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Consecutive permutation patterns in trees and mappings

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We initiate an enumerative study of consecutive permutation patterns in rooted labelled trees by analysing the number of trees of a certain size that avoid a single consecutive permutation pattern of length 3, and the corresponding number of trees that contain this pattern a specified number of times. Using a generating functions approach based on combinatorial decompositions with respect to the node with smallest label in the tree, we are able to characterize for all three classes of permutation patterns of length 3 the corresponding generating functions solutions. Via methods of analytic combinatorics applied to these generating functions we can provide asymptotic results for the number of trees avoiding a certain pattern and central limit theorems for the number of occurrences of a pattern. Moreover, we extend our analysis from trees to mappings and carry out corresponding enumerative studies concerning avoidance and occurrence of a single consecutive permutation pattern of length 3 for functional digraphs of mappings $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Close connections between the study of certain patterns in trees and mappings are also shown bijectively.

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1. Introduction

In the combinatorial and probabilistic literature various studies of quantities related to the labelling of trees, where the vertices of objects of size n are labelled with distinct integers of $[n] := \{1, 2, \ldots, n\}$, can be found. As two examples, we mention here work on rooted labelled trees or forests concerning proper vertices [12] (also called leaders, i.e., nodes x with largest label in the whole subtree rooted at x) and records [14] (i.e., nodes x with largest label amongst all vertices on the path from x to the root). Very recently, studies concerning avoidance [2] or occurrence [1] of classical permutation patterns in families of labelled trees and forests have been initiated.

Analyses of certain consecutive permutation patterns also appear in literature, in particular, occurrences of the pattern 12, i.e., ascents, in rooted labelled trees have been treated in [6], and there are a huge number of results for trees avoiding the pattern 21, so-called increasing trees. Furthermore, alternating (or intransitive) trees, which avoid the set of patterns $\{123, 321\}$, have been treated for several tree families [3, 15, 19], and we also mention the recent study [16] on ascending runs, i.e., maximal occurrences of patterns $\{12\dots k,\ k \geq 1\}$ in rooted labelled trees and mappings, i.e., functions $f:\ [n] \to [n]$. We further mention another line of research concerning occurrences and avoidance of structural patterns in trees, where one considers occurrences of a given contiguous tree pattern in the tree, see, e.g., [11, 20].

Whereas consecutive patterns in permutations have been analysed in detail via various methods, see, e.g., [7, 17], it seems that for trees, apart from the previously mentioned work, almost no further results on the occurrence or avoidance of consecutive permutation patterns of length 3 or higher are available. Here we initiate such a study by treating the enumeration problem when avoiding a single pattern of length 3, and analysing the number of occurrences of a single pattern of length 3, for rooted labelled unordered trees (i.e., the subtrees of any node are not ordered from left to right), also called Cayley-trees (the enumeration formula n^{n-1} for the number of rooted labelled trees of size n is attributed to A. Cayley). We assume here edges in the tree as oriented towards the root node and the occurrence of a consecutive permutation pattern $\sigma = \sigma(1)\dots\sigma(k) \in S_k$ of length k (with S_k the symmetric group on $[k]$) corresponds to a directed path $p = (x_1, x_2, \dots, x_k)$ of k vertices, whose sequence of labels is order-isomorphic to σ, i.e., $x_i < x_j \Leftrightarrow \sigma(i) < \sigma(j)$, for all $1 \leq i < j \leq k$. Note that throughout this work by tree we always mean a rooted labelled unordered tree and we identify a node with its label.

Moreover, we consider n-mappings, i.e., functions $f:\ [n] \to [n]$ and the corresponding functional digraphs $G_f = (V, E)$, i.e., the directed graph with vertex-set $V = [n]$ and edge-set $E = \{(i, f(i))\ :\ i \in [n]\}$, and extend the notion of consecutive permutation pattern occurrence/avoidance to them. Although structural properties of the functional digraphs of random mappings, where one of the n^n n-mappings is chosen with equal probability, have widely been studied (see, e.g., [9]), there seem to exist only few results concerning label patterns in mappings. Besides the analysis of runs [16], we want to mention the study [18] of alternating mappings, i.e., functions f, for which the iteration sequences $i = f^0(i), f^1(i), f^2(i), \dots$ are always forming an alternating sequence, i.e., where the functional digraph G_f avoids the

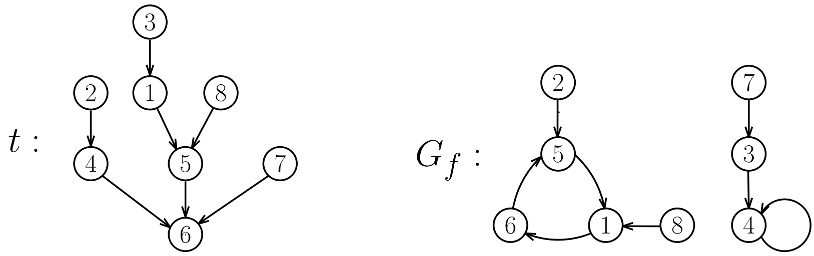

Figure 1: A tree t of size 8 and the functional digraph G_f of an 8-mapping f. There are 2 occurrences of 123 (paths $(1,5,6)$ and $(2,4,6)$) and one of 213 (path $(3,1,5)$) in t, whereas G_f avoids these two patterns.

set of patterns $\{123, 321\}$. Note that throughout this work we consider a function f and its functional digraph G_f as synonyms and do not strictly distinguish between nodes i in G_f and elements i in f. Figure 1 illustrates the occurrence and avoidance of certain consecutive permutation patterns in a labelled tree and a mapping.

The structure of the functional digraph of a mapping is rather simple and is well described in [10]: the weakly connected components of such graphs are just cycles of Cayley trees. This connection, although slightly more involved when taking into account the labels of the nodes, also allows one to gain results concerning consecutive permutation patterns in mappings from corresponding results in trees. Furthermore we want to mention that due to the combinatorial description of forests as sets of rooted labelled trees all results for trees immediately give corresponding results for forests, but we omit to state them here.

Throughout this work we use $X \stackrel{(d)}{=} Y$ to denote equality in distribution of random variables (r.v. for short) X and Y, whereas $X_n \xrightarrow{(d)} X$ means weak convergence, i.e., convergence in distribution, of the sequence of r.v. X_n to the r.v. X. $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2. With $W = W(z) := \sum_{n \ge 1} n^{n-1} \frac{z^n}{n!}$ we denote the so-called tree function, i.e., the exponential generating function of the number of Cayley trees of size-n, which satisfies the functional equation $W = ze^W$. With $f(n) \ll g(n)$ we denote, for sequences $f(n)$ and $g(n)$, that $f(n)$ is asymptotically smaller than $g(n)$, i.e., $f(n) = o(g(n))$, for $n \to \infty$. We further note that in cases where this does not cause ambiguity, for a better readability, we omit for functions and sequences the superscripts specifying the pattern σ.

2. Results

Due to obvious symmetry arguments, the permutation patterns $\sigma = \sigma(1)\ldots\sigma(k)$ and $\tilde{\sigma} = \tilde{\sigma}(1)\ldots\tilde{\sigma}(k)$, with $\tilde{\sigma}(j) = k+1-\sigma(j)$, for $1 \le j \le k$, are strongly consecutive-Wilf equivalent, i.e., for each m, n, the number of objects (trees or mappings) of size n with exactly m occurrences of the pattern σ always matches with the corresponding number of objects for the pattern $\tilde{\sigma}$. For patterns σ of length 3 we thus get three equivalence classes $123 \cong 321$, $132 \cong 312$, and $231 \cong 213$, and it suffices to state only results for the patterns 123, 132, and 231.

Results for avoiding a pattern of length 3

Theorem 1. *The exponential generating functions* $T^{[\sigma]}(z) := \sum_{n\ge1} T_n^{[\sigma]} \frac{z^n}{n!}$ *of the number* $T_n^{[\sigma]}$ *of rooted labelled trees of size* n *that avoid a given consecutive pattern* σ *of length* 3 *are all characterized as solutions of certain functional equations given below. Moreover, the exponential generating functions* $M^{[\sigma]}(z) := \sum_{n\ge0} M_n^{[\sigma]} \frac{z^n}{n!}$ *of the number* $M_n^{[\sigma]}$ *of* n*-mappings that avoid the corresponding pattern* σ *can be expressed via the function* $T^{[\sigma]}(z)$ *as stated below.*

Pattern σ	$T := T^{[\sigma]}(z)$	$M := M^{[\sigma]}(z)$
123	$z = e^{-T}\int_0^T \frac{e^t}{1+t}dt$	$M = \frac{1}{1-z(1+T)}$
132	$z = \int_0^T e^{-t-(T-t)e^{-t}}dt$	$M = \frac{e^{T-1+e^{-T}}}{1-e^T\int_0^T e^{-2t-(T-t)e^{-t}}dt}$
231	$z = e^{-T}\int_0^T e^{t-1+e^{-t}}dt$	$M = \frac{1}{1-ze^{1-e^{-T}}}$

Theorem 2. *The numbers* $T_n^{[\sigma]}$ *and* $M_n^{[\sigma]}$ *of rooted labelled trees of size* n *and* n*-mappings, respectively, that avoid a given consecutive pattern* σ *of length* 3 *are asymptotically, for* $n \to \infty$, *given as follows:*

$$T_n^{[\sigma]} \sim c_T \cdot \gamma^n \cdot n^{n-1}, \quad M_n^{[\sigma]} \sim c_M \cdot \gamma^n \cdot n^n,$$

with $\gamma = \frac{1}{e\rho}$, *where* ρ *is the radius of convergence of the corresponding generating function* $T^{[\sigma]}(z)$ *characterised via solutions of certain functional equations, and where* c_T, c_M *are some computable constants. Numerical values* *of the occurring constants are given below.*

Pattern σ	ρ	γ	c_T	c_M
123	0.42718536...	0.86117050...	1.53000135...	1.53000135...
132	0.44084481...	0.83448739...	1.74299311...	1.83550666...
231	0.44922576...	0.81891883...	2.23735314...	2.23735314...

