Journal of Combinatorics
Volume 13, Number 1, 41–78, 2022

Supercards, sunshines and caterpillar graphs

Paul Brown and Trevor Fenner

The vertex-deleted subgraph $G - v$, obtained from the graph G by deleting the vertex v and all edges incident to v, is called a *card* of G. The *deck* of G is the multiset of its unlabelled cards. The *number of common cards* $b(G, H)$ of G and H is the cardinality of the multiset intersection of the decks of G and H. A *supercard* G^+ of G and H is a graph whose deck contains at least one card isomorphic to G and at least one card isomorphic to H. We show how maximum sets of common cards of G and H correspond to certain sets of permutations of the vertices of a supercard, which we call *maximum saturating sets*. We apply the theory of supercards and maximum saturating sets to the case when G is a sunshine graph and H is a caterpillar graph. We show that, for large enough n, there exists some maximum saturating set that contains at least $b(G, H) - 2$ automorphisms of G^+, and that this subset is always isomorphic to either a cyclic or dihedral group. We prove that $b(G, H) \leq \frac{2(n+1)}{5}$ for large enough n, and that there exists a unique family of pairs of graphs that attain this bound. We further show that, in this case, the corresponding maximum saturating set is isomorphic to the dihedral group.

Keywords and phrases: Graph reconstruction, reconstruction numbers, vertex-deleted subgraphs, supercards, graph automorphisms, sunshine graph, caterpillar graph.

1. Introduction

In this paper all graphs are finite, undirected and contain no loops or multiple edges. Any graph-theoretic terminology and notation not explicitly explained below can be found in Bondy and Murty's text [5]. For more information on the action of a permutation group on the vertices of a graph, we refer the reader to the book by Lauri and Scapellato [16].

Let G be a graph of order n and let $u, v \in V(G)$. We denote the group of all permutations of $V(G)$ by $S_{V(G)}$ and the identity permutation of $S_{V(G)}$

arXiv: 2010.08266v1

by $1_{V(G)}$. A *transposition* of $V(G)$ is a permutation that swaps two vertices in $V(G)$ and leaves the rest unchanged.

The *neighbourhood* of v in G is the set $N_G(v)$ consisting of all vertices of G adjacent to v. The cardinality of this set is the *degree* of v in G, i.e., $d_G(v) = |N_G(v)|$. A *leaf* of G is a vertex of degree 1, and an *isolated vertex* of G is a vertex of degree 0. We denote the number of vertices of degree k in G by $d_k(G)$, so $\sum_i d_i(G) = n$. A *non-trivial* component of a graph is one of order at least two.

Suppose that H is another graph and that γ is a bijection from $V(G)$ to $V(H)$. For any $Z \subseteq V(G)$, we write the image of Z under γ as $\gamma(Z)$. When γ is, moreover, an isomorphism from G to H, i.e., xy is an edge of G if and only if $\gamma(x)\gamma(y)$ is an edge of H, we write $\gamma(G) = H$. We write $G \cong H$ to indicate that G and H are isomorphic. The group of all automorphisms of G, i.e., isomorphisms from G to itself is denoted by $\mathrm{Aut}(G)$. We note that any transposition of $V(G)$ that swaps a pair of leaves adjacent to the same vertex is in $\mathrm{Aut}(G)$.

Now let $Z \subseteq V(G)$. The Z-deleted subgraph $G - Z$ is obtained from G by deleting all the vertices of Z together with all edges of G incident to a vertex in Z. So $d_{G-Z}(v) = d_G(v) - |N_G(v) \cap Z|$, for all $v \in V(G - Z)$. When $Z = \{v\}$ or $Z = \{u, v\}$, we write $G - Z$ as $G - v$ or $G - u - v$, respectively. The *vertex-deleted subgraph* $G - v$ is also known as a *card* of G, and the multi-set of all n unlabelled cards of G is called the *deck* of G, which we denote by $\mathcal{D}(G)$. If Z is the set of leaves and isolated vertices of G then $G - Z$ is called the *skeleton* of G, denoted by $\mathrm{skel}(G)$.

Clearly, if $G \cong H$ then $\mathcal{D}(G) = \mathcal{D}(H)$. The *Reconstruction Conjecture*, first proposed by Kelly and Ulam in 1941 [13, 14, 22], asserts that, when $n > 2$, the converse also holds, i.e., G is isomorphic to H if and only if G has the same collection of n unlabelled cards as H. However, despite the efforts of many graph theorists, the status of the sufficiency of the condition remains unresolved. Surveys on the reconstruction problem can be found in [3, 4, 16].

Since the conjecture remains unresolved, attention has focused on related reconstruction problems. One such area is proving that certain classes of graphs are *reconstructible* (i.e., that the conjecture is true when G and H belong to that class of graphs), or even *recognisable* (i.e., that membership of the class can be determined from the deck). Many classes of graphs have been shown to be reconstructible, including *trees* by Kelly [14] and also Bondy [2], *disconnected graphs* by Greenwell and Hemminger [11], and also Manvel [18], *unicyclic graphs* by Manvel [17], and *maximal planar graphs* by Lauri [15].

Another area of research has been to consider how many cards are required to reconstruct a graph – either its existential (ally) or universal (adversary) *reconstruction number* (see [6, 20]) – or even just to recognise that it is a member of a particular class. An equivalent approach to finding universal reconstruction numbers is to consider the *maximum number of common cards* of two graphs. A *common card* of G and H is any card in the multiset intersection $\mathcal{D}(G) \cap \mathcal{D}(H)$, and the number of common cards of G and H, denoted by $b(G, H)$, is the cardinality of this multiset intersection. The Reconstruction Conjecture can then be reformulated as follows: if G and H are not isomorphic then $b(G, H) < n$ when $n > 2$.

Until a few years ago, there were no known families of pairs of non-isomorphic graphs that had $b(G, H) > \frac{n}{2} + \frac{1}{8}(3 + \sqrt{8n+9})$. However, Bowler, Brown and Fenner [6] showed that there are, in fact, several infinite families of pairs of non-isomorphic graphs G and H with $b(G, H) = 2\left\lfloor \frac{n-1}{3} \right\rfloor$. Moreover, they conjectured that $b(G, H)$ is bounded above by $\frac{2(n-1)}{3}$ for large enough n. Results for small graphs, i.e., for $n \leq 11$, have been provided by Baldwin [1], McMullen [19] and Rivshin [21].

In a subsequent paper [7], Bowler, Brown, Fenner and Myrvold showed that if G is disconnected and H is connected then $b(G, H) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$, i.e., the connectedness of a graph can be recognised from any $\left\lfloor \frac{n}{2} \right\rfloor + 2$ of its cards. They also characterised all pairs of graphs that attain this bound (most of these infinite families can also be found in [6]).

A similar recognition question is to ask how many cards are required to recognise whether a graph is a tree. Since H is a tree in some of the families in [7], it follows that at least $\left\lfloor \frac{n}{2} \right\rfloor + 2$ cards may be required. To fully answer the question, it is necessary to determine how many cards are required to distinguish a non-tree G from a tree H when it is known that G is connected. It is easy to show that, in this case, $b(G, H) \leq 2$ when G contains more than one cycle. Furthermore, if $G - v \cong H - t$, for some $t \in V(H)$, then v must be lie on every cycle of G. We can therefore restrict our attention to graphs where G is unicyclic and the length of its cycle is reasonably large. For the maximum value of $b(G, H)$ to be attained, it has been conjectured that G must be a *sunshine* graph (a connected graph for which $\mathrm{skel}(G)$ is a cycle) and H must be a *caterpillar* (a connected graph for which $\mathrm{skel}(H)$ is a path); see [6, 10]. Support for restricting our attention to sunshine graphs and caterpillars is the observation that $b(G, H) \leq 6$ when either G is a sunshine graph and H is a non-caterpillar tree, or when G is a non-sunshine unicyclic graph and H is a caterpillar.

In [9], Brown gave an intricate proof that, for large n, the number of common cards between a sunshine graph and a caterpillar of order n is at most $2\left\lfloor \frac{n+1}{5} \right\rfloor$ and, moreover, that this bound is only attained by a unique family of pairs of graphs for which $n \equiv 4 \pmod 5$. In this paper, we prove this result using *supercards*, a new approach to the study of the maximum number of common cards that we introduced in [8].

A *supercard* of G and H is a graph G^+, the deck of which contains at least one card isomorphic to G and at least one card isomorphic to H. In [8], we showed the existence of certain subsets of $S_{V(G^+)}$ of cardinality $b(G,\, H)$, the elements of which are in one-to-one correspondence with the elements of $\mathcal{D}(G) \cap \mathcal{D}(H)$. We called these subsets *maximum saturating sets.* We further showed that, for many families of pairs of graphs with a large number of common cards, there exist maximum saturating sets containing a large number of automorphisms of G^+.

We use supercards to investigate the case when G is a sunshine graph and H is a caterpillar. We show that, when $b(G,\, H) \geq 2\left\lfloor \sqrt{2n+1} \right\rfloor + 4$, we may construct supercards G^+ of G and H, and suitable maximum saturating sets for which almost all of the elements are automorphisms of G^+. We further show that, in each case, the subset of elements that are automorphisms forms a group isomorphic to either a cyclic or dihedral group. We present several families of sunshine-caterpillar pairs that have such supercards and a large number of common cards. Finally, we show that $b(G,\, H) \leq \frac{2(n+1)}{5}$ when $n \geq 62$, and we exhibit the unique family of pairs of graphs that attain this bound. In this case, the maximum saturating set is a subgroup of $\mathrm{Aut}(G^+)$ isomorphic to a dihedral group.

2. Supercards and maximum saturating sets

We now recall the main definitions and results in the theory of supercards. Detailed explanations and proofs of the results can be found in [8].

Lemma 2.1. *Let G and H be graphs, and let γ be a bijection from $V(G)$ to $V(H)$. Suppose that there is some vertex v of G such that $\gamma(G-v) = H-\gamma(v)$ and $\gamma(N_G(v)) = N_H(\gamma(v))$. Then $\gamma(G) = H$.*

Corollary 2.2. *Let G be a graph and let $v \in V(G)$. Suppose that $\gamma \in S_{V(G)}$. Then $\gamma \in \mathrm{Aut}(G)$ if and only if*

$$\gamma(G - v) = G - \gamma(v) \qquad \textit{and} \qquad \gamma(N_G(v)) = N_G(\gamma(v)).$$

Definition 2.3. *A* supercard *of G is any graph of order $n + 1$ whose deck contains a card isomorphic to G.*

Definition 2.4. *A* common supercard *of G and H is any graph that is a supercard of both G and H, i.e., a graph whose deck contains some card $\widehat{G}$ isomorphic to G and another card $\widehat{H}$ isomorphic to H. For brevity, we refer to such graphs as* supercards of G and H.

Lemma 2.5. *There exists a graph G^+ that is a supercard of G and H if and only if $b(G, H) \geq 1$.*

It is easy to verify that, if $v \in V(G)$, $t \in V(H)$ and γ is an isomorphism such that $\gamma(G - v) = H - t$, then the graph G^* defined by

$$\begin{aligned} &V(G^*) = V(G) \cup \{w\}\,, \\ (1) \qquad &E(G^*) = E(G) \cup \{xw \mid x \in V(G-v) \text{ and } \gamma(x)t \in E(H)\}\,, \end{aligned}$$

for some $w \notin V(G) \cup V(H)$, is a supercard of G and H. In this case, $\widehat{G} = G^* - w = G$ and $\widehat{H} = G^* - v \cong H$ (see Lemma 3.3 in [8].)

For the rest of this section we assume that $b(G, H) \geq 1$. We then let G^+ be some supercard of G and H, and let v and w be vertices of G^+ such that $\widehat{G} = G^+ - w \cong G$ and $\widehat{H} = G^+ - v \cong H$.

Definition 2.6. *The set of* active permutations *of G^+ with respect to v and w, denoted by $B_{vw}(G^+)$, is the subset of $S_{V(G^+)}$ defined by*

$$\begin{aligned} (2) \quad B_{vw}(G^+) &= \{\lambda \in S_{V(G^+)} \mid \lambda((G^+ - w) - \lambda^{-1}(v)) = (G^+ - v) - \lambda(w)\} \\ &= \{\lambda \in S_{V(G^+)} \mid \lambda(\widehat{G} - \lambda^{-1}(v)) = \widehat{H} - \lambda(w)\}. \end{aligned}$$

We note that $1_{V(G^+)} \in B_{vw}(G^+)$, and that if $\lambda \in B_{vw}(G^+)$ then $\lambda(w) \neq v$, since G and H are not isomorphic.

Definition 2.7. *$B^G_{vw}(G^+)$ is the subset of $B_{vw}(G^+)$ defined by*

$$(3) \qquad B^G_{vw}(G^+) = \{\lambda \in B_{vw}(G^+) \mid \lambda(\widehat{G}) = G^+ - \lambda(w)\}.$$

Definition 2.8. *A* maximum saturating set *of $B_{vw}(G^+)$ is a subset $X \subseteq B_{vw}(G^+)$ that satisfies the following three properties:*

(a) $1_{V(G^+)} \in X$;
(b) if λ and π are distinct permutations in X then $\lambda^{-1}(v) \neq \pi^{-1}(v)$ and $\lambda(w) \neq \pi(w)$;
(c) there is no σ in $B_{vw}(G^+) \setminus X$ such that $X \cup \{\sigma\}$ satisfies (b).

We note that, for any pair of distinct permutations λ and π in X, (b) guarantees that $G^+ - \lambda^{-1}(v) \neq G^+ - \pi^{-1}(v)$ and $G^+ - \lambda(w) \neq G^+ - \pi(w)$, although either pair of graphs could be isomorphic.

Although condition (c) only ensures that X is *maximal* with respect to (a) and (b), it follows from Theorem 2.11 below that all maximum saturating sets have the same cardinality. This implies that such sets are in fact of *maximum* cardinality with respect to (a) and (b), and justifies our terminology in Definition 2.8.

Definition 2.9. *Let X be a maximum saturating set of $B_{vw}(G^+)$. Then $X_G = X \cap B^G_{vw}(G^+)$.*

Definition 2.10. *A G^+-*optimum saturating set *of $B_{vw}(G^+)$ is a maximum saturating set X of $B_{vw}(G^+)$ for which $|X_G|$ takes its maximum possible value. We define $\chi(G^+) = |X_G|$ for any G^+-optimum saturating set X.*

Theorem 2.11. *Let $Y \subseteq B_{vw}(G^+)$ satisfy properties (a) and (b) of Definition 2.8.*

- *(a) If Y is not a maximum saturating set of $B_{vw}(G^+)$ then $|Y| < b(G, H)$.*
- *(b) If $|Y| < b(G, H)$ then there is a maximum saturating set X such that $Y \subset X$ (so Y is not a maximum saturating set).*
- *(c) Y is a maximum saturating set of $B_{vw}(G^+)$ if and only if $|Y| = b(G, H)$.*

We make frequent use of the fact that every maximum saturating set of $B_{vw}(G^+)$ has cardinality $b(G, H)$ without explicitly quoting this theorem.

3. Sunshine graphs and caterpillars

We recall that skel(G), the skeleton of the graph G, is the graph $G - X$, where X is the set of leaves and isolated vertices of G. A *sunshine graph* is a connected graph whose skeleton is a cycle and a *caterpillar* is a connected graph whose skeleton is a path. Clearly, all sunshine graphs are unicyclic and all caterpillars are trees. We shall use supercards to investigate the number of common cards between pairs of such graphs.

We denote the *diameter*, i.e., the length of a longest path, of a connected graph G by diam(G). If G consists of a tree T plus a collection of isolated vertices, then we define diam(G) = diam(T). A leaf at the end of any longest path in a graph is called a *peripheral* leaf. For any vertex v of G, we denote the number of vertices in $N_G(v)$ of degree 2 by $\tau_G(v)$. A *d-leaf* of G is a leaf w (in a component of order at least three) for which $\tau_G(w) = 0$, i.e., the degree of its neighbour is at least 3.

We are interested in pairs of sunshine graphs U and caterpillars T with a large number of common cards relative to their order n. We therefore assume, *for the rest of this paper*, that all the pairs U and T that we consider have $b(U, T) \geq 5$. By inspection, it is easy to show that, for such pairs, the unique cycle of U is of length at least 6, and that U has at least one leaf.

We use the following conventions for any sunshine graph S with skeleton $x_0x_1 \ldots x_{c-1}x_0$: for any integer k, we interpret x_k to be the vertex x_i, where i is the unique integer such that $0 \leq i \leq c-1$ and $k \equiv i \pmod{c}$; in addition, for $0 \leq b < a \leq c-1$, we abbreviate the path $x_a x_{a+1} \ldots x_{c-1}x_0x_1 \ldots x_b$ on $\mathrm{skel}(S)$ to $x_a x_{a+1} \ldots x_b$.

We make frequent use of the following easy result concerning the cards of a sunshine graph.

