On typical triangulations of a convex n-gon

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Let f_n be a function assigning weight to each possible triangle whose vertices are chosen from vertices of a convex polygon P_n of n sides. Suppose $\mathcal{T}_n$ is a random triangulation, sampled uniformly out of all possible triangulations of P_n. We study the sum of weights of triangles in $\mathcal{T}_n$ and give a general formula for average and variance of this random variable. In addition, we look at several interesting special cases of f_n in which we obtain explicit forms of generating functions for the sum of the weights. For example, among other things, we give new proofs for already known results such as the degree of a fixed vertex and the number of ears in $\mathcal{T}_n$, as well as, provide new results on the number of "blue" angles and refined information on the distribution of angles at a fixed vertex. We note that our approach is systematic and can be applied to many other new examples while generalizing the existing results.

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1. Introduction

We consider a convex polygon P_n with n vertices and label the vertices $V_n := \{v_{n,j}\}_{\{1 \leq j \leq n\}}$ in clockwise order. A triangulation is a set of $n-3$ non-crossing diagonals $v_{n,i}v_{n,j}$ with $1 \leq i \neq j \leq n$ which partitions P_n into $n-2$ triangles. Euler showed the number of possible triangulations for P_n is C_{n-2} where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. Triangulation has been extended to general point sets residing in various spaces and manifolds and also found many applications in computer science, computer graphics, and mathematics. We refer to [6, 10] and references within for a comprehensive review. The theme of this paper is with respect to the properties of a typical triangulation $\mathcal{T}_n$ of P_n. Studying $\mathcal{T}_n$ was initiated in a paper of Polyá [13] published in American Math Monthly in 1956. Among of large literature published on the subject, we refer to [1, 2, 5, 7, 8, 15] where, among other things, several aspects of $\mathcal{T}_n$ including the maximum degree of vertices, the longest diagonal, the number of ears, the number of triangles with a side

parallel to a fix side of P_n are studied. Our objective in this paper is to develop a somewhat systematic approach to address similar questions on $\mathcal{T}_n$. To that end, we first formalize the property of interest by defining a function that assigns weights to the triangles of each triangulation. Through a simple constructive algorithm that samples a uniform triangulation of P_n, we next derive a system of recursive equations for the generating functions corresponding to that function. We then leverage certain invariance properties of the function of interest to reduce the generating functions to solvable forms. By obtaining explicit information on these generating functions, we are finally able to describe the random triangulation with respect to the property of interest. To elaborate our approach, we give new proofs for already known results, and in addition, discuss a few new examples.

We start with stating a few notations. Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ refer to the set of all real and complex numbers. Let $P_{n,l,r}$ be the convex-hull of vertices $V_{n,l,r} := \{v_{n,j}\}_{\{l \leq j \leq r\}}$. With this notation, $P_n := P_{n,1,n}$ is the polygon of interest with n vertices and $P_{n,l,r}$ is a convex polygon with $m := r - l + 1$ sides. Let

$$T_{n,l,r} := \{T_{n,l,r,1}, \cdots, T_{n,l,r,C_{m-2}}\}$$

be the set of all triangulations of $P_{n,l,r}$. Suppose that we choose a triangulation $\mathcal{T}_n$ out of C_{n-2} triangulations in the set T_n (set $T_n := T_{n,1,n}$) with uniform probability $\mathbb{P} : T_n \to [0, 1]$. In the following, we use $\mathbb{E}$ and $\mathbb{V}\text{ar}$ to refer to the expectation and the variance with respect to $\mathbb{P}$. Let Γ_n be the set of all triangles whose vertices are in V_n. Define $f_n : \Gamma_n \to \mathbb{C}$ to be a function assigning weights to triangles in Γ_n. See Figure 1 for an example of $\mathcal{T}_n$ and how f_n assigns weights.
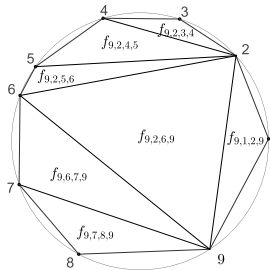

Figure 1: A random triangulation of an irregular P_9. The function f_9 assigns weights to all the triangles in $\mathcal{T}_9$.

Let $\mathcal{T}_{n,l,r}$ be a random triangulation drawn from $T_{n,l,r}$ with probability C_{r-l-1}^{-1} and

$$\mathcal{S}_{n,l,r} = \sum_{\Delta \in \mathcal{T}_{n,r,l}} f_n(\Delta),$$

to be the sum of weights of triangles in $\mathcal{T}_{n,l,r}$. We define the generating function of $\mathcal{S}_{n,l,r}$ as

$$g_{n,l,r}(z) = \mathbb{E}(z^{\mathcal{S}_{n,l,r}}) \quad \text{for} \quad z \in \mathbb{C}.$$

Clearly, $\mathcal{T}_n$ is $\mathcal{T}_{n,1,n}$. In the following, we set $\mathcal{S}_n := \mathcal{S}_{n,1,n}$, $g_n(z) := g_{n,1,n}(z)$, and $f_{n,l,j,r} := f_n(\Delta_{n,l,j,r})$ where we use $\Delta_{n,l,j,r}$ to refer to the triangle with three vertices $v_{n,l}, v_{n,j}, v_{n,r} \in V_n$. In our presentation, we always sort the indexes such that $l < j < r$.

Our first result gives the expectation $E(\mathcal{S}_n)$ and variance $\mathbb{V}\mathrm{ar}(\mathcal{S}_n)$ for a large class of functions f_n.

Theorem 1. *Suppose f_n is a function where $f_{n,l,j,r}$ depends only on $r - j$, $r - l$, $j - l$ and possibly n. For all $n \geq 2$,*

1. $\mathbb{E}(\mathcal{S}_{n,l,n}) = \frac{1}{C_{n-l-1}} \sum_{j=l}^{n-2} \beta_{n,j} \binom{2j-2l}{j-l}$, *where*

   (1) $$\beta_{n,j} = \sum_{s=j+1}^{n-1} f_{n,j,s,n} C_{s-j-1} C_{n-j-1}.$$

   When $l = 1$, (1) gives us $E(\mathcal{S}_n)$.

2. $\mathbb{V}\mathrm{ar}(\mathcal{S}_n) = \frac{1}{C_{n-2}} \sum_{j=1}^{n-2} \lambda_{n,j} \binom{2j-2}{j-1} - \frac{1}{C_{n-2}} \left(\sum_{j=1}^{n-2} \beta_{n,j} \binom{2j-2}{j-1} \right)^2$, *where $\beta_{n,j}$ is given by (1) and*

   (2)
   $$\lambda_{n,s} = \sum_{j=s+1}^{n-1} C_{j-s-1} C_{n-j-1} \bigg(f_{n,s,j,n}^2 + 2 f_{n,s,j,n}(E(\mathcal{S}_{n,s+n-j,n})$$
   $$+ E(\mathcal{S}_{n,j,n})) + 2 E(\mathcal{S}_{n,s+n-j,n}) E(\mathcal{S}_{n,j,n}) \bigg).$$

This general result can be applied to various interesting geometrical examples including the cases where f_n is the perimeter, the area, or the radius of the inscribed circle of the input triangle. In the first case, $\mathcal{S}_n$ is

related to the minimum-weight triangulation problem also known as optimal triangulation in computational geometry. Optimal triangulation is the problem of finding a triangulation of minimal total edge length where an input polygon must be subdivided into triangles that meet edge-to-edge and vertex-to-vertex, in such a way as to minimize the sum of the perimeters of the triangles [10, 19]. The two later cases are related to Japanese theorem [9] (See Chapter 4, p. 193), which indicates that if f_n is radius of inscribed circle of the input triangle, then $\mathcal{S}_n$ is constant. In addition, when n grows to infinity this sum approaches the diameter of circumscribed circle of the circular polygon P_n.

