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Maximum $\mathcal{H}$-free subgraphs

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Given a family of hypergraphs $\mathcal{H}$, let $f(m, \mathcal{H})$ denote the largest size of an $\mathcal{H}$-free subgraph that one is guaranteed to find in every hypergraph with m edges. This function was first introduced by Erdős and Komlós in 1969 in the context of union-free families, and various other special cases have been extensively studied since then. In an attempt to develop a general theory for these questions, we consider the following basic issue: which sequences of hypergraph families $\{\mathcal{H}_m\}$ have bounded $f(m, \mathcal{H}_m)$ as $m \to \infty$? A variety of bounds for $f(m, \mathcal{H}_m)$ are obtained which answer this question in some cases. Obtaining a complete description of sequences $\{\mathcal{H}_m\}$ for which $f(m, \mathcal{H}_m)$ is bounded seems hopeless.

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1. Introduction

A hypergraph H on vertex set $V(H)$ is a subset of $2^{V(H)}$. H is an ℓ-uniform hypergraph, or simply, an ℓ-graph, if $H \subseteq \binom{V(H)}{\ell}$. All hypergraphs in this paper have finitely many vertices (and edges). Given a family of hypergraphs $\mathcal{H}$, a hypergraph F is said to be $\mathcal{H}$-free if F contains no copy of any member of $\mathcal{H}$ as a (not necessarily induced) subgraph. Given a hypergraph F and a family $\mathcal{H}$, let $\mathrm{ex}(F, \mathcal{H})$ be the maximum size of an $\mathcal{H}$-free subgraph of F. Define

$$f(m, \mathcal{H}) := \min_{|F|=m} \mathrm{ex}(F, \mathcal{H}).$$

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Note that $f(m,\mathcal{H}) \geq c$ means that every F with m edges contains an $\mathcal{H}$-free subgraph $F' \subseteq F$ with $|F'| = c$. When the family $\mathcal{H}$ consists of a single hypergraph H, we abuse notation and write $f(m,H)$ instead of $f(m,\{H\})$.

This function was introduced by Erdős and Komlós in 1969 [1], who considered the case when $\mathcal{H}$ is the (infinite) family of hypergraphs A, B, C with $A \cup B = C$. The problem was further studied by Kleitman [2], and later by Erdős and Shelah [3], and finally settled by Fox, Lee and Sudakov [4] who proved that

$$f(m,\mathcal{H}) = \left\lfloor \sqrt{4m+1} \right\rfloor - 1.$$

Erdős and Shelah also considered the case when $\mathcal{H}$ is the family of hypergraphs A_1, A_2, A_3, A_4 with $A_1 \cup A_2 = A_3$ and $A_1 \cap A_3 = A_4$. They called this family B_2, proved that $f(m, B_2) \leq (3/2)m^{2/3}$ and conjectured that this bound is asymptotically tight. This conjecture was settled by Barát, Füredi, Kantor, Kim and Patkós in 2012 [5], who also considered more general problems (see [4] for further work).

The same problem has been studied in the special case when $\mathcal{H}$ is a family of graphs. Let $f_2(m,\mathcal{H})$ denote the maximum size of an $\mathcal{H}$-free subgraph that every graph with m edges is guaranteed to contain. These investigations began with a question of Erdős and Bollobás [6] in 1966 about $f_2(m, C_4)$, followed up by a conjecture of Erdős in [7]. Consequently the problem of determining $f_2(m,H)$ for various graphs has received considerable attention in the recent years [8, 9, 10]. The authors of [9, 10] also considered the problem in the case of ℓ-graphs.

In the hope of obtaining a general theory for these problems, we investigate the following basic question:

$$\begin{array}{c} \text{For which sequence of families } \{\mathcal{H}_m\}_{m=1}^{\infty} \\ \text{is } f(m,\mathcal{H}_m) \text{ bounded (as } m \to \infty)\text{?} \end{array} \tag{1}$$

Question (1) is too general to solve completely, so we focus on special cases. In subsection 2.1 we state our results for constant $\{\mathcal{H}_m\}_{m=1}^{\infty}$, and in subsection 2.2 we consider non-constant $\{\mathcal{H}_m\}_{m=1}^{\infty}$.

2. Our results

2.1. Constant sequences

Suppose $\{\mathcal{H}_m\}_{m=1}^{\infty}$ is a sequence such that $\mathcal{H}_m = \mathcal{H}$ for every m. First, we note that if $\mathcal{H}$ consists of finitely many members, then the answer to

Question (1) is given by the following characterization. A q-sunflower is a hypergraph $\{A_1, \ldots, A_q\}$ such that $A_i \cap A_j = \bigcap_{s=1}^{q} A_s$ for all $i \neq j$. This common intersection is referred to as the *core* of the sunflower.

Theorem 2.1. *Fix a family of hypergraphs $\mathcal{H}$ with finitely many members. If $\mathcal{H}$ contains a q-sunflower with sets of equal size, then $f(m, \mathcal{H}) \leq q - 1$. Otherwise, $f(m, \mathcal{H}) \to \infty$ as $m \to \infty$.*

Next, in the same spirit as the properties of being union-free and having no B_2, if the (infinite) family $\mathcal{H}$ specifies the intersection type of k sets (i.e. whether they are empty or not), then a characterization can be obtained in the form of Theorem 2.3. Before stating the theorem, we first define what we call an ℓ-even hypergraph and an ℓ-uneven hypergraph. A k-edge hypergraph is a hypergraph with k edges.

Definition 2.2 (ℓ-even and ℓ-uneven hypergraphs)**.** A k-edge hypergraph $H = \{A_1, \ldots, A_k\}$ is said to be *ℓ-even* for some $1 \leq \ell \leq k$ if for every subset $I \subseteq [k]$,

$$\bigcap_{i \in I} A_i \neq \varnothing \text{ iff } |I| \leq \ell.$$

It is said to be *ℓ-uneven* if there exist $I, J \in \binom{[k]}{\ell}$ such that

$$\bigcap_{i \in I} A_i \neq \varnothing \text{ but } \bigcap_{j \in J} A_j = \varnothing.$$

Theorem 2.3. *Let $1 \leq \ell < k$. Let $\mathcal{H}$ be the (infinite) family of all ℓ-uneven k-edge hypergraphs. Then, $f(m, \mathcal{H}) \to \infty$ as $m \to \infty$. Conversely, if $\mathcal{H}$ is the family of all ℓ-even k-edge hypergraphs, we have $f(m, \mathcal{H}) = k - 1$.*

2.2. Non-constant sequences

As a first step towards understanding the general problem in (1), we focus on the case when for every $m \geq 1$, $\mathcal{H}_m = \{H_m\}$ for a single hypergraph H_m, and further assume that all these hypergraphs H_m have the same number of edges. Thus we ask the following question:

$$\text{For which sequence of } k\text{-edge hypergraphs } \{H_m\}_{m=1}^{\infty} \text{ is } f(m, H_m) \text{ bounded (as } m \to \infty)? \tag{2}$$

We are unable to answer question (2) completely, even for $k = 3$. Our main results provide several necessary, or sufficient conditions that partially answer (2). Before presenting them, we introduce the following crucial definition:

Definition 2.4 (Equal Intersection Property)**.** For $k \geq 2$, Let $\mathbf{EIP}_k$ denote the set of all k-edge hypergraphs $H = \{A_1, \dots, A_k\}$ such that for every $1 \leq \ell \leq k$ and $I, J \in \binom{[k]}{\ell}$, we have $\left|\bigcap_{i \in I} A_i\right| = \left|\bigcap_{j \in J} A_j\right|$.

Since every two edges of a hypergraph form a 2-sunflower, we observe that the case $k = 2$ follows immediately from the construction in Theorem 2.1.

Proposition 2.5. *Let H_m be a 2-edge hypergraph for each $m \geq 1$. Then $f(m, H_m)$ is bounded as $m \to \infty$ if and only if $H_m \in \mathbf{EIP}_2$ for all but finitely many m.*

We may therefore assume in what follows that $k \geq 3$.

Let us now fix a hypergraph $H = \{A_1, \dots, A_k\}$ in $\mathbf{EIP}_k$. H can be encoded by k parameters $(b_1, \dots, b_k)$, corresponding to the k distinct sizes appearing in the Venn diagram of H. More precisely, for $1 \leq \ell \leq k$, and for all $I \in \binom{[k]}{\ell}$, let

$$b_\ell := \left| \bigcap_{i \in I} A_i \setminus \bigcup_{i \in [k] \setminus I} A_i \right|.$$

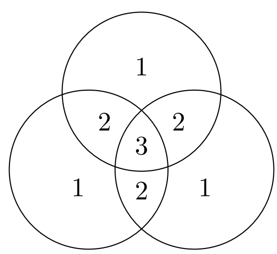

Figure 1: An example: $H(1, 2, 3) \in \mathbf{EIP}_3$.

By inclusion-exclusion, $b_1, \dots, b_k$ are well-defined for hypergraphs in $\mathbf{EIP}_k$. We denote $H \in \mathbf{EIP}_k$ with parameters $b_1, \dots, b_k \geq 0$ by $H(\vec{b})$, where $\vec{b} = (b_1, \dots, b_k)$. We shall see later (Lemma 4.1) that every sequence of k-edge hypergraphs $\{H_m\}$ such that $f(m, H_m)$ is bounded can only have finitely many members not in $\mathbf{EIP}_k$. For sequences $\{H_m\}_{m=1}^{\infty}$ such that $H_m \in \mathbf{EIP}_k$ for every $m \geq 1$, we obtain a sequence of length k vectors $\{\vec{b}(m)\}_{m=1}^{\infty}$, where $\vec{b}(m) = (b_1(m), \dots, b_k(m))$. We use boldface and write $\vec{\mathbf{b}}$ for the sequence $\{\vec{b}(m)\}_{m=1}^{\infty}$.