Remark 1.

- *The numbers* $T_n^{[\sigma]}$ *are for* $n \in [8]$ *given as follows.*

Pattern σ	$T_n^{[\sigma]}$, $n \in [8]$							
	1	2	3	4	5	6	7	8
123	1	2	8	50	426	4606	60418	932282
132	1	2	8	49	407	4280	54537	816905
231	1	2	8	49	406	4248	53740	797786

- *Only the enumeration sequence of* $T_n^{[123]}$ *occurs in OEIS as sequence A225052, but without giving a combinatorial meaning. Now we can provide such one as rooted labelled trees without double-ascents.*
- *One might compare the results for the exponential growth rates* γ *with the corresponding ones for unrestricted labelled trees:* $\gamma = 1$*, and for* 21*-avoiding labelled trees (so-called recursive trees, see [10]):* $\gamma = 1/e = 0.3678\ldots$*, as summarized in the following table.*

21	231	132	123	*unrestricted*
0.3678...	0.8189...	0.8344...	0.8611...	1

- *According to Theorem 2 one obtains, for* $n \to \infty$*, the following asymptotic relation for the enumeration sequences* $T_n^{[\sigma]}$*:*

$$T_n^{[231]} \ll T_n^{[132]} \ll T_n^{[123]}.$$

For two of the three patterns of length 3 we get a close connection between pattern avoidance in trees and mappings as subsumed in the following corollary.

Corollary 1. *For the patterns* $\sigma = 123$ *and* $\sigma = 231$ *the corresponding numbers in trees and mappings are related via*

$$M_n^{[\sigma]} = nT_n^{[\sigma]}.$$

Note that this relation does not hold, not even asymptotically, for the pattern $\sigma = 132$, for which we get instead $M_n^{[\sigma]} \sim 1.0530\ldots \cdot nT_n^{[\sigma]}$.

Results for occurrences of patterns of length 3

Theorem 3. *The exponential generating functions* $F^{[\sigma]}(z,v) := \sum_{n\ge1}\sum_{m\ge0} F^{[\sigma]}_{n,m}\frac{z^n v^m}{n!}$ *of the number* $F^{[\sigma]}_{n,m}$ *of rooted labelled trees of size* n *with* m *occurrences of a given consecutive pattern* σ *of length* 3 *are characterized as solutions of certain functional equations given below. Moreover, the exponential generating functions* $G^{[\sigma]}(z,v) := \sum_{n\ge0}\sum_{m\ge0} G^{[\sigma]}_{n,m}\frac{z^n v^m}{n!}$ *of the number* $G^{[\sigma]}_{n,m}$ *of* n*-mappings with* m *occurrences of the corresponding pattern* σ *can be expressed via the functions* $F^{[\sigma]}(z,v)$ *as stated below.*

Pattern σ	$F := F^{[\sigma]}(z,v)$	$G := G^{[\sigma]}(z,v)$
123	$z = e^{-F}\int_0^F e^t(1-(v-1)t)^{\frac{1}{v-1}}dt$	$\frac{1}{1-z(1-(v-1)F)^{-\frac{1}{v-1}}}$
132	$z = \int_0^F e^{-t-(F-t)e^{(v-1)t}}dt$	$\frac{e^{\frac{(v-1)F+1-e^{(v-1)F}}{v-1}}}{1-e^F\int_0^F e^{(v-2)t-(F-t)e^{(v-1)t}}dt}$
231	$z = e^{-F}\int_0^F e^{\frac{(1-v)t-1+e^{(v-1)t}}{1-v}}dt$	$\frac{1}{1-ze^{\frac{1-e^{(v-1)F}}{1-v}}}$

Theorem 4. *Let* $X_n^{[\sigma]}$ *and* $Y_n^{[\sigma]}$ *be the random variables counting the number of occurrences of the pattern* σ *of length* 3 *in a randomly chosen size-*n *tree or* n*-mapping, respectively. Then mean and variance of these r.v. are given as follows:*

<table>
<tr><th></th><th>123</th><th>132</th><th>231</th></tr>
<tr><td>$\mathbb{E}(X_n^{[\sigma]})$</td><td colspan="3" rowspan="2">$\frac{n}{6}-\frac{1}{2}+\frac{1}{3n}\sim\frac{1}{6}n$</td></tr>
<tr><td>$\mathbb{E}(Y_n^{[\sigma]})$</td></tr>
<tr><td>$\mathbb{V}(X_n^{[\sigma]})$</td><td rowspan="2">$\frac{n}{5}-\frac{2}{3}+\frac{1}{3n}+\frac{2}{3n^2}-\frac{8}{15n^3}$</td><td>$\frac{2n}{15}-\frac{1}{3}+\frac{1}{3n^2}-\frac{2}{15n^3}$</td><td rowspan="2">$\frac{7n}{60}-\frac{1}{6}-\frac{7}{12n}+\frac{7}{6n^2}-\frac{8}{15n^3}$</td></tr>
<tr><td>$\mathbb{V}(Y_n^{[\sigma]})$</td><td>$\frac{2n}{15}-\frac{1}{4}-\frac{1}{2n}+\frac{5}{4n^2}-\frac{19}{30n^3}$</td></tr>
</table>

Furthermore, after suitable normalization, the r.v. $X_n^{[\sigma]}$ *and* $Y_n^{[\sigma]}$ *converge in distribution to a standard normal distribution* $\mathcal{N}(0,1)$*, i.e.,* $\frac{X_n^{[\sigma]}-\mathbb{E}(X_n^{[\sigma]})}{\sqrt{\mathbb{V}(X_n^{[\sigma]})}} \xrightarrow{(d)} \mathcal{N}(0,1)$*, analogous for* $Y_n^{[\sigma]}$*.*

Remark 2. *As expected, the r.v.* $X_n^{[\sigma]}$ *and* $Y_n^{[\sigma]}$ *satisfy a central limit theorem with linear mean and variance. However, interestingly the variance, and thus the normalization constants, are different for the three pattern classes of* *length* 3*:*

$$\mathbb{V}(X_n^{[123]}) = \mathbb{V}(Y_n^{[123]}) \sim \frac{1}{5}n, \quad \mathbb{V}(X_n^{[132]}) \sim \mathbb{V}(Y_n^{[132]}) \sim \frac{2}{15}n,$$
$$\mathbb{V}(X_n^{[231]}) = \mathbb{V}(Y_n^{[231]}) \sim \frac{7}{60}n.$$

Again, for two of the three patterns of length 3 we get a close connection between pattern occurrences in trees and mappings.

Corollary 2. *For the patterns* $\sigma = 123$ *and* $\sigma = 231$ *the corresponding numbers in trees and mappings are related via*

$$G_{n,m}^{[\sigma]} = nF_{n,m}^{[\sigma]}, \quad \text{and thus} \quad Y_n^{[\sigma]} \stackrel{(d)}{=} X_n^{[\sigma]}.$$

Note that for the pattern 132 this exact relation does not hold, but the r.v. $X_n^{[\sigma]}$ and $Y_n^{[\sigma]}$ have the same limiting distribution behaviour.

3. Generating functions for permutation patterns in trees

3.1. Pattern 123

In order to count the number of occurrences of the consecutive pattern 123, i.e., double-ascents, in rooted labelled trees, we will use the decomposition of a tree with respect to the vertex labelled 1. However, to exploit this decomposition we have to introduce an auxiliary quantity and also count the number of occurrences of the pattern 12, thus ascents, in the tree.

Formally we will consider bicoloured trees: a node x will be coloured blue, i.e., it gets a marker B, if x is the starting node of the consecutive pattern 12, and it will be coloured red, thus gets a marker R, if it is the starting node of the consecutive pattern 123. Note that vertices can be coloured red and blue simultaneously; actually, if a node is coloured red it is also coloured blue. Let us denote by $\mathcal{F}$ the family of rooted labelled trees with vertices coloured in the previously described manner. The decomposition of a tree $t \in \mathcal{F}$ with respect to node 1 yields (after order-preserving relabellings) a (possibly empty) set of subtrees $t_1, \dots, t_k$ originally attached to node 1 and, in case that 1 is not the root of t, a subtree t_0, where node 1 is originally linked to a node $x \in t_0$. The following three cases might occur; see Figure 2. Note that attaching the subtrees $t_1, \dots, t_k$ to node 1 does not change the number of patterns 12 and 123.

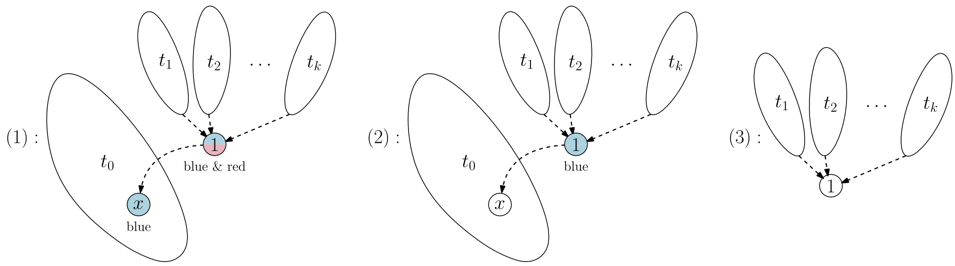

Figure 2: Pattern 123: Decomposition of a tree t with respect to the node with smallest label, where the three cases described below may occur. Starting nodes of occurrences of the pattern 12 and 123 relevant to this decomposition are coloured blue and red, respectively.

1. Node 1 is not the root of t and x is in t_0 coloured blue: in this case, the link from 1 to x creates a new pattern 123 and also a pattern 12 and thus node 1 will be coloured red and blue. Therefore the number of red vertices and blue vertices in t is one plus the sum of the number of corresponding vertices in $t_0, t_1, \dots, t_k$.
2. Node 1 is not the root of t and x is not blue in t_0: in that case, the link from 1 to x only creates a new pattern 12, but no pattern 123, and thus node 1 will be coloured blue. The number of red vertices in t is equal to the sum of the number of red vertices in $t_0, t_1, \dots, t_k$, whereas the number of blue vertices in t is one plus the sum of the blue vertices in these subtrees.
3. Node 1 is the root of t: here the number of blue vertices and red vertices in t is equal to the sum of the corresponding vertices in $t_1, \dots, t_k$.

