Then $\mathcal{M}(a) \neq \emptyset$ if and only if $a \in \mathbb{M}$. Furthermore, we note that $\lfloor a_p(n) \rfloor_t + \frac{1}{t}$ is the smallest number in $\mathbb{M}$ that is more than $a_p(n)$. Since $\gamma^{\frac{s}{t}}(G) \geq \gamma^{\frac{1}{t}}(G)$ for all graphs G, it suffices to show that $\gamma^{\frac{1}{t}}(G(n,p)) > a_p(n)$ with a probability that tends to 1 as $n \to \infty$.

For $m_1, m_2 \in \mathbb{Z}^+ \cup \{0\}$, let X_{m_1,m_2} be the random variable counting the number of $\frac{1}{t}$-SDFs $f : [n] \to \{0, 1, \frac{1}{t}\}$ of $G(n,p)$ with $|\{u \in [n] : f(u) = 1\}| = m_1$ and $|\{u \in [n] : f(u) = \frac{1}{t}\}| = m_2$. For $a \in \mathbb{M}$, let $X_a = \sum_{(m_1,m_2)\in\mathcal{M}(a)} X_{m_1,m_2}$.

For a graph G, an ordered pair (S_1, S_2) of subsets of $V(G)$ with $S_1 \cap S_2 = \emptyset$ is called a $\frac{1}{t}$-*self dominating pair* of G if the function $f : V(G) \to \{0, 1, \frac{1}{t}\}$ with

$$f(u) = \begin{cases} 0 & (u \in V(G) \setminus (S_1 \cup S_2)) \\ 1 & (u \in S_1) \\ \frac{1}{t} & (u \in S_2) \end{cases}$$

is a $\frac{1}{t}$-SDF of G. Let $\mathcal{S}_{m_1,m_2} = \left\{(S_1, S_2) \in \binom{[n]}{m_1} \times \binom{[n]}{m_2} : S_1 \cap S_2 = \emptyset\right\}$, and for $(S_1, S_2) \in \mathcal{S}_{m_1,m_2}$, let I_{S_1,S_2} be the random variable satisfying

$$I_{S_1,S_2} = \begin{cases} 1 & ((S_1, S_2) \text{ is a } \frac{1}{t}\text{-self dominating pair of } G(n,p)) \\ 0 & (\text{otherwise}). \end{cases}$$

Note that $X_{m_1,m_2} = \sum_{(S_1,S_2)\in\mathcal{S}_{m_1,m_2}} I_{S_1,S_2}$. The following claim plays a key role in our argument.

Claim 3.1. *For non-negative integers m_1 and m_2,*

$$E(X_{m_1,m_2}) = \frac{n!}{(n - m_1 - m_2)!\ m_1!\ m_2!}(1 - (1-p)^{m_1})^{n-m_1-m_2}.$$

Proof. For $(S_1, S_2) \in \mathcal{S}_{m_1,m_2}$, since $\Pr(N_G(u) \cap S_1 \neq \emptyset) = 1 - (1-p)^{m_1}$ for each $u \in [n] \setminus (S_1 \cup S_2)$,

$$\Pr(I_{S_1,S_2} = 1) = \prod_{u\in[n]\setminus(S_1\cup S_2)} \Pr(N_G(u) \cap S_1 \neq \emptyset) = (1 - (1-p)^{m_1})^{n-m_1-m_2}.$$

Since $X_{m_1,m_2} = \sum_{(S_1,S_2)\in\mathcal{S}_{m_1,m_2}} I_{S_1,S_2}$, it follows that

$$\begin{aligned} E(X_{m_1,m_2}) &= \sum_{(S_1,S_2)\in\mathcal{S}_{m_1,m_2}} E(I_{S_1,S_2}) \\ &= \sum_{(S_1,S_2)\in\mathcal{S}_{m_1,m_2}} \Pr(I_{S_1,S_2}=1) \\ &= \binom{n}{m_1}\binom{n-m_1}{m_2}(1-(1-p)^{m_1})^{n-m_1-m_2}, \end{aligned}$$

as desired. □

Since $\frac{1}{1-p} > 1$, the value $h_0 = \min\{h \in \mathbb{Z}^+ : t - \frac{1}{(1-p)^a} < 0$ for all $a \geq h\}$ is a well-defined constant (depending on p and t only). For $x \in \mathbb{R}^+$, let $L(x) = \log_{1/(1-p)} x$. Note that $a_p(n) = \log_{1/(1-p)} \frac{n}{\log_{1/(1-p)} n \ln n} = L(\frac{n}{L(n)\ln n})$. In the remainder of this proof, we consider $G(n,p)$ for sufficiently large n. Thus, for example, we may assume that $L(L(n)) > 0$, $n > ta_p(n)$, $a_p(n) > h_0$, etc.