Lemma 3.1. *Let S be a sunshine graph with skeleton $x_0x_1 \ldots x_{c-1}x_0$ and let x_i be in $V(\mathrm{skel}(S))$. Then $S - x_i$ consists of a caterpillar Q of diameter $c - \tau_S(x_i)$, together with $d_S(x_i) - 2$ isolated vertices. In addition:*

- *(a)* $\mathrm{skel}(Q)$ *is $x_{i+1}x_{i+2} \ldots x_{i-1}$ if and only if $d_S(x_{i+1}) \geq 3$ and $d_S(x_{i-1}) \geq 3$, i.e., $\tau_S(x_i) = 0$;*
- *(b)* $\mathrm{skel}(Q)$ *is $x_{i+1}x_{i+2} \ldots x_{i-2}$ if and only if $d_S(x_{i+1}) \geq 3$ and $d_S(x_{i-1}) = 2$, i.e., $\tau_S(x_i) = 1$;*
- *(c)* $\mathrm{skel}(Q)$ *is $x_{i+2}x_{i+3} \ldots x_{i-1}$ if and only if $d_S(x_{i+1}) = 2$ and $d_S(x_{i-1}) \geq 3$, i.e., $\tau_S(x_i) = 1$;*
- *(d)* $\mathrm{skel}(Q)$ *is $x_{i+2}x_{i+3} \ldots x_{i-2}$ if and only if $d_S(x_{i+1}) = d_S(x_{i-1}) = 2$, i.e., $\tau_S(x_i) = 2$.*

Moreover, x_{i+1} is a peripheral leaf of Q adjacent to x_{i+2} when $d_S(x_{i+1}) = 2$, i.e., in cases (c) and (d). Similarly, x_{i-1} is a peripheral leaf of Q adjacent to x_{i-2} when $d_S(x_{i-1}) = 2$, i.e., in cases (b) and (d).

Proof. Since x_{i+1} and x_{i-1} are the only non-leaves in $N_S(x_i)$ and x_i is on the unique cycle of S, clearly $S - x_i$ consists of a tree Q together with $d_S(x_i) - 2$ isolated vertices. Moreover, since $\mathrm{skel}(S)$ is a cycle, $\mathrm{skel}(Q)$ is a path, so Q is a caterpillar. It is easy to see that x_{i+1} is a peripheral leaf of Q when $d_S(x_{i+1}) = 2$; otherwise x_{i+1} is one of the two vertices of $\mathrm{skel}(Q)$ that is adjacent to a peripheral leaf. A similar observation holds for x_{i-1}. Cases (a) to (d) then follow immediately. Finally, it follows from (a) to (d) that $\mathrm{diam}(Q) = c - \tau_S(x_i)$ as $\mathrm{diam}(Q) = |V(\mathrm{skel}(Q))| + 1$. □

We also make the following easy observation about the possible cards of a caterpillar.

Lemma 3.2. *Let T be a caterpillar with skeleton $y_1y_2\dots y_p$.*

(a) $T - y_i$ contains precisely one non-trivial component if and only if $i \in \{1, p\}$.
(b) If t is a leaf of T that is not a d-leaf then t is a peripheral leaf, and is adjacent to either y_1 or y_p. Moreover, t is the only leaf that is adjacent to y_1 or y_p, respectively.
(c) There exist at most two leaves of T that are not d-leaves.

Proof. The results follow immediately by considering the structure of T. □

These two results yield the following important lemma.

Lemma 3.3. *Let U be a sunshine graph and T be a caterpillar. Then there exists a supercard of U and T that is a sunshine graph.*

Proof. Since U is a sunshine graph, clearly no card of U, and hence no common card of U and T, can contain more than one non-trivial component. So, since $b(U, T) \geq 5$, it follows from Lemma 3.2 that there exists a vertex v of U, a d-leaf t of T, and an isomorphism γ such that $\gamma(U - v) = T - t$.

Let s be the unique vertex of T adjacent to t, and let U^* be the supercard of U and T constructed as in equation (1). Now, since t is a leaf of T, clearly $\gamma^{-1}(s)$ is the only vertex of U^* adjacent to w. Moreover, since t is a d-leaf of T, it is easy to see that $d_U(\gamma^{-1}(s)) \geq 2$, so $\gamma^{-1}(s)$ is on the unique cycle of U. Thus U^* must be a sunshine graph. □

Let U be a sunshine graph and T a caterpillar. By Lemma 3.3, there exists some supercard U^+ of U and T that is a sunshine graph. So, for any such supercard U^+, let w and v be vertices of U^+ such that $\widehat{U} = U^+ - w \cong U$ and $\widehat{T} = U^+ - v \cong T$. Clearly, w is a leaf of U^+ as U is a sunshine graph. In addition, since T is a tree, v must be on the cycle of U^+ and $d_{U^+}(v) = 2$. We therefore label the skeleton of U^+ as $x_0x_1\dots x_{c-1}x_0$ (so the cycle is of length $c \geq 6$), where x_0 is adjacent to w, x_ν is v, for some ν, $1 \leq \nu \leq c-1$, and $d_{U^+}(x_{\nu-1}) \geq d_{U^+}(x_{\nu+1})$. We further arbitrarily label all the leaves of U^+, so that each distinct leaf adjacent to x_i is labelled x_i^j for some unique j, $1 \leq j \leq d_{U^+}(x_i) - 2$, where w is labelled x_0^1. Our supercard U^+ thus satisfies

$$V(U^+) = V(U) \cup \{w\} \quad \text{and} \quad E(U^+) = E(U) \cup \{x_0w\}, \tag{4}$$

assuming the above labelling of U^+.

For the rest of this section, we assume that U^+ is the supercard of U and T specified in (4). Clearly, any supercard of U and T is also a supercard of any pair of graphs isomorphic to U and T, respectively. So, for *ease* *of notation*, we shall write $U = U^+ - w$ instead of $\widehat{U} = U^+ - w$, and $T = U^+ - x_\nu$ instead of $\widehat{T} = U^+ - x_\nu$. We note that $\mathrm{skel}(U) = \mathrm{skel}(U^+)$. Applying Lemma 3.1 to U^+ and x_ν yields the following result.

Lemma 3.4. *We have the following possibilities for the skeleton of T.*

(a) If $\tau_{U^+}(x_\nu) = 0$ then $\mathrm{skel}(T)$ is $x_{\nu+1}x_{\nu+2}\dots x_{\nu-1}$.
(b) If $\tau_{U^+}(x_\nu) = 1$ then $\mathrm{skel}(T)$ is $x_{\nu+2}x_{\nu+3}\dots x_{\nu-1}$.
(c) If $\tau_{U^+}(x_\nu) = 2$ then $\mathrm{skel}(T)$ is $x_{\nu+2}x_{\nu+3}\dots x_{\nu-2}$.

It follows that $x_{\nu+1}$ is a peripheral leaf of T adjacent to $x_{\nu+2}$ when $\tau_{U^+}(x_\nu) \geq 1$, and $x_{\nu-1}$ is a peripheral leaf of T adjacent to $x_{\nu-2}$ when $\tau_{U^+}(x_\nu) = 2$.

Proof. This follows immediately by Lemma 3.1 with $S = U^+$ and $i = \nu$, noting that case (b) of that lemma cannot occur as $d_{U^+}(x_{\nu-1}) \geq d_{U^+}(x_{\nu+1})$. □

We recall that $B_{vw}(U^+)$ is the set of active permutations of U^+ with respect to v and w, i.e.,

(5) $$B_{vw}(U^+) = B_{x_\nu x_0^1}(U^+) = \{\lambda \in S_{V(U^+)} \mid \lambda(U - \lambda^{-1}(x_\nu)) = T - \lambda(w)\}.$$

Lemma 3.5. *Let $\lambda \in B_{vw}(U^+)$. Then exactly one of the following holds:*

(a) $\lambda(w)$ is a d-leaf of T, in which case $\mathrm{skel}(T - \lambda(w)) = \mathrm{skel}(T)$ and $\mathrm{diam}(T - \lambda(w)) = \mathrm{diam}(T)$;
(b) $\lambda(w)$ is a peripheral leaf of T that is not a d-leaf, in which case $\mathrm{diam}(T - \lambda(w))) = \mathrm{diam}(T) - 1$;
(c) $\lambda(w)$ is adjacent to a peripheral leaf of T, in which case $\mathrm{diam}(T - \lambda(w)) \leq \mathrm{diam}(T) - 1$.

Proof. By Lemmas 3.1 and 3.2(a), $\lambda(w)$ is either a leaf or adjacent to a peripheral leaf. Cases (a), (b) and (c) then follow easily by considering the structure of T. □

Lemma 3.6. *Let $\lambda \in B_{vw}(U^+)$. Then $\lambda^{-1}(x_\nu) \in V(\mathrm{skel}(U))$, i.e., $\lambda^{-1}(x_\nu)$ is x_μ for some μ, $0 \leq \mu \leq c - 1$, and $d_U(x_\mu) = d_T(\lambda(w)) + 1$. In addition, $U - x_\mu$, and therefore also $T - \lambda(w)$, consists of a caterpillar of diameter $c - \tau_U(x_\mu)$, together with $d_U(x_\mu) - 2$, equivalently $d_T(\lambda(w)) - 1$, isolated vertices.*

Proof. Since T is a tree, clearly $\lambda^{-1}(x_\nu) \in V(\mathrm{skel}(U))$, so $\lambda^{-1}(x_\nu)$ is x_μ, for some μ. In addition, $d_U(x_\mu) = d_T(\lambda(w)) + 1$ as $|E(U)| = |E(T)| + 1$. Finally, by Lemma 3.1, $U - x_\mu$ consists of a caterpillar of diameter $c - \tau_U(x_\mu)$, together with $d_U(x_\mu) - 2$ isolated vertices. □

From Lemma 3.6 and Theorem 2.11, it immediately follows that $b(U, T) \le c$.

For brevity, we frequently use Lemma 3.6 in the rest of this paper without explicit reference. Moreover, given any $\lambda \in B_{uv}(U^+)$, unless otherwise stated, we let $x_\mu = \lambda^{-1}(x_\nu)$. So, by (5),

$$\lambda(U - x_\mu) = \lambda((U^+ - w) - x_\mu) = (U^+ - x_\nu) - \lambda(w) = T - \lambda(w). \tag{6}$$

We write the skeletons of the caterpillars in $U - x_\mu$ and $T - \lambda(w)$ as

$$\mathrm{skel}(U - x_\mu) : x_a x_{a+1} \ldots x_b \qquad \text{and} \qquad \mathrm{skel}(T - \lambda(w)) : x_r x_{r+1} \ldots x_s, \tag{7}$$

respectively. On applying Lemma 3.1 with $S = U$ and $i = \mu$, we have the following result.

Corollary 3.7. *Let* $\mathrm{skel}(U - x_\mu)$ *be as in (7). Then* $x_a \in \{x_{\mu+1}, x_{\mu+2}\}$ *and* $x_b \in \{x_{\mu-1}, x_{\mu-2}\}$.

Since $c \ge 6$, it immediately follows that $a \ne b$.

Lemma 3.8. *Let* $\lambda \in B_{vw}(U^+)$. *Then* $b - a \equiv s - r \pmod{c}$. *Moreover, either*

(a) $\lambda(x_i) = x_{(r-a)+i}$ *for all* $x_i \in V(\mathrm{skel}(U - x_\mu))$, *or*
(b) $\lambda(x_i) = x_{(s+a)-i}$ *for all* $x_i \in V(\mathrm{skel}(U - x_\mu))$.

We note that $\lambda(x_b) = x_s$ *in (a) and* $\lambda(x_b) = x_r$ *in (b).*

Proof. λ maps the skeleton of $U - x_\mu$ onto the skeleton of $T - \lambda(w)$. So $b - a \equiv s - r \pmod{c}$, and either $\lambda(x_a) = x_r$ and $\lambda(x_b) = x_s$, or $\lambda(x_a) = x_s$ and $\lambda(x_b) = x_r$. It is then easy to see that either (a) or (b) must hold. □

Lemma 3.9. *Let* $\lambda \in B_{vw}(U^+)$. *Then*

$$\begin{aligned} c - \tau_{U^+}(x_\nu) &= \mathrm{diam}(T) \ge \mathrm{diam}(T - \lambda(w)) = \mathrm{diam}(U - x_\mu) \\ &= c - \tau_U(x_\mu) \ge c - 2. \end{aligned} \tag{8}$$

Proof. This follows easily by applying Lemma 3.1 to U^+ and x_ν, and then U and x_μ. □

Lemma 3.10. *Let* $\lambda \in B_{vw}(U^+)$ *be such that* $\lambda(w)$ *is a d-leaf of* T. *Suppose that* $\lambda(x_{\mu+2}) \notin \{x_{\nu-2}, x_{\nu+2}\}$. *Then* $\tau_U(x_\mu) = \tau_{U^+}(x_\nu) = 1$. *Moreover,* $\mathrm{skel}(U - x_\mu)$ *is either*

(a) $x_{\mu+1}x_{\mu+2}\ldots x_{\mu-2}$, *in which case* $\lambda(x_i) = x_{(\nu-\mu+1)+i}$ *for all* x_i *in* $V(\mathrm{skel}(U - x_\mu))$, *or*

(b) $x_{\mu+2}x_{\mu+3}\ldots x_{\mu-1}$, *in which case* $\lambda(x_i) = x_{(\nu+\mu+1)-i}$ *for all* x_i *in* $V(\mathrm{skel}(U - x_\mu))$.

Proof. $\mathrm{skel}(T - \lambda(w)) = \mathrm{skel}(T)$ by Lemma 3.5(a). So it follows from Corollary 3.4 that the possible skeletons of $T - \lambda(w)$ are determined by $\tau_{U^+}(x_\nu)$. In addition, $\tau_U(x_\mu) = \tau_{U^+}(x_\nu)$ by (8). It then follows from Lemma 3.1 that the possible skeletons of $U - x_\mu$ are also determined by $\tau_{U^+}(x_\nu)$. Using Lemma 3.8, it is now straightforward to determine all the possibilities for $\lambda(x_{\mu+2})$, for each of the three values of $\tau_{U^+}(x_\nu)$.

It is easy to see that $\lambda(x_{\mu+2}) \in \{x_{\nu-2}, x_{\nu+2}\}$ when $\tau_{U^+}(x_\nu) \neq 1$. Since this excluded by assumption, it follows that $\tau_{U^+}(x_\nu) = \tau_U(x_\mu) = 1$. In this case, x_r is $x_{\nu+2}$ and x_s is $x_{\nu-1}$ by Corollary 3.4(b). So, since $\lambda(x_{\mu+2}) \notin \{x_{\nu-2}, x_{\nu+2}\}$, it is straightforward to show that either (i) x_a is $x_{\mu+1}$, x_b is $x_{\mu-2}$ and Lemma 3.8(a) holds, or (ii) x_a is $x_{\mu+2}$, x_b is $x_{\mu-1}$ and Lemma 3.8(b) holds. In case (i), we have $r - a \equiv \nu - \mu + 1 \pmod{c}$, and in case (ii), we have $s + a \equiv \nu + \mu + 1 \pmod{c}$. Cases (a) and (b) of the lemma then immediately follow from Lemma 3.8. □

Definition 3.11. *We define* $\widetilde{B}_{vw}(U^+)$ *to be the subset of* $B_{vw}(U^+)$ *containing those permuations* λ *such that* $\lambda(w)$ *is a leaf of* U^+ *and a d-leaf of* T*. We further define* $\widetilde{X} = X \cap \widetilde{B}_{vw}(U^+)$ *for any maximum saturating set* X *of* $B_{vw}(U^+)$.

Lemma 3.12. *Let* $\lambda \in B_{vw}(U^+) \setminus \widetilde{B}_{vw}(U^+)$.

(a) *If* $\tau_{U^+}(x_\nu) = 0$ *then* $\lambda(w)$ *is not a d-leaf of* T*. Moreover,*
  (i) *if* $\lambda(w)$ *is a leaf of* T *then either* $\lambda(w) = x^1_{\nu+1}$ *and* $d_{U^+}(x_{\nu+1}) = 3$*, or* $\lambda(w) = x^1_{\nu-1}$ *and* $d_{U^+}(x_{\nu-1}) = 3$*;*
  (ii) *if* $\lambda(w)$ *is a cut-vertex of* T *then* $\lambda(w) \in \{x_{\nu+1}, x_{\nu-1}\}$.

(b) *If* $\tau_{U^+}(x_\nu) = 1$ *then*
  (i) *if* $\lambda(w)$ *is a d-leaf of* T *then* $\lambda(w) = x_{\nu+1}$ *and* $d_{U^+}(x_{\nu+2}) \geq 3$*;*
  (ii) *if* $\lambda(w)$ *is a leaf but not a d-leaf of* T *then either* $\lambda(w) = x_{\nu+1}$ *and* $d_{U^+}(x_{\nu+2}) = 2$*, or* $\lambda(w) = x^1_{\nu-1}$ *and* $d_{U^+}(x_{\nu-1}) = 3$*;*
  (iii) *if* $\lambda(w)$ *is a cut-vertex of* T *then* $\lambda(w) \in \{x_{\nu+2}, x_{\nu-1}\}$

(c) *If* $\tau_{U^+}(x_\nu) = 2$ *then* $\lambda(w)$ *is a d-leaf of* T *and* $\lambda(w) \in \{x_{\nu+1}, x_{\nu-1}\}$.

Proof. Since $\lambda \notin \widetilde{B}_{vw}(U^+)$, either $\lambda(w)$ is a d-leaf of T that is not a leaf of U^+, or it is not a d-leaf of T, in which case (b) or (c) of Lemma 3.5 must hold.

(a) Suppose that $\tau_{U^+}(x_\nu) = 0$. Then skel(T) is given by Corollary 3.4(a), so every leaf of T is a leaf of U^+. It immediately follows that $\lambda(w)$ cannot be a d-leaf of T. Now, if case (b) of Lemma 3.5 holds then it is easy to see that either $\lambda(w) = x^1_{\nu+1}$ and $d_{U^+}(x_{\nu+1}) = 3$, or $\lambda(w) = x^1_{\nu-1}$ and $d_{U^+}(x_{\nu-1}) = 3$. On the other hand, if case (c) holds then $\lambda(w)$ is either $x_{\nu+1}$ or $x_{\nu-1}$.