We remark that it is easy to show that $\mathcal{S}_n$ is a constant if and only if for all quadrilateral components $v_{n,l}v_{n,j}v_{n,i}v_{n,r}$ with $1 \leq l < j < i < r \leq n$ we have $f_{n,l,j,i} + f_{n,l,i,r} = f_{n,l,j,r} + f_{n,j,i,r}$. This follows by a repeated application of the rule, which "flips" one diagonal, will generate all the possible triangulations from any given triangulation, with each "flip" preserving the sum. See Figure 2, where the triangulation (Left) is flipped to (Right) by flipping $v_{9,2}v_{9,9}$ to $v_{9,1}v_{9,6}$.
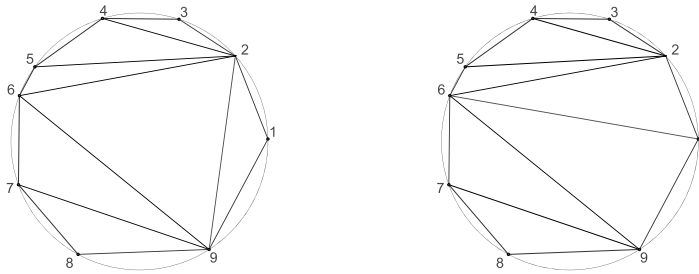

Figure 2: (Left) can be flipped to the (Right) by flipping $v_{9,2}v_{9,9}$ to $v_{9,1}v_{9,6}$.

We now present our examples. For these examples we will not apply Theorem 1. We instead show most of our results by deriving an explicit form for generating function $g_n(z)$. We remark, however, that application of Theorem 1, when appropriate, can provide a different expression for $\mathbb{E}$ and $\mathbb{V}\text{ar}$ which may result in new identities for Catalan numbers in particular. For the first two examples, the results hold true for all convex polygons. For the rest of examples, we assume, in addition, the polygon is regular.

1.0.0.1. Triangles with one side on P_n One would ask how many of the triangles in the random triangulation $\mathcal{T}_n$ have exactly one side in common

with perimeter of P_n. To answer this question we define $f_{n,l,j,r}$ as follows:

$$
(3) \qquad f_{n,l,j,r} = \begin{cases} 1 & \text{if } l > 1, \quad j = l + 1, \quad r > j + 1, \quad r \leq n \\ 1 & \text{if } l > 1, \quad j > l + 1, \quad r = j + 1, \quad r \leq n \\ 1 & \text{if } 2 < j < n - 1, \quad l = 1, \quad r = n \\ 1 & \text{if } l = 1, \quad j = 2, \quad 3 < r < n \\ 1 & \text{if } l = 1, \quad j > 2, \quad r = j + 1, \quad r < n \\ 0 & \text{o.w.} \end{cases}
$$

With this function, $\mathcal{S}_n$ counts the number of triangles of interest. The following lemma provides some information for $\mathcal{S}_n$.

Lemma 2. *We have*
(I) For all $n \geq 4$,

$$
g_n(z) = \frac{1}{C_{n-2}} \sum_{j=0}^{n-2} C_j \left[2 \binom{j+2}{n-2-j} - \binom{j+1}{n-2-j} \right] z^{2j+4-n}(1-z^2)^{n-2-j}.
$$

(II) For all $n \geq 4$, $\mathbb{E}(\mathcal{S}_n) = \frac{n(n-4)}{2n-5}$.
(III) For all $n \geq 5$, $\mathrm{Var}(\mathcal{S}_n) = \frac{2n(n-1)(n-4)(n-5)}{(2n-5)^2(2n-7)}$.

In the next result, we extend the previous example to slightly more general case where f_n is define as

$$
(4) \qquad f_{n,l,j,r} = \frac{1}{2}(w^{j-l} + w^{r-j}) \times \text{Eq } (3).
$$

In particular, we have

Lemma 3.

$$
E(\mathcal{S}_n) = -\frac{(n-1)(2w^{n-2} + 3w)}{2(2n-5)}
$$
$$
+ \frac{3w}{C_{n-2}} \sum_{j=0}^{n-3} w^j C_j \binom{2n-6-2j}{n-3-j} - \frac{w}{2C_{n-2}} \sum_{j=0}^{n-3} w^j C_j \binom{2n-4-2j}{n-2-j},
$$

for all $n \geq 4$.

We remark that by using simple identities

$$
\sum_{j=0}^{n-3} C_j \binom{2n-6-2j}{n-3-j} = \binom{2n-5}{n-2}
$$

and

$$\sum_{j=0}^{n-3} C_j \binom{2n-4-2j}{n-2-j} = \binom{2n-3}{n-1} - C_{n-2}.$$

we can show this Lemma gives the same result when $w = 1$ as Lemma 2. This is another example

1.0.0.2. Triangles with two sides on P_n (Ears) Next example is similar to the previous case with the exception that, in this example, we would ask how many of the triangles in $\mathcal{T}_n$ have at least two sides residing on the perimeter of P_n. To that end, we let $f_{n,l,j,r}$ to be as follows:

$$(5) \qquad f_{n,l,j,r} = \begin{cases} 1 & \text{if } 1 \leq l, j = l+1, \ r = l+2, \ r < n \\ 1 & \text{if } l = 1, \ j = n-1, \ r = n \\ 1 & \text{if } l = 1, \ j = 2, \ r = n \\ 0 & \text{o.w.} \end{cases}$$

Next lemma provides detailed information on $\mathcal{S}_n$ which counts the number of triangles of interest in $\mathcal{T}_n$:

Lemma 4. *For all $n \geq 4$, we have*

(I) $g_n(z) = 1 + \frac{1}{C_{n-2}} \sum_{j=0}^{n-3} C_j \left(\binom{j+1}{n-2-j} + 2\binom{j+1}{n-3-j} \right) (z-1)^{n-2-j}.$

(II) $\mathbb{E}(\mathcal{S}_n) = \frac{n(n-1)}{2(2n-5)}.$

(III) $\mathbb{V}\mathrm{ar}(\mathcal{S}_n) = \frac{n(n-1)(n-4)(n-5)}{2(2n-5)^2(2n-7)}$ *for $n \geq 6$.*