Claim 3.2. *Let m_1 and m_2 be non-negative integers with $a_p(n) - 1 < m_1 + \frac{m_2}{t} \leq a_p(n)$. Then the following are satisfied.*

- (i) *We have $E(X_{m_1,m_2}) < \exp\left[(m_1+m_2)(\ln n + 2) - \frac{L(n)\ln n}{(1-p)^{a_p(n)-m_1}}\right]$.*
- (ii) *If $0 \leq m_1 \leq a_p(n) - h_0$, then $E(X_{m_1,m_2}) < \exp[t(2L(n) - L(L(n)\ln n) \times \ln n)]$.*

Proof. (i) By Lemma 2.3, if $m_1 \geq 1$ and $m_2 \geq 1$, then

$$\begin{aligned} \frac{n!}{(n-m_1-m_2)!\, m_1!\, m_2!} &\leq n^{m_1+m_2} \cdot \frac{1}{\sqrt{2\pi m_1}\left(\frac{m_1}{e}\right)^{m_1}} \cdot \frac{1}{\sqrt{2\pi m_2}\left(\frac{m_2}{e}\right)^{m_2}} \\ &< (en)^{m_1+m_2}; \end{aligned}$$

if $m_i = 0$ for some $i \in \{1,2\}$, then $m_{3-i} \geq 1$, and hence

$$\begin{aligned} \frac{n!}{(n-m_1-m_2)!\, m_1!\, m_2!} &\leq n^{m_{3-i}} \cdot \frac{1}{\sqrt{2\pi m_{3-i}}\left(\frac{m_{3-i}}{e}\right)^{m_{3-i}}} \\ &< (en)^{m_{3-i}} = (en)^{m_1+m_2}. \end{aligned}$$

In either case,

$$\frac{n!}{(n-m_1-m_2)!\, m_1!\, m_2!} < (en)^{m_1+m_2}. \tag{1}$$

Furthermore, we have

(2)

$$n(1-p)^{m_1} = \frac{n(1-p)^{L(\frac{n}{L(n)\ln n})}}{(1-p)^{a_p(n)-m_1}} = \frac{n \cdot \frac{L(n)\ln n}{n}}{(1-p)^{a_p(n)-m_1}} = \frac{L(n)\ln n}{(1-p)^{a_p(n)-m_1}}.$$

By Claim 3.1, Lemma 2.4, (1), and (2),

$$\begin{aligned} E(X_{m_1,m_2}) &= \frac{n!}{(n-m_1-m_2)!\; m_1!\; m_2!}(1-(1-p)^{m_1})^{n-m_1-m_2} \\ &< (en)^{m_1+m_2}\left(e^{-(1-p)^{m_1}}\right)^{n-m_1-m_2} \\ &= \exp[(m_1+m_2)+(m_1+m_2)\ln n - n(1-p)^{m_1} \\ &\quad + (m_1+m_2)(1-p)^{m_1}] \\ &\le \exp\left[2(m_1+m_2)+(m_1+m_2)\ln n - \frac{L(n)\ln n}{(1-p)^{a_p(n)-m_1}}\right]. \end{aligned}$$

(ii) By the definitions of m_1 and m_2, we have

(3) $$m_1+m_2 \le t\left(m_1+\frac{m_2}{t}\right) \le ta_p(n) = t(L(n)-L(L(n)\ln n)).$$

Since $a_p(n)-m_1 \ge h_0$, it follows from the definition of h_0 that $(t - \frac{1}{(1-p)^{a_p(n)-m_1}})L(n)\ln n < 0$. This together with (i) and (3) implies that

$$\begin{aligned} E(X_{m_1,m_2}) &\le \exp\left[(m_1+m_2)(\ln n+2) - \frac{L(n)\ln n}{(1-p)^{a_p(n)-m_1}}\right] \\ &\le \exp\left[t(L(n)-L(L(n)\ln n))(\ln n+2) - \frac{L(n)\ln n}{(1-p)^{a_p(n)-m_1}}\right] \\ &= \exp\left[\left(t-\frac{1}{(1-p)^{a_p(n)-m_1}}\right)L(n)\ln n \right. \\ &\quad \left. +t(2L(n)-L(L(n)\ln n)\ln n - 2L(L(n)\ln n))\right] \\ &< \exp[t(2L(n)-L(L(n)\ln n)\ln n)], \end{aligned}$$

as desired. □

Claim 3.3. *Let $a \in \mathbb{M}$ be a number with $a_p(n)-1 < a \le a_p(n)$. Then $E(X_a) \to 0$ if $n \to \infty$.*