Definition 2.6 ($\alpha(\vec{\mathbf{b}})$)**.** For every sequence of length k vectors $\vec{\mathbf{b}} = \{\vec{b}(m)\}_{m=1}^{\infty}$ and $m \geq 1$, let

$$\alpha(\vec{\mathbf{b}})(m) := \min_{1\leq i\leq k-2}\left(\frac{b_i(m)}{mb_{i+1}(m)}\right).$$

Now we state our main results. To simplify notation we will often write b_i instead of $b_i(m)$ and $\alpha(\vec{\mathbf{b}})$ instead of $\alpha(\vec{\mathbf{b}})(m)$.

Theorem 2.7. *Let $k \geq 3$. Suppose the sequence of length k vectors $\vec{\mathbf{b}}$ satisfies $b_1, \ldots, b_{k-2} > 0$, $b_{k-1}, b_k \geq 0$ for every m. Then, for $m \geq 6$,*

$$\left(\frac{1}{2\left(\alpha(\vec{\mathbf{b}})+\frac{1}{m}\right)\binom{b_{k-1}+b_k}{b_k}}\right)^{\frac{1}{k}} \leq f(m, H(\vec{\mathbf{b}})) \leq \frac{k(k-1)}{\alpha(\vec{\mathbf{b}})} + k - 1.$$

Theorem 2.7 implies that when $\binom{b_{k-1}+b_k}{b_k}$ is bounded from above, $f(m, H(\vec{\mathbf{b}}))$ is bounded from above if and only if the sequence $\alpha(\vec{\mathbf{b}})$ is bounded away from zero.

We also have the following additional lower bound on $f(m, H(\vec{\mathbf{b}}))$:

Theorem 2.8. *Fix $k \geq 3$. Let $\vec{\mathbf{b}} = \{\vec{b}(m)\}_{m=1}^{\infty}$ be such that $b_k(m) = b_k$ for every m. Then, for $m \geq 6$,*

$$f(m, H(\vec{\mathbf{b}})) \geq \begin{cases} m^{\frac{1}{k(b_k+1)}}\left(\frac{b_{k-1}}{4(b_{k-2}+2b_{k-1})}\right)^{\frac{1}{k}}, & k \geq 4, \\ m^{\frac{1}{b_3+2}}\left(\frac{b_2}{4(b_1+2b_2)}\right)^{\frac{b_3+1}{b_3+2}}, & k = 3. \end{cases}$$

We now focus on $k = 3$. In this case $\alpha(\vec{\mathbf{b}}) = b_1/mb_2$ and Theorem 2.7 reduces to

$$\left(\frac{1}{2\left(\frac{b_1}{mb_2}+\frac{1}{m}\right)\binom{b_2+b_3}{b_3}}\right)^{\frac{1}{3}} \leq f(m, H(\vec{\mathbf{b}})) \leq \frac{6mb_2}{b_1} + 2. \tag{3}$$

When $b_3 = 0$, (3) implies that $f(m, H_3(b_1, b_2, 0))$ is bounded if and only if $b_1 = \Omega(mb_2)$. We now turn to $b_3 = 1$ which already seems to be a very interesting special case that is related to an open question in extremal graph theory (see Problem 7.3 in Section 7). Here (3) and Theorem 2.8 yield the following.

Corollary 2.9. Let $m \to \infty$. Then $f(m, H_3(b_1, b_2, 1))$ is bounded when $b_1 = \Omega(mb_2)$ and it is unbounded when either $b_1 + b_2 = o(m)$ or $b_1 = o(\sqrt{m}\, b_2)$.

Corollary 2.9 can be summarized in Figure 2. The light region corresponds to a bounded $f(m, H(\vec{\mathbf{b}}))$, and the dark region corresponds to unbounded $f(m, H(\vec{\mathbf{b}}))$. White regions correspond to areas where we do not know if $f(m, H(\vec{\mathbf{b}}))$ is bounded or not.

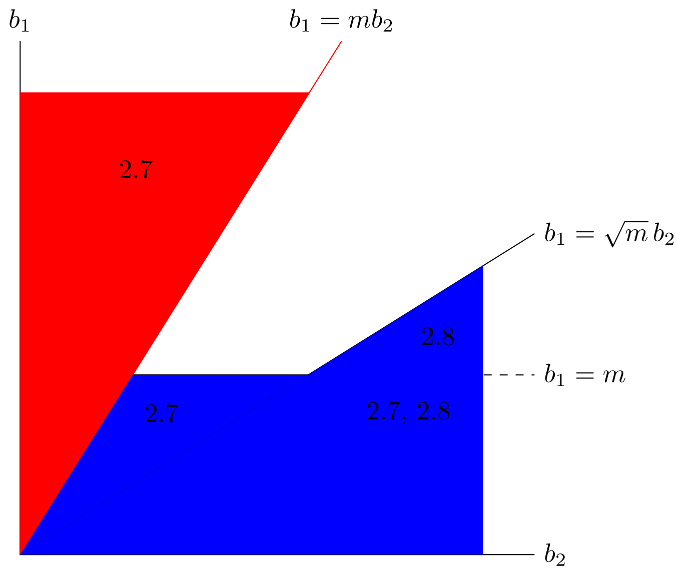

Figure 2: Theorems 2.7 and 2.8 for $\vec{\mathbf{b}} = (b_1, b_2, 1)$.

We are able to refine our results slightly via the following result.

Theorem 2.10. *For every odd prime power q we have*

$$f(q^2 + 1, H(q^2 - q - 1, q, 1)) = 2.$$

For functions $f(m)$ and $g(m)$, we write $f \gg g$ iff $g = o(f)$. Later, we shall show that Theorem 2.10 implies the following.

Corollary 2.11. When $b_1 \geq b_2^2$, $b_2 \geq \sqrt{m}$ and b_2 is a prime power,

$$f(m, H_3(b_1, b_2, 1)) = 2. \tag{4}$$

Further, when $b_1 \gg b_2^2$ and $b_2 \geq m^{0.68}$,

$$f(m, H_3(b_1, b_2, 1)) = 2. \tag{5}$$

Corollary 2.11 yields the following improvement on Figure 2. Note that we are using the parabola $b_1 = b_2^2$ as an asymptotic approximation of Corollary 2.11. By (4), $f(m, H_3(b_1, b_2, 1)) = 2$ infinitely often on this parabola, figuratively represented by vertical stripes in the interval $\sqrt{m} \le b_2 \le m^{0.68}$. We shall see later, by virtue of Theorem 7.2, that in the white region to the right of $b_1 = b_2^2$ and between the lines $b_1 = mb_2$ and $b_1 = \sqrt{m}\, b_2$, we have $f(m, H_3(b_1, b_2, 1)) > 2$.

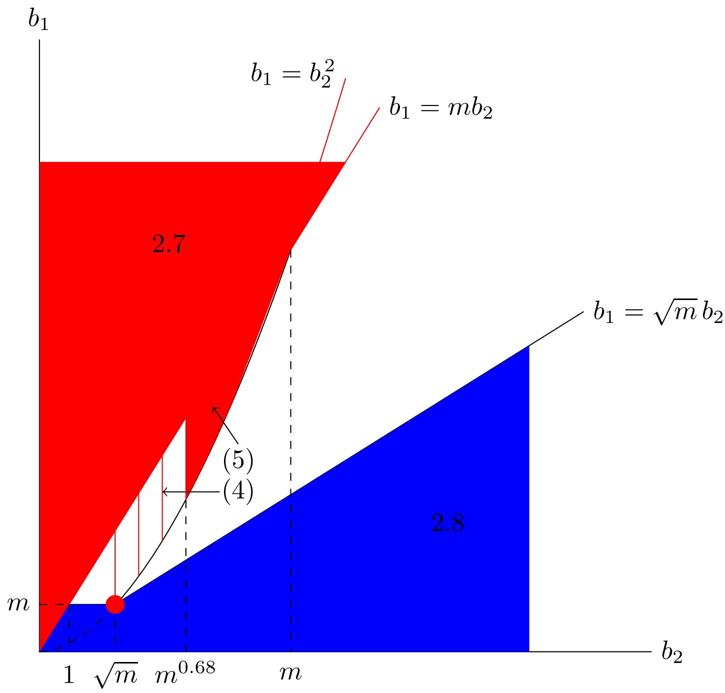

Figure 3: $\vec{\mathbf{b}} = (b_1, b_2, 1)$.

3. Proofs of Theorems 2.1 and 2.3

In this section, we prove Theorems 2.1 and 2.3, which answer question (1) for constant sequences. We use the following well-known facts about sunflowers and diagonal hypergraph Ramsey numbers.

Recall that a q-sunflower is a hypergraph $\{A_1, \ldots, A_q\}$ such that $A_i \cap A_j = \bigcap_{s=1}^{q} A_s$. The celebrated Erdős-Rado sunflower Lemma [11] states the following.

Lemma 3.1 (Erdős-Rado)**.** *Let H be an r-graph with $|H| = r!(\alpha-1)^r$. Then, H contains an α-sunflower.*

Next, recall that the hypergraph Ramsey number $r_\ell(s,t)$ is the minimum N such that any ℓ-graph on N vertices, admits a clique of size s or an independent set of size t. The following is a well-known theorem of Erdős, Hajnal and Rado [12]:

Theorem 3.2. *There are absolute constants $c(\ell), c'(\ell)$ such that*

$$twr_{\ell-1}(c't^2) < r_\ell(t,t) < twr_\ell(ct).$$

Here the tower function $twr_k(x)$ is defined by $twr_0(x) = 1$ and $twr_{i+1}(x) = 2^{twr_i(x)}$.

The right side of this theorem can be rewritten as follows:

(6) Let F be any ℓ-graph on n vertices. Then there is an absolute constant c_ℓ such that there is a subgraph $F' \subset F$ with $|V(F')| \geq c_\ell \cdot \log_{(\ell)}(n)$, which is either a clique or an independent set. Here $\log_{(\ell)}$ denotes iterated logarithms.