(b) Suppose that $\tau_{U^+}(x_\nu) = 1$. Then skel(T) is given by Corollary 3.4(b), so the only possible d-leaf of T that is not a leaf of U^+ is $x_{\nu+1}$; in this case, clearly $d_{U^+}(x_{\nu+2}) \geq 3$. Now, if case (b) of Lemma 3.5 holds then it is easy to see that either $\lambda(w) = x_{\nu+1}$ and $d_{U^+}(x_{\nu+2}) = 2$, or $\lambda(w) = x^1_{\nu-1}$ and $d_{U^+}(x_{\nu-1}) = 3$. On the other hand, if case (c) holds then $\lambda(w)$ is either $x_{\nu+2}$ or $x_{\nu-1}$.

(c) Suppose that $\tau_{U^+}(x_\nu) = 2$. Then skel(T) is given by Corollary 3.4(c), so the only possible d-leaves of T that are not leaves of U^+ are $x_{\nu+1}$ and $x_{\nu-1}$. Since equality holds throughout (8), clearly diam$(T - \lambda(w)) =$ diam(T) and, therefore, neither case (b) nor case (c) of Lemma 3.5 holds. □

Corollary 3.13. *Let X be a maximum saturating set of $B_{vw}(U^+)$.*

(a) If $\tau_{U^+}(x_\nu) = 0$ then $|X \setminus \widetilde{X}| \leq 4$.
(b) If $\tau_{U^+}(x_\nu) = 1$ then $|X \setminus \widetilde{X}| \leq 4$.
(c) If $\tau_{U^+}(x_\nu) = 2$ then $|X \setminus \widetilde{X}| \leq 2$.

Proof. This follows immediately from Lemma 3.12 and part (b) of Definition 2.8. □

We recall from Definition 2.10 that a U^+-optimum saturating set X of $B_{vw}(U^+)$ is a maximum saturating set of $B_{vw}(U^+)$ such that $|X_U| = \chi(U^+)$.

Corollary 3.14. *If there exists a U^+-optimum saturating set X of $B_{vw}(U^+)$ such that $\widetilde{X} \subseteq X_U$ then $b(U, T) \leq \chi(U^+) + 4$.*

We now consider the permutations in $B^U_{vw}(U^+)$, i.e., those permutations λ in $B_{vw}(U^+)$ such that $\lambda(U) = U^+ - \lambda(w)$.

Lemma 3.15. *Let $\lambda \in B^U_{vw}(U^+)$. Then $\lambda(w)$ is a leaf of U^+ adjacent to a vertex of degree $d_{U^+}(x_0)$, $d_U(x_\mu) = 2$, and either*

(a) $\lambda(x_i) = x_{(\nu-\mu)+i}$ for all x_i (a rotation), or
(b) $\lambda(x_i) = x_{(\nu+\mu)-i}$ for all x_i (a reflection).

Proof. Since $\lambda(U) = U^+ - \lambda(w)$, clearly $\lambda(w)$ must be a leaf of U^+ adjacent to a vertex of degree $d_{U^+}(x_0)$. Thus $d_U(x_\mu) = 2$ by Lemma 3.6. Now λ must preserve the cycle structure of U. So, since $\lambda(x_\mu) = x_\nu$, it follows that either $\lambda(x_{\mu+i}) = x_{\nu+i}$ and $\lambda(x_{\mu-i}) = x_{\nu-i}$ for all i, or $\lambda(x_{\mu+i}) = x_{\nu-i}$ and $\lambda(x_{\mu-i}) = x_{\nu+i}$ for all i. Cases (a) and (b) then follow immediately. □

The following lemma gives a methodology for replacing permutations in $\widetilde{B}_{vw}(U^+)$ by "equivalent" permutations in $B^U_{vw}(U^+)$.

Lemma 3.16. *Let $\lambda \in \widetilde{B}_{vw}(U^+)$. Suppose that $\lambda(x_{\mu+2}) \in \{x_{\nu-2}, x_{\nu+2}\}$. Then there exists $\widehat{\lambda} \in B^U_{vw}(U^+)$ such that $\widehat{\lambda}^{-1}(x_\nu) = \lambda^{-1}(x_\nu) = x_\mu$ and $\widehat{\lambda}(w) = \lambda(w)$.*

Proof. By Corollary 3.7, $x_a \in \{x_{\mu+1}, x_{\mu+2}\}$ and $x_b \in \{x_{\mu-1}, x_{\mu-2}\}$; so $x_{\mu+2} \in \{x_a, x_{a+1}\}$. Since $\lambda(w)$ is a d-leaf of T, it follows from Lemma 3.5(a) that $\operatorname{skel}(T-\lambda(w)) = \operatorname{skel}(T)$. So $x_r \in \{x_{\nu+1}, x_{\nu+2}\}$ and $x_s \in \{x_{\nu-1}, x_{\nu-2}\}$ by Lemma 3.4. We now assume that $\lambda(x_{\mu+2}) = x_{\nu+2}$ and prove the result in this case. The case when $\lambda(x_{\mu+2}) = x_{\nu-2}$ can be proved in a similar manner.

Suppose that Lemma 3.8(b) holds. Then $\{\lambda(x_a), \lambda(x_{a+1})\} \subseteq \{x_s, x_{s-1}\} \subseteq \{x_{\nu-1}, x_{\nu-2}, x_{\nu-3}\}$. Since $\lambda(x_{\mu+2}) = x_{\nu+2}$ and $x_{\mu+2} \in \{x_a, x_{a+1}\}$, this implies that $x_{\nu+2} \in \{x_{\nu-1}, x_{\nu-2}, x_{\nu-3}\}$. This is impossible as $c \geq 6$. Therefore Lemma 3.8(a) must hold. Hence $r - a \equiv s - b \equiv \nu - \mu \pmod{c}$, and $\lambda(x_i) = x_{(\nu-\mu)+i}$ for all x_i in $V(\operatorname{skel})(U - x_\mu))$.

Let θ be the transposition of $V(T-\lambda(w))$ that swaps $\lambda(x_{\mu+1})$ and $x_{\nu+1}$. We show that $\theta \in \operatorname{Aut}(T-\lambda(w))$. If $\lambda(x_{\mu+1}) = x_{\nu+1}$ then θ is $1_{V(T-\lambda(w))}$, so there is nothing to prove. Suppose therefore that $\lambda(x_{\mu+1}) \neq x_{\nu+1}$. It is then easy to show that $x_{\mu+1} \notin V(\operatorname{skel}(U - x_\mu))$, so x_a is $x_{\mu+2}$ and $x_{\mu+1}$ is a leaf of $U - x_\mu$ adjacent to x_a. Hence, correspondingly, $\lambda(x_{\mu+1})$ is a leaf of $T - \lambda(w)$ adjacent to x_r. Furthermore, x_r is $x_{\nu+2}$, so $x_{\nu+1}$ is also a leaf of $T - \lambda(w)$ adjacent to x_r. Thus θ swaps a pair of leaves adjacent to $x_{\nu+2}$, so $\theta \in \operatorname{Aut}(T - \lambda(w))$ in this case also.

By considering the vertices x_b and x_s, it is easy to show that the transposition θ' of $V(T-\lambda(w))$ that swaps $\lambda(x_{\mu-1})$ and $x_{\nu-1}$ is also in $\operatorname{Aut}(T-\lambda(w))$. Let us define $\widehat{\lambda} \in S_{V(U^+)}$ by $\widehat{\lambda}(x_\mu) = x_\nu$, $\widehat{\lambda}(w) = \lambda(w)$ and $\widehat{\lambda}(u) = \theta'\theta\lambda(u)$ for all other vertices u of U^+. Then

$$\widehat{\lambda}(U - x_\mu) = \theta'\theta\lambda(U - x_\mu) = \theta'\theta(T - \lambda(w)) = T - \lambda(w),$$

so $\widehat{\lambda} \in B_{vw}(U^+)$. Moreover, since $\widehat{\lambda}(x_{\mu+1}) = x_{\nu+1}$ and $\widehat{\lambda}(x_{\mu-1}) = x_{\nu-1}$, it follows that $\widehat{\lambda}(N_U(x_\mu)) = \{x_{\nu-1}, x_{\nu+1}\}$ as $d_U(x_\mu) = 2$. Hence $\widehat{\lambda}(U) = U^+ - \lambda(w)$ by Lemma 2.1, i.e., $\widehat{\lambda} \in B^U_{vw}(U^+)$. □

Lemma 3.17. *Let X be a maximum saturating set of $B_{vw}(U^+)$ and let L be the subset of X defined by $L = \{\lambda \in \widetilde{X} \mid \lambda(x_{\mu+2}) \in \{x_{\nu-2}, x_{\nu+2}\}\}$. Then*

(a) $|L| \leq \chi(U^+)$;
(b) if X is a U^+-optimum saturating set of $B_{vw}(U^+)$ then $X_U \cap \widetilde{X} = L$.

Proof. For each $\lambda \in L$, we let $\widehat{\lambda}$ be the permutation in $B^U_{vw}(U^+)$ as defined in Lemma 3.16. We then define $\widehat{L}$ to be the set of all such $\widehat{\lambda}$. Since X is a maximum saturating set of $B_{vw}(U^+)$, clearly $X \setminus L$ and $\widehat{L}$ are disjoint, and the set $\widehat{X}$ defined by $\widehat{X} = (X \setminus L) \cup \widehat{L}$ is also a maximum saturating set. Moreover $\widehat{X}_U = (X_U \setminus L) \cup \widehat{L}$. Therefore $|L| = |\widehat{L}| \leq |\widehat{X}_U| \leq \chi(U^+)$.

Now suppose that X is U^+-optimum. If $\lambda \in X_U \cap \widetilde{X}$ then $\lambda \in L$ by Lemma 3.15. Thus $X_U \cap \widetilde{X} \subseteq L$. So suppose that $L \not\subseteq X_U$. Then $|\widehat{X}_U| = |X_U \setminus L| + |\widehat{L}| > |X_U|$. This is impossible since X is U^+-optimum. So $L \subseteq X_U$, and therefore $X_U \cap \widetilde{X} = L$. □

Corollary 3.18. *Let X be a U^+-optimum saturating set of $B_{vw}(U^+)$. If there exists $\lambda \in \widetilde{X} \setminus X_U$ then $\tau_{U^+}(x_\nu) = 1$ and $\{\lambda(x_{\mu+2}), \lambda(x_{\mu-2})\} = \{x_{\nu+3}, x_{\nu-1}\}$.*

Proof. This follows easily from Lemma 3.17(b) and Lemma 3.10. □

It follows from this result that $\widetilde{X} \subseteq X_U$ when $\tau_{U^+}(x_\nu) \neq 1$.

If there exists a U^+-optimum saturating set X of $B_{vw}(U^+)$ such that $\widetilde{X} \subseteq X_U$ then $b(U, T) \leq \chi(U^+) + 4$ by Corollary 3.14. When there is no such set, we must construct another supercard of U and T.

Let $\sigma \in \widetilde{B}_{vw}(U^+)$, and let $x_\xi = \sigma^{-1}(x_\nu)$, so $\sigma(U - x_\xi) = T - \sigma(w)$. Let u be the unique vertex of T adjacent to $\sigma(w)$. Since $\sigma(w)$ is a d-leaf of T, clearly $d_{T-\sigma(w)}(u) \geq 2$, and thus $d_{U-x_\xi}(\sigma^{-1}(u)) \geq 2$. Hence $\sigma^{-1}(u)$ is x_η for some η, $0 \leq \eta \leq c-1$. We note that x_η cannot be x_ξ.

We now define a new sunshine graph U^+_σ constructed from U^+. First we delete from U^+ the edge x_0w and add an additional edge $x_\eta w$. We then relabel the skeleton of U^+ as $z_0z_1 \ldots z_{c-1}z_0$, where x_η is relabelled as z_0, x_ξ as z_ζ, and the other vertices on the cycle in the natural way, reversing the labelling around the cycle if necessary in order to ensure that $d_{U^+_\sigma}(z_{\zeta-1}) \geq d_{U^+_\sigma}(z_{\zeta+1})$. Finally, as before, we relabel all the leaves of U^+ so that each distinct leaf adjacent to z_i is labelled z_i^j for some unique j, $1 \leq j \leq d_{U^+_\sigma}(z_i) - 2$, where w is labelled z_0^1. We note this labelling is analogous to the original labelling of U^+.

Since $|V(U^+)| = |V(U^+_\sigma)|$, we may define a bijection θ from $V(U^+)$ to $V(U^+_\sigma)$ that encapsulates the relabelling described above, i.e., $\theta(x_i) = z_{i-\eta}$ if the order of the labels around the cycle did not need reversing, and $\theta(x_i) = z_{\eta-i}$ if it did. We also specify that $\theta(w) = w$, and that θ maps the remaining leaves of U^+ so that those adjacent to x_i map to leaves adjacent to $\theta(x_i)$ for each x_i. We note that $\theta(x_\eta) = z_0$ and $\theta(x_\xi) = z_\zeta$.

Clearly, $d_{U_\sigma^+}(z_\zeta) = 2$, as $d_U(x_\xi) = 2$ and x_η is not x_ξ. Let $U_\sigma = U_\sigma^+ - w$ and $T_\sigma = U_\sigma^+ - z_\zeta$. Then U_σ^+ is a supercard of U_σ and T_σ that satisfies

$$(9) \qquad V(U_\sigma^+) = V(U_\sigma) \cup \{w\} \quad \text{and} \quad E(U_\sigma^+) = E(U_\sigma) \cup \{z_0 w\}.$$

For ease of notation, we write $B_{vw}(U_\sigma^+)$ for the set of active permutations of U_σ^+ with respect to z_ζ and w, i.e.,

$$B_{vw}(U_\sigma^+) = \{\pi \in S_{V(U_\sigma^+)} \mid \pi(U_\sigma - \lambda^{-1}(z_\zeta)) = T_\sigma - \pi(w)\}.$$

Clearly, U_σ^+ is a sunshine graph with skeleton $z_0 z_1 \dots z_{c-1} z_0$, labelled analogously to the labelling of U^+, and U_σ is a sunshine graph. Furthermore, since $d_{U_\sigma^+}(z_\zeta) = 2$, it follows from Lemma 3.1 that T_σ is a caterpillar. Hence U_σ^+ is a supercard of the sunshine graph U_σ and the caterpillar T_σ. We may therefore use results corresponding to Lemma 3.4 to Corollary 3.18 by substituting U_σ^+, U_σ, T_σ and $B_{vw}(U_\sigma^+)$ for U^+, U, T and $B_{vw}(U^+)$, respectively.

We now show that U_σ^+ is also a supercard of U and T, and thence relate the maximum saturating sets of $B_{vw}(U^+)$ and $B_{vw}(U_\sigma^+)$. In each of the following four lemmas, we define U_σ^+ to be the supercard in equation (9) for the given permutation σ, satisfying $\sigma(U - x_\xi) = T - \sigma(w)$, where $x_\xi = \sigma^{-1}(x_\nu)$. As above, θ denotes the corresponding map from $V(U^+)$ to $V(U_\sigma^+)$.

Lemma 3.19. *Let $\sigma \in \widetilde{B}_{vw}(U_\sigma^+)$. Then $\theta^{-1}(U_\sigma) = U$, $\sigma\theta^{-1}(T_\sigma) = T$, so U_σ^+ is a supercard of U and T.*

Proof. Since $\theta(w) = w$, the restriction of θ to U is clearly a relabelling of the vertices of U that preserves neighbourhoods. So $\theta(U) = U_\sigma$. It now follows that

$$\sigma\theta^{-1}(U_\sigma^+ - w - z_\zeta) = \sigma(U - \theta^{-1}(z_\zeta)) = \sigma(U - x_\xi) = T - \sigma(w).$$

So, since $\sigma\theta^{-1}(N_{U_\sigma^+ - z_\zeta}(w)) = \{\sigma\theta^{-1}(z_0)\} = \{\sigma(x_\eta)\} = N_T(\sigma(w))$, it follows from Lemma 2.1 that $\sigma\theta^{-1}(U_\sigma^+ - z_\zeta) = T$. □

For any $\lambda \in B_{vw}(U^+)$, we define $\lambda_\sigma \in S_{V(U_\sigma^+)}$ by $\lambda_\sigma = \theta\sigma^{-1}\lambda\theta^{-1}$.

Lemma 3.20. *Let $\sigma \in \widetilde{B}_{vw}(U_\sigma^+)$ and let $\lambda \in B_{vw}(U^+)$. Then $\lambda_\sigma^{-1}(z_\zeta) = \theta(x_\mu)$ and $\lambda_\sigma \in B_{vw}(U_\sigma^+)$.*

Proof. $\lambda_\sigma^{-1}(z_\zeta) = \theta\lambda^{-1}\sigma\theta^{-1}(z_\zeta) = \theta(x_\mu)$. In addition, since $\theta^{-1}(U_\sigma) = U$, $\sigma\theta^{-1}(T_\sigma) = T$ and $\lambda(U - x_\mu) = T - \lambda(w)$, we have

$$\begin{aligned}\lambda_\sigma(U_\sigma - \lambda_\sigma^{-1}(z_\zeta)) &= \lambda_\sigma(U_\sigma - \theta(x_\mu)) = \theta\sigma^{-1}\lambda(U - x_\mu)\\ &= \theta\sigma^{-1}(T - \lambda(w)) = T_\sigma - \theta\sigma^{-1}\lambda\theta^{-1}(w),\end{aligned}$$

as $\theta(w) = w$. Hence $\lambda_\sigma(U_\sigma - \lambda_\sigma^{-1}(z_\zeta)) = T_\sigma - \lambda_\sigma(w)$, so $\lambda_\sigma \in B_{vw}(U_\sigma^+)$. □

Lemma 3.21. *Let X be a maximum saturating set of $B_{vw}(U^+)$ and suppose that $\sigma \in \widetilde{X}$. Then the set X_σ defined by $X_\sigma = \{\lambda_\sigma \mid \lambda \in X\}$ is a maximum saturating set of $B_{vw}(U_\sigma^+)$.*

Proof. By Lemma 3.20, $X_\sigma \subseteq B_{vw}(U_\sigma^+)$. Moreover, since X is a maximum saturating set of $B_{vw}(U^+)$ that contains σ, it is straightforward to show that X_σ satisfies conditions (a) and (b) of Definition 2.8. So, since $|X_\sigma| = |X|$, it follows from Theorem 2.11(c) that X_σ is a maximum saturating set of $B_{vw}(U_\sigma^+)$. □

For the final lemma in this section, we make use of the fact that if $\theta(x_i) = z_j$ then $\theta(x_{i+k}) \in \{z_{j-k}, z_{j+k}\}$, and $\theta^{-1}(z_{j+k}) \in \{x_{i-k}, x_{i+k}\}$ for all k.