Using this decomposition we obtain a symbolic equation for the family $\mathcal{F}$ by applying combinatorial constructions (see [10]). Besides basic operations as the disjoint union $+$, the partition product $*$ and the set-construction SET of labelled families, we require the boxed-product $\mathcal{A}^{\Box} * \mathcal{B}$ of families $\mathcal{A}$ and $\mathcal{B}$, which only contains those objects, where the smallest label 1 is contained in the $\mathcal{A}$-component. With $\mathcal{Z}$ we denote an atomic element, i.e., a vertex. Moreover, we use marking-operators: $\Theta_{\mathcal{Z}}(\mathcal{A})$ contains all structures obtained by distinguishing (i.e., marking) one node in an object of $\mathcal{A}$; to mark a blue vertex or a red vertex we use the markers B and R, respectively, and $\Theta_B(\mathcal{A})$ contains all structures obtained by distinguishing a blue node in an object of $\mathcal{A}$.

With these constructions the above decomposition can be described formally as follows, where the summands in the formal equation correspond to the cases occurring:

$$\begin{aligned}\mathcal{F} &= \mathcal{Z}^{\square} * \Theta_B(\mathcal{F}) * \textsc{Set}(\mathcal{F}) \times \{B, R\} \\ &\quad + \mathcal{Z}^{\square} * \big(\Theta_{\mathcal{Z}}(\mathcal{F}) \setminus \Theta_B(\mathcal{F})\big) * \textsc{Set}(\mathcal{F}) \times \{B\} + \mathcal{Z}^{\square} * \textsc{Set}(\mathcal{F}).\end{aligned} \tag{1}$$

We introduce the trivariate generating function $\tilde{F}(z, w, v)$ via

$$\begin{aligned}\tilde{F}(z, w, v) &:= \sum_{t \in \mathcal{T}} \frac{z^{|t|}\, w^{\sharp \text{ blue vertices in } t}\, v^{\sharp \text{ red vertices in } t}}{|t|!} \\ &= \sum_{n \geq 1} \sum_{\ell \geq 0} \sum_{m \geq 0} \tilde{F}_{n,\ell,m} \frac{z^n w^\ell v^m}{n!},\end{aligned}$$

where $\tilde{F}_{n,\ell,m}$ denotes the number of trees of size n with ℓ occurrences of the pattern 12 and m occurrences of the pattern 123. An application of the so-called symbolic method (see [10]) to the formal equation (1) yields then the following partial differential equation (PDE) for $\tilde{F} = \tilde{F}(z, w, v)$:

$$\tilde{F}_z = v w^2 e^{\tilde{F}} \tilde{F}_w + w e^{\tilde{F}} (z \tilde{F}_z - w \tilde{F}_w) + e^{\tilde{F}}.$$

Note that the boxed-product $\mathcal{C} = \mathcal{A}^{\square} * \mathcal{B}$ yields the equation $C_z = A_z \cdot B$ at the level of generating functions. Moreover, since the marking operators $\Theta_{\mathcal{Z}}$ and Θ_B applied to $\mathcal{F}$ generate $n\tilde{F}_{n,\ell,m}$ and $\ell\tilde{F}_{n,\ell,m}$ different trees of size n with ℓ blue vertices and m red vertices, respectively, this leads to expressions $z\tilde{F}_z$ and $w\tilde{F}_w$ in the equation stated.

Rearranging above equation yields the following first-order quasi-linear PDE for $\tilde{F}(z, w, v)$ with initial condition $\tilde{F}(0, w, v) = 0$:

$$(1 - wze^{\tilde{F}})\tilde{F}_z + (1 - v) w^2 e^{\tilde{F}} \tilde{F}_w - e^{\tilde{F}} = 0. \tag{2}$$

Equation (2) can be solved by applying the method of characteristics for first-order quasi-linear PDEs, see, e.g., [8]. Here we will omit these computations, but remark that we give a sketch of the application of this method and of the corresponding derivations (which are very similar to the ones for (2)) for the pattern 132 in the next section. However, it can be checked easily by taking partial derivatives and plugging them into (2) that the solution $\tilde{F} = \tilde{F}(z, w, v)$ of the PDE (2) is given implicitly as solution of the following functional equation:

$$z = \int_0^{\tilde{F}} e^{-s}(1-(v-1)w(\tilde{F}-s))^{\frac{1}{v-1}} ds. \tag{3}$$

We eliminate the auxiliary parameter by setting $w = 1$ and introduce the generating function

$$F(z,v) := \tilde{F}(z,1,v) = \sum_{n\geq 1}\sum_{m\geq 0} F_{n,m}\frac{z^n v^m}{n!},$$

with $F_{n,m}$ the number of size-n trees with m occurrences of the pattern 123. With $w = 1$ and the substitution $t = F - s$ we get from (3) the characterization of $F = F(z,v)$ stated in Theorem 3; namely, $F(z,v)$ satisfies the functional equation

$$z = e^{-F}\int_0^{F} e^{t}(1-(v-1)t)^{\frac{1}{v-1}} dt. \tag{4}$$

Furthermore by setting $v = 0$ we get the result for avoiding the pattern 123 stated in Theorem 1. To this aim we introduce the generating function

$$T = T(z) := F(z,0) = \sum_{n\geq 1} T_n \frac{z^n}{n!},$$

with $T_n = F_{n,0}$ the number of 123-avoiding size-n trees. Then, from (4) we get that $T(z)$ is given as solution of the following functional equation, which is also stated in Theorem 1:

$$z = e^{-T}\int_0^{T} \frac{e^t}{1+t} dt. \tag{5}$$

3.2. Pattern 132

When counting the number of occurrences of the consecutive pattern 132 by a recursive approach based on the decomposition with respect to the node labelled 1 we have to take into account as auxiliary parameter also the number of occurrences of the consecutive pattern 21, i.e., descents, in the tree.

Again we consider bicoloured trees, where a node x will be coloured blue, i.e., it gets a marker B, if x is the starting node of the consecutive pattern

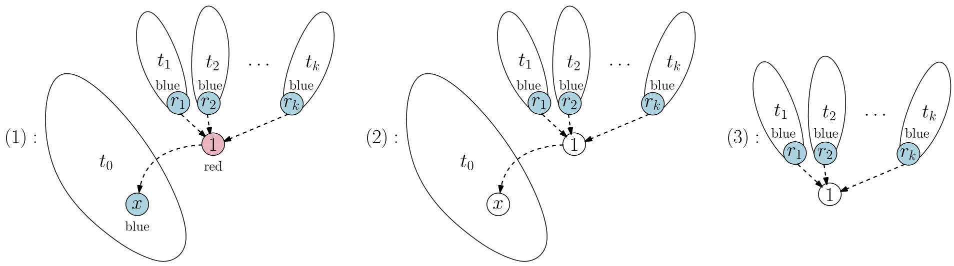

Figure 3: Pattern 132: Decomposition of a tree t with respect to the node with smallest label, where the three cases described below may occur. Starting nodes of occurrences of the pattern 21 and 132 relevant to this decomposition are coloured blue and red, respectively.

21, and it will be coloured red, thus gets a marker R, if it is the starting node of the consecutive pattern 132. With $\mathcal{F}$ we denote the family of rooted labelled trees with vertices coloured as described before. The decomposition of a tree $t \in \mathcal{F}$ with respect to node 1 yields (after order-preserving relabellings) a (possibly empty) set of subtrees $t_1, \dots, t_k$ with respective root-nodes $r_1, \dots, r_k$ originally attached to node 1 and, in case that 1 is not the root of t, a subtree t_0, where node 1 is originally linked to a node $x \in t_0$. The following three cases might occur, see Figure 3, where we also take into account that attaching the root r_j of the subtree t_j, $1 \le j \le k$, to node 1 creates a consecutive pattern 21 in t, thus each vertex r_j will be coloured blue, whereas attaching the subtrees to 1 does not change the number of occurrences of the pattern 132.

1. Node 1 is not the root of t and x is in t_0 coloured blue: in this case, the link from 1 to x creates in t a new pattern 132 and node 1 will be coloured red. Thus the number of red vertices in t is one plus the sum of the number of red vertices in $t_0, t_1, \dots, t_k$, whereas the number of blue vertices in t is k (for each of the nodes attached to 1) plus the sum of the number of blue vertices in $t_0, t_1, \dots, t_k$.
2. Node 1 is not the root of t and x is not blue in t_0: in that case, the link from 1 to x neither creates a pattern 21 nor a pattern 132, and thus node 1 remains uncoloured. The number of red vertices in t is equal to the sum of the number of red vertices in $t_0, t_1, \dots, t_k$, and the number of blue vertices in t is k plus the sum of the number of blue vertices in $t_0, t_1, \dots, t_k$.

3. Node 1 is the root of t: here the number of red vertices in t is equal to the sum of the red vertices in $t_1, \ldots, t_k$, and the number of blue vertices in t is k plus the sum of the number of blue vertices in $t_1, \ldots, t_k$.