Recall that there is a well-known bijection between binary trees with $n-2$ nodes and triangulations of P_n. See [10] for a review of various interesting bijections of similar nature. In [8], Hurtado and Noy use this bijection to give a combinatorial proof for section (I) and (II) of Lemma (4). We remark that our method has the capability of generalizing this result to cases such as the one described in (4), while it is not clear how a combinatorial argument can provide such extension in a straightforward manner. Having the last two examples, one can also provide the exact distribution on the number of triangles with no side on the perimeter of P_n also know as internal triangles. One final remark is that Lemma 4-(II) and Lemma 2-(II) imply that the average number of nodes with degree two (resp. one) in a uniformly sampled binary trees of $n - 2$ nodes is $\frac{n(n-4)}{2n-5}$ (resp. $\frac{n(n-1)}{2(2n-5)}$). See Figure 3 for an example.
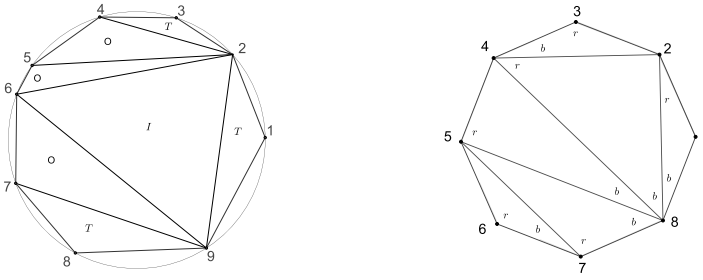
Figure 3: (left) A triangulation of an irregular P_9. The triangles with one side (resp. two sides) on the perimeter of P_n are marked by 'O' (resp. 'T'). There is also one internal triangle marked with I. (right) A triangulation of P_8 with marked angles.

In the next few examples, we assume $v_{n,j} := (\cos\theta_{n,j}, \sin\theta_{n,j})$, where $\theta_{n,i} := \frac{2\pi(i-1)}{n}$ for $1 \le i \le n$. In other words, P_n is a regular polygon inscribed in the unit circle.

1.0.0.3. Degree of a vertex   Our objective in this example is to obtain some information on how a typical vertex of $\mathcal{T}_n$ looks. Let $\mathcal{D}_{n,i}$ be the number diagonals incident with i-th vertex in $\mathcal{T}_n$. As it was shown in [7], any triangulation can be fully characterized by the sequence of degrees of the polygon vertices. Note that (a) by symmetry all $\mathcal{D}_{n,i}$ have identical distributions. (b) $\sum_{i=1}^{n} \mathcal{D}_{n,i} = 2(n-3)$. Therefore, we have $E(\mathcal{D}_{n,1}) = \cdots = E(\mathcal{D}_{n,n}) = \frac{2(n-3)}{n}$. By item (b), however, $\mathcal{D}_{n,i}$ are dependent. Hence, in order to obtain the full description of $\mathcal{D}_{n,1}$, we need to do a bit more work. Note that Bernasconi et al. [1] provided an elegant means to study the vertices of $\mathcal{T}_n$ in a very general sense. This is done by designing a Boltzmann sampler that reduces the study of $\mathcal{D}_{n,i}$s to properties of sequences of independent and identical distributed random variables. At this point we are not able to extend their approach to our model, however, we believe that the proposed approach in [1] and [3] might be proven to be useful in our case as well.

To that end, we let

$$f_{n,l,j,r} = \begin{cases} 1 & \text{if } l = 1 \\ 0 & \text{o.w.} \end{cases}$$

With this function, $\mathcal{S}_n$ is indeed $\mathcal{D}_{n,1} + 1$. Then, we get

Lemma 5. *For $n \geq 4$, we have*

1. $g_n(z) = \frac{1}{C_{n-2}} \sum_{s=1}^{n-2} \frac{s(2n-s-5)!}{(n-s-2)!(n-2)!} z^s$.
2. $\mathbb{E}(\mathcal{S}_n) = \frac{3(n-2)}{n}$ *for $n \geq 2$.*
3. $\mathbb{V}\mathrm{ar}(\mathcal{S}_n) = \frac{2(2n-3)(n-2)(n-3)}{n^2(n+1)}$ *for $n \geq 2$.*

In addition to [1], Devroye et al. [2] also studied the maximum of this sequence namely $\max_{1 \leq i \leq n} \mathcal{D}_{n,i}$ where they obtained same result for $\mathcal{D}_{n,1}$ (See Lemma 1 of [2]). Their proof is purely combinatorial while ours is based on derivation of the generating function $g_n(z)$.

Our main result for this example is to characterize the distribution of the portfolio of angles at the vertex 1. More precisely,

Theorem 6. *Let $\mathcal{A}_{n,i}$ be the number of angles of size $\frac{2\pi i}{n}$ at vertex 1 of $\mathcal{T}_n$. Then, for a fix sequence $0 \leq k_1, \cdots, k_{n-2} \leq n-2$ with $\sum_{i=1}^{n-2} ik_i = n-2$, we have*

$$\mathbb{P}(\mathcal{A}_{n,1} = k_1, \cdots, \mathcal{A}_{n,n-2} = k_{n-2})$$
$$= \frac{K(2n-K-5)!}{Z_{n,K}C_{n-2}(n-K-2)!(n-2)!} \binom{K}{k_1, \cdots, k_{n-2}} C_0^{k_1} C_1^{k_2} \cdots C_{n-3}^{k_{n-2}},$$

where $K := \sum_{i=1}^{n-2} k_i$ and

$$(6) \qquad Z_{n,K} := \sum_{\substack{\sum_{j=1}^{n-2} jp_j = n-2 \\ \sum_{j=1}^{n-2} p_j = K \\ 0 \leq p_j \leq n-2}} \binom{K}{p_1, \cdots, p_{n-2}} C_0^{p_1} \cdots C_{n-3}^{p_{n-2}}.$$

1.0.0.4. Blue angles Suppose for all $1 \leq l < j < r \leq n$ we mark the triangle $\Delta_{n,l,j,r}$ such that $\angle v_{n,l} v_{n,j} v_{n,r}$ is red, $\angle v_{n,j} v_{n,l} v_{n,r}$ is green, and $\angle v_{n,j} v_{n,r} v_{n,l}$ is blue. In the next two examples we focus on various properties of blue angles. Similar results hold for the other two colors by symmetrical arguments therefore we will not present them. See Figure 3 for an example on how the marking process works. We note that the total sum of blue angles in $\mathcal{T}_n$ can be studied by defining

$$(7) \qquad\qquad\qquad f_{n,l,j,r} = j - l.$$

Then it is easy to show

Lemma 7. $\mathbb{E}(\mathcal{S}_n) = \frac{2^{2n-5} - \binom{2n-5}{n-2}}{C_{n-2}}.$

Next, we count the number of blue angles equal to $\frac{2\pi p}{n}$ for a fixed $1 \leq p \leq n-1$. To that goal, we define

$$(8) \qquad f_{n,l,j,r} = \begin{cases} 1 & \text{if } j - l = p \\ 0 & \text{o.w.} \end{cases}$$

Here, we only report the result for $p = 1$ and leave the general case to reader with an understanding the general case follows from the same argument with a slight modification in the initial conditions.

Theorem 8. *Fix $p = 1$. For $n \geq 4$, we have*

1. *$g_n(z) = \frac{1}{C_{n-2}} \sum_{j=1}^{n-2} N_{n-2,j} z^j$, where $N_{n,k} := \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$s are Narayana numbers.*
2. *$\mathbb{E}(\mathcal{S}_n) = \frac{n-1}{2}$.*
3. *$\mathrm{Var}(\mathcal{S}_n) = \frac{(n-1)(n-2)(n-3)}{2(2n-5)}$.*

For more information on Narayana numbers, see the sequence A001263 in [16] and Exercise 6.36 in [17].