Now we are prepared to prove Theorems 2.1 and 2.3. Recall that a hypergraph is uniform if all its edges have the same size, otherwise it is non-uniform.

Proof of Theorem 2.1. Fix a family of hypergraphs $\mathcal{H}$ with n members, $\mathcal{H} = \{H_1, \ldots, H_n\}$. Let $H_i \in \mathcal{H}$ be an r-uniform q-sunflower with core W. For every $m \geq q$, let F be an r-uniform m-sunflower with core W. Then every subset of F of size q is isomorphic to H_i, thus proving $f(m,\mathcal{H}) \leq q-1$.

Suppose now that $\mathcal{H}$ consists of ℓ many uniform hypergraphs labeled $H_1, \ldots, H_\ell$ (none of which are sunflowers), and $(n-\ell)$ many non-uniform hypergraphs labeled $H_{\ell+1}, \ldots, H_n$. For $1 \leq i \leq \ell$, let r_i be the uniformity of H_i. Given any hypergraph F with m edges, we find a large $\mathcal{H}$-free subgraph as follows. First, since H_n is non-uniform, it contains a set of size a and a set of size $b \neq a$. Clearly, at least half of the edges of F have size $\neq a$, or at least half of them have size $\neq b$. Take the appropriate subgraph $F_1 \subset F$ of size $\geq \frac{m}{2}$. By successively halving the sizes, we obtain a chain of hypergraphs $F_{n-\ell} \subset F_{n-\ell-1} \subset \cdots \subset F_1 \subset F$ such that $F_{n-\ell}$ is $\{H_{\ell+1}, \ldots, H_n\}$-free, and $|F_{n-\ell}| \geq \frac{m}{2^{n-\ell}}$.

We now deal with the uniform part of $\mathcal{H}$. Notice that by Lemma 3.1, any r-graph G with $|G| = m$ contains an α-sunflower, as long as $m > r!\alpha^r$.

Taking $\alpha = \lfloor c_r m^{1/r} \rfloor$ where $c_r = ((2r)!)^{-1/r}$, satisfies the required condition. So, every r-graph G of size m contains a sunflower of size $\lfloor c_r m^{1/r} \rfloor$.

Since H_ℓ is r_ℓ-uniform, we note that either $F_{n-\ell}$ contains a subgraph of size $\frac{1}{2}|F_{n-\ell}|$ which has no sets of size r_ℓ (and hence is H_ℓ-free), or there is a subgraph of size $\frac{1}{2}|F_{n-\ell}|$ which is r_ℓ-uniform. In the second case, using Lemma 3.1 on this subgraph, we obtain an H_ℓ-free subgraph of $F_{n-\ell}$ of size at least $c_{r_\ell}\left(\frac{m}{2^{n-\ell+1}}\right)^{\frac{1}{r_\ell}}$. Thus, in either case, we conclude that there exists an H_ℓ-free subgraph $F'_{n-\ell+1} \subset F_{n-\ell}$ such that

$$|F'_{n-\ell+1}| \geq \min\left\{\frac{m}{2^{n-\ell+1}}\ ,\ c_{r_\ell}\left(\frac{m}{2^{n-\ell+1}}\right)^{\frac{1}{r_\ell}}\right\} \geq c'_{\mathcal{H}} \cdot m^{\frac{1}{r_\ell}}.$$

We iterate the same argument $\ell - 1$ more times, to finally obtain a constant $C_{\mathcal{H}}$ and a subgraph $F'_\ell \subset F_{n-\ell}$ such that F'_ℓ is $\mathcal{H}$-free, and

$$|F'_\ell| \geq C_{\mathcal{H}} \cdot m^{\frac{1}{r_1 \cdots r_\ell}}. \qquad \square$$

Proof of Theorem 2.3. Let $F = \{F_1, \ldots, F_m\}$ have size m. Suppose $1 \leq \ell < k$, and $\mathcal{H}$ is the family of all ℓ-uneven k-graphs. Then, there are distinct subsets $I, J \in \binom{[k]}{\ell}$, such that for every $H = \{A_1, \ldots, A_k\} \in \mathcal{H}$, $\bigcap_{i\in I} A_i = \varnothing$ and $\bigcap_{j \in J} A_j \neq \varnothing$. Then, we construct an ℓ-graph G with vertex set F, and hyperedges $\{\{F_1, \ldots, F_\ell\} :\ F_1 \cap \cdots \cap F_\ell = \varnothing\}$. By (6), there is a a constant c_ℓ and a subset $F' \subseteq F$ of size $\geq c_\ell \cdot \log_{(\ell)}(m)$, such that F' is either a clique or an independent set in G. In either case, F' is $\mathcal{H}$-free.

On the other hand, suppose $\mathcal{H}$ is such that for some $1 \leq \ell \leq k$ and any $I \subseteq [k]$, $\bigcap_{i\in I} A_i \neq \varnothing$ iff $|I| \leq \ell$. For every $m \geq k$, we construct a hypergraph $F = \{F_1, \ldots, F_m\}$ in the following manner. Consider the bipartite graph $B = \left([m], \binom{[m]}{\ell}\right)$ where $x \in [m]$ is adjacent to $y \in \binom{[m]}{\ell}$ iff $x \in y$. Let F_i be the set of neighbors in B of the vertex $i \in [m]$. Notice that for any $I \subseteq [k]$,

$$\bigcap_{i \in I} F_i = \begin{cases} \varnothing, & |I| > \ell, \\ \neq \varnothing, & |I| \leq \ell. \end{cases}$$

This construction therefore shows that $f(m, \mathcal{H}) = k - 1$. $\square$

4. Proof of Theorem 2.7

In this section, we prove Theorem 2.7. We begin with some preliminary analysis of the family $\mathbf{EIP}_k$.

First, we make the crucial observation regarding question (2) that every sequence of k-edge hypergraphs $\{H_m\}$ such that $f(m, H_m)$ is bounded, can only have finitely many members not in $\mathbf{EIP}_k$. This follows immediately from Lemma 4.1. Furthermore, for any $H(\vec{b}) \in \mathbf{EIP}_k$, one can explicitly determine the relation between the intersection sizes and the parameters $b_1, \ldots, b_k$ by inclusion-exclusion. We state this relation in Lemma 4.2.

Lemma 4.1. *Suppose $H = \{A_1, \ldots, A_k\}$ satisfies the following for some $1 \le \ell \le k$: there are two sets of indices $I, J \in \binom{[k]}{\ell}$ such that $|\bigcap_{i\in I} A_i| = a$ and $|\bigcap_{j\in J} A_j| = b$ with $a \neq b$. Then there is a constant c_ℓ such that $f(m, H) \ge c_\ell \cdot \log_{(\ell)}(m)$.*

Proof of Lemma 4.1. Let F be any hypergraph with m edges. Construct an ℓ-graph G with F as its vertex set, and hyperedges

$$\{\{B_1, \ldots, B_\ell\} : |B_1 \cap \cdots \cap B_\ell| = a\}\,.$$

By (6), there exists a subset $F' \subseteq F$ of size $c_\ell \cdot \log_{(\ell)}(m)$ which is either a clique or an independent set in G. In either case, H cannot be contained in F'. ■

Lemma 4.1 implies that if there are infinitely many m such that $H_m \notin \mathbf{EIP}_k$, then for each such non-EIP hypergraphs we have $f(m, H_m) \ge c' \cdot \log_{(k)}(m)$, where c' is the absolute constant $c' = \min\{c_1, \ldots, c_k\}$. This is an infinite subsequence of $\{H_m\}$. Therefore, if $f(m, H_m)$ is bounded, then by looking at the tail of $\{H_m\}$, we may assume WLOG that $H_m \in \mathbf{EIP}_k$ for every $m \ge 1$.

Recall that hypergraphs $H \in \mathbf{EIP}_k$ are characterized by the length k-vector $\vec{b}$, and for every sequence of hypergraphs $\{H_m\}_{m=1}^{\infty}$, we have a corresponding sequence of length k vectors $\vec{\mathbf{b}}$.

We now state the relation between the intersection sizes and the parameters $b_1, \ldots, b_k$ for $H(\vec{b}) \in \mathbf{EIP}_k$.

Lemma 4.2. *Let $H(\vec{b}) \in \mathbf{EIP}_k$, and $a_i = |A_1 \cap \cdots \cap A_i|$, for each $1 \le i \le k$. Then,*

$$b_i = a_i - \binom{k-i}{1} a_{i+1} + \binom{k-i}{2} a_{i+2} - \cdots + (-1)^{k-i} \binom{k-i}{k-i} a_k. \tag{7}$$

Before proving Theorem 2.7, we prove an auxiliary upper bound in Lemma 4.3, which provides a better upper bound on $f(m, H(\vec{\mathbf{b}}))$ with tighter constraints on $\vec{\mathbf{b}}$.

Lemma 4.3. *Suppose $\vec{\mathbf{b}} = (b_1, \ldots, b_k)$ is such that $b_i \geq 0$, and for every $1 \leq i \leq k-1$,*

$$\sum_{j=i}^{k-1} (-1)^{j-i} \binom{m-k+j-i-1}{j-i} b_j \geq 0. \tag{8}$$

Then $f(m, H(b_1, \ldots, b_k)) = k-1$.