Combining these cases leads to the following recursive description of the family $\mathcal{F}$, where we use the combinatorial constructions already defined in Section 3.1:

$$\begin{aligned}\mathcal{F} &= \mathcal{Z}^{\square} * \Theta_B(\mathcal{F}) * \textsc{Set}(\mathcal{F} \times \{B\}) \times \{R\} \\ &\quad + \mathcal{Z}^{\square} * \big(\Theta_{\mathcal{Z}}(\mathcal{F}) \setminus \Theta_B(\mathcal{F})\big) * \textsc{Set}(\mathcal{F} \times \{B\}) + \mathcal{Z}^{\square} * \textsc{Set}(\mathcal{F} \times \{B\}). \end{aligned} \tag{6}$$

When introducing the trivariate generating function $\tilde{F}(z, w, v)$ via

$$\begin{aligned}\tilde{F}(z, w, v) &:= \sum_{t \in \mathcal{T}} \frac{z^{|t|} \, w^{\sharp \text{ blue vertices in } t} \, v^{\sharp \text{ red vertices in } t}}{|t|!} \\ &= \sum_{n \geq 1} \sum_{\ell \geq 0} \sum_{m \geq 0} \tilde{F}_{n,\ell,m} \frac{z^n w^\ell v^m}{n!},\end{aligned}$$

where $\tilde{F}_{n,\ell,m}$ denotes the number of trees of size n with ℓ occurrences of the pattern 21 and m occurrences of the pattern 132, an application of the symbolic method to (6) yields the following first-order quasi-linear PDE for $\tilde{F} = \tilde{F}(z, w, v)$:

$$\tilde{F}_z = vwe^{w\tilde{F}} \tilde{F}_w + e^{w\tilde{F}} (z\tilde{F}_z - w\tilde{F}_w) + e^{w\tilde{F}},$$

with initial condition $\tilde{F}(0, w, v) = 0$. It can be written as follows:

$$(1 - ze^{w\tilde{F}})\tilde{F}_z - (v-1)we^{w\tilde{F}} \tilde{F}_w - e^{w\tilde{F}} = 0. \tag{7}$$

We sketch the application of the method of characteristics yielding the solution of (7), since later when considering the corresponding problem for mappings we require an intermediate result. Introducing a function $f = f(z, w, \tilde{F})$ and assuming $f(z, w, \tilde{F}(z, w)) = \text{const.}$ (we consider here v as a parameter), we obtain after taking partial derivatives the following PDE for f:

$$\left(1 - ze^{w\tilde{F}}\right) f_z - (v-1)we^{w\tilde{F}} f_w + e^{w\tilde{F}} f_{\tilde{F}} = 0.$$

To find solutions of the PDE we consider the system of characteristic equations (by assuming that the variables occurring are dependent on a parameter t, $z = z(t)$, $w = w(t)$, $\tilde{F} = \tilde{F}(t)$, and using the notation $\dot{z} = \frac{dz}{dt}$, etc.):

$$\dot{z} = 1 - ze^{w\tilde{F}}, \quad \dot{w} = -(v-1)we^{w\tilde{F}}, \quad \dot{\tilde{F}} = e^{w\tilde{F}}. \tag{8}$$

From the second and third characteristic equation (8) we obtain the separable differential equation (DEQ) $\frac{dw}{d\tilde{F}} = -(v-1)w$, whose solution gives the first integral

$$we^{(v-1)\tilde{F}} = C_1 = \text{const.} \tag{9}$$

The first and third characteristic equation (8), after substituting $w = C_1 e^{-(v-1)\tilde{F}}$, gives the first order linear DEQ $\frac{dz}{d\tilde{F}} = -z + e^{-C_1 \tilde{F} e^{-(v-1)\tilde{F}}}$, and the general solution of it leads to the first integral $ze^{\tilde{F}} - \int_0^{\tilde{F}} e^{t(1-we^{(v-1)(\tilde{F}-t)})}dt = C_2 = \text{const}$. Considering $C_2 = g(C_1) = \text{const.}$, with $g(x)$ an arbitrary differentiable function, we obtain the general solution of (7) in the implicit form

$$ze^{\tilde{F}} - \int_0^{\tilde{F}} e^{t(1-we^{(v-1)(\tilde{F}-t)})}dt = g(we^{(v-1)\tilde{F}}).$$

Taking into account the initial condition $\tilde{F}(0, w, v) = 0$ characterizes the function $g(x)$ as $g(x) = 0$ and one gets the required solution of the PDE (7) for $\tilde{F} = \tilde{F}(z, w, v)$ as solution of the functional equation

$$z = \int_0^{\tilde{F}} e^{-t-w(\tilde{F}-t)e^{(v-1)t}} dt. \tag{10}$$

Introducing the generating function

$$F(z, v) := \tilde{F}(z, 1, v) = \sum_{n\geq 1}\sum_{m\geq 0} F_{n,m}\frac{z^n v^m}{n!},$$

with $F_{n,m}$ the number of size-n trees with m occurrences of the pattern 132, and setting $w = 1$ in (10) yields the characterization of $F = F(z, v)$ stated in Theorem 3 as solution of the functional equation

$$z = \int_0^{F} e^{-t-(F-t)e^{(v-1)t}} dt. \tag{11}$$

Moreover, introducing the generating function

$$T(z) := F(z, 0) = \sum_{n\geq 1} T_n \frac{z^n}{n!},$$

with $T_n = F_{n,0}$ the number of 132-avoiding size-n trees, and evaluation (11) at $v = 0$ shows the result stated in Theorem 1, namely that $T = T(z)$ is characterized as solution of the functional equation

$$z = \int_0^T e^{-t-(T-t)e^{-t}} dt. \tag{12}$$

3.3. Pattern 231

For the pattern 231 the situation is slightly different from the ones previously treated, since here the decomposition of a tree with respect to the smallest labelled vertex 1 has an influence to the last element in the permutation pattern; this requires a different auxiliary quantity and leads to another kind of equation at the level of generating functions. Namely, for a recursive treatment of the number of occurrences of 231 in a tree t we also take into consideration the number of consecutive patterns 12 ending at the root of t (i.e., ascents ending at the root). Formally we will again consider bicoloured trees, where each starting node x of the consecutive pattern 231 is coloured red, and thus gets a marker R, and where each starting node x of the consecutive pattern 12 ending at the root is coloured blue, thus gets a marker B; the familiy of rooted labelled trees with vertices coloured in this way is denoted by $\mathcal{F}$.

The decomposition of a tree $t \in \mathcal{F}$ with respect to node 1 yields (after order-preserving relabellings) a (possibly empty) set of subtrees $t_1, \dots, t_k$ with respective root-nodes $r_1, \dots, r_k$ originally attached to node 1 and, in case that 1 is not the root of t, a subtree t_0, where node 1 is originally linked to a node $x \in t_0$. We will distinguish the following three cases, see Figure 4. Of particular importance is the observation that each pattern 12 ending at the root r_j in a subtree t_j, $1 \leq j \leq k$, creates a new pattern 231 in t when attaching the subtrees to node 1.

1. Node 1 is not the root of t and x is not the root of t_0: in this case, the number of blue vertices in t is equal to the number of blue vertices in t_0, whereas the number of red vertices in t is given by the sum of the number of red vertices in $t_0, t_1, \dots, t_k$ plus the sum of the number of blue vertices in $t_1, \dots, t_k$.
2. Node 1 is not the root of t and x is the root of t_0: in that case, the link from 1 to x creates an additional pattern 12 ending at the root of t and thus 1 is coloured blue. Therefore, the number of blue vertices in t is one plus the number of blue vertices in t_0. Furthermore the number of red vertices in t is given by the sum of the number of red vertices in $t_0, t_1, \dots, t_k$ plus the sum of the number of blue vertices in $t_1, \dots, t_k$.

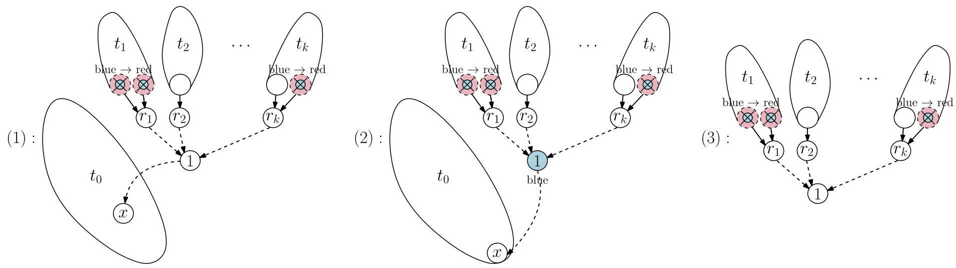

Figure 4: Pattern 231: Decomposition of a tree t with respect to the node with smallest label, where the three cases described below may occur. Starting nodes of occurrences of the pattern 12 ending at the root (of subtrees) and the pattern 231 relevant to this decomposition are coloured blue and red, respectively.

3. Node 1 is the root of t: here t does not contain any pattern 12 ending at the root of t, thus there are no blue vertices in t. Furthermore the number of red vertices in t is given by the sum of the number of red vertices in $t_1, \dots, t_k$ plus the sum of the number of blue vertices in $t_1, \dots, t_k$.