Proof. By the definition of X_a,

$$\begin{aligned} E(X_a) &= E\left(\sum_{(m_1,m_2)\in\mathcal{M}(a)} X_{m_1,m_2}\right) \\ &= \sum_{\substack{(m_1,m_2)\in\mathcal{M}(a)\\ 0\le m_1\le a_p(n)-h_0}} E(X_{m_1,m_2}) + \sum_{\substack{(m_1,m_2)\in\mathcal{M}(a)\\ a_p(n)-h_0<m_1\le a}} E(X_{m_1,m_2}). \end{aligned}$$

Note that the number of $m_1 \in \mathbb{Z}^+$ satisfying $a_p(n) - h_0 < m_1 \le a$ is at most h_0 because $a \le a_p(n)$. Hence $\sum_{\substack{(m_1,m_2)\in\mathcal{M}(a)\\ a_p(n)-h_0<m_1\le a}} E(X_{m_1,m_2})$ is a sum with a constant number of terms. Thus it suffices to prove the following:

(A1) $\sum_{\substack{(m_1,m_2)\in\mathcal{M}(a)\\ 0\le m_1\le a_p(n)-h_0}} E(X_{m_1,m_2}) \to 0$ $(n\to\infty)$, and

(A2) for each $(m_1,m_2)\in\mathcal{M}(a)$, if $a_p(n)-h_0 < m_1 \le a$, then $E(X_{m_1,m_2}) \to 0$ $(n\to\infty)$.

By Claim 3.2(ii),

$$\begin{aligned} \sum_{\substack{(m_1,m_2)\in\mathcal{M}(a)\\ 0\le m_1\le a_p(n)-h_0}} E(X_{m_1,m_2}) &< (a_p(n)-h_0+1)\exp[t(2L(n)-L(L(n)\ln n)\ln n)] \\ &\le a_p(n)\exp[t(2L(n)-L(L(n)\ln n)\ln n)] \\ &= \exp[\ln a_p(n) + t(2L(n)-L(L(n)\ln n)\ln n)] \\ &< \exp[\ln L(n) + t(2L(n)-L(L(n)\ln n)\ln n)] \\ &\to 0 \quad (n\to\infty), \end{aligned}$$

which proves (A1).

Next, we assume that $(m_1,m_2)\in\mathcal{M}(a)$ satisfies $a_p(n)-h_0 < m_1 \le a$, and prove (A2). We have

$$\begin{aligned} m_1+m_2 = t\left(m_1+\frac{1}{t}m_2\right) - (t-1)m_1 &< ta_p(n) - (t-1)(a_p(n)-h_0) \\ &= a_p(n) + (t-1)h_0. \end{aligned}$$

Note that $\alpha := (t-1)h_0$ is a constant depending solely on p and t. We further remark that $L(L(n)\ln n)\ln n > L(L(n)\ln n) \gg \max\{L(n),\ln n\}$. Hence it follows from Claim 3.2(i) that

$$\begin{aligned}E(X_{m_1,m_2}) &\le \exp\left[(m_1+m_2)(\ln n+2) - \frac{L(n)\ln n}{(1-p)^{a_p(n)-m_1}}\right]\\ &< \exp[(a_p(n)+\alpha)(\ln n+2) - L(n)\ln n]\\ &= \exp[-L(L(n)\ln n)\ln n + 2L(n) - 2L(L(n)\ln n) + \alpha\ln n + 2\alpha]\\ &\to 0 \quad (n\to\infty),\end{aligned}$$

which proves (A2). □

Let $A_n = \{a \in \mathbb{M} : a_p(n) - 1 < a \le a_p(n)\}$. Then $|A_n| \le t$. In particular, $\sum_{a\in A_n} E(X_a)$ is a sum with a constant number of terms. Consequently, it follows from Lemma 2.1 and Claim 3.3 that

$$\Pr(\gamma^{\frac{1}{t}}(G(n,p)) \le a_p(n)) \le \sum_{a\in A_n} \Pr(X_a \ge 1) \le \sum_{a\in A_n} E(X_a) \to 0 \quad (n\to\infty),$$

and so $\Pr(\gamma^{\frac{1}{t}}(G(n,p)) > a_p(n)) \to 1$ $(n\to\infty)$.