This paper is organized as follows. In Section 2 we introduced the main tools and prove Theorem 1. Section 3 includes the proof of results for the examples.

2. An algorithm and structure of $g_n(z)$

We begin this section with describing an algorithm that generates a uniformly sampled random triangulation of P_n. We note that are currently various paradigms in the literature for sampling of a random triangulation. We refer to [2] and [4] for algorithmic instances, to [11, 12, 14] for random walk based samplers, and to [1] and [3] for Boltzmann samplers. Due to its constructive recursive nature, we choose the following simple algorithm belonging to the community folklore. For a given $1 \leq l < r \leq n$, we define the function $\mu_{n,l,r}$ such that

$$(9) \qquad \mu_{n,l,r}(j) = \frac{C_{j-l-1} C_{r-j-1}}{C_{r-l-1}}.$$

Note $\mu_{n,l,r}$ is indeed a probability distribution on integer numbers between l and r since by Catalan recursive identity we have

$$C_0 = 1, \quad \text{and} \quad C_{m+1} = \sum_{s=0}^{m} C_s C_{m-s} \quad \text{for} \quad m \geq 0.$$

Next, we define our sampling algorithm. With an abuse of notation we refer to this algorithm also as $\mathcal{T}_{n,l,r}$. It should be clear from the context whether we intend the algorithm or the triangulation itself.

2.0.0.1. Sampling algorithm: $\mathcal{T}_{n,r,l}$

1. Generate random integer $\mathcal{J} = j$ between l and r with probability $\mu_{n,l,r}$.
2. If $r > l + 2$, then return $\mathcal{T}_{n,l,j} \cup \mathcal{T}_{n,j,r} \cup \Delta_{n,l,j,r}$.
3. If $r = l + 2$, then return $\Delta_{n,l,l+1,l+2}$.
4. If $r < l + 2$, then return empty.

Note that for each fixed triangle $\Delta_{n,l,j,r}$ there are exactly $C_{l-j-1}C_{r-j-1}$ triangulations with $\Delta_{n,l,j,r}$ among their triangles. Therefore, the probability that a uniformly sampled triangulation from $T_{n,l,r}$ has the triangle $\Delta_{n,l,j,r}$ is exactly $C_{l-j-1}C_{r-j-1}C_{r-l-1}^{-1}$. Given that $\mathcal{T}_{n,l,j}$ and $\mathcal{T}_{n,j,r}$ are independent, an inductive argument implies that $\mathcal{T}_n$ is uniformly distributed on T_n.

We are now ready to study $g_n(z)$ as the main tool in this paper. To that end, we note that by the algorithm $\mathcal{T}_{n,l,r}$, we have

$$(10) \qquad\qquad \mathcal{S}_{n,l,r} = f_{n,l,\mathcal{J},r} + \mathcal{S}_{n,l,\mathcal{J}} + \mathcal{S}_{n,\mathcal{J},r},$$

for $r, l \in [n]$ with $r - l > 2$. Similarly,

$$(11) \qquad\qquad \mathcal{S}_{n,l,l+2} = f_{n,l,l+1,l+2}, \quad \text{and} \quad \mathcal{S}_{n,l,l+1} = 0.$$

Recall (1) and (9). Define $h_{n,l,r}(z) = C_{r-l-1}g_{n,l,r}(z)$. By the recursive equations (10) and (11), we have

$$(12) \qquad\qquad h_{n,l,r}(z) = \sum_{j=l+1}^{r-1} z^{f_{n,l,j,r}} h_{n,l,j}(z) h_{n,j,r}(z)$$

with $h_{n,l,l+1}(z) = 1$ and $h_{n,l,l+2}(z) = z^{f_{n,l,l+1,l+2}}$.

We first give the following lemma 9 that indicates, for a certain class of functions f_n, rotation and shifts do not effect the form of $h_{n,l,r}(z)$.

Lemma 9. *Suppose $f_{n,l,j,r}$ is a function of $r - j$, $r - l$, $j - l$ and possibly n. Then*

(I) For all $1 \leq l < r \leq n - 1$, $h_{n,l,r}(z) = h_{n,l+1,r+1}(z)$.

(II) Suppose $n \geq 4$. Additionally, assume $f_{n,l,j,r}$ is independent of n. Then

$$h_{n,l,n}(z) = h_{n-1,l-1,n-1}(z)$$

for all $l = 3, 4, \ldots, n-1$.

Proof of (I). Since $f_{n,l,j,r}$ is merely a function of $r - j$, $r - l$, $j - l$ and possibly n, we have that $f_{n,l+1,j+1,r+1} = f_{n,l,j,r}$ for all $1 \leq l < r \leq n - 2$. We proceed the proof by induction on $p := r - l$, that is, we show that $h_{n,l,l+p}(z) = h_{n,l+1,l+p+1}(z)$ for all $1 \leq p \leq n-1$ with an understanding that $1 \leq l < l + p \leq n$. By (12), we have that $h_{n,l,l+1}(z) = h_{n,l+1,l+2}(z) = 1$ and $h_{n,l,l+2}(z) = h_{n,l+1,l+3}(z)$, which implies that the lemma hold for $p = 1, 2$. Next, we assume that the lemma holds for $p = 1, 2, \cdots, s - 1$ and prove it also holds for $p = s$. In other words, we show $h_{n,l+1,l+s+1}(z) = h_{n,l,l+s}(z)$. To that end, by (12), we obtain

$$
\begin{aligned}
h_{n,l+1,l+s+1}(z) &= \sum_{j=l+2}^{l+s} z^{f_{n,l+1,j,l+s+1}} h_{n,l+1,j}(z) h_{n,j,l+s+1}(z) \\
&= \sum_{j=l+2}^{l+s} z^{f_{n,l,j-1,l+s}} h_{n,l,j-1}(z) h_{n,j-1,l+s}(z) \\
&= \sum_{j=l+1}^{l+s-1} z^{f_{n,l,j,l+s}} h_{n,l,j}(z) h_{n,j,l+s}(z) = h_{n,l,l+s}(z)
\end{aligned}
$$

where for the second equality we used the induction hypothesis.   □

Proof of (II). By the assumption $f_{n,s,j,n} = f_{n-1,s-1,j-1,n-1}$ for all $1 < s < j \leq n$. We proceed the proof by induction on $l = n - 1, n - 2, \ldots, 3$. By (12), we have that $h_{n,n-1,n}(z) = h_{n-1,n-2,n-1}(z) = 1$ and $h_{n,n-2,n}(z) = h_{n-1,n-3,n-1}(z)$, which shows that the claim holds for $l = n - 1, n - 2$. We assume that the claim holds for $l = n - 1, n - 2, \ldots, s + 1$ and show that it also holds for $l = s$. By (12) and Lemma 9-(I), we have

$$
\begin{aligned}
h_{n,s,n}(z) &= \sum_{j=s+1}^{n-1} z^{f_{n,s,j,n}} h_{n,s,j}(z) h_{n,j,n}(z) \\
&= \sum_{j=s+1}^{n-1} z^{f_{n,s,j,n}} h_{n,s+n-j,n}(z) h_{n,j,n}(z),
\end{aligned}
$$