In order to describe this decomposition of the family $\mathcal{F}$ formally we require, in addition to constructions already introduced in previous sections, a replacement-operator for markers: let $\Phi_{B\to R}$ be the operator that when applied to a combinatorial family replaces every marker B by a marker R in these objects. We obtain then the following formal equation for the family $\mathcal{F}$:

$$\begin{aligned}\mathcal{F} &= \mathcal{Z}^{\Box} * \big(\Theta_{\mathcal{Z}}(\mathcal{F}) \setminus \mathcal{F}\big) * \textsc{Set}\big(\Phi_{B\to R}(\mathcal{F})\big) \\ &\quad + \mathcal{Z}^{\Box} * \mathcal{F} * \textsc{Set}\big(\Phi_{B\to R}(\mathcal{F})\big) \times \{B\} + \mathcal{Z}^{\Box} * \textsc{Set}\big(\Phi_{B\to R}(\mathcal{F})\big).\end{aligned} \tag{13}$$

Let us define the trivariate generating function $\tilde{F}(z, w, v)$ via

$$\begin{aligned}\tilde{F}(z,w,v) &:= \sum_{t\in\mathcal{T}} \frac{z^{|t|}\, w^{\sharp \text{ blue vertices in } t}\, v^{\sharp \text{ red vertices in } t}}{|t|!} \\ &= \sum_{n\geq 1}\sum_{\ell\geq 0}\sum_{m\geq 0} \tilde{F}_{n,\ell,m} \frac{z^n w^\ell v^m}{n!},\end{aligned}$$

where $\tilde{F}_{n,\ell,m}$ denotes the number of trees of size n with ℓ occurrences of the pattern 12 ending at the root of the tree and m occurrences of the pattern 231. When applying the symbolic method to (13), we have to take into account that at the level of generating functions the operator $\Phi_{B\to R}$ corresponds to a replacement of the variable w (counting the marker B) by the variable v (counting the marker R), thus $\Phi_{B\to R}(\mathcal{F})$ gives $\tilde{F}(z,v,v)$. Therefore we get the following DEQ for the function $\tilde{F}(z,w,v)$, where also evaluations at $w=v$, i.e., $\tilde{F}(z,v,v)$, occur:

$$\tilde{F}_z(z,w,v) = \big(z\tilde{F}_z(z,w,v) - \tilde{F}(z,w,v)\big)e^{\tilde{F}(z,v,v)} + w\tilde{F}(z,w,v)e^{\tilde{F}(z,v,v)} + e^{\tilde{F}(z,v,v)},$$

together with the initial condition $\tilde{F}(0,w,v) = 0$. It can be rewritten as follows:

$$(1 - ze^{\tilde{F}(z,v,v)})\tilde{F}_z(z,w,v) + (1-w)e^{\tilde{F}(z,v,v)}\tilde{F}(z,w,v) - e^{\tilde{F}(z,v,v)} = 0. \tag{14}$$

To solve this equation we first set $w=v$ in (14) and study the auxiliary function $A = A(z,v) := \tilde{F}(z,v,v)$, which satisfies the first-order non-linear DEQ

$$(1 - ze^{A})A_z - (1-(1-v)A)e^{A} = 0, \tag{15}$$

with initial condition $A(0,v)=0$. Equation (15) can be solved by standard means, e.g., when considering it as a quasi-linear first-order PDE and applying the method of characteristics. This characterizes $A = A(z,v)$ implicitly as solution of the following functional equation:

$$z = (1-(1-v)A)^{\frac{1}{1-v}} \int_0^A (1-(1-v)s)^{\frac{v-2}{1-v}} e^{-s} ds. \tag{16}$$

We omit these calculations, but it can be checked easily that (16) indeed satisfies (15) as well as the stated initial condition. Plugging $A(z,v) = \tilde{F}(z,v,v)$ into (14) yields a first-order linear DEQ for $\tilde{F}(z,w,v)$; however, since variable w only encodes an auxiliary quantity, which is no more of further interest, we set $w=1$ and introduce the generating function

$$F(z,v) := \tilde{F}(z,1,v) = \sum_{n\ge 1}\sum_{m\ge 0} F_{n,m}\frac{z^n v^m}{n!},$$

with $F_{n,m}$ the number of size-n trees with m occurrences of the pattern 231. This gives the following DEQ for $F = F(z,v)$, with $A = A(z,v)$ as in (16):

$$(1 - ze^{A})F_z - e^{A} = 0.$$

Thus, together with the initial condition $F(0, v) = 0$, we obtain the solution

$$F(z, v) = \int_0^z \frac{e^{A(t,v)}}{1 - te^{A(t,v)}} dt. \tag{17}$$

However, we want to get rid of the appearance of $A(z, v)$ in the solution of $F(z, v)$, which can be done by some manipulations. First we rewrite the integrand of (17). Namely, by partial integration of (16), we obtain

$$\begin{aligned} z &= (1 - (1 - v)A)^{\frac{1}{1-v}} \\ &\quad \times \left((1 - (1 - v)s)^{-\frac{1}{1-v}} e^{-s} \Big|_0^A + \int_0^A (1 - (1 - v)s)^{-\frac{1}{1-v}} e^{-s} ds \right) \\ &= e^{-A} - (1 - (1 - v)A)^{\frac{1}{1-v}} \cdot \left(1 - \int_0^A (1 - (1 - v)s)^{-\frac{1}{1-v}} e^{-s} ds \right) \end{aligned} \tag{18}$$

and further, after simple manipulations,

$$\frac{e^A}{1 - ze^A} = \frac{1}{(1 - (1 - v)A)^{\frac{1}{1-v}} \cdot \left(1 - \int_0^A (1 - (1 - v)s)^{-\frac{1}{1-v}} e^{-s} ds \right)}. \tag{19}$$

Moreover, taking derivates of (18) gives

$$\frac{dz}{dA} = (1 - (1 - v)A)^{\frac{v}{1-v}} \cdot \left(1 - \int_0^A (1 - (1 - v)s)^{-\frac{1}{1-v}} e^{-s} ds \right). \tag{20}$$

Thus, using (19) and (20), the substitution $x = A(t, v)$ in (17) leads to an integral-free representation of $F(z, v)$ in terms of $A(z, v)$:

$$F(z, v) = \int_0^{A(z,v)} \frac{1}{1 - (1 - v)x} dx = \frac{1}{1 - v} \ln \left(\frac{1}{1 - (1 - v)A(z, v)} \right),$$

and thus to the following relation between the auxiliary function $A = A(z, v)$ and the required $F = F(z, v)$:

$$A = \frac{1 - e^{-(1-v)F}}{1 - v}. \tag{21}$$

Using (21) we first obtain from (16) the equation

$$z = e^{-F} \int_0^A (1 - (1 - v)s)^{\frac{v-2}{1-v}} e^{-s} ds,$$

which leads after substituting $s = \frac{1-e^{-(1-v)t}}{1-v}$ to the representation of $F = F(z,v)$ stated in Theorem 3 as solution of the functional equation

$$z = e^{-F} \int_0^F e^{\frac{(1-v)t-1+e^{-(1-v)t}}{1-v}} dt. \tag{22}$$

Moreover, by setting $v = 0$, we get the result stated in Theorem 1 for the generating function

$$T(z) := F(z,0) = \sum_{n \geq 1} T_n \frac{z^n}{n!}$$

of the number $T_n = F_{n,0}$ of 231-avoiding size-n trees, i.e., that $T = T(z)$ is given as solution of the functional equation

$$z = e^{-T} \int_0^T e^{t-1+e^{-t}} dt. \tag{23}$$

4. Asymptotic results for permutation patterns in trees

4.1. Pattern-avoiding trees

We will show here the asymptotic results for the number $T_n^{[\sigma]}$ of labelled trees avoiding a consecutive pattern σ of length 3 stated in Theorem 2. We do this by considering the corresponding generating functions $T = T^{[\sigma]}(z)$ computed in Section 3, which are characterized implicitly via functional equations of the form $z = \phi(T)$. Deducing the asymptotic behaviour of the coefficients of such implicitly given generating functions via methods of analytic combinatorics is well-established and nicely described in [10]; thus we may give the computations in a somewhat condensed form. In order to characterize the dominant singularity (or singularities, but for the generating functions occurring in this work, the singularity of smallest modulus is always unique) of $T(z)$ we consider the function $h(z,T) := z - \phi(T)$. According to the implicit function theorem, this equation cannot be resolved with respect to T in a locally unique way for points $(z,T) = (\rho,\tau)$ satisfying $h(\rho,\tau) = 0$ and $h_T(\rho,\tau) = 0$.

4.1.1. Pattern 123 Due to the functional equation (5) that satisfies T, we consider

$$h(z,T) = z - e^{-T} \int_0^T \frac{e^t}{1+t} dt,$$

and search for points $(z, T) = (\rho, \tau)$ solving the equations

(24a) $$h(\rho, \tau) = \rho - e^{-\tau} \int_0^{\tau} \frac{e^t}{1+t} dt = 0,$$

(24b) $$h_T(\rho, \tau) = e^{-\tau} \int_0^{\tau} \frac{e^t}{1+t} dt - \frac{1}{1+\tau} = 0.$$

Equation (24b) characterizes τ as solution of

(25) $$\frac{e^{\tau}}{1+\tau} = \int_0^{\tau} \frac{e^t}{1+t} dt.$$

Plugging this into (24a) gives the relation

(26) $$\rho = \frac{1}{1+\tau}.$$

Since the coefficients of the series $T(z) = \sum_{n \geq 1} T_n \frac{z^n}{n!}$ are all non-negative, according to the theorem of Pringsheim (see, e.g., [10, Theorem IV.6]), a dominant singularity of $T(z)$ has to be located on the positive real axis. Thus, let us proceed by defining τ as the uniquely determined positive real solution of equation (25). Using the exponential integral, defined as the Cauchy principal value $\mathrm{Ei}(x) := (CPV) \int_{-\infty}^{x} \frac{e^t}{t} dt$, this solution can be characterized as non-negative real solution of $\mathrm{Ei}(1+\tau) - \mathrm{Ei}(1) = \frac{e^{1+\tau}}{1+\tau}$ and is numerically given as $\tau = 1.34090414\ldots$. The corresponding ρ obtained via (26), which gives the dominant singularity and also the radius of convergence of the function $T(z)$, thus can also be described by means of the exponential integral as positive real solution of the equation $\mathrm{Ei}(\frac{1}{\rho}) - \mathrm{Ei}(1) = \rho e^{\frac{1}{\rho}}$ and is numerically given as $\rho = 0.42718536\ldots$.

For such kind of problem where the function $T(z)$ is characterized implicitly by some functional equation, the only difficulty in determining the asymptotic behaviour of the coefficients that may remain is to justify that there are no further dominant singularities, i.e., singularities on the circle of convergence, other than ρ. This could be done simply by a numerical search for pairs $(\tilde{\rho}, \tilde{\tau})$ on the compact set $\{(\tilde{\rho}, \tilde{\tau}) \in \mathbb{C}^2 : |\tilde{\rho}| \leq \rho \text{ and } |\tilde{\tau}| \leq \tau\}$ satisfying equations (24a) and (24b) (see the derivation for the pattern 132); note that indeed only $\tilde{\tau}$ with $|\tilde{\tau}| \leq \tau$ are of relevance, since due to the positivity of the coefficients of $T(z)$ there holds for any $\tilde{\rho}$ with $|\tilde{\rho}| \leq \rho$ that $|\tilde{\tau}| = |T(\tilde{\rho})| \leq T(|\tilde{\rho}|) \leq T(\rho) = \tau$. However, in the present case also an analytic argument can be used to show that ρ is the unique dominant singularity.