Lemma 3.22. *Let X be a U^+-optimum saturating set of $B_{vw}(U^+)$. Suppose there exists some $\sigma \in \widetilde{X} \setminus X_U$. Then $|\widetilde{X}| \leq 2\max(\chi(U^+), \chi(U_\sigma^+)) + 1$.*

Proof. Let $X_\sigma = \{\pi_\sigma \mid \pi \in X\}$, and let P be the subset of X_σ defined by $P = \{\pi_\sigma \mid \pi \in \widetilde{X} \setminus X_U\}$. We show that $|P| \leq \chi(U_\sigma^+) + 1$. As $|P| = |\widetilde{X} \setminus X_U|$, it will then follow that $|\widetilde{X}| \leq \chi(U^+) + \chi(U_\sigma^+) + 1 \leq 2\max(\chi(U^+), \chi(U_\sigma^+)) + 1$. We note that, since X_σ is a maximum saturating set of $B_{vw}(U_\sigma^+)$ by Lemma 3.21, we may define $\widetilde{X}_\sigma = X_\sigma \cap \widetilde{B}_{vw}(U_\sigma^+)$ as in Definition 3.11.

It follows from the definition of P that, given any $\pi_\sigma \in P$, there exists a corresponding $\pi \in \widetilde{X} \setminus X_U$. We next show that if $\lambda_\sigma \in P \cap \widetilde{X}_\sigma$ then $\lambda_\sigma(z_{\alpha+2}) \in \{z_{\zeta-2}), z_{\zeta+2}\}$, where $z_\alpha = \lambda_\sigma^{-1}(z_\zeta)$. So $P \cap \widetilde{X}_\sigma \subseteq L_\sigma$, where L_σ is the subset of X_σ that corresponds to the subset L of X in Lemma 3.17. Using Lemma 3.17(a) for X_σ, it will immediately follow that $|P \cap \widetilde{X}_\sigma| \leq \chi(U_\sigma^+)$.

Let $\lambda_\sigma \in P \cap \widetilde{X}_\sigma$. On using Corollary 3.18 first for λ and then for σ, we see that $\{\lambda(x_{\mu-2}), \lambda(x_{\mu+2})\} = \{\sigma(x_{\xi-2}), \sigma(x_{\xi+2})\}$. Thus $\{\sigma^{-1}\lambda(x_{\mu-2}), \sigma^{-1}\lambda(x_{\mu+2})\} = \{x_{\xi-2}, x_{\xi+2}\}$. So, since $z_\alpha = \theta(x_\mu)$ by Lemma 3.20, it follows that $\theta^{-1}(z_{\alpha+2}) \in \{x_{\mu-2}, x_{\mu+2}\}$, and hence $\sigma^{-1}\lambda\theta^{-1}(z_{\alpha+2}) \in \{x_{\xi-2}, x_{\xi+2}\}$. Therefore $\lambda_\sigma(z_{\alpha+2}) \in \{\theta(x_{\xi-2}), \theta(x_{\xi+2}\} = \{z_{\zeta-2}, z_{\zeta+2}\}$ as required.

It remains to be shown that $|P \setminus \widetilde{X}_\sigma| \leq 1$. So suppose that there exists $\lambda_\sigma \in P \setminus \widetilde{X}_\sigma$. Now $\lambda(w)$ is a d-leaf of T as $\lambda \in \widetilde{X}$. So, since $\theta\sigma^{-1}(T) = T_\sigma$ by Lemma 3.19 and $\lambda_\sigma = \theta\sigma^{-1}\lambda\theta^{-1}$, it follows that $\lambda_\sigma(w)$ must be a d-leaf of T_σ. As $\lambda_\sigma \notin \widetilde{X}_\sigma$, this implies that, although $\lambda_\sigma(w)$ is a leaf of T_σ, it is not a leaf of U_σ^+. We now show that $z_{\zeta+1}$ is the unique leaf of T_σ that is not a leaf of U^+_σ. Since $\pi_\sigma(w)$ must be distinct for each $\pi_\sigma \in X_\sigma$, this will imply that $P \setminus \widetilde{X}_\sigma = \{\lambda_\sigma\}$, and therefore $|P \setminus \widetilde{X}_\sigma| \leq 1$.

Now $\operatorname{diam}(T) = \operatorname{diam}(T_\sigma)$ as $T \cong T_\sigma$. So, by applying Lemma 3.1 to U^+ and x_ν, and then to U^+_σ and z_ζ, it is easy to see that $\tau_{U^+}(x_\nu) = \tau_{U^+_\sigma}(z_\zeta)$. Now, since X is U^+-optimum and $\sigma \in \widetilde{X} \setminus X_U$, it follows from Corollary 3.18 that $\tau_{U^+}(x_\nu) = 1$, and thus $\tau_{U^+_\sigma}(z_\zeta) = 1$. On applying Lemma 3.4 to U^+_σ, T_σ and z_ζ, it then follows that $z_{\zeta+1}$ is the only leaf of T_σ that is not a leaf of U^+_σ. This completes the proof. □

Theorem 3.23. *Let U^+ be a supercard of U and T that is a sunshine graph that has the largest possible value of $\chi(U^+)$ over all supercards of U and T that are sunshine graphs. Then $b(U, T) \leq 2\chi(U^+) + 5$.*

Proof. Let X be a U^+-optimum saturating set of $B_{vw}(U^+)$. Now, if $\widetilde{X} \subseteq X_U$ then the result holds immediately by Corollary 3.14. So suppose that there exists $\sigma \in \widetilde{X} \setminus X_U$, and let U^+_σ be the supercard of U and T as defined in equation (9). Then $|\widetilde{X}| \leq 2\max(\chi(U^+), \chi(U^+_\sigma)) + 1$ by Lemma 3.22. So, since U^+_σ is a sunshine graph, $b(U, T) \leq 2\chi(U^+) + 5$ by Corollary 3.13. □

4. The set $B^U_{vw}(U^+)$

Let U be a sunshine graph and T be a caterpillar. *For the rest of this paper, we shall assume that U^+ is a supercard of U and T that satisfies the conditions of Theorem 3.23.* In light of the bound in this theorem, we now concentrate on the set $B^U_{vw}(U^+)$. For ease of notation, we write B_U instead of $B^U_{vw}(U^+)$.

Let $\lambda \in B_U$. By Lemma 3.15, $\lambda(w)$ is a leaf of U^+ adjacent to a vertex of degree $d_{U^+}(x_0)$. Moreover, λ is either a rotation of the cycle $\operatorname{skel}(U)$ and $\lambda(x_i) = x_{\nu-\mu+i}$ for each x_i, or λ is a reflection of the cycle $\operatorname{skel}(U)$ and $\lambda(x_i) = x_{\nu+\mu-i}$ for each x_i. We may therefore partition B_U into the *rotations* $\operatorname{Rot}(B_U)$, and the *reflections* $\operatorname{Ref}(B_U)$.

We make frequent use of the following well-known results concerning the rotations and reflections of a cycle.

Lemma 4.1. *Let $\lambda, \pi \in B_U$, and let $x_\alpha = \lambda(x_0)$ and $x_\beta = \pi(x_0)$. The following results holds for all x_i.*

(a) If $\lambda \in \operatorname{Rot}(B_U)$ then $\lambda(x_i) = x_{\alpha+i}$ and $\lambda^{-1}(x_i) = x_{i-\alpha}$.
(b) If $\lambda \in \operatorname{Ref}(B_U)$ then $\lambda(x_i) = \lambda^{-1}(x_i) = x_{\alpha-i}$.
(c) If $\lambda, \pi \in \operatorname{Rot}(B_U)$ then $\lambda\pi(x_i) = \pi\lambda(x_i) = x_{(\alpha+\beta)+i}$.
(d) If $\lambda, \pi \in \operatorname{Ref}(B_U)$ then $\pi\lambda(x_i) = x_{(\beta-\alpha)+i}$ and $\lambda\pi(x_i) = x_{(\alpha-\beta)+i}$.

(e) *If* $\lambda \in \mathrm{Rot}(B_U)$ *and* $\pi \in \mathrm{Ref}(B_U)$ *then* $\lambda\pi(x_i) = x_{(\beta+\alpha)-i}$*, and* $\pi\lambda(x_i) = x_{(\beta-\alpha)-i}$*.*

Corollary 4.2. *Let* $\lambda \in B_U$*. Suppose there exists* $\sigma \in B_U$ *such that* $\sigma(x_i) = \lambda^{-1}(x_i)$ *for all* x_i*. Then either* λ *and* σ *are both in* $\mathrm{Rot}(B_U)$*, or they are both in* $\mathrm{Ref}(B_U)$*.*

Corollary 4.3. *Let* λ*,* π *be in* B_U*. Suppose there exists* $\sigma \in B_U$ *such that* $\sigma(x_i) = \lambda\pi(x_i)$ *for all* x_i*. Then* $\sigma \in \mathrm{Rot}(B_U)$ *if and only if either* λ *and* π *are both in* $\mathrm{Rot}(B_U)$*, or they are both in* $\mathrm{Ref}(B_U)$*; otherwise* $\sigma \in \mathrm{Ref}(B_U)$*.*

We call any $\lambda \in \mathrm{Rot}(B_U)$ such that $\lambda(x_0) = x_0$, a *trivial rotation* (equivalently, $\lambda(x_i) = x_i$ for all x_i by Lemma 4.1(a)). Clearly, $\lambda^{-1}(x_\nu) = x_\nu$ for every trivial rotation λ in B_U. It then immediately follows from Definition 2.8 that $1_{V(U^+)}$ is the only trivial rotation in any maximum saturating set X of $B_{vw}(U^+)$.

Definition 4.4. *Let* X *be a maximum saturating set of* $B_{vw}(U^+)$*. We define* $X_{\mathrm{Rot}} = X \cap \mathrm{Rot}(B_U)$*,* $X_{\mathrm{Ref}} = X \cap \mathrm{Ref}(B_U)$ *and* $X_{\mathrm{Aut}} = X \cap \mathrm{Aut}(U^+)$*.*

Lemma 4.5. *Let* X *be a maximum saturating set of* $B_{vw}(U^+)$*, and let* λ *and* π *be distinct permutations in* X*. If* λ *and* π *are both in* X_{Rot} *then* $\lambda(x_i) \neq \pi(x_i)$ *for all* x_i*. Similarly, if* λ *and* π *are both in* X_{Ref} *then* $\lambda(x_i) \neq \pi(x_i)$ *for all* x_i*.*

Proof. Let $x_\alpha = \lambda(x_0)$ and $x_\beta = \pi(x_0)$. Suppose that there exists x_k such that $\lambda(x_k) = \pi(x_k)$. Now, if both λ and π are in $\mathrm{Rot}(B_U)$ then it follows from Lemma 4.1(a) that $\alpha = \beta$, so $\lambda^{-1}(x_\nu) = \pi^{-1}(x_\nu)$. Similarly, if both λ and π are in $\mathrm{Ref}(B_U)$ then it follows from Lemma 4.1(b) that $\lambda^{-1}(x_\nu) = \pi^{-1}(x_\nu)$. Either contradicts property (b) of Definition 2.8. □

Corollary 4.6. *Let* X *be a maximum saturating set of* $B_{vw}(U^+)$*, and let* $Z = \{\lambda(x_0) \mid \lambda \in X_U\}$*. Then* $|X_U| \leq 2|Z|$*.*

Proof. For each $z \in Z$, it follows from Lemma 4.5 that there is at most one $\lambda \in X_{\mathrm{Rot}}$ and at most one $\pi \in X_{\mathrm{Ref}}$ such that $\lambda(x_0) = \pi(x_0) = z$. □

We now make some further observations about B_U.

Lemma 4.7. *Let* $\theta \in B_U$ *and let* $u \in V(U^+)$*. Then*

(a) $d_{U^+}(u) = d_{U^+}(\theta(u)) + 1$ *if and only if* u *is* x_0 *and* $\theta(w)$ *is not adjacent to* $\theta(x_0)$*;*

(b) $d_{U^+}(u) = d_{U^+}(\theta(u)) - 1$ *if and only if* u *is not* x_0 *and* $\theta(w)$ *is adjacent to* $\theta(u)$*;*

(c) $d_{U^+}(u) = d_{U^+}(\theta(u))$ *otherwise.*

Proof. Since $\theta(U) = U^+ - \theta(w)$, it follows that $d_U(u) = d_{U^+-\theta(w)}(\theta(u))$ for any $u \in V(U)$. It is now easy to show that one of (a), (b) and (c) must hold as w is adjacent to x_0 in U^+. □

Corollary 4.8. *Let $\theta \in B_U$ be such that $\theta(w)$ is adjacent to x_0. If x_i is not x_0 and $\theta(x_i) \neq x_0$ then $d_{U^+}(x_i) = d_{U^+}(\theta(x_i))$.*

Corollary 4.9. *Let $\theta \in B_U$.*

(a) $d_{U^+}(x_0) = d_{U^+}(\theta(x_0)) + 1$ if and only if $\theta \notin \mathrm{Aut}(U^+)$.
(b) $d_{U^+}(x_0) = d_{U^+}(\theta(x_0))$ if and only if $\theta \in \mathrm{Aut}(U^+)$.
(c) If $\theta(x_0) = x_0$ then $\theta \in \mathrm{Aut}(U^+)$.

Proof. By Corollary 2.2, $\theta \in \mathrm{Aut}(U^+)$ if and only if $\theta(w)$ is adjacent to $\theta(x_0)$. Parts (a) and (b) then follow from Lemma 4.7 when u is x_0. Part (c) is immediate from part (b). □

It follows from Corollary 4.9(c) that any trivial rotation in B_U is an automorphism of U^+.

Lemma 4.10. *Let $\lambda \in \mathrm{Aut}(U^+)$ and $\pi \in B_U \setminus \mathrm{Aut}(U^+)$. Then $\lambda\pi \in B_U \setminus \mathrm{Aut}(U^+)$ and $d_{U^+}(\lambda\pi(x_0)) = d_{U^+}(x_0) - 1$.*

Proof. Since $\pi \in B_U$ and $\lambda \in \mathrm{Aut}(U^+)$, clearly $\lambda\pi(U) = \lambda(U^+ - \pi(w)) = U^+ - \lambda\pi(w)$, so $\lambda\pi \in B_U$. Now, if $\lambda\pi \in \mathrm{Aut}(U^+)$ then $\pi = \lambda^{-1}\lambda\pi \in \mathrm{Aut}(U^+)$ as $\mathrm{Aut}(U^+)$ is a group. Since this is impossible, $\lambda\pi \in B_U \setminus \mathrm{Aut}(U^+)$. So $d_{U^+}(\lambda\pi(x_0)) = d_{U^+}(x_0) - 1$ by Corollary 4.9(a). □

Lemma 4.11. *Let $\lambda \in B_U \setminus \mathrm{Aut}(U^+)$ be such that $\lambda^2(x_0) = x_0$. Then $\lambda(w)$ is adjacent to x_0.*

Proof. $d_{U^+}(x_0) = d_{U^+}(\lambda(x_0)) + 1$ by Lemma 4.9(a). So, since $\lambda^2(x_0) = x_0$, clearly $d_{U^+}(\lambda(x_0)) = d_{U^+}(\lambda^2(x_0)) - 1$. Therefore $\lambda(w)$ is adjacent to $\lambda^2(x_0)$, i.e. x_0, by Lemma 4.7(b). □

Corollary 4.12. *Let $\lambda \in \mathrm{Ref}(B_U) \setminus \mathrm{Aut}(U^+)$. Then $\lambda(w)$ is adjacent to x_0.*

Proof. As $\lambda \in \mathrm{Ref}(B_U)$, we have $\lambda^2(x_0) = x_0$ by Lemma 4.1(b). □

We next show that if any non-trivial rotation is an automorphism of U^+ then $B_U = \mathrm{Aut}(U^+)$.

Lemma 4.13. *Suppose there exists $\lambda \in \mathrm{Rot}(B_U) \cap \mathrm{Aut}(U^+)$ such that $\lambda(x_0) \neq x_0$. Then $B_U = \mathrm{Aut}(U^+)$.*

Proof. Let us assume, to the contrary, that there exists $\pi \in B_U \setminus \mathrm{Aut}(U^+)$. Suppose first that $\pi \in \mathrm{Rot}(B_U)$. Then, by Lemma 4.1(c) and Lemma 4.10, $d_{U^+}(\pi\lambda(x_0)) = d_{U^+}(\lambda\pi(x_0)) = d_{U^+}(x_0)-1$. Thus $d_{U^+}(\lambda(x_0)) = d_{U^+}(\pi\lambda(x_0))$ $+1$ as $\lambda \in \mathrm{Aut}(U^+)$. However, using Lemma 4.7(a) with $\theta = \pi$ and $u = \lambda(x_0)$ implies that $\lambda(x_0)$ is x_0, which is impossible.