and

$$h_{n-1,s-1,n-1}(z) = \sum_{j=s}^{n-2} z^{f_{n-1,s-1,j,n-1}} h_{n-1,s-1,j}(z) h_{n-1,j,n-1}(z)$$

$$= \sum_{j=s}^{n-2} z^{f_{n-1,s-1,j,n-1}} h_{n-1,n-j+s-2,n-1}(z) h_{n-1,j,n-1}(z)$$

$$= \sum_{j=s+1}^{n-1} z^{f_{n,s,j,n}} h_{n-1,n-j+s-1,n-1}(z) h_{n-1,j-1,n-1}(z)$$

$$= \sum_{j=s+1}^{n-1} z^{f_{n,s,j,n}} h_{n,s+n-j,n}(z) h_{n,j,n}(z).$$

Where we used the induction hypothesis for the last equality. Therefore, we have shown $h_{n,s,n}(z) = h_{n-1,s-1,n-1}(z)$, which completes the induction. $\quad\square$

Recall that $h_n(z) = C_{n-2}g_n(z)$. Therefore, $\mathbb{E}$ and $\mathbb{V}\mathrm{ar}$ follow from $h_n(z)$:

$$(13) \quad \mathbb{E}(\mathcal{S}_n) = \frac{1}{C_{n-2}} h_n', \text{ and } \mathbb{V}\mathrm{ar}(\mathcal{S}_n) = \frac{1}{C_{n-2}}(h_n' + h_n'') - \frac{1}{C_{n-2}^2}(h_n')^2,$$

where

$$h_n' := \frac{d}{dz} h_n(z) \mid_{z=1} \text{ and } h_n'' := \frac{d^2}{dz^2} h_n(z) \mid_{z=1}.$$

Similarly, we define $h_{n,l,r}'$ and $h_{n,l,r}''$.

Proof of Theorem 1. Suppose $f_{n,l,j,r}$ is a function of $r-j$, $r-l$, $j-l$ and possibly n. We will calculate $h_{n,l,n}'$ and $h_{n,l,n}''$ to prove Theorem 1. To that end, note that Lemma 9-(I) reduces the calculation $h_{n,l,r}(z)$ to that of $h_{n,l,n}(z)$. In other words, equation (12) is reduced to

$$(14) \qquad h_{n,l,n}(z) = \sum_{j=l+1}^{n-1} z^{f_{n,l,j,n}} h_{n,l+n-j,n}(z) h_{n,j,n}(z)$$

with $h_{n,n-1,n}(z) = 1$ and $h_{n,n-2,n}(z) = z^{f_{n,n-2,n-1,n}}$.

By $h_{n,l,n}(1) = C_{n-l-1}$, we rewrite (14) as

$$h_{n,l,n}'(z) = \sum_{j=l+1}^{n-1} f_{n,l,j,n} C_{j-l-1} C_{n-j-1}$$

$$(15) \qquad\qquad + \sum_{j=l+1}^{n-1} (h'_{n,l+n-j,n} C_{n-j-1} + C_{j-l-1} h'_{n,j,n})$$

with $h'_{n,n-1,n} = 0$ and $h'_{n,n-2,n} = f_{n,n-2,n-1,n}$. Now, define $\mathbf{M}_n$ to be the matrix $(m_{ij})_{1 \leq i,j \leq n-1}$ where

$$m_{ij} = \begin{cases} 1 & \text{if } 1 \leq i = j \leq n-1 \\ -2C_{j-i-1} & \text{if } 1 \leq i < j \leq n-1 \\ 0 & \text{if } 1 \leq j < i \leq n-1. \end{cases}$$

Recall (1). Then, the recurrence (15) can be written as

$$(16) \qquad \mathbf{M}_n (h'_{n,1,n}, \ldots, h'_{n,n-1,n})^T = (\beta_{n,1}, \ldots, \beta_{n,n-2}, 0)^T.$$

To solve this system of equations, we define the matrix $\mathbf{D}_n = (d_{ij})_{1 \leq i,j \leq n-1}$, where

$$d_{ij} = \begin{cases} \binom{2j-2i}{j-i} & \text{if } 1 \leq i \leq j \leq n-1 \\ 0 & \text{if } 1 \leq j < i \leq n-1. \end{cases}$$

Recall the generating function of Catalan numbers:

$$(17) \qquad C(t) = \sum_{n \geq 0} C_n t^n = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^n = \frac{1 - \sqrt{1-4t}}{2t}.$$

Since the matrices $\mathbf{M}_n$ and $\mathbf{D}_n$ are upper triangular with diagonal ones, we have that $\sum_{j=1}^{n-1} m_{ij} d_{jl} = 0$ for all $1 \leq l < i \leq n-1$ and $\sum_{j=1}^{n-1} m_{ij} d_{ji} = 1$ for all $1 \leq i \leq n-1$. Suppose $1 \leq i < l \leq n-1$. We observe that from the convolution

$$\frac{1}{2x\sqrt{1-4x}} - \frac{1}{2x} = C(x) \times \frac{1}{\sqrt{1-4x}} = \sum_{n \geq 0} \sum_{j=0}^{n} C_j \binom{2j}{j} x^n,$$

we obtain

$$2 \sum_{j=0}^{n} C_j \binom{2s-2j}{s-j} = \binom{2s+2}{s+1}.$$

Hence, $\sum_{j=1}^{n-1} m_{ij} d_{jl} = 0$ for all $1 \leq i < l \leq n-1$. This shows that for all $n \geq 2$, $\mathbf{M}_n \mathbf{D}_n = I_{n-1}$, where I_n is the $(n \times n)$ identity matrix. Similarly,

(14) implies

$$h''_{n,l,n} = \sum_{j=l+1}^{n-1} f_{n,l,j,n}(f_{n,l,j,n} - 1)C_{j-l-1}C_{n-j-1}$$

$$+ \sum_{j=l+1}^{n-1} f_{n,l,j,n}(h'_{n,l+n-j,n}C_{n-j-1} + h'_{n,j,n}C_{j-l-1})$$

$$(18) \qquad + \sum_{j=l+1}^{n-1} (h''_{n,l+n-j,n}C_{n-j-1} + 2h'_{n,l+n-j,n}h'_{n,l,n} + h''_{n,j,n}C_{j-l-1})$$

with $h''_{n,n-1,n} = 0$ and $h''_{n,n-2,n}(z) = f_{n,n-2,n-1,n}(f_{n,n-2,n-1,n} - 1)$. With notation

$$\gamma_{n,l} = \sum_{j=l+1}^{n-1} \Bigg(\quad f_{n,l,j,n}(f_{n,l,j,n} - 1)C_{j-l-1}C_{n-j-1}$$

$$+2f_{n,l,j,n}(h'_{n,l+n-j,n}C_{n-j-1} + h'_{n,j,n}C_{j-l-1})$$

$$+2h'_{n,l+n-j,n}h'_{n,l,n} \Bigg),$$

(18) can be written as

$$\mathbf{M}_n(h''_{n,1,n}, \ldots, h''_{n,n-1,n})^T = (\gamma_{n,1}, \ldots, \gamma_{n,n-2}, 0)^T.$$