Namely, assume that $\tilde{\rho} = \rho e^{i\varphi}$, with $0 < \varphi < 2\pi$, is also a dominant singularity and thus $\tilde{\rho}, \tilde{\tau}$, with $\tilde{\tau} = T(\tilde{\rho})$, are solutions of (24a) and (24b). As a consequence these numbers satisfy the relation (26) and thus $\tilde{\tau} = (1+\tau)e^{-i\varphi} - 1$. Simple manipulations show that $|\tilde{\tau}|^2 = \tau^2 + 2(\tau+1)(1-\cos\varphi)$. Since $\cos\varphi < 1$, for $0 < \varphi < 2\pi$, it follows that $|\tilde{\tau}| > \tau$, which is a contradiction.

In order to apply singularity analysis we require the local behaviour of $T = T(z)$ in a complex neighbourhood (actually in a so-called Δ-domain, see [10]) of the dominant singularity $z = \rho$. This can be obtained easily by carrying out a series expansion of the defining equation (5) around $z = \rho$ and $T = \tau$. Defining $\phi(x) := e^{-x} \int_0^x \frac{e^t}{1+t} dt$ we have to consider

$$\rho + (z-\rho) = \phi(\tau) + \phi'(\tau)(T-\tau) + \phi''(\tau)\frac{(T-\tau)^2}{2} + \mathcal{O}\big((T-\tau)^3\big).$$

With

$$\phi'(x) = \frac{1}{1+x} - e^{-x}\int_0^x \frac{e^t}{1+t}dt \quad \text{and} \quad \phi''(x) = -\frac{2+x}{(1+x)^2} + e^{-x}\int_0^x \frac{e^t}{1+t}dt,$$

and by using (24a) and (26), we obtain

$$\phi(\tau) = \rho, \quad \phi'(\tau) = 0, \quad \text{and} \quad \phi''(\tau) = -\rho^2.$$

Thus, after simple manipulations, we get

$$T = \tau \pm \sqrt{\tfrac{2}{\rho}}\sqrt{1-\frac{z}{\rho}} \cdot \big(1 + \mathcal{O}(T-\tau)\big).$$

Using that for real z, $z \to \rho_-$, there holds $T(z) < T(\rho) = \tau$, thus $T \to \tau_-$, this characterizes the correct sign and we get the required local expansion of $T(z)$ in a slit neighbourhood of $z = \rho$ (where we additionally used that $\tau = \frac{1-\rho}{\rho}$):

$$T(z) = \frac{1-\rho}{\rho} - \sqrt{\tfrac{2}{\rho}} \cdot \sqrt{1-\frac{z}{\rho}} + \mathcal{O}\big(1-\tfrac{z}{\rho}\big).$$

A standard application of singularity analysis, together with Stirling's formula for the factorials,

$$n! = \frac{n^n}{e^n}\sqrt{2\pi n} \cdot \left(1 + \mathcal{O}(n^{-1})\right),$$

for $n \to \infty$, and by using $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, shows then the asymptotic behaviour of the coefficients T_n of $T(z)$ stated in Theorem 2:

$$T_n = n![z^n]T(z) = -n!\sqrt{\tfrac{2}{\rho}}\,\frac{n^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})}\frac{1}{\rho^n}\cdot\left(1+\mathcal{O}(n^{-1})\right)$$
$$= \frac{1}{\sqrt{\rho}}\cdot\frac{n^{n-1}}{(e\rho)^n}\cdot\left(1+\mathcal{O}(n^{-1})\right).$$

Remark 3. *As mentioned earlier, the sequence T_n occurs in OEIS as sequence A225052, but without mentioning a concrete combinatorial interpretation. There also the asymptotic behaviour of T_n, for $n \to \infty$, as given in Theorem 2 has been noticed. Furthermore, differentiating the functional equation (5) with respect to z shows that $T(z)$ satisfies the differential equation*

$$T'(z) = \frac{1+T(z)}{1-z(1+T(z))}.$$

From this equation one easily gets the following recurrence formula for the numbers T_n, where we additionally set $T_0 := 1$:

$$T_{n+1} = T_n + n\sum_{k=0}^{n-1}\binom{n-1}{k}T_kT_{n-k}, \quad n \ge 0.$$

These results can be found also in OEIS as entries for the sequence A225052.

4.1.2. Pattern 231 This pattern can be treated completely analogous to the previous one. Due to the functional equation (23) for $T(z)$ we consider

$$h(z,T) = z - e^{-T}\int_0^T e^{t-1+e^{-t}}dt,$$

and search for points $(z,T) = (\rho,\tau)$ solving the equations

$$h(\rho,\tau) = \rho - e^{-\tau}\int_0^\tau e^{t-1+e^{-t}}dt = 0, \tag{27a}$$

$$h_T(\rho,\tau) = e^{-\tau}\int_0^\tau e^{t-1+e^{-t}}dt - e^{-1+e^{-\tau}} = 0. \tag{27b}$$

Equation (27b) characterizes τ as solution of

$$e^{\tau-1+e^{-\tau}} = \int_0^\tau e^{t-1+e^{-t}}dt, \quad \text{or} \quad e = \int_0^\tau e^{e^{-t}}dt, \tag{28}$$

where the second equation can be obtained from the first one by partial integration. Plugging this into (27a) gives the relation

$$\rho = e^{-1+e^{-\tau}}, \quad \text{or} \quad \tau = -\ln(1+\ln\rho). \tag{29}$$

The uniquely determined positive real solution of equation (28) can be characterized as follows by using the exponential integral, $\mathrm{Ei}(e^{-\tau}) = \mathrm{Ei}(1) - e$, and is numerically given by $\tau = 1.61058707\ldots$. Thus also the dominant singularity ρ, related to τ via (29), can be characterized via the exponential integral, $\mathrm{Ei}(1+\ln\rho) = \mathrm{Ei}(1) - e$, and is numerically given as $\rho = 0.44922576\ldots$.

Again it is not difficult to justify that there are no other dominant singularities than ρ. Namely, let us assume that there is also the dominant singularity $\tilde{\rho} = \rho e^{i\varphi}$, with some $0 < \varphi < 2\pi$. The corresponding values $\tau = T(\rho)$ and $\tilde{\tau} = T(\tilde{\rho})$ have to satisfy, according to (29), $e^{-\tau} = 1 + \ln\rho$ and $e^{-\tilde{\tau}} = 1 + \ln\tilde{\rho} = 1 + \ln\rho + i\varphi = e^{-\tau} + i\varphi$. From this equation we further get

$$|e^{\tilde{\tau}}| = \frac{e^{\tau}}{\sqrt{1+\varphi^2 e^{2\tau}}} > e^{\tau}, \quad \text{for } \varphi \neq 0.$$

However, the relation $|e^z| > e^x$, for a complex z and positive real $x > 0$, implies $\Re z > x$ and thus $|z| > x$. Therefore we get that $|\tilde{\tau}| > \tau$, which is a contradiction, since the coefficients of $T(z)$ are all non-negative implying $|\tilde{\tau}| \le \tau$.

The local expansion of $T(z)$ in a complex neighbourhood of the dominant singularity ρ can be obtained by a series expansion of (23) around $z = \rho$ and $T = \tau$, which gives (we omit these straightforward computations)

$$T(z) = \tau - \frac{\sqrt{2}}{\sqrt{1+\ln\rho}} \sqrt{1 - \frac{z}{\rho}} + \mathcal{O}\big(1 - \tfrac{z}{\rho}\big),$$

from which, by a standard application of singularity analysis, one gets the required asymptotic expansion of the coefficients T_n stated in Theorem 2:

$$T_n = n![z^n]T(z) = \frac{1}{\sqrt{1+\ln\rho}} \cdot \frac{n^{n-1}}{(e\rho)^n} \cdot \big(1 + \mathcal{O}(n^{-1})\big).$$

Remark 4. *Differentiating the functional equation (23) multiple times gives a differential equation for $T(z)$, from which a recurrence formula for the coefficients T_n can be deduced. However, since the recurrence obtained is much less appealing than the corresponding one for 123-avoiding trees, we omit to state it here.*

4.1.3. Pattern 132 Here the functional equation (23) for $T(z)$ leads to a study of the function

$$h(z,T) = z - \int_0^T e^{-t-(T-t)e^{-t}} dt,$$

and to a search for points $(z,T) = (\rho,\tau)$ solving the equations

$$h(\rho,\tau) = \rho - \int_0^\tau e^{-t-(\tau-t)e^{-t}} dt = 0, \tag{30a}$$

$$h_T(\rho,\tau) = -e^{-\tau} + \int_0^\tau e^{-2t-(\tau-t)e^{-t}} dt = 0. \tag{30b}$$

The positive real solution of (30b) is numerically given by $\tau = 1.45820126\ldots$, and plugging this into (30a) one gets the dominant singularity and radius of convergence ρ of $T(z)$, which is numerically given by $\rho = 0.44084481\ldots$. Unlike for the patterns previously studied, here we are not aware of a simple analytic argument showing that there are no further singularities on the circle of convergence. However, since we only have to examine the compact set $\{(z,T) \in \mathbb{C}^2 : |z| \le \rho, |T| \le \tau\}$ and since according to (30a) and (30b) only continuous functions are involved, the absence of further singularities can be justified easily by numerical evidence. Moreover, when considering the image of the mapping $T \mapsto \phi(T) = \int_0^T e^{-t-(T-t)e^{-t}} dt = z$, it is apparent that this mapping is injective for $|T| \le \tau$ and the cusp at $T = \tau$ with $\phi(\tau) = \rho$ illustrates the unique dominant singularity of $T(z)$; see Figure 5.

