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A 2-regular graph has a prime labeling if and only if it has at most one odd component

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A graph G is prime if the vertices can be distinctly labeled with the integers $1, 2, \dots, |V(G)|$ so that adjacent vertices have relatively prime labels. In this paper, we prove the long-standing conjecture that a 2-regular graph G is prime if and only if G has at most one odd component.

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1. Introduction

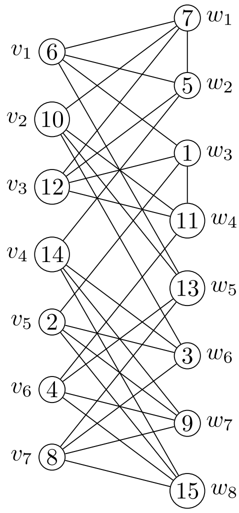
For a simple graph G with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, a *prime labeling* of G is a bijection $f : V(G) \to \{1, 2, \dots, n\}$ such that $f(v_i)$ and $f(v_j)$ are relatively prime for all $\{v_i, v_j\} \in E(G)$ (see Figure 1). A graph is called *prime* if it admits a prime labeling. This concept was formally introduced in 1982 by Tout, Dabboucy and Howalla [13]. That seminal paper contained two conjectures that have driven much of the study of prime labelings; the first concerns trees and was originally made by Entriger in 1980.

Conjecture 1.1 (Entriger, see [13]). *Every tree has a prime labeling.*

This conjecture remains open, though there is strong evidence to suggest it is true. Pikhurto showed that all trees with at most 50 vertices are prime [8, 9] and, together with Haxell and Taraz, that all sufficiently large trees are prime [7]. Moreover, a number of special families of trees are known to be prime, including complete binary trees [5]; spiders, palm trees, banana trees, and binomial trees [10]; and, trivially, paths and stars.

The second conjecture from [13] proposes a classification for the primality of 2-regular graphs.

Conjecture 1.2 (Tout, Dabboucy and Howalla [13]). *A 2-regular graph G has a prime labeling if and only if G has at most one odd component.*



$L(2) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

$L(3) = \{v_1, v_3, w_6, w_7, w_8\}$

$L(5) = \{v_2, w_2, w_8\}$

$L(7) = \{v_4, w_1\}$

$L(11) = \{w_4\}$

$L(13) = \{w_5\}$

Figure 1: A prime labeling of a 4-regular graph on 15 vertices. The sets $L(p)$ for each prime less than 15 are also given.

Since a 2-regular graph is a disjoint union of cycles, this is equivalent to saying that $G = \bigcup_{i=1}^{m} C_{n_i}$ is prime if and only if at most one n_i is odd. We note that for a 2-regular graph, a prime labeling is equivalent to a *vertex prime labeling* (i.e. a labeling of the edges of a graph with distinct integers $1, 2, \ldots, |E(G)|$ such that for each vertex of degree at least 2, the greatest common divisor of the labels on its incident edges is 1), and this conjecture has been considered through that lens as well (cf. [4]).

In this paper, we prove Conjecture 1.2.

2. Previous results

Given a labeling of the vertices of a graph G with distinct integers from the set $L = \{1, 2, \ldots, |V(G)|\}$, let $L(p) \subseteq V(G)$ denote the subset of vertices of G labeled with a multiple of p. See Figure 1 for an example of a prime labeling together with the sets $L(p)$ for each relevant prime p. If $L(p)$ is an independent set for every prime $p \in L$, then the labeling is a prime labeling. We will use this as the basis for a prime labeling algorithm in Section 3.

An immediate observation is that for a graph G with a prime labeling, $\alpha(G) \geq |L(2)| = \lfloor |V(G)|/2 \rfloor$, where $\alpha(G)$ denotes the independence number of G. Let $G = \bigcup_{i=1}^{m} C_{n_i}$ be a 2-regular graph, and set $N = \sum_{i=1}^{m} n_i$. It is easy to see that if two or more n_i's are odd, then $\alpha(G) \leq (N-2)/2 < \lfloor N/2 \rfloor$, so G cannot have a prime labeling.