Proof of Lemma 4.3. Let $\vec{\mathbf{b}}$ satisfy the restrictions given in (8). Note that we need to construct a hypergraph sequence $\{F_m\}_{m=1}^{\infty}$, such that every k-edge subgraph of F_m is isomorphic to $H(\vec{b})$. To achieve this, we define the following general construction:

Construction 4.4 ($F_m^{d_1,\ldots,d_k}$)**.** Given $d_1, \ldots, d_k \geq 0$ and $m \geq k$, let $B = ([m], Y)$ be the bipartite graph with parts $[m]$ and Y, where Y is defined as follows. For $1 \leq \ell \leq k$ and $1 \leq j \leq d_\ell$, let

$$Y_j^\ell = \left\{ \begin{array}{cc} \{v_j^S : S \in \binom{[m]}{\ell}\}, & \ell < k \\ \{w_j\}, & \ell = k \end{array} \right\},$$

where $v_j^S \neq v_{j'}^{S'}$ for every $(j, S) \neq (j', S')$ and $w_j \neq w_{j'}$ for every $j \neq j'$. Then

$$Y = \bigcup_{\ell=1}^{k} \bigcup_{j=1}^{d_\ell} Y_j^\ell.$$

For $x \in [m]$ and $v_j^S \in Y$, let $(x, v_j^S) \in E(B)$ iff $x \in S$, and let $(x, w_j) \in E(B)$ for every $x \in [m]$ and $w_j \in Y$. Then, define $F_m^{d_1,\ldots,d_k} = \{A_1, \ldots, A_m\}$, where $A_i = N_B(i) \subset Y$ for $i = 1, \ldots, m$. ■

For example, the construction $F_4^{1,2,3}$ is given by:

$$\left\{ \begin{array}{l} A_1 = \{v_1^1; v_1^{12}, v_2^{12}, v_1^{13}, v_2^{13}, v_1^{14}, v_2^{14}; w_1, w_2, w_3\} \\ A_2 = \{v_1^2; v_1^{12}, v_2^{12}, v_1^{23}, v_2^{23}, v_1^{24}, v_2^{24}; w_1, w_2, w_3\} \\ A_3 = \{v_1^3; v_1^{13}, v_2^{13}, v_1^{23}, v_2^{23}, v_1^{34}, v_2^{34}; w_1, w_2, w_3\} \\ A_4 = \{v_1^4; v_1^{14}, v_2^{14}, v_1^{24}, v_2^{24}, v_1^{34}, v_2^{34}; w_1, w_2, w_3\} \end{array} \right\}.$$

Informally, in this example, A_i consists of one vertex v_1^i corresponding to $\{i\}$, two vertices v_1^{ij} and v_2^{ij} corresponding to two-element subsets $\{i, j\}$, and three vertices w_1, w_2, w_3 that are in the common intersection of all the A_i's, $1 \leq i \leq 4$.

We observe the following property of the intersection sizes of the edges of $F_m^{d_1,\ldots,d_k}$.

Claim 4.5. For $1 \le i \le k$ and any i-edge subgraph $\{A_{r_1}, \ldots, A_{r_i}\} \subset F_m^{d_1,\ldots,d_k}$, the size of the common intersection $a_i := |A_{r_1} \cap \cdots \cap A_{r_i}|$ is given by

$$a_i = d_i + \binom{m-i}{1} d_{i+1} + \cdots + \binom{m-i}{k-1-i} d_{k-1} + d_k. \tag{9}$$

Proof of Claim 4.5. Suppose $G = \{A_{r_1}, \ldots, A_{r_i}\} \subset F_m^{d_1,\ldots,d_k}$. We shall now count $|A_{r_1} \cap \cdots \cap A_{r_i}|$. For a fixed hypergraph $F_m^{d_1,\ldots,d_k} \supseteq G' \supseteq G$, let $U_{G'}$ denote the set of all vertices of $F_m^{d_1,\ldots,d_k}$ which are in all the edges of G' but none of the edges of $F_m^{d_1,\ldots,d_k} \setminus G'$. Notice that $A_{r_1} \cap \cdots \cap A_{r_i}$ is a disjoint union of $U_{G'}$'s, $G' \supseteq G$. Therefore,

$$a_i = |A_{r_1} \cap \cdots \cap A_{r_i}| = \sum_{G' \supseteq G} |U_{G'}| = \sum_{G' \supseteq G} \left| \bigcap_{X \in G'} X \setminus \bigcup_{X \notin G'} X \right|. \tag{10}$$

Fix a $G' \supseteq G$. Let $G' = \{A_{r_1}, \ldots, A_{r_i}, A_{s_1}, \ldots, A_{s_{|G'|-i}}\}$. We observe that,

- For $i \le |G'| < k$, $U_{G'}$ consists exactly of the vertices
$$\left\{ v_j^{\{r_1,\ldots,r_i,s_1,\ldots,s_{|G'|-1}\}} : 1 \le j \le d_{|G'|} \right\}.$$
- For $k \le |G'| < m$, $\bigcap_{X \in G'} X = \{w_1, \ldots, w_{d_k}\} \subseteq \bigcup_{X \notin G'} X$, thus
$$U_{G'} = \varnothing.$$
- For $|G'| = m$, $U_{G'} = \bigcap_{X \in G'} X = \{w_1, \ldots, w_{d_k}\}$.

Therefore,

$$|U_{G'}| = \begin{cases} d_{|G'|}, & i \le |G'| < k, \\ 0, & k \le |G'| < m, \\ d_k, & |G'| = m. \end{cases}$$

Plugging back these values into (10), we get

$$a_i = d_i + \binom{m-i}{1} d_{i+1} + \cdots + \binom{m-i}{k-1-i} d_{k-1} + d_k$$

for every $1 \le i \le k$. ∎

Now we return to the proof of Lemma 4.3. Given a length k vector $\vec{b} \geq 0$ which satisfies (8) for $1 \leq i \leq k-1$, let d_i be the left hand side of (8), i.e.,

$$d_i := \sum_{j=i}^{k-1} (-1)^{j-i} \binom{m-k+j-i-1}{j-i} b_j,$$

and let $d_k = b_k$. Now, we look at the construction $F_m = F_m^{d_1,\ldots,d_k}$, and pick any k-edge subgraph $G \subset F_m$. Observe that $G \in \mathbf{EIP}_k$, and therefore there is a length k vector $\vec{g}$ such that $G = H(\vec{g})$. It suffices to check that $\vec{g} = \vec{b}$.

Suppose $G = \{A_1, \ldots, A_k\}$. For $1 \leq i \leq k$, let $a_i := |A_1 \cap \cdots \cap A_i|$. Recall that Lemma 4.2 gave us a way of computing $\vec{g}$ in terms of $\vec{a}$, and Claim 4.5 computes $\vec{a}$ in terms of $\vec{d}$. In order to precisely write down these relations, we introduce a few matrices.

Notation. Let us define the following quantities for arbitrary $m \geq k \geq 1$.

- Let $a_{ij}^{(m)} = \binom{m-i}{j-i}$ and $b_{ij}^{(m)} = (-1)^{j-i}\binom{m-i}{j-i}$.[1] Then, denote by $A_{k,m}$ and $B_{k,m}$ the upper triangular matrices

$$A_{k,m} = (a_{ij}^{(m)})_{1 \leq i,j \leq k}, \text{ and } B_{k,m} = (b_{ij}^{(m)})_{1 \leq i,j \leq k},$$

- Let $\vec{\mathbf{1}}$ denote the all-one vector, and $\vec{\mathbf{0}}$ the all-zero vector.
- Define $D_{k-1,m} := \begin{bmatrix} A_{k-1,m} & \vec{\mathbf{1}} \\ \vec{\mathbf{0}}^\intercal & 1 \end{bmatrix}$.
- Let $W_{k-1,m}$ be the $(k-1) \times (k-1)$ matrix given by

$$W_{k-1,m} = (w_{ij}^{(m)})_{1 \leq i,j \leq k-1},$$

  where $w_{ij}^{(m)} = (-1)^{j-i}\binom{m-k+j-i-1}{j-i}$.
- Define $W'_{k-1,m} := \begin{bmatrix} W_{k-1,m} & \vec{\mathbf{0}} \\ \vec{\mathbf{0}}^\intercal & 1 \end{bmatrix}$. ■

First, we observe that the assertion of Lemma 4.2 can be rephrased as,

$$\vec{g} = B_{k,k}\vec{a}. \tag{11}$$

Next, in terms of matrices, equality (9) reads

$$\vec{a} = D_{k-1,m}\vec{d}. \tag{12}$$

[1] By our convention, $\binom{x}{y} = 0$ if $y < 0$. Thus $a_{ij}^{(m)} = b_{ij}^{(m)} = 0$ whenever $j < i$.

Finally, by the definition of $\vec{d}$, we have

$$\vec{d} = W'_{k-1,m}\vec{b}. \tag{13}$$

Putting together Equations (11, 12, 13), we obtain:

$$\vec{g} = B_{k,k}D_{k-1,m}W'_{k-1,m} \cdot \vec{b}.$$

By Proposition A.2 from the Appendix, we know that the product matrix $B_{k,k}D_{k-1,m}W'_{k-1,m}$ is I_k, and this concludes the proof of Lemma 4.3. ■

We now have gathered all the equipment required to complete the proof of Theorem 2.7.