A straightforward expansion of (23) around $z = \rho$ and $T = \tau$ leads then to the following local expansion of $T(z)$ around the dominant singularity ρ,

$$T(z) = \tau - \sqrt{\kappa}\sqrt{1-\frac{z}{\rho}} + \mathcal{O}\big(1-\tfrac{z}{\rho}\big), \quad \text{with} \quad \kappa = \frac{2\rho}{e^{-\tau}+e^{-2\tau}-\int_0^\tau e^{-3t-(\tau-t)e^{-t}}dt},$$

and a standard application of singularity analysis gives the asymptotic result for the coefficients T_n stated in Theorem 2:

$$T_n = n![z^n]T(z) = \sqrt{\frac{\kappa}{2}} \cdot \frac{n^{n-1}}{(e\rho)^n} \cdot \big(1 + \mathcal{O}(n^{-1})\big).$$

4.2. Pattern-occurrences in trees

We prove here the results concerning the distribution of the r.v. $X_n^{[\sigma]}$ counting the number of occurrences of the consecutive pattern σ of length 3 in a

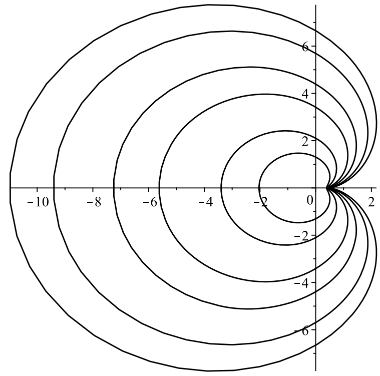

Figure 5: A complex plot of the image of the mapping $T \mapsto \phi(T) = \int_0^T e^{-t-(T-t)e^{-t}} dt = z$ for $|T| \in \{0.8, 1.0, 1.2, 1.3, 1.4, \tau\}$.

randomly chosen labelled tree of size n stated in Theorem 4. They can be obtained from the generating functions $F = F^{[\sigma]}(z, v)$ of the number $F_{n,m}^{[\sigma]}$ of trees of size n with exactly m occurrences of the pattern σ computed in Section 3 by methods of analytic combinatorics. First one easily gets exact and asymptotic results for the expectation and the variance of X_n by expanding the functional equations characterizing F around $v = 1$ and extracting coefficients. Second in order to show a central limit theorem of X_n we use the concept of singularity perturbation analysis, see [10], by studying the local behaviour around the dominant singularity of the generating function $F(z, v)$, where one considers v as a fixed parameter chosen in a complex neighbourhood of 1, and apply the so-called quasi-power theorem of Hwang, see [13], that guarantees the stated limiting distribution behaviour. Again, in order to characterize the dominant singularity of $F(z, v)$ we consider the function $h(z, F) := z - \phi(F)$ and use that, according to the implicit function theorem, this equation cannot be resolved with respect to F in a locally unique way for points $(z, F) = (\rho(v), \tau(v))$ satisfying $h(\rho(v), \tau(v)) = 0$ and $h_F(\rho(v), \tau(v)) = 0$.

4.2.1. Pattern 123 Since $F(z, v) = \sum_{n\geq1} \sum_{m\geq0} n^{n-1} \mathbb{P}\{X_n = m\} \frac{z^n v^m}{n!}$, the k-th factorial moments $\mathbb{E}(X_n^{\underline{k}})$ of X_n can be obtained by expanding $F(z, v)$ around $v = 1$. Thus, when setting $u = v - 1$, we get

$$F(z, v) = \sum_{k\geq0} F_k(z) \frac{u^k}{k!}, \quad \text{with} \quad F_k(z) = \sum_{n\geq1} \frac{n^{n-1} \mathbb{E}(X_n^{\underline{k}}) z^n}{n!}.$$

When expanding the functional equation (4) characterizing $F(z,v)$ we get

$$z = e^{-F_0}F_0 + \frac{e^{-F_0}}{6}\left(-F_0^3 - 6F_0F_1 + 6F_1\right)u + \frac{e^{-F_0}}{120}\cdot q_2u^2 + \mathcal{O}(u^3),$$

with $q_2 = 3F_0^5 - 10F_0^4 + 20F_0^3F_1 - 60F_0^2F_1 + 60F_0F_1^2 - 60F_0F_2 - 120F_1^2 + 60F_2$.

Comparing coefficients of the powers of u characterizes $F_0(z)$ as the tree function $W = W(z)$ (which is known a priori), and successively

$$F_1(z) = \frac{W^3}{6(1-W)}, \quad F_2(z) = \frac{W^4(-4W^3 + 18W^2 - 39W + 30)}{180(1-W)^3}.$$

Extracting coefficients of $F_1(z)$ and $F_2(z)$ by using

$$W = \sum_{n\ge1} n^{n-1}\frac{z^n}{n!}, \quad \frac{W}{1-W} = zW'(z) = \sum_{n\ge1} n^n\frac{z^n}{n!},$$
$$\frac{W}{(1-W)^3} = z(zW')' = \sum_{n\ge1} n^{n+1}\frac{z^n}{n!},$$

gives the exact and asymptotic results for the first and second factorial moment, and thus also for the variance, of X_n stated in Theorem 4; we omit these straightforward computations.

For the limiting distribution behaviour we consider again the functional equation (4) for $F = F(z,v)$, where we treat v as a parameter chosen in a complex neighbourhood of 1. We study

$$h(z,F) = z - e^{-F}\int_0^F e^t(1-(v-1)t)^{\frac{1}{v-1}}\,dt,$$

and search for points $(z,F) = (\rho,\tau)$, with $\rho = \rho(v)$, $\tau = \tau(v)$ depending on v, which are solutions of the equations

$$h(\rho,\tau) = \rho - e^{-\tau}\int_0^\tau e^t(1-(v-1)t)^{\frac{1}{v-1}}\,dt = 0, \tag{31a}$$

$$h_F(\rho,\tau) = e^{-\tau}\int_0^\tau e^t(1-(v-1)t)^{\frac{1}{v-1}}\,dt - (1-(v-1)\tau)^{\frac{1}{v-1}} = 0. \tag{31b}$$

Thus τ is characterized as solution of the equation

$$e^\tau(1-(v-1)\tau)^{\frac{1}{v-1}} = \int_0^\tau e^t(1-(v-1)t)^{\frac{1}{v-1}}\,dt, \tag{32}$$

and ρ is related to τ via

$$\rho = (1-(v-1)\tau)^{\frac{1}{v-1}}. \tag{33}$$

Note that, also according to the implicit function theorem, since $h_{FF}(\rho(1), \tau(1)) = h_{FF}(e^{-1}, 1) \neq 0$, there is a uniquely determined analytic function $\tau(v)$ (and thus also $\rho(v)$) around $v = 1$ characterized by these equations. A series expansion of the functional equation (4) around $F = \tau$ and $z = \rho$ gives after easy computations the following local expansion of $F(z, v)$ around $z = \rho(v)$:

$$F = \tau - \sqrt{2}\,\rho^{\frac{v-1}{2}} \sqrt{1 - \frac{z}{\rho}} + \mathcal{O}\big(1 - \tfrac{z}{\rho}\big). \tag{34}$$

An application of singularity analysis to (34) shows the following asymptotic behaviour of the coefficients of $F(z, v)$, and thus of the probability generating function of the r.v. X_n:

$$\mathbb{E}\left(v^{X_n}\right) = \frac{n!}{n^{n-1}}[z^n]F(z,v) = \rho^{\frac{v-1}{2}} \cdot \frac{1}{(e\rho)^n} \cdot \left(1 + \mathcal{O}(n^{-1})\right).$$

Setting $v = e^s$ we obtain an asymptotic expansion of the moment generating function:

$$\mathbb{E}\left(e^{sX_n}\right) = e^{U(s)\cdot n + V(s)} \cdot \left(1 + \mathcal{O}(n^{-1})\right), \tag{35}$$

with functions U and V given as follows:

$$U(s) = -\big(1 + \ln\big(\rho(e^s)\big)\big), \qquad V(s) = \frac{e^s - 1}{2} \cdot \ln\big(\rho(e^s)\big).$$

This is exactly the setting of the quasi-power theorem due to Hwang [13, 10] from which we can deduce $\mathbb{E}(X_n) = U'(0)n + \mathcal{O}(1)$, $\mathbb{V}(X_n) = U''(0)n + \mathcal{O}(1)$ and a central limit theorem for the normalized r.v. $\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}}$. Note that from (32) and (33), via an expansion around $v = 1$, we easily get the local expansion $U(s) = -1 - \ln \rho(e^s) = \frac{1}{6}s + \frac{1}{10}s^2 + \mathcal{O}(s^3)$ around $s = 0$, which again gives the asymptotic behaviour of the expectation $\mathbb{E}(X_n) \sim \frac{1}{6}n$ and of the variance $\mathbb{V}(X_n) \sim \frac{1}{5}$ already obtained from our exact results. Altogether this completes the proof of Theorem 4 for the pattern 123.

4.2.2. Pattern 132 and 231 The computations for the patterns 132 and 231, where one treats the functional equations (11) and (22), respectively, are completely analogous to the ones carried out for 123 in the previous section; thus we may omit them here.

5. Permutation patterns in mappings

One useful aspect of the chosen treatment of pattern avoidance/occurrences for labelled trees is that this approach can be extended without much effort to a study of corresponding quantities in mappings. Based on a combinatorial decomposition of mappings with respect to the smallest element 1 after introducing auxiliary quantities, we obtain (partial) differential equations that relate the corresponding generating functions for trees and mappings. Solving these equations, we are able to express the g.f. for mappings by the corresponding ones for trees, as stated in Theorem 1 and Theorem 3. Interestingly, one observes from these results that for the patterns 123 and 231 studies of pattern avoidance and pattern occurrences in trees and mappings are closely related, which is quantified in Corollary 1 and Corollary 2. We will provide a combinatorial explanation of these results via a concrete bijection. Only for the remaining pattern 132, where the quantities in trees and mappings do not seem to be related in a direct way, we carry out the before-mentioned generating functions approach.