For the remainder of this paper, we will let $G = \bigcup_{i=1}^{m} C_{n_i}$ be a 2-regular graph with at most one n_i odd. Deretsky, Lee and Mitchem proved Conjecture 1.2 for $m \leq 4$ [4]. Borosh, Hensley and Hobbs extended this to $m \leq 7$ in the case that none of the n_i's are odd [3]. They also proved the existence of an integer n_0 such that G is prime for an arbitray m when $n_1 = n_2 = \cdots = n_m$ is an even integer at most 300,000 or at least n_0. Additional special cases of Conjecture 1.2 can be found in [1, 3, 4].

The current author proved the following theorem in [11].

Theorem 2.1 (Theorem 1 in [11]). *Every cubic bipartite graph has a prime labeling except $K_{3,3}$.*

An immediate consequence of this is that every 2-regular bipartite graph is prime.

Corollary 2.2 (Corollary 2, Part 1 in [11]). *Every 2-regular graph with all even components has a prime labeling.*

Thus for the rest of the paper we can assume that G is a 2-regular graph with precisely one odd component; without loss of generality, take n_m to be odd. Theorem 2.1 also leads to the following result.

Corollary 2.3 (Corollary 2, Part 3 in [11]). *Let $G = \bigcup_{i=1}^{m} C_{n_i}$ be a 2-regular graph with n_i even for $i = 1, 2, \ldots, m-1$ and n_m odd. If $n_m = 2^x + p^y$, where $x \geq 1$ and p is an odd prime that is relatively prime to $2^x - 1$, then G has a prime labeling.*

The smallest odd integer that does not satisfy the condition in Corollary 2.3 is $n_m = 149$, so the smallest counterexample to Conjecture 1.2, if it exists, must have at least 149 vertices.

3. Proof of Conjecture 1.2

We use a modification of the algorithm employed in [11] to prove the remaining cases. Some auxiliary results will be needed. Let $\Phi(n,p)$ count the number of positive integers at most n with no prime factors less than p. The author proved the following bound in [12].

Theorem 3.1 (Theorem 1.1 in [12]). *If $p \geq 11$ is prime and $n \geq 2p$, then*

$$\Phi(n,p) \geq \left\lfloor \frac{2n}{p} \right\rfloor + 1.$$

Theorem 3.1 was used in [11] to inductively label a cubic bipartite graph. For 2-regular graphs, we weaken and extend Theorem 3.1 to the following bound. Note that for any integer n, $\lfloor n/p \rfloor$ counts the number of even multiples of p at most $2n+1$, and $\lfloor (2n+p+1)/2p \rfloor$ counts the number of odd multiples of p at most $2n+1$. We make extensive use of the inequality $x/y \geq \lfloor x/y \rfloor \geq x/y - 1$ throughout this section.

Corollary 3.2. *Suppose $p \geq 5$ is prime. Then for all $n \geq \max\{p, 74\}$,*

$$\Phi(2n+1, p) \geq 2\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{2n+p+1}{2p} \right\rfloor.$$

Proof. We derive the inequality directly for $p \in \{5, 7\}$. First,

$$\Phi(2n+1, 5) = n + 1 - \left\lfloor \frac{n+2}{3} \right\rfloor \geq n + 1 - \frac{n+2}{3} = \frac{2n+1}{3}$$

and

$$2\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n+3}{5} \right\rfloor \leq \frac{2n}{5} + \frac{n+3}{5} = \frac{3n+3}{5}.$$

Since $(2n+1)/3 > (3n+3)/5$ for all $n \geq 74$, the bound is established for $p = 5$.

Similarly,

$$\begin{aligned}
\Phi(2n+1, 7) &= n + 1 - \left\lfloor \frac{n+2}{3} \right\rfloor - \left\lfloor \frac{n+3}{5} \right\rfloor + \left\lfloor \frac{n+8}{15} \right\rfloor \\
&\geq n + 1 - \frac{n+2}{3} - \frac{n+3}{5} + \frac{n+8}{15} - 1 = \frac{8n-11}{15}
\end{aligned}$$

and

$$2\left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n+4}{7} \right\rfloor \leq \frac{2n}{7} + \frac{n+4}{7} = \frac{3n+4}{7}.$$

Since $(8n-11)/15 > (3n+4)/7$ for all $n \geq 74$, the bound is established for $p = 7$.