Suppose, on the other hand, that $\pi \in \mathrm{Ref}(B_U)$. Then, by Lemma 4.10 and Corollary 4.3, $\lambda\pi \in \mathrm{Ref}(B_U) \setminus \mathrm{Aut}(U^+)$. So $\lambda\pi(w)$ is adjacent to x_0 by Corollary 4.12. By Lemma 4.1(b), $\lambda\pi(\pi(x_0)) = \lambda(x_0)$. So, since $\pi(x_0) \neq x_0$ by Corollary 4.9(c) and $\lambda(x_0) \neq x_0$ by assumption, it follows from Corollary 4.8 with $\theta = \lambda\pi$ and $x_i = \pi(x_0)$ that $d_{U^+}(\pi(x_0)) = d_{U^+}(\lambda\pi(\pi(x_0))) =$ $d_{U^+}(\lambda(x_0))$. However, Corollary 4.9(a) and (b) applied to π and λ respectively, implies that $d_{U^+}(\pi(x_0)) \neq d_{U^+}(\lambda(x_0))$, contradicting the previous statement. Hence $B_U = \mathrm{Aut}(U^+)$. □

Corollary 4.14. *Let X be a U^+-optimum saturating set of $B_{vw}(U^+)$.*

- *(a) If $B_U \neq \mathrm{Aut}(U^+)$ then $X_{\mathrm{Aut}} \subseteq X_{\mathrm{Ref}} \cup \{1_{V(U^+)}\}$.*
- *(b) If $|X_{\mathrm{Aut}}| \geq 3$ then there exists a non-trivial rotation in $\mathrm{Aut}(U^+)$ and $B_U = \mathrm{Aut}(U^+)$.*
- *(c) If $b(U,T) \geq 10$ and $B_U = \mathrm{Aut}(U^+)$ then there exists a non-trivial rotation in $\mathrm{Aut}(U^+)$.*

Proof. (a) $1_{V(U^+)}$ is the only trivial rotation in X. So, if there exists another rotation in X_{Aut} then $B_U = \mathrm{Aut}(U^+)$ by Lemma 4.13.

(b) Suppose that λ and π are distinct permutations in $X_{\mathrm{Aut}} \setminus \{1_{V(U^+)}\}$. By Lemma 4.13, we may assume that both λ and π are reflections as $1_{V(U^+)}$ is the only trivial rotation in X. Now, since $\mathrm{Aut}(U^+)$ is a group, it follows from Corollary 4.3 that $\lambda\pi$ is a rotation in $\mathrm{Aut}(U^+)$. Moreover, since $\lambda(x_0) \neq \pi(x_0)$ by Lemma 4.5, $\lambda\pi(x_0) \neq x_0$ by Lemma 4.1(d). Therefore $B_U = \mathrm{Aut}(U^+)$ by Lemma 4.13.

(c) $|X_{\mathrm{Aut}}| = \chi(U^+)$ as X is U^+-optimum and $B_U = \mathrm{Aut}(U^+)$. The result now follows from Theorem 3.23 and part (b). □

We now consider those rotations that are not automorphisms of U^+.

Lemma 4.15. *Suppose there exists $\lambda, \pi \in \mathrm{Rot}(B_U) \setminus \mathrm{Aut}(U^+)$ such that $\lambda(x_0) \neq \pi(x_0)$. Then both $\lambda(w)$ and $\pi(w)$ are adjacent to x_0.*

Proof. Let $x_\alpha = \lambda(x_0)$ and $x_\beta = \pi(x_0)$, so $\lambda(x_i) = x_{\alpha+i}$ and $\pi(x_i) = x_{\beta+i}$ by Lemma 4.1(a). Now, by Corollary 4.9(c), $\alpha \neq 0$ as $\lambda \notin \mathrm{Aut}(U^+)$. So, by Lemma 4.7, $d_{U^+}(x_{-\alpha}) \leq d_{U^+}(\pi(x_{-\alpha})) = d_{U^+}(x_{\beta-\alpha})$.

Suppose that $\lambda(w)$ is not adjacent to x_0, i.e. $\lambda(x_{-\alpha})$. Then, since $\alpha \neq 0$, it follows from Lemma 4.7 that $d_{U^+}(x_{-\alpha}) = d_{U^+}(\lambda(x_{-\alpha})) = d_{U^+}(x_0)$. So

$d_{U^+}(x_{\beta-\alpha}) \geq d_{U^+}(x_0)$. Now, by Corollary 4.9(a), we have that $d_{U^+}(x_0) = d_{U^+}(\pi(x_0)) + 1$ as $\pi \notin \text{Aut}(U^+)$. Thus

$$d_{U^+}(x_{\beta-\alpha}) \geq d_{U^+}(x_0) > d_{U^+}(\pi(x_0)) = d_{U^+}(x_\beta) = d_{U^+}(\lambda(x_{\beta-\alpha})).$$

Hence, $x_{\beta-\alpha}$ must be x_0 by Lemma 4.7(a), and therefore, $\beta = \alpha$, which is impossible. So $\lambda(w)$ must be adjacent to x_0. By symmetry, clearly $\pi(w)$ is also adjacent to x_0. ☐

Corollary 4.16. *Let X be a maximum saturating set of $B_{vw}(U^+)$. If $|X_{\text{Rot}} \setminus X_{\text{Aut}}| \geq 2$ then $\lambda(w)$ is adjacent to x_0 for all $\lambda \in X_{\text{Rot}} \setminus X_{\text{Aut}}$.*

Proof. Suppose that there exist distinct rotations λ and π in $X_{\text{Rot}} \setminus X_{\text{Aut}}$. Then $\lambda(x_0) \neq \pi(x_0)$ by Lemma 4.5, so the result follows immediately from Lemma 4.15. ☐

The following lemma shows that there exists a correspondence between those permutations λ for which $\lambda(w)$ is adjacent to x_0, and a subset of $\text{Aut}(U)$.

Lemma 4.17. *We have the following results.*

- *(a) For each $\lambda \in B_U$ for which the leaf $\lambda(w)$ of U^+ is adjacent to x_0, there exists $\lambda^* \in \text{Aut}(U)$ such that $\lambda^*(x_i) = \lambda(x_i)$ for all x_i.*
- *(b) For each $\lambda^* \in \text{Aut}(U)$, there exists $\lambda \in B_U$ such $\lambda(w) = w$ and $\lambda(x_i) = \lambda^*(x_i)$ for all x_i.*

Proof. (a) Let $\lambda \in B_U$ be such that $\lambda(w)$ is adjacent to x_0. Let ϕ be the transposition that swaps the leaves w and $\lambda(w)$, and let $\lambda^* = \phi\lambda$. Then $\lambda^*(U) = \phi\lambda(U) = \phi(U^+ - \lambda(w)) = U$, as $\phi \in \text{Aut}(U^+)$.

(b) Let $\lambda^* \in \text{Aut}(U)$. Then the permutation λ defined by $\lambda(w) = w$ and $\lambda(u) = \lambda^*(u)$ for all $u \in V(U)$ clearly has the required properties. ☐

Lemma 4.18. *Suppose there exists $\lambda, \pi \in \text{Ref}(B_U) \setminus \text{Aut}(U^+)$ such that $\lambda(x_0) \neq \pi(x_0)$. Then there exists $\sigma \in \text{Rot}(B_U) \setminus \text{Aut}(U^+)$ such that $\sigma(w) = w$ and $\sigma(x_i) = \lambda\pi(x_i)$ for all x_i.*

Proof. Since $\lambda(w)$ and $\pi(w)$ are both adjacent to x_0 by Corollary 4.12, it follows from Lemma 4.17(a) that there exist λ^* and π^* in $\text{Aut}(U)$ such that $\lambda^*(x_i) = \lambda(x_i)$ and $\pi^*(x_i) = \pi(x_i)$ for all x_i. Since $\text{Aut}(U)$ is a group, $\lambda^*\pi^* \in \text{Aut}(U)$. So, by Lemma 4.17(b), there exists $\sigma \in B_U$ such that $\sigma(w) = w$ and $\sigma(x_i) = \lambda\pi(x_i)$ for all x_i.

Now, by Corollary 4.3, $\sigma \in \text{Rot}(B_U)$. Moreover, since $\lambda(x_0) \neq \pi(x_0)$, it follows from Lemma 4.1(d) that $\sigma(x_0) \neq x_0$. As $B_U \neq \text{Aut}(U^+)$, it immediately follows from Lemma 4.13 that $\sigma \notin \text{Aut}(U^+)$. ☐

Lemma 4.19. *Suppose there exists* $\lambda, \pi \in \mathrm{Ref}(B_U) \setminus \mathrm{Aut}(U^+)$ *such that* $\lambda(x_0) \neq \pi(x_0)$*. Then* $\psi(w)$ *is adjacent to* x_0 *for all* $\psi \in B_U$*.*

Proof. Let σ be the rotation from Lemma 4.18 and let $\psi \in B_U$. If $\psi(x_0) = x_0$ then $\psi \in \mathrm{Aut}(U^+)$ by Corollary 4.9(c), so $\psi(w)$ is adjacent to x_0. We therefore assume that $\psi(x_0) \neq x_0$. We first show that $\psi \notin \mathrm{Aut}(U^+)$.

Suppose that $\psi \in \mathrm{Aut}(U^+)$. Since $\psi(x_0) \neq x_0$ and $B_U \neq \mathrm{Aut}(U^+)$, it follows from Lemma 4.13 that ψ must be a reflection. By Lemma 4.10 and Corollary 4.3, we see that $\psi\sigma \in \mathrm{Ref}(B_U) \setminus \mathrm{Aut}(U^+)$. So $\psi\sigma(w)$ is adjacent to x_0 by Corollary 4.12. However, $\psi\sigma(w) = \psi(w)$ as $\sigma(w) = w$, so $\psi\sigma(w)$ must be adjacent to $\psi(x_0)$ as $\psi \in \mathrm{Aut}(U^+)$. Since $\psi(x_0) \neq x_0$ this is impossible. This contradiction shows that $\psi \notin \mathrm{Aut}(U^+)$.

It remains to be shown that $\psi(w)$ is adjacent to x_0. If $\psi \in \mathrm{Ref}(B_U)$ then the result follows from Corollary 4.12. So suppose that $\psi \in \mathrm{Rot}(B_U) \setminus \mathrm{Aut}(U^+)$. Since the result follows from Lemma 4.15 when $\sigma(x_0) \neq \psi(x_0)$, we may assume that $\sigma(x_0) = \psi(x_0)$, thus $\psi^{-1}(x_0) = \sigma^{-1}(x_0)$ by Lemma 4.1(a). Hence $\sigma^{-1}(x_0) \neq x_0$. As $\sigma(w) = w$, it now follows from Lemma 4.7(b) with $\theta = \sigma$ and $u = \sigma^{-1}(x_0)$, that $d_{U^+}(\sigma^{-1}(x_0)) = d_{U^+}(\sigma\sigma^{-1}(x_0)) - 1$. So $d_{U^+}(\sigma^{-1}(x_0)) = d_{U^+}(x_0) - 1$, and thus, $d_{U^+}(\psi^{-1}(x_0)) = d_{U^+}(x_0) - 1$. The result now follows by again applying Lemma 4.7(b) to ψ and $\psi^{-1}(x_0)$. □

Corollary 4.20. *Let* X *be a maximum saturating set of* $B_{vw}(U^+)$*. If* $|X_{\mathrm{Ref}} \setminus X_{\mathrm{Aut}}| \geq 2$ *then* $\lambda(w)$ *is adjacent to* x_0 *for all* $\lambda \in B_U$*.*

Proof. Suppose that λ and π are two distinct reflections in $X_{\mathrm{Ref}} \setminus X_{\mathrm{Aut}}$. Then $\lambda(x_0) \neq \pi(x_0)$ by Lemma 4.5, so the result follows immediately from Lemma 4.19. □

We note that Corollary 4.20 is a stronger result than the analogous result for rotations in Corollary 4.16, since the conclusions of Corollary 4.20 applies to all permuations in B_U.

Lemma 4.21. *Suppose that* $B_U \neq \mathrm{Aut}(U^+)$ *and* $\chi(U^+) \geq 5$*. Let* X *be a* U^+*-optimum saturating set of* $B_{vw}(U^+)$*. Then*

(a) for each $\lambda \in X_U$*, there exists a distinct leaf of* U^+*, namely* $\lambda(w)$*, adjacent to* x_0*;*
(b) $d_{U^+}(x_0) \geq \chi(U^+) + 2$*;*
(c) for each $\lambda \in X_U$*, there exists* $\lambda^* \in \mathrm{Aut}(U)$ *such that* $\lambda^*(x_0) = \lambda(x_0)$*.*

Proof. Part (b) follows immediately from part (a). In addition, part (c) follows from part (a) by Lemma 4.17(a). Now, by property (b) of Definition 2.8 and Lemma 3.15, $\lambda(w)$ is a distinct leaf of U^+ for each distinct $\lambda \in X_U$. To complete the proof, we now show that each such $\lambda(w)$ is adjacent to x_0.

By Corollary 4.20, there is nothing to prove when $|X_{\mathrm{Ref}} \setminus X_{\mathrm{Aut}}| \geq 2$; so we assume that $|X_{\mathrm{Ref}} \setminus X_{\mathrm{Aut}}| \leq 1$. Furthermore, since $B_U \neq \mathrm{Aut}(U^+)$, it follows from Corollary 4.14 that $X_{\mathrm{Aut}} \subseteq X_{\mathrm{Ref}} \cup \{1_{V(U^+)}\}$ and $|X_{\mathrm{Aut}}| \leq 2$. Since $|X_U| \geq 5$, this implies that $|X_{\mathrm{Rot}} \setminus X_{\mathrm{Aut}}| \geq 2$. It now follows from Corollaries 4.16 and 4.12 that $\lambda(w)$ is adjacent to x_0 for each $\lambda \in X_U \setminus X_{\mathrm{Aut}}$. As $1_{V(U^+)}(w)$ is clearly also adjacent to x_0, this concludes the proof in the case that $X_{\mathrm{Aut}} = \{1_{V(U^+)}\}$.

Suppose then there exists $\lambda \in X_{\mathrm{Ref}} \cap X_{\mathrm{Aut}}$, and let ϕ and η be permutations in $X_{\mathrm{Rot}} \setminus X_{\mathrm{Aut}}$. By Lemma 4.10 and Corollary 4.3, both $\lambda\phi$ and $\lambda\eta$ are in $\mathrm{Ref}(B_U) \setminus \mathrm{Aut}(U^+)$. Furthermore, $\lambda\phi(x_0) \neq \lambda\eta(x_0)$ as $\phi(x_0) \neq \eta(x_0)$ by Corollary 4.5. By applying Lemma 4.19 to $\lambda\phi$ and $\lambda\eta$, this implies that $\psi(w)$ is adjacent to x_0 for all $\psi \in B_U$. □

Theorem 4.22. *Suppose that $B_U \neq \mathrm{Aut}(U^+)$ and $n \geq 12$. Then $b(U, T) \leq 2\left\lfloor \sqrt{2n+1} \right\rfloor + 3$.*

Proof. We recall that $d_1(U)$ and $d_2(U)$ are the number of vertices of U of degrees 1 and 2, respectively. Let $d_q(U) = n - d_1(U) - d_2(U)$, i.e., the number of vertices of U of degree 3 or more.

Now, if $\chi(U^+) \leq 4$ then $b(U, T) \leq 13$ by Theorem 3.23, thus the bound holds. We may therefore assume that $\chi(U^+) \geq 5$, so the conclusions of Lemma 4.21 hold. It follows from part(b) of that lemma that $d_U(x_0) = d_{U^+}(x_0) - 1 \geq \chi(U^+) + 1 \geq 6$.

Let X be a U^+-optimum saturating set of U^+, and let $Z = \{\lambda(x_0) \mid \lambda \in X_U\}$. By Lemma 4.21(c), there exists a subset Λ^* of $\mathrm{Aut}(U)$ such that $\{\lambda^*(x_0) \mid l\lambda^* \in \Lambda^*\} = Z$. This implies that U contains at least $|Z|$ vertices of degree $d_U(x_0)$, and hence $d_q(U) \geq |Z|$. In addition, since each vertex x_i in Z is adjacent to precisely $d_U(x_0) - 2$ leaves, it follows that $d_1(U) \geq |Z|(d_U(x_0) - 2)$. Therefore $d_1(U) \geq |Z|(\chi(U^+) - 1)$.

Now, by property (b) of Definition 2.8 and Lemma 3.15, for each distinct λ in X_U there exists a distinct vertex of U of degree 2, namely $\lambda^{-1}(x_\nu)$. So $d_2(U) \geq |X_U| = \chi(U^+)$. Since each vertex in Z has degree at least 6 in U, it therefore follows that

$$\begin{aligned} n = |V(U)| = d_1(U) + d_2(U) + d_q(U) &\geq |Z|(\chi(U^+) - 1) + \chi(U^+) + |Z| \\ &= (|Z| + 1)\chi(U^+). \end{aligned}$$

So, since $\chi(U^+) = |X_U| \leq 2|Z|$ by Corollary 4.6, we have $2n \geq \chi(U^+)(\chi(U^+) + 2)$. Solving for $\chi(U^+)$, yields $\chi(U^+) \leq \sqrt{2n+1} - 1$. Therefore $b(U, T) \leq 2\left\lfloor \sqrt{2n+1} - 1 \right\rfloor + 5$ by Theorem 3.23, yielding the bound. □

5. The case when $B^U_{vw}(U^+) = \mathrm{Aut}(U^+)$

For the whole of this section, we assume that that there exists some non-trivial rotation in $\mathrm{Aut}(U^+)$. So $B^U_{vw}(U^+) = \mathrm{Aut}(U^+)$ by Lemma 4.13, and thus $X_U = X_{\mathrm{Aut}} = X_{\mathrm{Rot}} \cup X_{\mathrm{Ref}}$, for any maximum saturating set X. We note that, if $B^U_{vw}(U^+) = \mathrm{Aut}(U^+)$ but there does not exist such a rotation, then $b(U, T) \le 9$ by Corollary 4.14(c). This motivates our assumption.