By (16) and the fact that $\mathbf{M}_n\mathbf{D}_n = I_{n-1}$ for $n \geq 2$, we obtain

$$(h'_{n,1,n}, \ldots, h'_{n,n-1,n})^T = \mathbf{D}_n(\beta_{n,1}, \ldots, \beta_{n,n-2}, 0)^T$$

and

$$(h''_{n,1,n}, \ldots, h''_{n,n-1,n})^T = \mathbf{D}_n(\gamma_{n,1}, \ldots, \gamma_{n,n-2}, 0)^T.$$

Thus, for all $l = 1, 2, \ldots, n - 2$,

$$h'_{n,l,n} = \sum_{j=l}^{n-2} \beta_{n,j}\binom{2j - 2l}{j - l} \text{ and } h''_{n,l,n} = \sum_{j=l}^{n-2} \gamma_{n,j}\binom{2j - 2l}{j - l},$$

which complete the proof of Theorem 1.   □

Example 10. *If $f_{n,l,j,r} = 1$ for all $1 \leq l < j < r \leq$, then*

$$\beta_{n,j} = \sum_{i=j+1}^{n-1} C_{i-j-1}C_{n-i-1} = \sum_{i=0}^{n-2-j} C_i C_{n-2-j-i} = C_{n-1-j},$$

which leads to

$$
\begin{aligned}
C_{n-2}E(\mathcal{S}_n) &= \sum_{j=1}^{n-2} \beta_{n,j} \binom{2j-2}{j-1} \\
&= \sum_{j-1}^{n-2} C_{n-1-j} \binom{2j-2}{j-1} = \binom{2n-3}{n-2} - \binom{2n-4}{n-2} \\
&= (n-2)C_{n-2},
\end{aligned}
$$

as expected.

Example 11. *Suppose f_n is a polynomial, where for a fixed $w \in \mathbb{C}$,*

$$f_{n,l,j,r} = \frac{1}{3}(w^{j-l} + w^{r-j} + w^{r-l}).$$

Then, for all $n \geq 4$, Theorem 1 implies

$$\mathbb{E}(\mathcal{S}_n) = w^{n-1} - \frac{(n-1)w}{3} + \frac{2w}{3C_{n-2}} \sum_{j=0}^{n-3} w^j C_j \binom{2n-4-2j}{n-2-j}.$$

3. Examples

The main idea for all the proofs in this section is as follows. We define two generating functions

$$H_2(t,z) = \sum_{n \geq 3} h_{n,2,n}(z)t^{n-3} \text{ and } H_1(t,z) = \sum_{n \geq 3} h_n(z)t^{n-3}.$$

Our end goal is to obtain H_1 as $h_n(z)$ can be easily obtained by extracting the coefficients of t^{n-3}. However, in most cases, we first obtain H_2 and then solve H_1 with respect to H_2. To that end, we first simplify (12) using certain properties of f_n at hand and then derive explicit equations for H_1 and H_2 through the application of recursion (12).

3.1. Triangles with only one side on P_n

In this subsection, we give the proof of Theorem 2. We Recall (3). Note that by (12) and Lemma 9, we have

$$h_{n,2,n}(z) = 2zh_{n-1,2,n-1}(z) + \sum_{j=4}^{n-2} h_{j,2,j}(z)h_{n-j+2,2,n-j+2}(z),$$

with $h_{3,2,3}(z) = h_{4,2,4}(z) = 1$. Multiplying by t^{n-3} and summing over $n \geq 5$, we obtain

$$H_2(t,z) - t - 1 = 2zt(H_2(t,z) - 1) + t(H_2(t,z) - 1)^2.$$

By solving this equation, we obtain

$$
\begin{aligned}
H_2(t,z) &= \frac{1 + 2(1-z)t - \sqrt{1 - 4t(z - z^2t + t)}}{2t} \\
&= 1 - z + (z + t - z^2t)C(t(z + t - z^2t)).
\end{aligned}
$$
(19)

Thus, by (17), for all $n \geq 4$,

$$h_{n,2,n}(z) = \sum_{j=0}^{n-3} C_j \binom{j+1}{n-3-j} z^{2j+4-n}(1-z^2)^{n-3-j}.$$

By (12) with using Lemma 9, we have

$$h_{n,1,n}(z) = 2h_{n,2,n}(z) + z\sum_{j=3}^{n-2} h_{j+1,2,j+1}(z)h_{n-j+2,2,n-j+2}(z)$$

with $h_{3,1,3}(z) = 1$. By multiplying by t^{n-3} and summing over $n \geq 4$, we obtain

$$H_1(t,z) = 1 + 2(H_2(t,z) - 1) + z(H_2(t,z) - 1)^2.$$

By (20), we obtain

(20) $H_1(t,z) = \frac{2tz^3 - 3tz - z^2 + t}{t} - \frac{(2tz^2 - 2t - z)(z + t - z^2t)}{t}C\left(t(z + t - z^2t)\right).$

where $C(.)$ is defined by (17). Hence, for all $n \geq 4$,

$$h_n(z) = \sum_{j=0}^{n-2} C_j \left[2\binom{j+2}{n-2-j} - \binom{j+1}{n-2-j}\right] z^{2j+4-n}(1-z^2)^{n-2-j}.$$

This finishes the proof of Theorem 2-(I).

Next, by (20), we have

$$H'_1(t,1) := \frac{\partial}{\partial z} H_1(t,z) \mid_{z=1} = \frac{t}{2} - 4t^2 + 3t^3 - \frac{t(20t^2 - 10t + 1)}{2\sqrt{1-4t}}.$$

The coefficient of t^{n-3} in $H'_1(t,1)$ is

$$h'_n = \frac{-1}{2}\binom{2n-2}{n-1} + 5\binom{2n-4}{n-2} - 10\binom{2n-6}{n-3}.$$

This completes the Proof of Theorem 2-(II). Similarly, (20) gives

$$\frac{\partial^2}{\partial z^2} H_1(t,z) \mid_{z=1} = 12t^3 - 4t^2 + \frac{4t^2(1 - 9t + 24t^2 - 14t^3)}{(1-4t)^{3/2}},$$

Extracting the coefficient of t^{n-3} gives

$$\frac{1}{C_{n-2}} h''_n = \frac{n(n-1)(n^2 - 9n + 20)}{(2n-5)(2n-7)}.$$

Therefore, $\mathbb{V}\mathrm{ar}(\mathcal{S}_n)$ is followed from (13).

3.1.0.1. A slight generalization   By similar arguments as in the beginning of this section, we have

$$h_{n,2,n}(z) = (z^w + z^{w^{n-3}})h_{n-1,2,n-1}(z) + \sum_{j=4}^{n-2} h_{j,2,j}(z)h_{n-j+2,2,n-j+2}(z),$$

$$h_{n,1,n}(z) = 2h_{n,2,n}(z) + \sum_{j=3}^{n-2} z^{w^{j-1}} h_{j+1,2,j+1}(z)h_{n-j+2,2,n-j+2}(z)$$

with $h_{4,2,4}(z) = h_{3,1,3}(z) = 1$. Differentiating $h_n(z)$ at $z = 1$ and using the fact $h_{n,l,r}(1) = C_{r-l-1}$, we obtain

$$h'_{n,2,n} = (w + w^{n-3})C_{n-4} + 2h'_{n-1,2,n-1} + 2\sum_{j=4}^{n-2} h'_{j,2,j}C_{n-j-1},$$

$$h'_{n,1,n} = 2h'_{n,2,n} + \sum_{j=3}^{n-2} w^{j-1}C_{j-2}C_{n-j-1} + 2\sum_{j=3}^{n-2} h'_{j+1,2,j+1}C_{n-j-1}$$

with $h'_{4,2,4} = h'_{3,1,3} = 0$.