Finally, let $p \geq 11$ be prime. Using Theorem 3.1 we have

$$\Phi(2n+1, p) \geq \left\lfloor \frac{4n+2}{p} \right\rfloor + 1 \geq \frac{4n+2}{p}.$$

Moreover, we know

$$2\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{2n+p+1}{2p} \right\rfloor \leq \frac{2n}{p} + \frac{2n+p+1}{2p} = \frac{6n+p+1}{2p},$$

and since $(4n+2)/p > (6n+p+1)/2p$ for all $n \geq p$, the bound is established for $p \geq 11$. $\square$

Corollary 3.2 will be used to assign the odd labels of p in our algorithm below; for the even labels of p, we need to count multiples of small primes. Let

$$E_p(n) = \{x \leq n : x \text{ is even and } q|x \text{ for some odd prime } q < p\}$$

and

$$O_p(n) = \{x \leq n : x \text{ is odd and } q|x \text{ for some odd prime } q < p\}.$$

Note that $n = |O_p(2n)| + \Phi(2n, p)$ counts all the odd positive integers less than $2n$ for any positive integers n and p. The following bound was obtained from Theorem 3.1 in [11].

Corollary 3.3 (Corollary 1 in [11])**.** *If $p \geq 11$ is prime and $n \geq p$, then*

$$n - |E_p(2n)| - 3\left|O_p\left(\frac{2n}{p}\right)\right| - \left\lfloor\frac{n}{p}\right\rfloor + \left|E_p\left(\frac{2n}{p}\right)\right| \geq 0.$$

Again we extend and weaken this result for use with 2-regular graphs.

Corollary 3.4. *If $p \geq 5$ is prime and $n \geq \max\{p, 74\}$, then*

$$n - |E_p(2n)| - 2\left|O_p\left(\frac{2n+1}{p}\right)\right| - \left\lfloor\frac{n}{p}\right\rfloor + \left|E_p\left(\frac{2n}{p}\right)\right| \geq 0.$$

Proof. We derive the inequality directly for $p \in \{5, 7\}$. First,

$$\begin{aligned}
& n - |E_5(2n)| - 2\left|O_5\left(\frac{2n+1}{5}\right)\right| - \left\lfloor\frac{n}{5}\right\rfloor + \left|E_5\left(\frac{2n}{5}\right)\right| \\
= \quad & n - \left\lfloor\frac{n}{3}\right\rfloor - 2\left\lfloor\frac{n+8}{15}\right\rfloor - \left\lfloor\frac{n}{5}\right\rfloor + \left\lfloor\frac{n}{15}\right\rfloor \\
\geq \quad & n - \frac{n}{3} - \frac{2n+16}{15} - \frac{n}{5} + \frac{n}{15} - 1 \\
= \quad & \frac{6n-31}{15} \geq 0
\end{aligned}$$

if $n \geq 74$, so the inequality holds for $p = 5$.

Similarly,

$$\begin{aligned}
& n - |E_7(2n)| - 2\left|O_7\left(\frac{2n+1}{7}\right)\right| - \left\lfloor\frac{n}{7}\right\rfloor + \left|E_7\left(\frac{2n}{7}\right)\right| \\
= \; & n - \left(\left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{n}{5}\right\rfloor - \left\lfloor\frac{n}{15}\right\rfloor\right) - 2\left(\left\lfloor\frac{n+11}{21}\right\rfloor + \left\lfloor\frac{n+18}{35}\right\rfloor\right. \\
& \left. - \left\lfloor\frac{n+53}{105}\right\rfloor\right) - \left\lfloor\frac{n}{7}\right\rfloor + \left(\left\lfloor\frac{n}{21}\right\rfloor + \left\lfloor\frac{n}{35}\right\rfloor - \left\lfloor\frac{n}{105}\right\rfloor\right) \\
\geq \; & n - \frac{n}{3} - \frac{n}{5} + \frac{n}{15} - 1 - \frac{2n+22}{21} - \frac{2n+36}{35} \\
& + \frac{2n+106}{105} - 1 - \frac{n}{7} + \frac{n}{21} - 1 + \frac{n}{35} - 1 - \frac{n}{105} \\
= \; & \frac{34n-532}{105} \geq 0
\end{aligned}$$

if $n \geq 74$, so the inequality holds for $p = 7$.