For clarity, we write $\mathrm{Rot}(\mathrm{Aut}(U^+))$ and $\mathrm{Ref}(\mathrm{Aut}(U^+))$ instead of $\mathrm{Rot}(B_U)$ and $\mathrm{Ref}(B_U)$, respectively. Since $\mathrm{Aut}(U^+)$ is a group, we can simplify Corollaries 4.2 and 4.3 as follows.

Corollary 5.1. *Let* $\lambda \in \mathrm{Aut}(U^+)$*. Then either* λ *and* λ^{-1} *are both in* $\mathrm{Rot}(\mathrm{Aut}(U^+))$*, or they are both in* $\mathrm{Ref}(\mathrm{Aut}(U^+))$*.*

Corollary 5.2. *Let* $\lambda, \pi \in \mathrm{Aut}(U^+)$*. Then* $\lambda\pi \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ *if and only if either* λ *and* π *are both in* $\mathrm{Rot}(\mathrm{Aut}(U^+))$*, or they are both in* $\mathrm{Ref}(\mathrm{Aut}(U^+))$*; otherwise* $\lambda\pi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$*.*

We recall that x_i^j is the j^{th} distinct leaf adjacent to x_i. The following two results are easy to prove.

Lemma 5.3. *There exists* $\lambda \in \mathrm{Rot}(\mathrm{Aut}(U^+))$*, where* $\lambda(x_i) = x_{\alpha+i}$ *and* $\lambda(x_i^j) = x_{\alpha+i}^j$ *for all* i *and* j*, if and only if* $d_{U^+}(x_i) = d_{U^+}(x_{\alpha+i})$ *for all* i*.*

Lemma 5.4. *There exists* $\lambda \in \mathrm{Ref}(\mathrm{Aut}(U^+))$*, where* $\lambda(x_i) = x_{\beta-i}$ *and* $\lambda(x_i^j) = x_{\beta-i}^j$ *for all* i *and* j*, if and only if* $d_{U^+}(x_i) = d_{U^+}(x_{\beta-i})$ *for all* i*.*

If $Z \subseteq \mathrm{Aut}(U^+)$, we define $Z(u) = \{\lambda(u) \mid \lambda \in Z\}$. We note that if Z is a subgroup of $\mathrm{Aut}(U^+)$ then $Z(u)$ is the *orbit* of u under the group action of Z on $V(U^+)$.

Lemma 5.5. *Let* $A = \{\beta \mid x_\beta \in \mathrm{Rot}(\mathrm{Aut}(U^+))(x_0)\}$*, and let* δ *be the smallest positive element in* A*. Then*

(a) δ *divides every element of* A*;*
(b) $2 \le \delta \le \frac{c}{2}$*;*
(c) there exists $\phi \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ *such that* $\phi^{\frac{c}{\delta}} = 1_{V(U^+)}$*,* $\phi(x_i) = x_{\delta+i}$ *and* $\phi(x_i^j) = x_{\delta+i}^j$ *for all* i*,* j*.*

Proof. By assumption, there exists some non-trivial rotation in $\mathrm{Aut}(U^+)$; so $0 < \delta < c$. Let $\sigma \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ be such that $\sigma(x_0) = x_\delta$.

(a) Suppose that there exists $\beta \in A$ such that δ does not divide β, and let $\pi(x_0) = x_\beta$. Let d be the highest common factor of δ and β. By the Euclidean algorithm, there exist integers a and b such that $b\delta + a\beta = d$.

Now, on using Lemma 4.1, and Corollaries 5.1 and 5.2 repeatedly, it is easy to see that $\sigma^b\pi^a \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ and $\sigma^b\pi^a(x_i) = x_{b\delta+a\beta+i} = x_{d+i}$ for all x_i. Therefore $x_d \in \mathrm{Rot}(\mathrm{Aut}(U^+))(x_0)$, and thus $d \in A$. Since $0 < d < \delta$, this contradicts the minimality of δ.

(b) δ divides c as $c \in A$, thus $\delta \leq \frac{c}{2}$. So suppose that $\delta = 1$. Then, on using Corollary 5.2 and Lemma 4.1(c) repeatedly, we see that $\sigma^\nu \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ and $\sigma^\nu(x_0) = x_\nu$. This is impossible as $d_{U^+}(x_0) \geq 3$ and $d_{U^+}(x_\nu) = 2$. Therefore $2 \leq \delta \leq \frac{c}{2}$.

(c) $d_{U^+}(x_i) = d_{U^+}(x_{\delta+i})$ for all x_i, as $\sigma \in \mathrm{Rot}(\mathrm{Aut}(U^+))$. Hence by Lemma 5.3, there exists $\phi \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ such that $\phi(x_i) = x_{\delta+i}$ and $\phi(x_i^j) = x_{\delta+i}^j$ for all i, j. As δ divides c, it is easy to see that $\phi^{\frac{c}{\delta}}(x_i) = x_i$ and $\phi^{\frac{c}{\delta}}(x_i^j) = x_i^j$ for all i, j. So $\phi^{\frac{c}{\delta}} = 1_{V(U^+)}$. □

By Corollary 5.2, the composition of two rotations is also a rotation. We may therefore make the following definition.

Definition 5.6. *Let δ be the positive integer and ϕ be the rotation from Lemma 5.5. We define Φ to be the cyclic subgroup of* $\mathrm{Aut}(U^+)$ *of order $\frac{c}{\delta}$ generated by ϕ, i.e., $\Phi = \{\phi^j \mid 0 \leq j < \frac{c}{\delta}\}$.*

For the rest of this section we assume that δ, ϕ and Φ are as in Definition 5.6. Since the cycle length $c = \delta|\Phi|$, and the number of leaves of U^+ is $d_1(U^+)$, we see that

$$n + 1 = \delta|\Phi| + d_1(U^+). \tag{10}$$

For any $\pi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$, we denote the right and left cosets of Φ with respect to π by $\Phi\pi$ and $\pi\Phi$, respectively. It follows from Corollary 5.2 that $\Phi\pi \subseteq \mathrm{Ref}(\mathrm{Aut}(U^+))$ and $\pi\Phi \subseteq \mathrm{Ref}(\mathrm{Aut}(U^+))$.

The orbit $\Phi(u)$ of a vertex u of U^+ under Φ is the set $\{\phi^j(u) \mid 0 \leq j < \frac{c}{\delta}\}$. It follows from a well-known result in Group Theory that, for any two vertices u and t of U^+, either $\Phi(u) = \Phi(t)$ or $\Phi(u) \cap \Phi(t) = \emptyset$. Thus t is in $\Phi(u)$ if and only if $\Phi(u) = \Phi(t)$. Now, for every vertex u of U^+, clearly $1_{V(U^+)}$ is the only element of Φ that fixes u. It therefore follows from the Orbit-Stabiliser and Lagrange Theorems [12] that $|\Phi(u)| = |\Phi| = |\Phi\pi| = \frac{c}{\delta}$.

Lemma 5.7. *Let $x_i \in V(\mathrm{skel}(U^+))$. Then*

(a) $\mathrm{Rot}(\mathrm{Aut}(U^+))(x_i) = \Phi(x_i)$;
(b) if $\lambda \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ *then* $\lambda^{-1}(x_i) \in \Phi(x_i)$;
(c) if $\pi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$ *then* $\Phi\pi(x_i) = \pi\Phi(x_i)$;
(d) if $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_i) \cap \Phi(x_i) \neq \emptyset$ *then* $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_i) \subseteq \Phi(x_i)$.

Proof. (a) Clearly $\Phi(x_i) \subseteq \mathrm{Rot}(\mathrm{Aut}(U^+))(x_i)$. So let $\lambda \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ and let $x_\alpha = \lambda(x_0)$. Now, by Lemma 5.5, there exists some positive integer k such that $\alpha = k\delta$. Moreover, on using Corollary 5.2 and Lemma 4.1, we see that $\lambda(x_i) = x_{k\delta+i} = \phi^k(x_i)$. Therefore $\lambda(x_i) \in \Phi(x_i)$, and thus $\mathrm{Rot}(\mathrm{Aut}(U^+))(x_i) \subseteq \Phi(x_i)$.

(b) This follows immediately from part (a) as $\lambda^{-1} \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ for all $\lambda \in \mathrm{Rot}(\mathrm{Aut}(U^+))$ by Corollary 5.1.

(c) Let $x_\beta = \pi(x_0)$, and let $0 \leq j < \frac{c}{\delta}$. Then, on using Corollary 5.2 and Lemma 4.1, we see that $\phi^j\pi(x_i) = x_{j\delta+\beta-i} = \pi\phi^{-j}(x_i)$. So $\phi^j\pi(x_i) \in \pi\Phi(x_i)$ by part (b), and therefore $\Phi\pi(x_i) \subseteq \pi\Phi(x_i)$. It similarly follows that $\pi\Phi(x_i) \subseteq \Phi\pi(x_i)$.

(d) Suppose there exists $\pi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$ such that $\pi(x_i) \in \Phi(x_i)$. Then $\Phi\pi(x_i) = \Phi(x_i)$, so $\pi\Phi(x_i) = \Phi(x_i)$ by part (c). Now let $\lambda \in \mathrm{Ref}(\mathrm{Aut}(U^+))$. Then $\pi\lambda(x_i) \in \Phi(x_i)$ by Corollary 5.2 and part (a). So $\pi\lambda(x_i) \in \pi\Phi(x_i)$, and therefore $\lambda(x_i) \in \Phi(x_i)$, yielding the result. □

We now show that we can always find some maximum saturating set X such that X_{Aut} is isomorphic to a subgroup of $\mathrm{Aut}(U^+)$.

Theorem 5.8. *Suppose that* $\mathrm{Aut}(U^+)$ *contains a non-trivial rotation. We consider the following two (not necessarily disjoint) cases:*

(i) $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_\nu) \subseteq \Phi(x_\nu)$*;*
(ii) $d_{U^+}(x_0) = 3$ *and* $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_0) \subseteq \Phi(x_0)$*.*

Then

(a) if either case (i) or case (ii) holds, there exists a U^+*-optimum saturating set* X *of* $B_{vw}(U^+)$ *such that* $X_{\mathrm{Aut}} = \Phi \cong C(\frac{c}{\delta})$*, the cyclic group of order* $\frac{c}{\delta}$*.*
(b) if neither case holds, there exists $\pi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$ *and a* U^+*-optimum saturating set* X *of* $B_{vw}(U^+)$ *such that* $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi \cong D(\frac{2c}{\delta})$*, the dihedral group of order* $\frac{2c}{\delta}$*.*

Proof. Since $\mathrm{Aut}(U^+)$ contains a non-trivial rotation, we may define Φ as in Definition 5.6. Now, by parts (a) and (b) of Lemma 5.7, $\lambda(x_0) \in \Phi(x_0)$ and $\lambda^{-1}(x_\nu) \in \Phi(x_\nu)$ for all $\lambda \in \mathrm{Rot}(\mathrm{Aut}(U^+))$. Also, since $1_{V(U^+)} \in \Phi$ and $|\Phi(x_\nu)| = |\Phi(w)| = |\Phi|$, clearly Φ satisfies properties (a) and (b) of Definition 2.8.

(a) It follows from Theorem 2.11 that there exists a maximum saturating set X of $B_{vw}(U^+)$ such that $\Phi \subseteq X_{\mathrm{Aut}}$. Let Y be any maximum saturating set of $B_{vw}(U^+)$. Suppose that case (i) holds. Then, since it follows from Corollary 5.1 and (i) that $\lambda^{-1}(x_\nu) \in \Phi(x_\nu)$ for all $\lambda \in \mathrm{Ref}(\mathrm{Aut}(U^+))$, we see that $\lambda^{-1}(x_\nu) \in \Phi(x_\nu)$ for all $\lambda \in \mathrm{Aut}(U^+)$. By Definition 2.8(b), the vertices $\lambda^{-1}(x_\nu)$ are distinct for each $\lambda \in Y$. So $|Y_{\mathrm{Aut}}| \leq |\Phi(x_\nu)| = |\Phi|$, and therefore $X_{\mathrm{Aut}} = \Phi$ and thus X is U^+-optimum.

Suppose instead that case (ii) holds. Then $\lambda(x_0) \in \Phi(x_0)$ for all $\lambda \in \mathrm{Aut}(U^+)$, and hence $\lambda(w) \in \Phi(w)$ for all $\lambda \in \mathrm{Aut}(U^+)$ as $d_{U^+}(x_0) = 3$. By Definition 2.8(b), the vertices $\lambda(w)$ are distinct for each $\lambda \in Y$. So $|Y_{\mathrm{Aut}}| \leq |\Phi(w)| = |\Phi|$, and therefore $X_{\mathrm{Aut}} = \Phi$ and thus X is U^+-optimum.

(b) We first show that there exists $\pi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$ such that $\Phi(x_\nu) \cap \Phi\pi(x_\nu) = \emptyset$ and $\Phi(w) \cap \Phi\pi(w) = \emptyset$.

Now, since (i) does not hold, we may choose $\sigma \in \mathrm{Ref}(\mathrm{Aut}(U^+))$ such that $\sigma(x_\nu) \notin \Phi(x_\nu)$. If $\sigma(x_0) \notin \Phi(x_0)$ then $\sigma(w) \notin \Phi(w)$, and we simply set $\pi = \sigma$. So suppose that $\sigma(x_0) \in \Phi(x_0)$ and let $x_\alpha = \sigma(x_0)$. By Lemma 5.7(d), $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_0) \subseteq \Phi(x_0)$, and hence $d_{U^+}(x_0) \geq 4$ since case (ii) does not hold. So $d_{U^+}(x_\alpha) \geq 4$ as $\sigma \in \mathrm{Aut}(U^+)$. We therefore define $\pi \in \mathbf{S}_{V(U^+)}$ such that $\pi(w) = x_\alpha^2$, $\pi(\sigma^{-1}(x_\alpha^2)) = \sigma(w)$, and $\pi(u) = \sigma(u)$ for all other vertices u of U^+. Clearly, $\pi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$. Furthermore, $\Phi(x_\nu) \cap \Phi\pi(x_\nu) = \emptyset$ as $\pi(x_\nu) = \sigma(x_\nu)$, and $\Phi(w) \cap \Phi\pi(w) = \emptyset$ by construction.

By Lemma 4.1(b), $\lambda^{-1}(x_\nu) = \lambda(x_\nu)$ for all $\lambda \in \mathrm{Ref}(\mathrm{Aut}(U^+))$. So $\{\lambda^{-1}(x_\nu) \mid \lambda \in \Phi\pi\} = \Phi\pi(x_\nu)$ as $\Phi\pi \subseteq \mathrm{Ref}(\mathrm{Aut}(U^+))$. It is now easy to show that the set of permutations $\Phi \cup \Phi\pi$ satisfies property (b) of Definition 2.8. As $1_{V(U^+)} \in \Phi$, it then follows from Theorem 2.11 that there exists a maximum saturating set X of $B_{vw}(U^+)$ such that $\Phi \subseteq X_{\mathrm{Rot}}$ and $\Phi\pi \subseteq X_{\mathrm{Ref}}$.

Let Y be any maximum saturating set of $B_{vw}(U^+)$. If $\lambda \in Y_{\mathrm{Rot}}$ then $\lambda^{-1}(x_\nu) \in \Phi(x_\nu)$ and if $\lambda \in Y_{\mathrm{Ref}}$ then $\lambda^{-1}(x_\nu) \in \Phi\pi(x_\nu)$. By Definition 2.8(b), the vertices $\lambda^{-1}(x_\nu)$ are distinct for each $\lambda \in Y$. Therefore $|Y_{\mathrm{Rot}}| \leq |\Phi|$ and $|Y_{\mathrm{Ref}}| \leq |\Phi\pi|$. Hence $X_{\mathrm{Rot}} = \Phi$, $X_{\mathrm{Ref}} = \Phi\pi$ and thus X is U^+-optimum. Clearly, $\Phi \cup \Phi\pi$ is isomorphic to the dihedral group of order $\frac{2c}{\delta}$, as Φ is isormorphic to the cyclic group of order $\frac{c}{\delta}$. □

The following lemma concerning the orbits of X_{Aut} follows from Definition 2.8, and the fact that the set of inverses of the elements of X_{Aut} is just X_{Aut}, since X_{Aut} is a group.

Lemma 5.9. *Let X be the U^+-optimum saturating set from Theorem 5.8.*

(a) $X_{\mathrm{Aut}}(x_\nu) = \{\lambda^{-1}(x_\nu) \mid \lambda \in X_{\mathrm{Aut}}\}$ and $|X_{\mathrm{Aut}}(x_\nu)| = |X_{\mathrm{Aut}}|$. In addition, $d_{U^+}(x_i) = 2$ and $\tau_{U^+}(x_i) = \tau_{U^+}(x_\nu)$ for each vertex x_i in $X_{\mathrm{Aut}}(x_\nu)$.

(b) $|X_{\mathrm{Aut}}(w)| = |X_{\mathrm{Aut}}|$, and each vertex in $X_{\mathrm{Aut}}(w)$ is a leaf adjacent to a vertex of degree $d_{U^+}(x_0)$.