Proof of Theorem 2.7. Recall that $\alpha = \min_{1\le i\le k-2}\left(\frac{b_i(m)}{mb_{i+1}(m)}\right)$, and we wish to prove that

$$f(m, H(\vec{\mathbf{b}})) \le \frac{k(k-1)}{\alpha} + k - 1.$$

Note that this bound is trivial if $\frac{k(k-1)}{\alpha} \ge m$, therefore we may assume that $\alpha m > k(k-1)$. From the definition of α, note that $b_i \ge \alpha m b_{i+1}$ for each $1 \le i \le k-2$. By successively applying these inequalities we obtain $b_i \ge \alpha m b_{i+1} \ge \alpha^2 m^2 b_{i+2} \ge \cdots \ge \alpha^{k-i-1}m^{k-i-1}b_{k-1}$. Thus,

$$\begin{aligned} b_i \ge \alpha m b_{i+1} &\ge \sum_{r=i+1}^{k-1} \frac{\alpha m}{k} \cdot b_{i+1} \\ &\ge \sum_{r=i+1}^{k-1} \frac{\alpha^{r-i}m^{r-i}}{k} \cdot b_r \\ &\ge \sum_{r=i+1}^{k-1} \left(\frac{\alpha m}{k}\right)^{r-i} b_r \\ &\ge \sum_{r=i+1}^{k-1} \binom{\lfloor \frac{\alpha m}{k} \rfloor}{r-i} b_r. \end{aligned} \tag{14}$$

The last inequality follows from $X^t \ge \binom{\lfloor X \rfloor}{t}$. Observe that the assumption $\frac{\alpha m}{k} > k-1$ implies $\lceil \frac{\alpha m}{k} \rceil \ge k$. Therefore, for $1 \le i \le k-2$ and $i+1 \le r \le k-1$, we have

$$\left\lfloor \frac{\alpha m}{k} \right\rfloor \ge \left\lceil \frac{\alpha m}{k} \right\rceil - k + r - i - 1 \ge 0.$$

Thus, (14) gives us

$$b_i \geq \sum_{r=i+1}^{k-1} \binom{\lfloor \frac{\alpha m}{k} \rfloor}{r-i} b_r \geq \sum_{r=i+1}^{k-1} \binom{\lceil \frac{\alpha m}{k} \rceil - k + r - i - 1}{r-i} b_r$$
$$\geq \sum_{r=i+1}^{k-1} (-1)^{r-i+1} \binom{\lceil \frac{\alpha m}{k} \rceil - k + r - i - 1}{r-i} b_r,$$

implying

$$b_i + \sum_{r=i+1}^{k-1} (-1)^{r-i} \binom{\lceil \frac{\alpha m}{k} \rceil - k + r - i - 1}{r-i} b_r \geq 0.$$

This is exactly the condition (8), with m replaced by $\lceil \frac{\alpha m}{k} \rceil$, so Lemma 4.3 gives us a hypergraph K on $\lceil \frac{\alpha m}{k} \rceil$ edges such that every k sets of K are isomorphic to $H(\vec{\mathbf{b}})$.

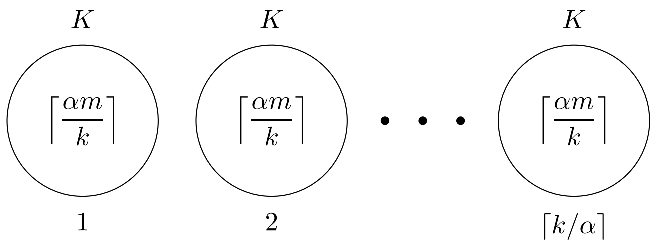

Figure 4: Constructing F_m from copies of K.

Now, consider a $\lceil \frac{k}{\alpha} \rceil$-fold disjoint union of K's. This hypergraph F_m has $\lceil \frac{k}{\alpha} \rceil \cdot \lceil \frac{\alpha m}{k} \rceil \geq m$ edges, and note that as long as we pick $1 + \lceil \frac{k}{\alpha} \rceil \cdot (k-1)$ edges, some k of them fall in the same copy of K. These k edges create a $H(\vec{\mathbf{b}})$ by construction of K. This shows $f(m, H(\vec{\mathbf{b}})) \leq \lceil \frac{k}{\alpha} \rceil \cdot (k-1)$, completing the proof of the upper bound.

Now we prove the lower bound. Recall that we are aiming to prove

$$f(m, H(\vec{\mathbf{b}})) \geq \max_{1 \leq i \leq k-2} \left(\frac{m b_{i+1}}{2(b_i + b_{i+1}) \binom{b_{k-1} + b_k}{b_k}} \right)^{\frac{1}{k}}. \tag{15}$$

Suppose F is a hypergraph on m edges. Either F has a subgraph F_1 of size $\frac{m}{2}$ which is of the same uniformity as $H(\vec{b})$, or it has a subgraph of size $\frac{m}{2}$ which is not of this uniformity. If the latter is true, then $\mathrm{ex}(F, H(\vec{b})) \geq \frac{m}{2}$.

Otherwise, we focus on the subgraph F_1. Let T be a $H(\vec{b})$-free subgraph in F_1 of maximum size, say $|T| = t$. Then, for every $S \in F_1 \setminus T$, there exist distinct $A_1, \dots, A_{k-1} \in T$ such that $\{A_1, \dots, A_{k-1}, S\}$ forms a $H(\vec{b})$. Therefore, there are fixed $A_1, \dots, A_{k-1} \in T$ and a subgraph $F_2 \subseteq F_1 \setminus T$ such that $\{A_1, \dots, A_{k-1}, S\}$ forms a $H(\vec{b})$ for every $S \in F_2$, where

$$|F_2| \geq \frac{\frac{m}{2} - t}{\binom{t}{k-1}}.$$

Further, note that $|A_1 \cap \cdots \cap A_{k-1} \cap S| = b_k$ for every $S \in F_2$, therefore there is a subgraph $F_3 \subseteq F_2$ such that every element $S \in F_3$ intersects $A_1 \cap \cdots \cap A_{k-1}$ in the exact same set, and

$$|F_3| \geq \frac{\frac{m}{2} - t}{\binom{t}{k-1}\binom{b_{k-1}+b_k}{b_k}}.$$

Finally, for any $1 \leq i \leq k-2$, let $X_i := A_1 \cap \cdots \cap A_i \setminus (A_{i+1} \cup \cdots \cup A_{k-1})$, and

$$h_i := |\{(x, B): \ x \in X_i, \ B \in F_3, \ x \in B\}|.$$

Let $D := \max_{x \in V(F_3)} \deg_{F_3}(x)$. As $\{A_1, \dots, A_{k-1}, B\}$ is an $H(\vec{b})$ for each $B \in F_3$,

$$|F_3| \cdot b_{i+1} = h_i \leq D \cdot |X_i|. \tag{16}$$

Now, for a fixed $S \in F_3$,

$$\begin{aligned}
|X_i| &= |S \cap X_i| + |X_i \setminus S| \\
&= \left| S \cap \bigcap_{j=1}^{i} A_j \setminus \left(\bigcup_{j=i+1}^{k-1} A_j \right) \right| + \left| \bigcap_{j=1}^{i} A_j \setminus \left(\bigcup_{j=i+1}^{k-1} A_j \cup S \right) \right| \\
&= b_{i+1} + b_i,
\end{aligned}$$

Therefore (16) implies

$$D \geq \frac{|F_3| \cdot b_{i+1}}{b_i + b_{i+1}} \geq \frac{(\frac{m}{2} - t) b_{i+1}}{\binom{t}{k-1}\binom{b_{k-1}+b_k}{b_k}(b_i + b_{i+1})}.$$

Note that the sets in F_3 that achieve the maximum degree D is $H(\vec{b})$-free. This is because if I is the common intersection of any set from F_3 with $A_1 \cap \cdots \cap A_{k-1}$, and if x is a vertex of degree D in F_3, then every edge through x contains $\{x\} \cup I$. This leads us to the inequality

$$t \geq \frac{(\frac{m}{2} - t)b_{i+1}}{\binom{t}{k-1}(b_i + b_{i+1})\binom{b_{k-1}+b_k}{b_k}},$$

i.e.,

$$t\binom{t}{k-1} \geq \frac{(\frac{m}{2} - t)b_{i+1}}{(b_i + b_{i+1})\binom{b_{k-1}+b_k}{b_k}}.$$

Since $m \geq 6$, note that if $t \geq \frac{m}{4}$, then $t \geq \left(\frac{m}{2}\right)^{\frac{1}{3}} \geq \left(\frac{m}{2}\right)^{\frac{1}{k}}$, which is larger than the right side of (15). So we may assume $t < \frac{m}{4}$, which would lead us to

$$t^k \geq 2t\binom{t}{k-1} \geq \frac{mb_{i+1}}{2(b_i + b_{i+1})\binom{b_{k-1}+b_k}{b_k}}. \tag{17}$$

As (17) holds for every $1 \leq i \leq k-2$, this gives the bound that we seek. $\square$

5. Proof of Theorem 2.8

In this section we prove Theorem 2.8. The proof is by induction on b_k, starting from $b_k = 0$. Notice that the lower bound of Theorem 2.7 gives us the following corollary, which serves as the base case for our induction argument:

Corollary 5.1. For $m \geq 6$,

$$f(m, H(b_1, \ldots, b_{k-1}, 0)) \geq \max_{1 \leq i \leq k-2} \left(\frac{mb_{i+1}}{2(b_i + b_{i+1})}\right)^{\frac{1}{k}}.$$

Further, one can asymptotically improve this bound when $k = 3$:

Proposition 5.2. *For $m \geq 4$,*

$$f(m, H(b_1, b_2, 0)) \geq \sqrt{\frac{mb_2}{2(b_1 + 2b_2)}}.$$

Proof. Let $|F| = m$ and $H = H(b_1, b_2, 0)$. Either F has a $(b_1 + 2b_2)$-uniform subgraph F_1 of size $\frac{m}{2}$, or it has a subgraph of size $\frac{m}{2}$ in which none of the edges have size $(b_1 + 2b_2)$. If the latter is true, then $\mathrm{ex}(F, H) \geq \frac{m}{2}$. Otherwise let us focus on F_1. Let T be an H-free subset of maximum size in F_1, and suppose $|T| = t$. Note that for any $B \in F_1 \setminus T$, there are sets $A_1, A_2 \in T$ such that (B, A_1, A_2) is a $H(b_1, b_2, 0)$. Suppose $V = \bigcup_{A \in T} A$, then we have $|B \cap V| \geq 2b_2$, and $|V| \leq t(b_1 + 2b_2)$. Let $D = \max_{x \in V} \deg_{F_1}(x)$. Then,

$$2b_2 \cdot |F_1 \setminus T| \leq |\{(x, B) :\ x \in V, B \in F_1 \setminus T, x \in B\}| \leq D \cdot |V|,$$

and

$$D \geq \frac{(m - 2t)b_2}{t(b_1 + 2b_2)}.$$

Let $x \in V$ have the maximum degree in F. Since the subgraph of size D containing x is H-free, we obtain

$$t \geq \frac{(m - 2t)b_2}{t(b_1 + 2b_2)}.$$

If $t \geq \frac{m}{4}$, then $t \geq \frac{1}{2}\sqrt{m} \geq \sqrt{\frac{mb_2}{2(b_1+2b_2)}}$. So assume $t < \frac{m}{4}$, and therefore $t^2 \geq \frac{mb_2}{2(b_1+2b_2)}$, as desired. $\square$

Before we prove Theorem 2.8 we require the following lemma from [13]:

Lemma 5.3. *Let $H = (V, E)$ be a k-graph on m vertices, and let $\alpha(H)$ denote the independence number of H. Then,*

$$\alpha(H) \geq \frac{k-1}{k} \cdot \left(\frac{m^k}{k|E(H)|}\right)^{\frac{1}{k-1}}.$$

Now we are prepared to prove Theorem 2.8.