For $p \geq 11$, we consider a few cases. First, if $2n+1$ is not a multiple of p, then $|O_p((2n+1)/p)| = |O_p(2n/p)|$ and the desired inequality follows directly from Corollary 3.3.

Assume now that $2n+1 = \alpha p$; if $\alpha \geq 7$, then $\{3, 5, 7\} \subset O_p((2n+1)/p)$, so $|O_p((2n+1)/p)| \geq 3$. Additionally, $|O_p((2n+1)/p)| \leq |O_p(2n/p)| + 1$, so it follows that $3|O_p(2n/p)| \geq 3|O_p((2n+1)/p)| - 3 \geq 2|O_p((2n+1)/p)|$. The desired inequality again follows from Corollary 3.3.

Finally, we consider the case $2n+1 = \alpha p$ for $1 \leq \alpha \leq 6$. Set

$$\begin{aligned}
f(\alpha) &= n - |E_p(2n)| - 2\left|O_p\left(\frac{2n+1}{p}\right)\right| - \left\lfloor\frac{n}{p}\right\rfloor + \left|E_p\left(\frac{2n}{p}\right)\right| \\
&= n - |E_p(2n)| - 2\left|O_p(\alpha)\right| - \left\lfloor\frac{\alpha p - 1}{2p}\right\rfloor + \left|E_p\left(\alpha - \frac{1}{p}\right)\right|,
\end{aligned}$$

and note that $n - |E_p(2n)|$ counts the number of even integers at most $2n$ that do not have a prime factor less than p. If $\alpha \in \{1, 2\}$, then

$$f(\alpha) = n - |E_p(2n)| - 2(0) - 0 + 0 \geq 0$$

since $n - |E_p(2n)|$ is clearly nonnegative. If $\alpha \in \{3, 4\}$, then $n \geq 3p \geq 33$ and

$$f(\alpha) = n - |E_p(2n)| - 2(1) - 1 + 0 \geq 0$$

since $\{2, 4, 8, 16, 32, 2p\}$ forms a subset of the integers counted by $n - |E_p(2n)|$. If $\alpha \in \{5, 6\}$, then $n \geq 5p \geq 55$ and

$$f(\alpha) = n - |E_p(2n)| - 2(2) - 2 + 0 \geq 0$$

since $\{2, 4, 8, 16, 32, 2p, 4p\}$ forms a subset of the integers counted by $n - |E_p(2n)|$. This completes the proof. □

We are now ready to prove that every 2-regular graph with exactly one odd component has a prime labeling. For a graph G and any subset of vertices $S \subseteq V(G)$, let $N(S)$ denote the open neighborhood of S.

Theorem 3.5. *Suppose $G = \bigcup_{i=1}^{m} C_{n_i}$ is a 2-regular graph with n_i even for $i = 1, 2, \ldots, m-1$ and $n_m \geq 149$ odd. Then G has a prime labeling.*

Proof. We know $|V(G)| = \sum_{i=1}^{m} n_i$ is odd, so let $|V(G)| = 2n + 1$. Since $|V(G)| \geq 149$, we know $n \geq 74$. Let $G' = \bigcup_{i=1}^{m-1} C_{n_i}$, and let $f : V(G') \to \{a, b\}$ be a proper 2-coloring of the vertices of G'. Additionally, let $g : V(C_{n_m}) \to \{a, b, c\}$ be a proper 3-coloring of the vertices of C_{n_m} with precisely one vertex colored c; call this vertex v_c. Let A and B be the subsets of vertices of G colored a and b, respectively; we know $|A| = |B| = n$. We describe an inductive algorithm that produces a prime labeling of G such that all the even labels are placed on vertices in A, 1 is placed on v_c, and the remaining odd labels are placed on vertices in B.