(c) If $\lambda \in X \setminus X_{\mathrm{Aut}}$ *then* $\lambda^{-1}(x_\nu) \notin X_{\mathrm{Aut}}(x_\nu)$ *and* $\lambda(w) \notin X_{\mathrm{Aut}}(w)$*.*

Lemma 5.10. *Let* X *be the* U^+*-optimum saturating set from Theorem 5.8 and let* $B = \{\lambda(w) \mid \lambda \in X \setminus X_{\mathrm{Aut}}$ *and* $d_{U^+}(\lambda(w)) = 1\}$*. Let* A *be any subset of* B *such that* $\Phi(a_1) \cap \Phi(a_2) = \emptyset$ *for all distinct* a_1 *and* a_2 *in* A*. Then*

(a) if $X_{\mathrm{Aut}} = \Phi$ *then* $|X_{\mathrm{Aut}}| \leq \left\lfloor \frac{n+1}{\delta+1+|A|} \right\rfloor$*;*

(b) if $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi$ *then* $|X_{\mathrm{Aut}}| \leq 2 \left\lfloor \frac{n+1}{\delta+2+|A|} \right\rfloor$*.*

When $|A| = 0$*, equality in either case (a) or case (b) holds if and only if* $d_1(U^+) = |X_{\mathrm{Aut}}|$*.*

Proof. Suppose there exists $a_1 \in A$. By Lemma 5.9(c), $a_1 \notin X_{\mathrm{Aut}}(w)$. So $a_1 \notin \Phi(w)$, and thus $\Phi(a_1) \cap \Phi(w) = \emptyset$. Similarly, $\Phi(a_1) \cap \Phi\pi(w) = \emptyset$ when $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi$. Therefore $\Phi(a_1) \cap X_{\mathrm{Aut}}(w) = \emptyset$. So, since $\Phi(a_1) \cap \Phi(a_2) = \emptyset$ for all $a_2 \in A \setminus \{a_1\}$, it follows that $d_1(U^+) \geq |A||\Phi| + |X_{\mathrm{Aut}}(w)|$. Hence $d_1(U^+) \geq |A||\Phi| + |X_{\mathrm{Aut}}|$ by Lemma 5.9(b). Together with equation (10), this yields $n+1 \geq |X_{\mathrm{Aut}}| + (|A|+\delta)|\Phi|$. Cases (a) and (b) immediately follow. Clearly, when $|A| = 0$, equality holds if and only if $d_1(U^+) = |X_{\mathrm{Aut}}|$. □

6. Upper bounds on $b(U, T)$

We now obtain upper bounds on $b(U, T)$ for the three possible values of $\tau_{U^+}(x_\nu)$, and the two possible cases for X_{Aut} from Theorem 5.8. We further show that, for each of the six cases, the maximum value of $b(U, T)$ is attained if and only if the structure of U^+ is as described below.

- (S0a) $n \equiv 2 \pmod 3$, $d_{U^+}(x_i) = 3$ when $i \equiv 0 \pmod 2$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 1 \pmod 2$.
- (S0b) $n \equiv 6 \pmod 7$, $d_{U^+}(x_i) = 4$ when $i \equiv 0 \pmod 4$, $d_{U^+}(x_i) = 3$ when $i \equiv 2 \pmod 4$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 1 \pmod 4$.
- (S1a) $n \equiv 3 \pmod 4$, $d_{U^+}(x_i) = 3$ when $i \equiv 0 \pmod 3$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 1 \pmod 3$.
- (S1b) $n \equiv 4 \pmod 5$, $d_{U^+}(x_i) = 4$ when $i \equiv 0 \pmod 3$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 1 \pmod 3$.
- (S2a) $n \equiv 4 \pmod 5$, $d_{U^+}(x_i) = 3$ when $i \equiv 0 \pmod 4$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 2 \pmod 4$.
- (S2b) $n \equiv 6 \pmod 7$, $d_{U^+}(x_i) = 4$ when $i \equiv 0 \pmod 5$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, either $\nu \equiv 2 \pmod 5$ or $\nu \equiv 3 \pmod 5$.

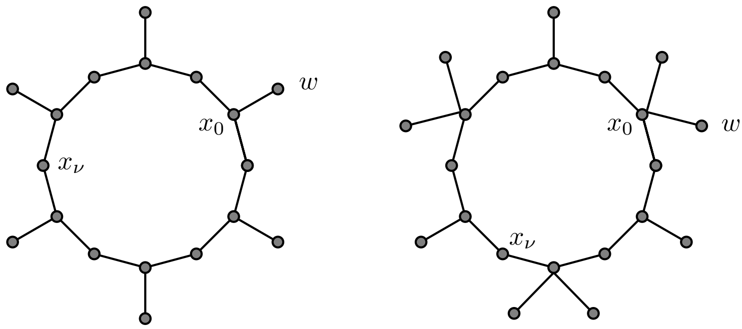

Figure 1: The supercard U^+ with structure (S0a) when $n = 17$ and structure (S0b) when $n = 20$.

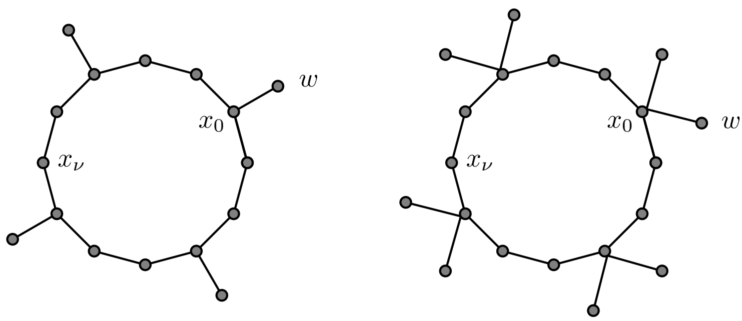

Figure 2: The supercard U^+ with structure (S1a) when $n = 15$ and structure (S1b) when $n = 19$.

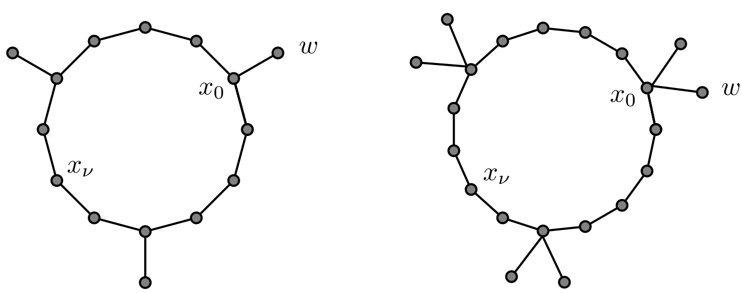

Figure 3: The supercard U^+ with structure (S2a) when $n = 14$ and structure (S2b) when $n = 20$.

We note that, $\tau_{U^+}(x_\nu) = 0$ when U^+ has structure (S0a) or (S0b); $\tau_{U^+}(x_\nu) = 1$ when U^+ has structure (S1a) or (S1b); $\tau_{U^+}(x_\nu) = 2$ when U^+ has structure (S2a) or (S2b). We will see that for structures (S0a), (S1a) and (S2a), the corresponding X_{Aut} is Φ, and for structures (S0b), (S1b) and (S2b), the corresponding X_{Aut} is $\Phi \cup \Phi\pi$.

Examples of the structures of the six possible supercards are shown in Figures 1, 2 and 3, where we assume that the indices of the vertices x_i increase in the clockwise direction.

We first prove that if U^+ has any of these structures then $B^U_{vw}(U^+) = \text{Aut}(U^+)$.

Lemma 6.1. *Suppose that U^+ has any of the six structures* (S0a) *through to* (S2b)*. Then there exists a non-trivial rotation in* $\text{Aut}(U^+)$ *and* $B^U_{vw}(U^+) = \text{Aut}(U^+)$*.*

Proof. It is easy to see that if U^+ has any of the six structures, then there exists a non-trivial rotation in $\text{Aut}(U^+)$. So $B^U_{vw}(U^+) = \text{Aut}(U^+)$ by Lemma 4.13, in each case. □

We now obtain the value of $b(U, T)$ in each of the six cases.

Lemma 6.2. *Let $n \geq 14$. Suppose that U^+ has any of the six structures* (S0a) *through to* (S2b)*.*

- *(a) If U^+ has structure* (S0a) *then* $b(U, T) = \frac{n+1}{3} + 2$*.*
- *(b) If U^+ has structure* (S0b) *then* $b(U, T) = \frac{2(n+1)}{7} + 1$*.*
- *(c) If U^+ has structure* (S1a) *then* $b(U, T) = \frac{n+1}{4} + 1$*.*
- *(d) If U^+ has structure* (S1b) *then* $b(U, T) = \frac{2(n+1)}{5}$*.*
- *(e) If U^+ has structure* (S2a) *then* $b(U, T) = \frac{n+1}{5}$*.*
- *(f) If U^+ has structure* (S2b) *then* $b(U, T) = \frac{2(n+1)}{7}$*.*

Proof. In each of the six cases, there exists a non-trivial rotation in $\text{Aut}(U^+)$ by Lemma 6.1, so we may apply the results from Section 5. In particular, we therefore define Φ and δ as in Definition 5.6.

Suppose that U^+ has structure (S0a), (S1a) or (S2a). Then clearly, case (ii) of Theorem 5.8 holds. So there exists a U^+-optimum saturating set X such that $X_{\text{Aut}} = \Phi$.

Suppose instead that U^+ has structure (S0b), (S1b) or (S2b). Then, it is easy to see that there exists $\lambda \in \text{Ref}(\text{Aut}(U^+))$ such that $\lambda(x_0) = x_0$, and to verify that $\lambda(x_\nu)$, i.e., $x_{c-\nu}$, is not in $\Phi(x_\nu)$. So case (i) of Theorem 5.8 cannot hold. Moreover, case (ii) of Theorem 5.8 cannot hold as $d_{U^+}(x_0) \geq 4$.

Therefore, there exists $\pi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$ and a U^+-optimum saturating set X such that $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi$.

(a) Since $d_1(U^+) = |\Phi|$, it follows from equation (10) that $|X_{\mathrm{Aut}}| = |\Phi| = \frac{n+1}{3}$. It is easy to see that $U - x_2 \cong U - x_{c-2} \cong T - x_{\nu+1} \cong T - x_{\nu-1}$, and to check using Lemma 3.12 that $\lambda(w) \in \{x_{\nu-1}, x_{\nu+1}\}$ for any $\lambda \in X \setminus X_{\mathrm{Aut}}$. Therefore $|X \setminus X_{\mathrm{Aut}}| = 2$, and $b(U, T) = |X| = \frac{n+1}{3} + 2$.

(b) Since $d_1(U^+) = 3|\Phi|$, it follows from equation (10) that $|X_{\mathrm{Aut}}| = 2|\Phi| = \frac{2(n+1)}{7}$. It is easy to see that $U - x_2 \cong U - x_{c-2} \cong T - x_{\nu+1}$, and to check using Lemma 3.12 that $\lambda(w) = x_{\nu+1}$ for any $\lambda \in X \setminus X_{\mathrm{Aut}}$. Therefore $|X \setminus X_{\mathrm{Aut}}| = 1$, and $b(U, T) = |X| = \frac{2(n+1)}{7} + 1$.

(c) Since $d_1(U^+) = |\Phi|$, it follows from equation (10) that $|X_{\mathrm{Aut}}| = |\Phi| = \frac{n+1}{4}$. It is easy to see that $U - x_2 \cong U - x_{c-2} \cong T - x_{\nu+1}$, and to check that to check using Lemma 3.12 that $\lambda(w) = x_{\nu+1}$ for any $\lambda \in X \setminus X_{\mathrm{Aut}}$. Therefore $|X \setminus X_{\mathrm{Aut}}| = 1$, and $b(U, T) = |X| = \frac{n+1}{4} + 1$.

For (d), (e), and (f), it is easy to check using Lemma 3.12 that $\lambda(w) \in V(\mathrm{skel}(U^+))$ for any $\lambda \in X \setminus X_{\mathrm{Aut}}$. Furthermore, it is easy to see that $U - x_i \ncong T - x_j$, for all i, j, and hence $X = X_{\mathrm{Aut}}$ in each case.

(d) Since $d_1(U^+) = 2|\Phi|$, it follows from equation (10) that $|X_{\mathrm{Aut}}| = 2|\Phi| = \frac{2(n+1)}{5}$. Therefore $b(U, T) = |X| = \frac{2(n+1)}{5}$.

(e) Since $d_1(U^+) = |\Phi|$, it follows from equation (10) that $|X_{\mathrm{Aut}}| = |\Phi| = \frac{n+1}{5}$. Therefore $b(U, T) = |X| = \frac{n+1}{5}$.

(f) Since $d_1(U^+) = 2|\Phi|$, it follows from equation (10) that $|X_{\mathrm{Aut}}| = 2|\Phi| = \frac{2(n+1)}{7}$. Therefore $b(U, T) = |X| = \frac{2(n+1)}{7}$. ☐

For each of the following four lemmas, we assume that *there exists a non-trivial rotation in* $\mathrm{Aut}(U^+)$, *and we let* X *be the* U^+*-optimum saturating set of* $B_{vw}(U^+)$ *from Theorem 5.8*. We recall that $\widetilde{X}$ is the subset of X containing those permuations λ such that $\lambda(w)$ is a leaf of U^+ and a d-leaf of T. We note that $X_U = X_{\mathrm{Aut}}$ by Lemma 4.13.

Lemma 6.3. *Let* $n \geq 56$ *and suppose that* $\tau_{U^+}(x_\nu) = 0$.

(a) If $X_{\mathrm{Aut}} = \Phi$ *then* $b(U, T) = |X| \leq \frac{n+1}{3} + 2$. *Furthermore, equality holds if and only if* U^+ *has structure* (S0a).

(b) If $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi$ *then* $b(U, T) = |X| \leq \frac{2(n+1)}{7} + 1$. *Furthermore, equality holds if and only if* U^+ *has structure* (S0b).

Proof. It follows from Corollary 3.18 that $\widetilde{X} \subseteq X_{\mathrm{Aut}}$, so $X \setminus X_{\mathrm{Aut}} \subseteq X \setminus \widetilde{X}$. Therefore $|X \setminus X_{\mathrm{Aut}}| \leq 4$ by Corollary 3.13(a).

(a) By Lemma 5.10(a), $|X_{\mathrm{Aut}}| \leq \left\lfloor \frac{n+1}{\delta+1} \right\rfloor$. Simple calculations then show that the bound holds for $\delta \geq 3$ with strict inequality.

So suppose that $\delta = 2$. If $d_{U^+}(x_0) \geq 4$, then $d_1(U^+) \geq 2|\Phi|$, so $|X_{\mathrm{Aut}}| = |\Phi| \leq \left\lfloor \frac{n+1}{4} \right\rfloor$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, if $d_{U^+}(x_0) = 3$ then U^+ has structure (S0a), so $b(U, T) = \frac{n+1}{3} + 2$ by Lemma 6.2(a). Therefore, the bound holds in all cases, and is attained if and only if U^+ has structure (S0a).

(b) By Lemma 5.9(a), U^+ contains $2|\Phi|$ vertices x_i with $\tau_{U^+}(x_i) = 0$. As $\tau_{U^+}(x_\nu) = 0$, this implies that $\delta \geq 4$. In addition, it is easy to see by inspection that $d_1(U^+) \geq 3|\Phi|$ when $\delta \geq 5$.

Suppose that $\delta \geq 5$. Then $|X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{\delta+3} \right\rfloor$ by (10). Simple calculations then show that the bound holds with strict inequality when $\delta \geq 6$ or $|X \setminus X_{\mathrm{Aut}}| = 3$. So suppose that $\delta = 5$ and $|X \setminus X_{\mathrm{Aut}}| = 4$. In this case, it is easy to see from Lemma 3.12(a) that $\{x^1_{\nu-1}, x^1_{\nu+1}\}$ is the set B of leaves defined in Lemma 5.10. Moreover, since $\delta > 2$, we may clearly put $A = B$ in the lemma. Hence $|X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{9} \right\rfloor$ by Lemma 5.10(b), and again simple calculations show that the bound holds with strict inequality.

So suppose that $\delta = 4$. Then $d_{U^+}(x_{\nu-1}) > d_{U^+}(x_{\nu+1}) \geq 3$, and $d_{U^+}(x_{\nu+2}) = 2$. It immediately follows from Lemma 3.12(a)(i) that $|X \setminus X_{\mathrm{Aut}}| \leq 3$ as $d_{U^+}(x_{\nu-1}) \geq 4$. Now, when $d_{U^+}(x_{\nu-1}) \geq 5$, we have that $d_1(U^+) \geq 4|\Phi|$, so $|X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{8} \right\rfloor$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, when $d_{U^+}(x_{\nu-1}) = 4$, clearly U^+ has structure (S0b), so $b(U, T) = \frac{2(n+1)}{7} + 1$ by Lemma 6.2(b). Hence the bound holds in all cases, and is attained if and only if U^+ has structure (S0b). □

Lemma 6.4. *Let $n \geq 60$ and suppose that $\tau_{U^+}(x_\nu) = 2$.*

- *(a) If $X_{\mathrm{Aut}} = \Phi$ then $b(U, T) \leq \frac{n+1}{5}$. Furthermore, equality holds in the bound if and only if U^+ has structure* (S2a).
- *(b) If $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi$ then $b(U, T) \leq \frac{2(n+1)}{7}$. Furthermore, equality holds in the bound if and only if U^+ has structure* (S2b).

Proof. It follows from Corollary 3.18 that $\widetilde{X} \subseteq X_{\mathrm{Aut}}$, so $X \setminus X_{\mathrm{Aut}} \subseteq X \setminus \widetilde{X}$. Therefore $|X \setminus X_{\mathrm{Aut}}| \leq 2$ by Corollary 3.13(c).

(a) By Lemma 5.10(a), $|X_{\mathrm{Aut}}| \leq \left\lfloor \frac{n+1}{\delta+1} \right\rfloor$. Simple calculations then show that the bound holds for $\delta \geq 5$ with strict inequality.