Proof of Theorem 2.8. Fix k and $\vec{\mathbf{b}}$. Recall that b_k is fixed, and we wish to show that for $m \geq 6$,

$$f(m, H(b_1, \ldots, b_k)) \geq \begin{cases} m^{\frac{1}{k(b_k+1)}} \left(\frac{b_{k-1}}{4(b_{k-2}+2b_{k-1})}\right)^{\frac{1}{k}}, & k \geq 4, \\ m^{\frac{1}{b_3+2}} \left(\frac{b_2}{4(b_1+2b_2)}\right)^{\frac{b_3+1}{b_3+2}}, & k = 3. \end{cases} \tag{18}$$

Suppose $|F| = m$. Then, either F has a subgraph F_1 of size at least $\frac{m}{2}$ which has uniformity the same as that of $H(\vec{b})$, or it does not. When the latter is true, we have $\mathrm{ex}(F, H(\vec{b})) \geq \frac{m}{2}$. Since $\frac{m}{2} \geq m^{\frac{1}{4}} \cdot \left(\frac{1}{8}\right)^{\frac{1}{4}}$ and $\frac{m}{2} \geq m^{\frac{1}{2}} \cdot \left(\frac{1}{8}\right)^{\frac{1}{2}}$, we may assume that the former is true. We wish to show that F_1 contains a $H(\vec{b})$-free subgraph of large size.

We proceed by induction on b_k. Notice that we already established the results for $b_k = 0$ in Corollary 5.1 (using $b_{k-1} \le 2b_{k-1}$) and Proposition 5.2.

Construct a k-graph G with vertex set F_1 and call $\{A_1, \ldots, A_k\}$ an edge in G iff $\{A_1, \ldots, A_k\} \cong H(\vec{b})$. Clearly, $t = \alpha(G)$ is a lower bound to our problem. By Lemma 5.3,

$$k|E(G)| \ge \left(\frac{k-1}{k}\right)^{k-1} \cdot \frac{(m/2)^k}{t^{k-1}}.$$

Given $1 \le i \le k$ and $B_1, \ldots, B_i \in F_1$, denote by $\deg_G(B_1, \ldots, B_i)$ the number of edges of G containing $\{B_1, \ldots, B_i\}$. As

$$\sum_{A_1,\ldots,A_{k-2} \in F_1} \deg_G(A_1, \ldots, A_{k-2}) = \binom{k}{2} |E(G)|,$$

we obtain

$$\begin{aligned}\sum_{A_1,\ldots,A_{k-2} \in F_1} \deg_G(A_1, \ldots, A_{k-2}) &\ge \frac{\binom{k}{2}}{k} \cdot \frac{(k-1)^{k-1}}{k^{k-1}} \cdot \frac{(m/2)^k}{t^{k-1}} \\ &= \frac{(k-1)^k}{2k^{k-1}} \cdot \frac{(m/2)^k}{t^{k-1}}.\end{aligned}$$

The sum on the left side has at most $\binom{m/2}{k-2} \le \frac{(m/2)^{k-2}}{(k-2)!}$ terms, therefore there exist distinct $A_1, \ldots, A_{k-2} \in F_1$ such that

$$\deg_G(A_1, \ldots, A_{k-2}) \ge \frac{(k-2)!(k-1)^k}{2k^{k-1}} \cdot \frac{(m/2)^2}{t^{k-1}}.$$

Note that $\frac{(k-2)!(k-1)^k}{2k^{k-1}} > \frac{1}{4}$ for every $k \ge 3$. Let $\mathcal{B}$ denote the set of all edges $B \in F_1$ which are covered by an edge through $\{A_1, \ldots, A_{k-2}\}$ in G. Then, $|\mathcal{B}|^2 \ge \deg_G(A_1, \ldots, A_{k-2})$, and so

$$|\mathcal{B}|^2 \ge \frac{1}{4} \cdot \frac{(m/2)^2}{t^{k-1}} = \frac{1}{16} \cdot \frac{m^2}{t^{k-1}}. \tag{19}$$

As $\{A_1, \ldots, A_{k-2}\}$ is a subgraph of $H(\vec{b})$, we have

$$|A_1 \cap \cdots \cap A_{k-2}| = b_{k-2} + 2b_{k-1} + b_k.$$

Also, for every $B \in \mathcal{B}$, $\{A_1, \ldots, A_{k-2}, B\}$ is a subgraph of $H(\vec{b})$. Thus,

$$|A_1 \cap \cdots \cap A_{k-2} \cap B| = b_{k-1} + b_k.$$

Now,

$$\begin{aligned}|\mathcal{B}| \cdot (b_{k-1} + b_k) &= |\{(x, B) :\ x \in A_1 \cap \cdots \cap A_{k-2}, B \in \mathcal{B}, x \in B\}| \\ &= \sum_{x \in A_1 \cap \cdots \cap A_{k-2}} \deg_{\mathcal{B}}(x).\end{aligned}$$

Let D be the maximum degree of a vertex in F_1. Then, by (19),

$$D \cdot (b_{k-2} + 2b_{k-1} + b_k) \geq |\mathcal{B}| \cdot (b_{k-1} + b_k) \geq \frac{1}{4}(b_{k-1} + b_k) \cdot \frac{m}{t^{\frac{k-1}{2}}}. \tag{20}$$

Also, note that

$$\frac{b_{k-1} + b_k}{b_{k-2} + 2b_{k-1} + b_k} \geq \frac{b_{k-1}}{b_{k-2} + 2b_{k-1}} \iff b_k(b_{k-2} + b_{k-1}) \geq 0.$$

Therefore (20) gives us,

$$D \geq \frac{1}{4} \cdot \frac{b_{k-1}}{b_{k-2} + 2b_{k-1}} \cdot \frac{m}{t^{\frac{k-1}{2}}}. \tag{21}$$

Now, we notice that if x is a vertex of degree D, then deleting it from the edges through x gives us a family of uniformity one less than that of F_1. By induction on b_k, this subfamily already contains a $H(b_1, \ldots, b_{k-1}, b_k - 1)$-free family of size $f(D, H(b_1, \ldots, b_{k-1}, b_k - 1))$, which is a natural lower bound to our problem. Therefore,

$$t \geq f(D, H(b_1, \ldots, b_{k-1}, b_k - 1))$$

We now split into two cases.

- **Case I:** $k \geq 4$**.** Now we use the inductive lower bound given by (18):

$$t \geq D^{\frac{1}{kb_k}} \left(\frac{b_{k-1}}{4(b_{k-2} + 2b_{k-1})} \right)^{\frac{1}{k}} \iff D \leq \left(\frac{4(b_{k-2} + 2b_{k-1})}{b_{k-1}} \right)^{b_k} \cdot t^{kb_k}.$$

  Combining this bound with (21), we get

$$\left(\frac{4(b_{k-2} + 2b_{k-1})}{b_{k-1}} \right)^{b_k} \cdot t^{kb_k} \geq \frac{b_{k-1}}{4(b_{k-2} + 2b_{k-1})} \cdot \frac{m}{t^{\frac{k-1}{2}}},$$

Which, on invoking $t^{\frac{k-1}{2}} \leq t^k$, leads us to

$$t^{k(b_k+1)} \geq m \left(\frac{b_{k-1}}{4(b_{k-2} + 2b_{k-1})} \right)^{b_k+1},$$

finishing off the induction step.

- **Case II:** $k = 3$**.** In this case we use the inductive lower bound in (18) of

$$t \geq D^{\frac{1}{b_3+1}} \left(\frac{b_2}{4(b_1 + 2b_2)} \right)^{\frac{b_3}{b_3+1}} \iff D \leq \left(\frac{4(b_1 + 2b_2)}{b_2} \right)^{b_3} \cdot t^{b_3+1}.$$

Again, combining this bound with (21), we obtain

$$\left(\frac{4(b_1 + 2b_2)}{b_2} \right)^{b_3} \cdot t^{b_3+1} \geq \frac{b_2}{4(b_1 + 2b_2)} \cdot \frac{m}{t}.$$

This implies $t \geq m^{\frac{1}{b_3+2}} \left(\frac{b_2}{4(b_1+2b_2)} \right)^{\frac{b_3+1}{b_3+2}}$, completing the induction step.

□

6. Proof of Theorem 2.10

In this section, we prove Theorem 2.10. For the proof, we rely upon the incidence structure of Miquelian inversive planes $\mathbf{M}(q)$ of order q. An inversive plane consists of a set of points $\mathcal{P}$ and a set of circles $\mathcal{C}$ satisfying three axioms [14]:

- Any three distinct points are contained in exactly one circle.
- If $P \neq Q$ are points and c is a circle containing P but not Q, then there is a unique circle b through P, Q and satisfying $b \cap c = \{P\}$.
- $\mathcal{P}$ contains at least four points not on the same circle.