Let $L = \{1, 2, \ldots, 2n + 1\}$ be the set of labels that need to be used in a prime labeling of G. Choose any subset $A_0 \subset A$ such that $|A_0| = \lfloor n/3 \rfloor$ and arbitrarily place the even multiples of 3 from L on vertices in A_0. It is easy to deduce that $|B \setminus N(A_0)| \geq n - 2\lfloor n/3 \rfloor \geq \lfloor (n+2)/3 \rfloor$, so choose any subset $B_0 \subseteq B \setminus N(A_0)$ such that $|B_0| = \lfloor (n+2)/3 \rfloor$ and arbitrarily place the odd multiples of 3 from L on vertices in B_0. Clearly $L(3) = A_0 \cup B_0$ is an independent set, and the sets $E_5(2n+1) \subset L$ and $O_5(2n+1) \subset L$ represent the even and odd labels, respectively, that have been used.

Now take $p \leq 2n + 1$ to be a prime greater than 3, and assume that all of the labels in $E_p(2n+1) \subset L$ have been placed on vertices in A and all of the labels in $O_p(2n+1) \subset L$ have been placed on vertices in B such that $L(q)$ is an independent set for every odd prime $q < p$. We extend this (partial) labeling to include all multiples of p. If $p > n$, then p is the only label in L that is a multiple of p, and this label can be arbitrarily placed on any unlabeled vertex in B. Thus, we can assume $p \leq n$.

We first show that we can place the even multiples of p. The labels that need to be placed are all the even multiples of p that do not have a smaller odd prime factor. If $x \leq 2n$ is an even multiple of p with a smaller odd prime factor, then $x/p \le 2n/p$ has an odd prime factor less than p. The number of such integers is $|E_p(2n/p)|$, so there are $\lfloor n/p \rfloor - |E_p(2n/p)|$ even multiples of p that need to be placed. We exclude all vertices already labeled with an even multiple of an odd prime less than p, of which there are $|E_p(2n)|$. We also exclude any neighbors of vertices already labeled with an odd multiple of p; i.e. any odd integer y such that y/p is less than $(2n+1)/p$ and has an odd prime factor less than p. There are $|O_p((2n+1)/p)|$ such vertices, so we exclude the $2|O_p((2n+1)/p)|$ potential neighbors of these vertices. This yields at least $n - |E_p(2n)| - 2|O_p((2n+1)/p)|$ available vertices in A, and $n-|E_p(2n)|-2|O_p((2n+1)/p)| \ge \lfloor n/p \rfloor - |E_p(2n/p)|$ by Corollary 3.4. Thus, we can arbitrarily place the remaining even multiples of p on the subset of available vertices.

We now place the odd multiples of p. Any odd positive integer at most $2n+1$ with a prime factor less than p will have already been assigned to a vertex in B, so the number of unused vertices in B is equal to the number of positive integers less than or equal to $2n+1$ that are not divisible by any prime less than p. If we let B_1 denote the vertices in B already labeled, then by Theorem 3.1 $|B \setminus B_1| = \Phi(2n+1, p)$. Let $A_1 \subset A$ denote the set of all vertices labeled with an even multiple of p, so that $|A_1| = \lfloor n/p \rfloor$ and $|N(A_1)| \le 2\lfloor n/p \rfloor$. Then the number of available vertices in B is at least $|B \setminus (B_1 \cup N(A_1))| \ge \Phi(2n+1, p) - 2\lfloor n/p \rfloor$. Some of the odd multiples of p may have been placed already, but there are only $\lfloor (2n+p+1)/2p \rfloor$ total odd multiples of p, and $\Phi(2n+1, p) - 2\lfloor n/p \rfloor \ge \lfloor (2n+p+1)/2p \rfloor$ by Corollary 3.2. Thus any remaining odd multiples of p can be arbitrarily placed on vertices in $B \setminus (B_1 \cup N(A_1))$. This ensures that $L(p)$ is an independent set.

After we have labeled G with all multiples of every odd prime at most $2n+1$, the only remaining unused labels from L are the powers of 2 less than $2n+1$ and 1. The powers of 2 can be arbitrarily placed on the only remaining unlabeled vertices in A, while 1 is placed on v_c. Since $L(2) = A$ is also an independent set, the resulting labeling is a prime labeling of G. $\square$

Corollaries 2.2 and 2.3 together with Theorem 3.5 complete the proof of Conjecture 1.2.

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