So suppose that $\delta \leq 4$. By Lemma 5.9(a), U^+ contains $|\Phi|$ vertices x_i with $\tau_{U^+}(x_i) = 2$. This implies that $\delta = 4$, and that every cut-vertex of U^+ is in $\Phi(x_0)$. Now, if $d_{U^+}(x_0) \geq 4$ then $d_1(U^+) \geq 2|\Phi|$, so $|X_{\mathrm{Aut}}| \leq \left\lfloor \frac{n+1}{6} \right\rfloor$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, if $d_{U^+}(x_0) = 3$ then U^+ has structure (S2a), so $b(U, T) = \frac{n+1}{5}$ by Lemma 6.2(e). Therefore, the bound holds in all cases, and is attained if and only if U^+ has structure (S2a).

(b) By Lemma 5.10(b), $|X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{\delta+2} \right\rfloor$. Simple calculations then show that the bound holds for $\delta \geq 6$ with strict inequality.

So suppose that $\delta \leq 5$. By Lemma 5.9(a), U^+ contains $2|\Phi|$ vertices x_i with $\tau_{U^+}(x_i) = 2$. This implies that $\delta = 5$, and that every cut-vertex of U^+ is in $\Phi(x_0)$. Since U^+ contains at least $2|\Phi|$ leaves by Lemma 5.9(b), clearly $d_{U^+}(x_0) \geq 4$. Now, if $d_{U^+}(x_0) \geq 5$ then $d_1(U^+) \geq 3|\Phi|$, so $|X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{8} \right\rfloor$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, if $d_{U^+}(x_0) = 4$ then U^+ has structure (S2b), so $b(U, T) = \frac{2(n+1)}{7}$ by Lemma 6.2(f). Therefore, the bound holds in all cases, and is attained if and only if U^+ has structure (S2b). $\square$

Since $\widetilde{X}$ may not be contained in X_{Aut} when $\tau_{U^+}(x_\nu) = 1$, we need an auxillary result in this case.

Lemma 6.5. *Suppose that $\lambda \in \widetilde{X} \setminus X_{\mathrm{Aut}}$ and that $\delta \leq \frac{c-3}{2}$. Then $\lambda(w)$ is adjacent to $x_{\nu+2}$ and $\lambda^{-1}(x_\nu) \in \{x_2, x_{c-2}\}$.*

Proof. Let $x_\mu = \lambda^{-1}(x_\nu)$. By Corollary 3.18, $\tau_{U^+}(x_\nu) = 1$ and $\{\lambda(x_{\mu+2}), \lambda(x_{\mu-2})\} = \{x_{\nu+3}, x_{\nu-1}\}$ as $\lambda \in \widetilde{X} \setminus X_{\mathrm{Aut}}$. Moreover, by Lemma 3.10, either

(a) $\mathrm{skel}(U - x_\mu)$ is $x_{\mu+1}x_{\mu+2}\ldots x_{\mu-2}$ and $\lambda(x_i) = x_{(\nu-\mu+1)+i}$ for all x_i in $V(\mathrm{skel}(U - x_\mu))$, or

(b) $\mathrm{skel}(U - x_\mu)$ is $x_{\mu+2}x_{\mu+3}\ldots x_{\mu-1}$ and $\lambda(x_i) = x_{(\nu+\mu+1)-i}$ for all x_i in $V(\mathrm{skel}(U - x_\mu))$.

We recall that $d_{U^+}(x_i) = d_{U^+}(x_{i+\delta}) = d_{U^+}(x_{i-\delta})$ for all x_i, as $\phi(x_i) = x_{i+\delta}$.

Case (a): We first note that $\{x_{\mu+1+\delta}, x_{\mu+1+2\delta}, x_{\mu-2-\delta}\} \subseteq V(\mathrm{skel}(U - x_\mu))$ as $\delta \leq \frac{c-3}{2}$. Suppose that $\lambda(w)$ is not adjacent to $x_{\nu+2}$. Then

$$\begin{aligned} d_{U^+}(x_{\mu+1}) > d_{U-x_\mu}(x_{\mu+1}) &= d_{T-\lambda(w)}(\lambda(x_{\mu+1})) \\ &= d_{T-\lambda(w)}(x_{\nu+2}) = d_{U^+}(x_{\nu+2}). \end{aligned}$$

So $d_{U^+}(x_{\mu+1+\delta}) > d_{U^+}(x_{\nu+2+\delta})$ and $d_{U^+}(x_{\mu+1+2\delta}) > d_{U^+}(x_{\nu+2+2\delta})$. However, since $\lambda(x_{\mu+1+\delta}) = x_{\nu+2+\delta}$ and $\lambda(x_{\mu+1+2\delta}) = x_{\nu+2+2\delta}$, we see that $d_{U-x_\mu}(x_{\mu+1+\delta}) = d_{T-\lambda(w)}(x_{\nu+2+\delta})$ and $d_{U-x_\mu}(x_{\mu+1+2\delta}) = d_{T-\lambda(w)}(x_{\nu+2+2\delta})$. As $\delta \leq \frac{c-3}{2}$, it follows that x_μ is not adjacent to $x_{\mu+1+\delta}$ or $x_{\mu+1+2\delta}$, so w must be adjacent to both of these vertices. This is impossible since w is a leaf. Therefore $\lambda(w)$ is adjacent to $x_{\nu+2}$.

Suppose now that x_μ is not x_2. Then w is not adjacent to $x_{\mu-2}$, and thus

$$d_{U^+}(x_{\nu-1}) > d_{T-\lambda(w)}(x_{\nu-1}) = d_{U-x_\mu}(\lambda^{-1}(x_{\nu-1})) \\ = d_{U-x_\mu}(x_{\mu-2}) = d_{U^+}(x_{\mu-2}).$$

So $d_{U^+}(x_{\nu-1-\delta}) > d_{U^+}(x_{\mu-2-\delta})$. However, since $\lambda(x_{\mu-2-\delta}) = x_{\nu-1-\delta}$, it follows that $d_{U-x_\mu}(x_{\mu-2-\delta}) = d_{T-\lambda(w)}(x_{\nu-1-\delta})$, which is impossible as neither $\lambda(w)$ nor x_ν are adjacent to $x_{\nu-1-\delta}$. Therefore x_μ is x_2.

Case (b) can be proved in a similar manner by replacing $x_{\mu+k}$ by $x_{\mu-k}$ for each k, and vice versa, and also substituting x_{c-2} for x_2. □

Lemma 6.6. *Let $n \geq 48$ and suppose that $\tau_{U^+}(x_\nu) = 1$.*

- *(a) If $X_{\mathrm{Aut}} = \Phi$ then $b(U, T) \leq \frac{n+1}{4} + 1$. Furthermore, equality holds if and only if U^+ has structure* (S1a)*.*
- *(b) If $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi$ then $b(U, T) \leq \frac{2(n+1)}{5}$. Furthermore, equality holds if and only if and U^+ has structure* (S1b)*.*

Proof. As $\tau_{U^+}(x_\nu) = 1$, we see that $\delta \geq 3$. When $\delta = 3$, it is easy to see that there exists a $\psi \in \mathrm{Ref}(\mathrm{Aut}(U^+))$ such that $\psi(x_\nu) = x_{\nu+1} \notin \Phi(x_\nu)$ and $\psi(x_0) \in \Phi(x_0)$. Thus $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_\nu) \not\subseteq \Phi(x_\nu)$, whereas $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_0) \subseteq \Phi(x_0)$ by Lemma 5.7(d).

Suppose that $\delta \geq \frac{c-2}{2}$, so $|\Phi| = \frac{c}{\delta} \leq 2$. Then, since $|X_{\mathrm{Aut}}| = \chi(U^+)$, it follows from Theorem 3.23 that $b(U, T) \leq 9$ when $X_{\mathrm{Aut}} = \Phi$, and $b(U, T) \leq 13$ when $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi$. Simple calculations show that both bounds hold with strict inequality.

We therefore assume that $\delta \leq \frac{c-3}{2}$. Now, if $\lambda \in \widetilde{X} \setminus X_{\mathrm{Aut}}$ then $\lambda^{-1}(x_\nu) \in \{x_2, x_{c-2}\}$ by Lemma 6.5; hence $|\widetilde{X} \setminus X_{\mathrm{Aut}}| \leq 2$ by Definition 2.8(b). Therefore, since $|X \setminus \widetilde{X}| \leq 4$ by Corollary 3.13(b), clearly $|X \setminus X_{\mathrm{Aut}}| \leq 6$.

Let B be the set of leaves defined in Lemma 5.10. Then, by Lemma 3.12(b) and Lemma 6.5, $B \subseteq \{x^1_{\nu-1}, x^j_{\nu+2}, x^k_{\nu+2}\}$, for some j, k. On the other hand, if $\lambda \in X \setminus X_{\mathrm{Aut}}$ but $\lambda(w) \notin B$ then $\lambda(w) \in \{x_{\nu-1}, x_{\nu+1}, x_{\nu+2}\}$, by Lemma 3.12(b). It follows that if $|X \setminus X_{\mathrm{Aut}}| \geq 4$, then there exists a subset A of B with $|A| \geq 1$ that satisfies the conditions of Lemma 5.10. In addition, if $\delta \geq 4$ and $|X \setminus X_{\mathrm{Aut}}| \geq 5$, it is easy to see that there is some subset A of B with $|A| \geq 2$ that satisfies the conditions of Lemma 5.10.

(a) Suppose that $\delta \geq 4$. Then, by Lemma 5.10(a), $|X_{\mathrm{Aut}}| \leq \lfloor \frac{n+1}{5} \rfloor$ when $|X \setminus X_{\mathrm{Aut}}| \leq 3$, $|X_{\mathrm{Aut}}| \leq \lfloor \frac{n+1}{6} \rfloor$ when $|X \setminus X_{\mathrm{Aut}}| = 4$, and $|X_{\mathrm{Aut}}| \leq \lfloor \frac{n+1}{7} \rfloor$ otherwise. Simple calculations then show that the bound holds with strict inequality.

So, suppose that $\delta = 3$. Then, since $X_{\mathrm{Aut}} = \Phi$ and $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_\nu) \not\subseteq \Phi(x_\nu)$, it follows from Theorem 5.8 that $d_{U^+}(x_0) = 3$. So U^+ has structure (S1a) and $b(U, T) = \frac{n+1}{4} + 1$ by Lemma 6.2(c). Therefore, the bound holds in all cases, and is attained if and only if U^+ has structure (S1a).

(b) Suppose that $\delta \geq 4$. Then, by Lemma 5.10(b), $|X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{6} \right\rfloor$ when $|X \setminus X_{\mathrm{Aut}}| \leq 3$, $|X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{7} \right\rfloor$ when $|X \setminus X_{\mathrm{Aut}}| = 4$, and $|X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{8} \right\rfloor$ otherwise. Simple calculations then show that the bound holds with strict inequality.

So, suppose that $\delta = 3$. Then, since $X_{\mathrm{Aut}} = \Phi \cup \Phi\pi$, $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_\nu) \not\subseteq \Phi(x_\nu)$ and $\mathrm{Ref}(\mathrm{Aut}(U^+))(x_0) \subseteq \Phi(x_0)$, it follows from Theorem 5.8 that $d_{U^+}(x_0) \geq 4$. Now, by Lemma 5.9(a), U^+ contains $|X_{\mathrm{Aut}}|$ vertices x_i with $d_{U^+}(x_i) = 2$. It then follows from Lemma 5.9(c) that if $\lambda \in X \setminus X_{\mathrm{Aut}}$ then $\lambda^{-1}(x_\nu)$ must be a cut-vertex of U^+. It is easy to see by inspection that there can be no such λ, and therefore $X = X_{\mathrm{Aut}}$. Now, if $d_{U^+}(x_0) \geq 5$, then $d_1(U^+) \geq 3|\Phi|$, so $|X| = |X_{\mathrm{Aut}}| \leq 2\left\lfloor \frac{n+1}{6} \right\rfloor$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, when $d_{U^+}(x_0) = 4$, it follows that U^+ has structure (S1b), and $b(U, T) = \frac{2(n+1)}{5}$ by Lemma 6.2(d). Therefore, the bound holds in all cases, and is attained if and only if U^+ has structure (S1b). □

By combining the above results with Theorem 4.22, we finally obtain a bound on $b(U, T)$ which holds in all cases.

Theorem 6.7. *Let U be a sunshine graph and T be a caterpillar of order n, where $n \geq 60$. Suppose there exists a sunshine graph U^+ that is a supercard of U and T such that $\mathrm{Aut}(U^+)$ contains a non-trivial rotation. Then $b(U, T) \leq \frac{2(n+1)}{5}$, with equality if and only if U^+ has structure* (S1b)*, in which case $n \equiv 4 \pmod 5$. Moreover, in all other cases, $b(U, T) \leq \frac{n+1}{3} + 3$.*

Proof. This follows immediately from Lemmas 6.1 to 6.6, and the proof of Lemma 6.6(b). □

We note that, with more work, we can show that this bound holds for smaller values of n (this is relatively straightforward for $n \geq 35$). However, since the proofs are slightly technical, in the interests of brevity, we have not included them in this paper.

Theorem 6.8. *Let U be a sunshine graph and T be a caterpillar, where $n \geq 62$. Then $b(U, T) \leq \frac{2(n+1)}{5}$, with equality if and only if there is a supercard of U and T that has structure* (S1b)*, in which case $n \equiv 4 \pmod 5$. Moreover, in all other cases, $b(U, T) \leq \frac{n+1}{3} + 3$ when $n \geq 65$.*

Proof. We may clearly assume that $b(U, T) \geq 10$. By Lemma 3.3, there exists a supercard U^+ of U and T that is a sunshine graph, and we may choose U^+ to have the largest possible value of $\chi(U^+)$ over all supercards of U and T that are sunshine graphs. If $B_U(U^+) = \mathrm{Aut}(U^+)$ then $\mathrm{Aut}(U^+)$ contains a non-trivial rotation by Corollary 4.14(c), so the results follow from Theorem 6.7. When $B_U(U^+) \neq \mathrm{Aut}(U^+)$, they follow from Theorem 4.22 by straightforward calculation. □

References

[1] J. Baldwin, *Graph reconstruction numbers.* MSc Dissertation, Rochester Institute of Technology (2004).

[2] J.A. Bondy, *Some uniqueness theorems in graph theory.* DPhil thesis, Oxford University (1968).

[3] J.A. Bondy, A graph reconstructor's manual. *Surveys in Combinatorics, London Mathematics Society Lecture Notes Series* **166** (1991), 221–252. MR1161466

[4] J.A. Bondy and R.L. Hemminger, Graph reconstruction – a survey. *Journal of Graph Theory* **1** (1977), 227–268. MR0480189

[5] J.A. Bondy and U.S.R. Murty, Graph Theory. *Graduate Texts in Mathematics*. Springer, Berlin (2008). MR2368647

[6] A. Bowler, P. Brown and T. Fenner, Families of pairs of graphs with a large number of common cards. *Journal of Graph Theory* **63** (2010), 146–163. MR2588730

[7] A. Bowler, P. Brown, T. Fenner and W. Myrvold, Recognising connectedness from vertex-deleted subgraphs. *Journal of Graph Theory* **67** (2011), 285–299. MR2839367

[8] P. Brown and T. Fenner, A new approach to graph reconstruction using supercards. *Journal of Combinatorics* **9** (2018), 95–118. MR3749186

[9] P. Brown, *On the maximum number of common cards between various classes of graphs.* PhD Thesis, Birkbeck, University of London (2008).

[10] M. Francalanza, *Adversary-reconstruction of trees: the case of caterpillars and sunshine graphs.* MSc thesis, University of Malta (1999).

[11] R.L. Greenwell and R.L. Hemminger, Reconstructing graphs. *The Many Facets of Graph Theory, Lecture Notes in Mathematics* **110** (1968), 91–114. MR0252258

[12] W. Ledermann, A.J. Weir, *Introduction to Group Theory (Second Edition).* Addison Wesley Longman, Harlow (1996).

[13] P.J. Kelly, *On isometric transformations.* PhD Thesis, University of Wisconsin (1942). MR2937555

[14] P.J. Kelly, A congruence theorem for trees. *Pacific Journal of Maths* **7** (1957), 961–968. MR0087949

[15] J. Lauri, The reconstruction of maximal planar graphs. *Journal of Combinatorial Theory (Ser. B)* **30** (1981), 196–214. MR0615314

[16] J. Lauri and R. Scapellato, *Topics in Graph Automorphisms and Reconstruction.* LMS Student Texts **54**, Cambridge University Press (2003). MR1971819

[17] B. Manvel, Reconstruction of unicyclic graphs, in *Proof Techniques in Graph Theory.* Academic Press, New York (1969). MR0256917

[18] B. Manvel, *On reconstruction of graphs.* PhD thesis, University of Michigan (1970). MR2619459

[19] B. McMullen, *Reconstruction numbers.* MSc thesis, Rochester Institute of Technology (2005).

[20] W. Myrvold, *Ally and adversary reconstruction problems.* PhD thesis, University of Waterloo (1988).

[21] D. Rivshin, *A computational investigation of graph reconstruction.* MSc thesis, Rochester Institute of Technology (2007). MR1858815

[22] S.M. Ulam, *A Collection of Mathematical Problems.* Wiley, New York (1960). MR0120127

Paul Brown
Department of Computer Science and Information Systems
Birkbeck, University of London
Malet Street
London, WC1E 7HX
United Kingdom
E-mail address: paulb@dcs.bbk.ac.uk

Trevor Fenner
Department of Computer Science and Information Systems
Birkbeck, University of London
Malet Street
London, WC1E 7HX
United Kingdom
E-mail address: trevor@dcs.bbk.ac.uk

Received January 24, 2019