Define $H_1'(t) = \sum_{n \geq 3} h_{n,1,n}' t^{n-3}$ and $H_2'(t) = \sum_{n \geq 3} h_{n,2,n}' t^{n-3}$. Then, the above recurrences can be rewritten in terms of $H_1'(t)$ and $H_2'(t)$ as

$$H_2'(t) = wt(C(t) - 1) + wt(C(wt) - 1) + 2tH_2'(t)C(t),$$
$$H_1'(t) = w(C(wt) - 1)(C(t) - 1) + 2H_2'(t)C(t).$$

Thus, $H_2'(t) = \frac{wt(C(t) + C(wt) - 2)}{\sqrt{1 - 4t}}$ and

$$H_1'(t) = w(C(wt) - 1)(C(t) - 1) + \frac{2wtC(t)(C(t) + C(wt) - 2)}{\sqrt{1 - 4t}}$$
$$= \frac{w(1 - 4t)C(wt)}{2t} + \frac{w(6t - 1)C(wt)}{2t\sqrt{1 - 4t}} - 3w(C(t) - 1).$$

Recall the generating function (17) and $\frac{1}{\sqrt{1-4t}} = \sum_{n \geq 0} \binom{2n}{n} t^n$. Thus, by extracting the coefficient of t^{n-3} in $H_1'(t)$ and by (13), we complete the proof of Lemma 3.

3.2. Triangles with two sides on P_n

In this subsection, we give the proof of Theorem 4. The proof is very similar to that of the previous section. Note that $h_{3,2,3}(z) = 1$ and $h_{4,2,4}(z) = z$. By (12), (5), and Lemma 9, for $n \geq 5$ we have

$$\text{(21)} \qquad h_{n,2,n}(z) = \sum_{j=3}^{n} h_{j,2,j}(z) h_{n-j+2,2,n-j+2}(z).$$

Multiplying by t^{n-3} and summing over all terms, we obtain

$$H_2(t, z) - zt - 1 = -t + t(H_2(t, z))^2.$$

Equivalently,

$$\text{(22)} \qquad H_2(t, z) = \frac{1 - \sqrt{1 - 4t + 4(1 - z)t^2}}{2t}.$$

Once again, (12) and Lemma 9 imply

$$\text{(23)} \qquad h_{n,1,n}(z) = 2z h_{n,2,n}(z) + \sum_{j=4}^{n-1} h_{j,2,j}(z) h_{n-j+3,2,n-j+3}(z),$$

with $h_{3,1,3}(z) = z$ and $h_{4,1,4}(z) = 2z^2$. Multiplying by t^{n-3} and summing over $n \geq 5$, we obtain

$$H_1(t, z) - 2z^2 t - z = 2z(H_2(t, z) - zt - 1) + (H_2(t, z) - 1)^2.$$

Solving for $H_1(t, z)$, and replacing $H_2(z, t)$ from (22), we have

$$H_1(t, z) = \frac{1 - 2(2 - z)t + 4(1 - z)t^2 - (1 - 2(1 - z)t)\sqrt{1 - 4t + 4(1 - z)t^2}}{2t^2}.$$

Recall the generating function (17). To extract the coefficients of H_1, we rewrite H_1 using $C(t)$ function as follows

$$H_1(t, z) = \frac{(1 - 2(1 - z)t)(1 - (1 - z)t)}{t}C(t(1 - (1 - z)t)) - \frac{1 - 2(1 - z)t}{t}$$

$$= (1 - 2(1 - z)t)\sum_{j \geq 0} C_j t^{j-1}(1 - (1 - z)t)^{j+1} - \frac{1 - 2(1 - z)t}{t}$$

$$(24) \qquad = (1 - 2(1 - z)t)\sum_{j \geq 0}\sum_{i=0}^{j+1} C_j \binom{j + 1}{i} t^{i+j+1}(z - 1)^i - \frac{1 - 2(1 - z)t}{t}.$$

Extracting the coefficient of t^{n-3}, we have completed the proof Theorem 4-(I).

From (24), we have

$$H_1'(t, 1) = \frac{\partial}{\partial z}H_1(t, z) |_{z=1} = -2 + \frac{1}{t} - \frac{1 - 5t}{t\sqrt{1 - 4t}},$$

which leads to $\frac{1}{C_{n-2}}h_n' = \frac{n(n-1)}{2(2n-5)}$, for all $n \geq 4$. Moreover,

$$H_1''(t, 1) = \frac{\partial^2}{\partial z^2}H_1(t, z) |_{z=1} = \frac{2t(2 - 7t)}{(1 - 4t)^{3/2}},$$

which shows $\frac{1}{C_{n-2}}h_n'' = \frac{n(n-1)(n-2)(n-3)}{4(2n-5)(2n-7)}$ for $n \geq 5$. Therefore, $\mathbb{E}(\mathcal{S}_n)$ and $\mathbb{V}\mathrm{ar}(\mathcal{S}_n)$ follow from (13).

3.3. Degree of vertex 1

Recall that conditions for Lemma 9 s not satisfied by f_n for this example, however, by a very similar type of argument we can show $h_{n,l,r}(z) =$

$h_{n,2,r-l+2}(z)$ for all $2 \le l < r \le n-1$ and $n \ge 3$. Then, (12) implies

$$h_{n,1,n}(z) = z \sum_{j=2}^{n-1} h_{j,1,j}(z) h_{n-j+2,2,n-j+1}(z),$$

and

$$h_{n,2,n}(z) = \sum_{j=3}^{n-1} h_{j,2,j}(z) h_{n-j+2,2,n-j+1}(z),$$

where $h_{2,1,2}(z) = h_{3,2,3}(z) = h_{4,2,4}(z) = 1$. By translating these recurrence in terms of generating functions $H_1(t,z)$ and $H_2(t,z)$, we obtain

$$H_1(t,z) = zH_2(t,z) + ztH_1(t,z)H_2(t,z) \text{ and } H_2(t,z) = 1 + t(H_2(t,z))^2.$$

Therefore,

$$H_1(t,z) = \frac{zC(t)}{1 - ztC(t)} \text{ and } H_2(t,z) = C(t).$$

Thus,

$$H_1(t,z) = \sum_{s \ge 1} z^s t^{s-1} C^s(t),$$

By Equation 2.5.16 [18], we obtain

$$t^3 H_1(t,z) = \sum_{s \ge 0} \sum_{j \ge 0} z^s \frac{s(2j+s-1)!}{j!(j+s)!} t^{j+s+2},$$

which leads to

$$h_n(z) = \sum_{s=0}^{n-2} z^s \frac{s(2n-s-5)!}{(n-s-2)!(n-2)!}$$

In addition, $\mathbb{E}(\mathcal{S}_n)$ and $\mathbb{V}\mathrm{ar}(\mathcal{S}_n)$ easily follow from (13). This completes the proof of Lemma 5.