Every inversive plane is a 3-$(n^2 + 1, n + 1, 1)$-design for some integer n, which is called its order. An inversive plane is called Miquelian if it satisfies Miquel's theorem [14]. The usefulness of Miquelian inversive planes lies in the fact that their automorphism groups are sharply 3-transitive (cf. pp 274–275, Section 6.4 of [15]). There are a few known constructions of $\mathbf{M}(q)$, one such construction is outlined here. The points of $\mathbf{M}(q)$ are elements of

$\mathbb{F}_q^2$ and a special point at infinity, denoted by ∞. The circles are the images of the set $K = \mathbb{F}_q \cup \{\infty\}$ under the permutation group $PGL_2(q^2)$, given by

$$x \mapsto \frac{ax^\alpha + c}{bx^\alpha + d},\ ad - bc \neq 0, \alpha \in \mathrm{Aut}(\mathbb{F}_q^2).$$

For further information on inversive planes and their constructions, the reader is referred to [15, 16, 17].

Now, we prove Theorem 2.10.

Proof of Theorem 2.10.. Recall that for every odd prime power q, we are required to demonstrate a hypergraph on q^2+1 edges with the property that every three edges form an $H(q^2-q-1, q, 1)$. Let $\mathbf{M}(q)$ be a Miquelian inversive plane, with points labeled $\{1, 2, \dots, q^2+1\}$. Then, we consider the (q^2+q)-graph $F = \{A_1, \dots, A_{q^2+1}\}$, whose vertex set $V(F)$ is the circles of $\mathbf{M}(q)$, and A_i is the collection of circles containing i. By the inversive plane axiom, any three distinct points have a unique circle through them. It suffices to show that any two distinct points P, Q in $\mathbf{M}(q)$ have $q+1$ distinct circles through them. By 2-transitivity of the Automorphism group, we know that any two points have the same number a_2 of circles through them. Now, for any $P \neq Q$,

$$\begin{aligned}(q^2+1-2)\cdot 1 &= |\{(R, c) : R \text{ is a point}, c \text{ is a circle through } P, Q, R\}| \\ &= a_2 \cdot (q+1-2),\end{aligned}$$

Thus $a_2 = q+1$. So, F is (q^2+q)-uniform, every two edges of F have an intersection of size $q+1$, and every three edges of F have an intersection of size 1. By inclusion-exclusion, they form a $H(q^2-q-1, q, 1)$. ☐

Now, we prove Corollary 2.11.

Proof of Corollary 2.11. First, we prove (4), which asserts that whenever $b_1 \geq b_2^2 \geq m$ and b_2 is a prime power, $f(m, H_3(b_1, b_2, 1)) = 2$. Initially we start with an inversive plane construction, which gives gives us b_2^2+1 sets such that any three of them are an isomorphic copy of $H(b_2^2-b_2-1, b_2, 1)$. As long as $b_2^2+1 \geq m$, we can take a subgraph of the construction and still obtain m sets satisfying the same property. Also note that as $b_1 \geq b_2^2$, we can create a $H_3(b_1, b_2, 1)$-construction by first creating an inversive plane F, which is a $H_3(b_2^2-b_2-1, b_2, 1)$-construction, and then adding $(b_1-b_2^2+b_2+1)$ new distinct points to each set in F. This proves (4).

To prove (5), we shall use the result of Baker, Harman and Pintz [18] on the density of primes, which states that for sufficiently large x there is a prime p such that

$$x - x^{0.525} < p < x.$$

Let $g(x)$ be the inverse of $x - x^{0.525}$ for large x. Then, $x < g(p)$. Using monotonicity of g, it can be shown that $g(p) < p + p^{0.529}$ for large p. Thus, for large enough m, there exists a prime p such that

$$p < x < p + p^{0.529}. \tag{22}$$

Now, let $b_1 \gg b_2$ and $b_2 \geq m^{0.68}$, as in the hypothesis. From (22), we get a prime number p with $p < b_2 < p + p^{0.529}$. Let $F = \{A_1, \cdots, A_{p^2+1}\}$ be the $H_3(p^2 - p - 1, p, 1)$-construction obtained from Theorem 2.10. Note that $m < b_2^{0.68^{-1}} = b_2^{1.4706} < p^2 + 1$. Let $F' = \{A_1, \cdots, A_m\}$. For every $1 \leq i < j \leq m$, add $b_2 - p$ many new vertices $v_1^{ij}, \ldots, v_{b_2-p}^{ij}$ to the sets A_i and A_j, i.e, let

$$B_i = A_i \sqcup \bigcup_{j \neq i} \{v_r^{ij} : 1 \leq r \leq b_2 - p\}.$$

Suppose $K = \{B_i : 1 \leq i \leq m\}$. Observe that for every i,

$$|B_i| = p^2 + p + m(b_2 - p),$$

and for every $i \neq j$,

$$|B_i \cap B_j| = p + b_2 - p = b_2.$$

Hence, K is a hypergraph such that any three edges form a $H_3(p^2 + p + m(b_2 - p), b_2, 1)$. Since

$$\begin{aligned} p^2 + p + m(b_2 - p) &< p^2 + p + p^{1.4706+0.529} \\ &= p^2 + p^{1.9996} + p \\ &< 3b_2^2 \ll b_1, \end{aligned}$$

we can add adequately many new vertices to every edge of K in order to get a hypergraph whose any three edges form a $H_3(b_1, b_2, 1)$. $\square$

7. Further problems

We discuss a few further problems that are of interest. Of course, the main open question is (2), which asks to characterize all sequences of k-edge hypergraphs H_m for which $f(m, H_m)$ is bounded. As we discussed, even the case $k = 3$ turns out to be quite challenging..

Let us focus on the case $k = 3$ and $\vec{\mathbf{b}} = (b_1, b_2, 1)$. The current state of affairs was summarized in Figure 3. Observe that all the upper bounds in the lightly shaded regions are actually upper bounds of 2. Therefore, one may ask the following question:

Problem 7.1. Characterize all values of (b_1, b_2) such that

$$f(m, H_3(b_1, b_2, 1)) = 2.$$

We cannot solve this problem completely. However, we can derive a necessary condition on b_1, b_2, b_3 for which $f(m, H_3(b_1, b_2, b_3)) = 2$ as follows. Suppose F is a hypergraph with $V(F) = \{1, \cdots, n\}$ such that any three edges of F form a $H_3(b_1, b_2, b_3)$. Let d_i denote the degree of vertex i in F. By double-counting arguments,

$$\begin{aligned}
\sum_{i=1}^{n} \tbinom{d_i}{3} &= \tbinom{m}{3} b_3, \\
\sum_{i=1}^{n} \tbinom{d_i}{2} &= \tbinom{m}{2} (b_2 + b_3), \\
\sum_{i=1}^{n} d_i &= m(b_1 + 2b_2 + b_3).
\end{aligned}$$

After algebraic manipulation of these expressions and using the Cauchy-Schwarz inequality $\sum_{i=1}^n d_i \cdot \sum_{i=1}^n d_i^3 \geq \left(\sum_{i=1}^n d_i^2\right)^2$ and large m, we obtain Theorem 7.2.

Theorem 7.2. *Suppose $f(m, H_3(b_1, b_2, b_3)) = 2$. Then, for large enough m,*

$$b_1 b_3 + \frac{b_1 b_2}{m} + \frac{b_2 b_3}{m} \geq b_2^2.$$

In particular, when $b_3 = 1$,

$$b_1 + \frac{b_1 b_2}{m} \geq b_2^2.$$

Theorem 7.2 gives more insight into Figure 3. Basically, there are two cases to consider. When b_1 is asymptotically larger than $\frac{b_1 b_2}{m}$, i.e. when $b_2 = o(m)$, this means that $b_1 \geq b_2^2$ is necessary for $f = 2$. When $b_2 \geq m$, this gives us $b_1 \geq mb_2$, which is exactly the construction in Lemma 4.3. Further, note that this transition occurs exactly at the intersection of the line $b_1 = mb_2$ and the parabola $b_1 = b_2^2$.

As a further special case of Problem 7.1, one can look at $\vec{\mathbf{b}} = (m, b_2, 1)$ where $1 \ll b_2 \ll \sqrt{m}$. We expect this range to be solvable via a construction, since there are constructions for $b_2 = 1$ (Theorem 2.7) and $b_2 = \sqrt{m}$ (Theorem 2.10). The problem is equivalent to constructing bipartite graphs with certain properties, as stated below.

Problem 7.3. Suppose $1 \ll b_2 \ll \sqrt{m}$. Is there a bipartite graph G with parts A, B, such that $|A| = m$, the degree of every vertex in A is asymptotic to m, the size of the common neighborhood of every pair in A is asymptotic to b_2, and every three vertices in A have a unique common neighbor in B?

If such a bipartite graph can be constructed, then we can let $F = \{N_G(u) : u \in A\}$. This hypergraph will testify for $f(m, H(m, b_2, 1)) = 2$. From the proof of Theorem 7.2, we know that if such a bipartite graph exists, it cannot be regular from B: a regular construction from B implies equality in the Cauchy-Schwarz inequality, which would imply $b_2 = \Theta(\sqrt{m})$. Therefore if such a graph is constructed, B needs to have vertices of different degrees.

Notice also that if the answer to Problem 7.3 is affirmative, then we can shade the small triangle in Figure 3 light. This is courtesy of the fact that any (b_1, b_2) in that region can be written as a sum $(x, y) + (m, z)$, with $x \geq my$. We can then take a $H_3(x, y, 0)$-construction $\{A_1, \ldots, A_m\}$ and a $H_3(m, z, 1)$-construction $\{A'_1, \ldots, A'_m\}$, and merge them together to obtain the $H_3(b_1, b_2, 1)$-construction $\{A_1 \cup A'_1, \ldots, A_m \cup A'_m\}$.