This completes the proof of Theorem 3.1. □

4. Graph-theoretical refinement of Theorem 3.1

In this section, we complete the proof of Theorem 1.3. Let s, t, and p be numbers as in Theorem 1.3. Let $\varepsilon \in \mathbb{R}^+$. Then by Theorem 3.1, there exists $N_0 \in \mathbb{Z}^+$ such that for every integer $n \ge N_0$,

$$\Pr\left(\gamma^{\frac{1}{t}}(G(n,p)) < \lfloor a_p(n)\rfloor_t + \frac{1}{t}\right) < \frac{\varepsilon}{2(s-1)}$$

and

$$\Pr\left(\gamma^{\frac{s}{t}}(G(n,p)) \notin \left[\lfloor a_p(n)\rfloor_t + \frac{1}{t},\ b_{p,t}(n)\right]\right) < \frac{\varepsilon}{2}.$$

Fix an integer $n \ge N_0$, and let i be an integer with $t-s+1 \le i \le t-1$. Since $b_{p,t}(n)$ is an integer, $b_{p,t}(n) - \frac{i}{t}$ is a non-integer. Furthermore, if a graph G satisfies $\gamma^{\frac{s}{t}}(G) = b_{p,t}(n) - \frac{i}{t}$, then

$$\lfloor \gamma^{\frac{s}{t}}(G)\rfloor = \left\lfloor b_{p,t}(n) - \frac{i}{t}\right\rfloor = b_{p,t}(n) - 1,$$

and hence

$$\gamma^{\frac{s}{t}}(G) = b_{p,t}(n) - \frac{i}{t} = \lfloor \gamma^{\frac{s}{t}}(G) \rfloor + 1 - \frac{i}{t} \le \lfloor \gamma^{\frac{s}{t}}(G) \rfloor + 1 - \frac{t-s+1}{t}$$
$$= \lfloor \gamma^{\frac{s}{t}}(G) \rfloor + \frac{s-1}{t}.$$

This together with Lemma 2.2 implies that if $\gamma^{\frac{s}{t}}(G) = b_{p,t}(n) - \frac{i}{t}$, then $\gamma^{\frac{1}{t}}(G) < \lfloor \gamma^{\frac{s}{t}}(G) \rfloor = b_{p,t}(n) - 1$. Hence we have $\Pr(\gamma^{\frac{1}{t}}(G(n,p)) < b_{p,t}(n) - 1) \ge \Pr(\gamma^{\frac{s}{t}}(G(n,p)) = b_{p,t}(n) - \frac{i}{t})$. On the other hand, since $b_{p,t}(n) - 1 = \lfloor \lfloor a_p(n) \rfloor_t + \frac{1}{t} \rfloor \le \lfloor a_p(n) \rfloor_t + \frac{1}{t}$,

$$\Pr\left(\gamma^{\frac{s}{t}}(G(n,p)) = b_{p,t}(n) - \frac{i}{t}\right) \le \Pr(\gamma^{\frac{1}{t}}(G(n,p)) < b_{p,t}(n) - 1)$$
$$\le \Pr\left(\gamma^{\frac{1}{t}}(G(n,p)) < \lfloor a_p(n) \rfloor_t + \frac{1}{t}\right)$$
$$< \frac{\varepsilon}{2(s-1)}.$$

Consequently,

$$\Pr\left(\gamma^{\frac{s}{t}}(G(n,p)) \in \left\{ b_{p,t}(n) - \frac{i}{t} : t-s+1 \le i \le t-1 \right\}\right) < \frac{\varepsilon}{2},$$

and hence

$$\Pr\left(\gamma^{\frac{s}{t}}(G(n,p)) \notin \left[\lfloor a_p(n) \rfloor_t + \frac{1}{t},\ b_{p,t}(n)\right]\right.$$
$$\left. \text{or } \gamma^{\frac{s}{t}}(G(n,p)) \in \left\{ b_{p,t}(n) - \frac{i}{t} : t-s+1 \le i \le t-1 \right\}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε is arbitrary, this completes the proof of Theorem 1.3.

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References

[1] S. Bermudo, H. Fernau and J.M. Sigarreta, The differential and the Roman domination number of a graph, *Appl. Anal. Discrete Math.* **8** (2014), 155–171. MR3289473

[2] A. Bonato and C. Wang, A note on domination parameters in random graphs, *Discuss. Math. Graph Theory* **28** (2008), 335–343. MR2477234

[3] E.J. Cockayne, P.A. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, Roman domination in graphs, *Discrete Math.* **278** (2004), 11–22. MR2035387

[4] M. Furuya, A continuous generalization of domination-like invariants, *J. Comb. Optim.* **41** (2021), 905–922. MR4264989

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs, *Marcel Dekker, Inc.*, New York (1998). MR1605684

[6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination inn Graphs: Advanced Topics, *Marcel Dekker, Inc.*, New York (1998). MR1605685

[7] M.A. Henning and A. Yeo, Total domination in graphs, *Springer Monographs in Mathematics. Springer*, New York (2013). MR3060714

[8] I. Stewart, Defend the Roman Empire! *Sci. Am.* **281** (1999), 136–139.

[9] B. Wieland and A.P. Godbole, On the domination number of a random graph, *Electron. J. Combin.* **8** (2001), #R37. MR1877656

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