Next, we give the proof of Theorem 6.

Proof of Theorem 6. Recall $K := \sum_{i=1}^{n-2} k_i$. Note that

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_{n,1} &= k_1, \cdots, \mathcal{A}_{n,n-2} = k_{n-2}) \\
&= \mathbb{P}(\mathcal{A}_{n,1} = k_1, \cdots, \mathcal{A}_{n,n-2} = k_{n-2}, \mathcal{S}_n = K) \\
&= \mathbb{P}(\mathcal{A}_{n,1} = k_1, \cdots, \mathcal{A}_{n,n-2} = k_{n-2} \mid \mathcal{S}_n = K) \mathbb{P}(\mathcal{S}_n = K)
\end{aligned}$$

The last term is given by Lemma 5. Hence, it is enough to calculate the first term of the right-hand side of the equality. Given $k_1, \cdots, k_{n-2}$ and K, we count how many triangulations have this portfolio at vertex 1. Note that there are $\binom{K}{k_1, \cdots, k_{n-2}}$ choices of these angles. For each of these $\binom{K}{k_1, \cdots, k_{n-2}}$ choices, we have $C_0^{k_1} C_1^{k_2} \cdots C_{n-3}^{k_{n-2}}$ triangulations that fit the description. Therefore,

$$\mathbb{P}\left(\mathcal{A}_{n,1} = k_1, \cdots, \mathcal{A}_{n,n-2} = k_{n-2} | \mathcal{S}_n = K\right)$$
$$= \frac{1}{Z_{n,K}} \binom{K}{k_1, \cdots, k_{n-2}} C_0^{k_1} C_1^{k_2} \cdots C_{n-3}^{k_{n-2}},$$

where $Z_{n,k}$ is the number of triangulations with K angles at vertex "1", defined by (6). This completes the proof. $\square$

3.4. Blue angles

Proof of Lemma 7. Recall (8). Lemma 9 along with (12) implies

$$h_{n,1,n}(z) = z \sum_{j=2}^{n-1} z^{j-2} h_{j,1,j}(z) h_{n-j+1,1,n-j+1}(z),$$

where $h_{2,1,2}(z) = 1$. Then by rewriting this recurrence in terms of the generating function $\tilde{H}_1 = \sum_{n \geq 2} h_{n,1,n}(z) t^{n-2}$, we have

$$\tilde{H}_1(t, z) = 1 + zt\tilde{H}_1(zt, z)\tilde{H}_1(t, z).$$

Hence $\tilde{H}_1$ has the following form

$$\tilde{H}_1(t, z) = \cfrac{1}{1 - \cfrac{zt}{1 - \cfrac{z^2 t}{1 - \cfrac{z^3 t}{\ddots}}}}.$$

Calculating the derivative at $z = 1$, we obtain $\mathbb{E}(\mathcal{S}_n) = \frac{2^{2n-5} - \binom{2n-5}{n-2}}{C_{n-2}}$ for $n \geq 3$ as claimed in Lemma 7. $\square$

Proof of Theorem 8. Recall (7) and let $p = 1$. (12) along with Lemma 9 implies

$$h_{n,1,n}(z) = z h_{n-1,1,n-1}(z) + \sum_{j=3}^{n-1} h_{j,1,j}(z) h_{n-j+1,1,n-j+1}(z),$$

where $h_{2,1,2}(z) = 1$ and $h_{3,1,3}(z) = z$. By multiplying by t^{n-3} and summing over $n \geq 4$, we obtain

$$tH_1(t,z) = z(tH_1(t,z) + 1) + t(tH_1(t,z) + 1)H_1(t,z).$$

By solving this equation, we obtain

$$H_1(t,z) = \frac{1 - t - zt - \sqrt{t^2(1-z)^2 - 2t(1+z) + 1}}{2t^2}.$$

Once again extracting the coefficient of t^{n-3} in $H_1(t,z)$ completes the proof of Theorem 8. □

References

[1] N. Bernasconi, K. Panagiotou, A. Steger, *On properties of random dissections and triangulations*, Combinatorica, 30 (6), 627–654, 2010. MR2789731

[2] L. Devroye, P. Flajolet, F. Hurtado, M. Noy, W. Steiger, *Properties of random triangulations and trees*, Discrete Comput Geom, 22 (1), 105–117, 1999. MR1692686

[3] P. Duchon, P. Flajolet, G. Louchard, G. Schaeffer, *Boltzmann samplers for the random generation of combinatorial structures*, Combinatorics, Probability, and Computing, 13 (4-5), 577–625, 2004. MR2095975

[4] P. Epstein, J.R. Sack, *Generating triangulations at random*, ACM Transactions on Modeling and Computer Simulation, 4 (3), 267–278, 1994.

[5] Z. Gao, N.C. Wormald, *The distribution of the maximum vertex degree in random planar maps*, J. Combinatorial Theory Series A, 89 (2), 201–230, 2000. MR1741015

[6] J.E. Goodman, J. O'Rourke, C.D. Tóth (ed.) *Handbook of Discrete and Computational Geometry*, Third Edition, CRC Press, 2017. MR3793131

[7] F. Hurtado, M. Noy, *Graph of triangulations of a convex polygon and tree of triangulations*, Computational Geometry, 13 (3), 179–188, 1999. MR1723053

[8] F. Hurtado, M. Noy, *Ears of triangulations and Catalan numbers*, Discrete Mathematics, 149 (1-3), 319–324, 1996. MR1375121

[9] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 1925. MR0120538

[10] J.A. De Loera, J Rambau, F. Santos, *Triangulations: Structures for Algorithms and Applications*, Springer, 2010. MR2743368

[11] L. McShine, P. Tetali, *On the mixing time of the triangulation walk and other Catalan structures,* Randomization Methods in Algorithm Design, DIMACS-AMS Vol. 43, 147–160, 1998. MR1660784

[12] M. Molloy, B. Reed, W. Steiger, *On the mixing rate of the triangulation walk*, Randomization Methods in Algorithm Design: DIMACS Workshop, 179–190, 1997. MR1660786

[13] G. Polyá, *On picture-writing*, American Math Monthly, 51, 689–697, 1956. MR0081865

[14] D. Randall, P. Tetali, *Analyzing Glauber dynamics by comparison of Markov chains*, Journal of Mathematical Physics, 41 (3), 15–98, 2000. MR1757972

[15] A. Regev, *Enumerating trianulations by parallel diagonals*, Journal of Integer Sequences, 15, Article 12.8.5, 2012. MR2998666

[16] N.J.A. Sloane, *The on-line encyclopedia of integer sequences*. Available at http://www.research.att.com/~njas/sequences.

[17] R.P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, Cambridge, 1999. MR1676282

[18] H.S. Wilf, *Generatingfunctionology*, Academic Press, Inc., 1990. MR1034250

[19] Y.F. Xu, *Minimum weight triangulations*, Handbook of Combinatorial Optimization, Vol 2, Boston, MA: Kluwer Academic Publishers, pp. 617–634, 1998. MR1665412

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