Appendix A

Our goal in this section is to prove the matrix identity asserted in Proposition A.2. Recall that the binomial coefficient $\binom{-a}{s}$ is interpreted as $(-1)^s \binom{a+s-1}{s}$. Observe that with this definition, the generalized binomial coefficients also satisfy Pascal's identity $\binom{a}{s} = \binom{a-1}{s} + \binom{a-1}{s-1}$. Before seeing the proof of Proposition A.2, we establish a useful identity in Lemma A.1.

Lemma A.1. *For integers $x \geq 0, y \geq z \geq 0$, we have*

$$\sum_{t=0}^{z} (-1)^t \binom{x}{t} \binom{y-t}{z-t} = (-1)^z \binom{x-y+z-1}{z}. \tag{23}$$

Proof of Lemma A.1. One can prove this identity using induction on y. Note that when $y = z$, the identity becomes

$$\sum_{t=0}^{z}(-1)^t\binom{x}{t} = (-1)^z\binom{x-1}{z},$$

which follows from applying Pascal's identity $\binom{x}{t} = \binom{x-1}{t} + \binom{x-1}{t-1}$ to each term and telescoping.

Now suppose that (23) holds for some y. Then,

$$\begin{aligned}\sum_{t=0}^{z}(-1)^t\binom{x}{t}\binom{y-t+1}{z-t} &= \sum_{t=0}^{z}(-1)^t\binom{x}{t}\binom{y-t}{z-t}\\ &\quad+\sum_{t=0}^{z-1}(-1)^t\binom{x}{t}\binom{y-t}{z-t-1}.\end{aligned}$$

By induction hypothesis, the first term is $(-1)^z\binom{x-y+z-1}{z}$ and the second term is $(-1)^{z-1}\binom{x-y+z-2}{z-1}$. Their sum is $(-1)^z\binom{x-y+z-2}{z}$, as desired. ■

We are now going to state and prove Proposition A.2. Recall the following notation:

$$a_{ij}^{(m)} = \binom{m-i}{j-i},\ b_{ij}^{(m)} = (-1)^{j-i}\binom{m-i}{j-i},$$
$$w_{ij}^{(m)} = (-1)^{j-i}\binom{m-k+j-i-1}{j-i},$$
$$A_{k,m} = (a_{ij}^{(m)})_{1\le i,j\le k},\ B_{k,m} = (b_{ij}^{(m)})_{1\le i,j\le k}, W_{k-1,m} = (w_{ij}^{(m)})_{1\le i,j\le k-1},$$

and,

$$D_{k-1,m} = \begin{bmatrix} A_{k-1,m} & \vec{\mathbf{1}} \\ \vec{\mathbf{0}}^\intercal & 1\end{bmatrix},\ W'_{k-1,m} = \begin{bmatrix} W_{k-1,m} & \vec{\mathbf{0}} \\ \vec{\mathbf{0}}^\intercal & 1\end{bmatrix}.$$

Proposition A.2.

$$B_{k,k}\cdot D_{k-1,m}\cdot W'_{k-1,m} = I_k.$$

Proof. Note that $B_{k,k} = \begin{bmatrix} B_{k-1,k} & \vec{v} \\ \vec{\mathbf{0}}^\intercal & 1\end{bmatrix}$, where $v_i = (-1)^{k-i}$, and therefore

$$B_{k,k}D_{k-1,m}W'_{k-1,m} = \begin{bmatrix} B_{k-1,k}A_{k-1,m}W_{k-1,m} & B_{k-1,k}\vec{\mathbf{1}}+v \\ \vec{\mathbf{0}}^\intercal & 1\end{bmatrix}.$$

We verify that $B_{k-1,k}\vec{\mathbf{1}} + v = \vec{\mathbf{0}}$ and $B_{k-1,k}A_{k-1,m}W_{k-1,m} = I_{k-1}$ in Claims A.3 and A.4, respectively.

Claim A.3. $B_{k-1,k}\vec{\mathbf{1}} + v = \vec{\mathbf{0}}$.

Proof of Claim A.3. Note that the i'th row of $B_{k-1,k}\vec{\mathbf{1}}$ is

$$\sum_{j=1}^{k-1} b_{ij}^{(k)} = \sum_{j=i}^{k-1}(-1)^{j-i}\binom{k-i}{j-i} = \sum_{j=0}^{k-i-1}(-1)^j\binom{k-i}{j} = 0 - (-1)^{k-i} = -v_i,$$

as desired. ∎

Claim A.4. $B_{k-1,k}A_{k-1,m}W_{k-1,m} = I_{k-1}$.

Proof of Claim A.4. Note that the (i,j)th entry of the product matrix is given by

$$\begin{aligned}
&\sum_{r=1}^{k-1}\sum_{s=1}^{k-1} b_{ir}^{(k)} a_{rs}^{(m)} w_{sj}^{(m)} \\
&= \sum_{r=1}^{k-1}\sum_{s=1}^{k-1}(-1)^{r-i+j-s}\binom{k-i}{r-i}\binom{m-r}{s-r}\binom{m-k+j-s-1}{j-s} \qquad (24)\\
&= \sum_{s=1}^{k-1}(-1)^{j-s}\binom{m-k+j-s-1}{j-s}\sum_{r=1}^{k-1}(-1)^{r-i}\binom{k-i}{r-i}\binom{m-r}{s-r}.
\end{aligned}$$

Observe that, using Lemma A.1 for $x = k-i, y = m-i, z = s-i$, we get

$$\begin{aligned}
\sum_{r=1}^{k-1}(-1)^{r-i}\binom{k-i}{r-i}\binom{m-r}{s-r} &= \sum_{r=i}^{s}(-1)^{r-i}\binom{k-i}{r-i}\binom{m-r}{s-r} \\
&= \sum_{r=0}^{s-i}(-1)^{r}\binom{k-i}{r}\binom{m-i-r}{s-i-r} \\
&= (-1)^{s-i}\binom{k-m+s-i-1}{s-i}.
\end{aligned}$$

Plugging this back into (24), we get that the (i,j)th entry of the product matrix is

$$\sum_{s=1}^{k-1}(-1)^{j-i}\binom{m-k+j-s-1}{j-s}\binom{k-m+s-i-1}{s-i} \qquad (25)$$

Notice that the sum in (25) only runs from $s = i$ to $s = j$, and therefore after the change of variable $s \mapsto s + i$, the expression reduces to

$$(-1)^{j-i} \sum_{s=0}^{j-i} \binom{m-k+j-s-i-1}{j-i-s} \binom{k-m+s-1}{s}. \tag{26}$$

Note that $\binom{s-(m-k)-1}{s} = (-1)^s \binom{m-k}{s}$, so (26) is the sum

$$(-1)^{j-i} \sum_{s=0}^{j-i} (-1)^s \binom{m-k}{s} \binom{m-k+j-i-1-s}{j-i-s},$$

which, on invoking Lemma A.1 for $x = m-k, y = m-k+j-i-1, z = j-i$, reduces to

$$(-1)^{j-i} \cdot (-1)^{j-i} \cdot \binom{m-k-m+k-j+i+1+j-i-1}{j-i} = \binom{0}{j-i}.$$

Clearly, this is 0 when $j \neq i$ and 1 when $j = i$. ■

This completes the proof of Proposition A.2. □

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References

[1] P. Erdős and J. Komlós. On a problem of Moser. *Combinatorial theory and its applications, I (Proc. Colloq., Balatonfüred, 1969)*, pages 365–367, 1970. MR0297582

[2] D. Kleitman. Review of the article [1].

[3] P. Erdős and S. Shelah. On problems of Moser and Hanson. *Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs)*, pages 75–79. Lecture Notes in Math., Vol. 303, 1972. MR0337646

[4] Jacob Fox, Choongbum Lee, and Benny Sudakov. Maximum union-free subfamilies. *Israel J. Math.*, 191(2):959–971, 2012. MR3011503

[5] János Barát, Zoltán Füredi, Ida Kantor, Younjin Kim, and Balázs Patkós. Large B_d-free and union-free subfamilies. *SIAM J. Discrete Math.*, 26(1):71–76, 2012. MR2902633

[6] Paul Erdős. Problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 361–362, 1968.

[7] P. Erdős. Some unsolved problems in graph theory and combinatorial analysis. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 97–109. Academic Press, London, 1971. MR0277392

[8] F. Foucaud, M. Krivelevich, and G. Perarnau. Large subgraphs without short cycles. *SIAM J. Discrete Math.*, 29(1):65–78, 2015. MR3295686

[9] David Conlon, Jacob Fox, and Benny Sudakov. Large subgraphs without complete bipartite graphs. *arXiv preprint arXiv:1401.6711*, 2014.

[10] David Conlon, Jacob Fox, and Benny Sudakov. Short proofs of some extremal results II. *J. Combin. Theory Ser. B*, 121:173–196, 2016. MR3548291

[11] P. Erdős and R. Rado. Intersection theorems for systems of sets. *J. London Math. Soc.*, 35:85–90, 1960. MR0111692

[12] P. Erdős, A. Hajnal, and R. Rado. Partition relations for cardinal numbers. *Acta Math. Acad. Sci. Hungar.*, 16:93–196, 1965. MR0202613

[13] Joel Spencer. Turán's theorem for k-graphs. *Discrete Math.*, 2:183–186, 1972. MR0297614

[14] P. Dembowski and D. R. Hughes. On finite inversive planes. *J. London Math. Soc.*, 40:171–182, 1965. MR0172156

[15] Peter Dembowski. *Finite Geometries: Reprint of the 1968 edition*. Springer Science & Business Media, 2012. MR0233275

[16] Mauro Biliotti and Alessandro Montinaro. On the automorphism group of inversive planes of odd order. *J. Algebra*, 354:49–70, 2012. MR2879222

[17] Gloria Rinaldi. Inversive planes, Minkowski planes and regular sets of points. *European J. Combin.*, 22(3):357–363, 2001. MR1822723

[18] R. C. Baker, G. Harman, and J. Pintz. The difference between consecutive primes. II. *Proc. London Math. Soc. (3)*, 83(3):532–562, 2001. MR1851081

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