5.1. Pattern 123 and 231

In [16] the authors give a bijection φ from the set of pairs (t, y) consisting of a tree t of certain size $n \geq 1$ and y a node in t to the set of n-mappings $f : [n] \to [n]$. As we will show, the function φ preserves occurrences of certain consecutive patterns, in particular of the patterns 123 and 213, from which we may deduce Corollary 1 and Corollary 2. For the sake of completeness we state bijection φ (together with a proof that it is indeed such one) also in the appendix of the present work as Theorem 6. To state the bijection and to prove its properties it is advantageous to consider a tree t as the functional digraph of a certain function $t : [n] \to [n]$. Namely, we will denote by $t(x)$ the out-neighbour of node x in the tree t. That is, for x a non-root node, $x \neq \mathsf{root}(t)$, $t(x)$ is the unique node such that $(x, t(x))$ is an edge in t. Furthermore, we define $t\,(\mathsf{root}(t)) := \mathsf{root}(t)$, i.e., think of an additional slope at the root of t.

The map φ is exemplified in Figure 6. Described in a nutshell, the nodes in the tree t lying on the path p from y to the root $\mathsf{root}(t)$ correspond to

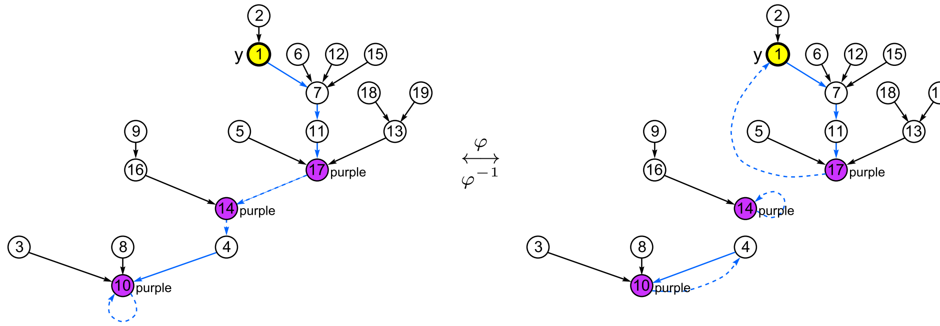

Figure 6: An example of the bijection φ described in Theorem 6 applied to the pair $(t,1)$, with t the tree of size 19 depicted on the left side yielding the 19-mapping f on the right side. The unique path from 1 to the root of t is marked by blue edges, with edges dashed that are reattached according to the bijection. Furthermore, right-to-left-maxima on this path, which correspond to cycle-leaders in f, are marked by purple nodes.

cyclic elements in the mapping f, i.e., elements i, such that there exists an $\ell \geq 1$ with $f^{\ell}(i) = i$, and the right-to-left maxima in the sequence of labels in p correspond to the "cycle-leaders" in f, i.e., the largest elements amongst all cyclic elements in the respective components. Furthermore, only outgoing edges from right-to-left maxima in p might be detached by φ, in other words, $t(x) \neq f(x)$ implies that x is in t a right-to-left maximum in the path p, or equivalently, that x is in f a cycle-leader.

We will show that φ preserves occurrences of a consecutive pattern σ, where the largest element in σ is at the last position.

Theorem 5. *Let $\sigma = \sigma(1)\sigma(2)\dots\sigma(k-1)k \in S_k$ be a permutation, where the largest element is at the last position. Then the bijection φ presented in Theorem 6 preserves the number of occurrences of the consecutive pattern σ, i.e., for a pair (t,y) of a labelled tree t and a node $y \in t$, and the mapping $f = \varphi(t,y)$ being the image of (t,y) under the map φ, there holds that t and f have the same number of occurrences of σ.*

Proof. First assume that $q = (x, t(x), \dots, t^{k-1}(x))$ is an occurrence of the consecutive pattern σ in the tree t. Since q is order-isomorphic to σ, it follows that $t^{k-1}(x) > t^i(x)$, for all $0 \leq i \leq k-2$. Thus none of the nodes $t^i(x)$, $0 \leq i \leq k-2$, could be a right-to-left maximum on the path p from y to the root of t, which implies that all edges $(t^i(x), t^{i+1}(x))$, $0 \leq i \leq k-2$, persist by applying the function φ. Therefore $f^i(x) = t^i(x)$, for $1 \leq i \leq k-1$, and thus $(x, f(x), \ldots, f^{k-1}(x))$ is also an occurrence of the consecutive pattern σ in the mapping f.

Next let us assume that $q' = (x, f(x), \ldots, f^{k-1}(x))$ is an occurrence of the consecutive pattern σ in the mapping $f = \varphi(t, y)$. Again, since q' is order-isomorphic to σ, we have $f^{k-1}(x) > f^i(x)$, for all $0 \le i \le k-2$, which implies that none of the elements $f^i(x)$, $0 \le i \le k-2$, could be a cycle-leader, i.e., the largest element amongst cyclic elements of a connected component of the directed digraph of f. Therefore, all edges $(f^i(x), f^{i+1}(x))$, $0 \le i \le k-2$, already occurred in t and persisted when applying the function φ. Thus $t^i(x) = f^i(x)$, for $1 \le i \le k-1$, and $(x, t(x), \ldots, t^{k-1}(x))$ formed an occurrence of the consecutive pattern σ in the tree t. ☐

This yields as a corollary the following correspondence between trees and mappings concerning avoidance and occurrences of certain consecutive patterns.

Corollary 3. *Let $\sigma = \sigma(1)\sigma(2)\ldots\sigma(k) \in S_k$ be a permutation, where the largest element or the smallest element is at the last position, i.e., where $\sigma(k) = k$ or $\sigma(k) = 1$. Then we have*

$$G_{n,m}^{[\sigma]} = nF_{n,m}^{[\sigma]}, \quad \text{and} \quad M_n^{[\sigma]} = nT_n^{[\sigma]}, \quad \text{for } n \ge 1.$$

Proof. If $\sigma(k) = k$ then Theorem 5 immediately shows that the statement is valid. If $\sigma(k) = 1$ then consider the complementary pattern $\tilde{\sigma} = \tilde{\sigma}(1)\ldots\tilde{\sigma}(k)$, defined via $\tilde{\sigma}(j) = k + 1 - \sigma(j)$, $1 \le j \le k$, which satisfies the assumptions of Theorem 5. Together with the obvious fact that $G_{n,m}^{[\sigma]} = G_{n,m}^{[\tilde{\sigma}]}$ and $F_{n,m}^{[\sigma]} = F_{n,m}^{[\tilde{\sigma}]}$, which follows again by complementing the labels of size-n trees or n-mappings via $i \mapsto n + 1 - i$, this also shows the result for such kind of patterns. ☐

Corollaries 1 and 2 thus follow from Corollary 3. Note that corresponding results also hold for sets of patterns, which either contain only patterns with largest label at the last position or contain only patterns with smallest label at the last position, but in general do not hold when considering sets that contain patterns with smallest and largest label at the last position.

We further remark that at the level of generating functions this correspondence between trees and mappings reads as $M(z) = 1 + zT'(z)$, and $G(z, v) = 1 + zF_z(z, v)$. It can be checked easily that the g.f. $M(z)$ and $G(z, v)$ presented in Theorem 1 and Theorem 3, respectively, satisfy these relations. As stated earlier, these generating functions have been computed first by extending the approach based on a combinatorial decomposition with respect to element 1 from trees to mappings.

5.2. Pattern 132

In order to study the consecutive pattern 132 in mappings we extend the generating functions approach that has been introduced for labelled trees in Section 3, which is based on a decomposition with respect to the smallest element 1. First we remark that such a study can be reduced to so-called connected mappings, i.e., it suffices to consider the weakly connected components of the functional digraph. Namely, in combinatorial terms, mappings and connected mappings are linked by the SET-construction and thus we can easily transfer results from one family to the other. We start by considering connected mappings for which, analogously to our previous analysis of labelled trees, we have to introduce as auxiliary parameter the number of occurrences of the pattern 21. Formally we consider bicoloured structures, where each node x is coloured blue, i.e., gets a marker B, if it is the starting node of the pattern 21, and x is coloured red, i.e., gets a marker R, if it is the starting node of the pattern 132. We introduce the combinatorial family $\mathcal{C}$ of connected mappings with nodes coloured as described before.

The decomposition of a connected mapping $c \in \mathcal{C}$ with respect to the node with smallest label 1 yields (after order-preserving relabellings) a (possibly empty) set $t_1, \ldots, t_k$ of $k \geq 0$ subtrees with respective root-nodes $r_1, \ldots, r_k$ originally attached to node 1 and, in case that 1 is not part of a loop, a structure g_0, where node 1 is originally linked to a node $x \in g_0$. Depending on whether 1 is contained in a cycle or not, g_0 itself is a tree or a connected mapping. Five cases might occur, see Figure 7, where we also take into account that attaching the root r_j of the subtree t_j, $1 \leq j \leq k$, to node 1 creates a consecutive pattern 21 in c, thus each vertex r_j will be coloured blue, whereas attaching the subtrees to 1 does not change the number of occurrences of the pattern 132.

1. Node 1 is not contained in a cycle of c and x is in g_0 coloured blue: in this case, the link from 1 to x creates in c a new pattern 132 and node 1 will be coloured red. Thus the number of red vertices in c is one plus the sum of the number of red vertices in $g_0, t_1, \ldots, t_k$, whereas the number of blue vertices in c is k (for each of the nodes attached to 1) plus the sum of the number of blue vertices in $g_0, t_1, \ldots, t_k$.
2. Node 1 is not contained in a cycle of c and x is not blue in g_0: in that case, the link from 1 to x neither creates a pattern 21 nor a pattern 132, and thus node 1 remains uncoloured. The number of red vertices in c is equal to the sum of the number of red vertices in $g_0, t_1, \ldots, t_k$, and the number of blue vertices in c is k plus the sum of the number of blue vertices in $g_0, t_1, \ldots, t_k$.

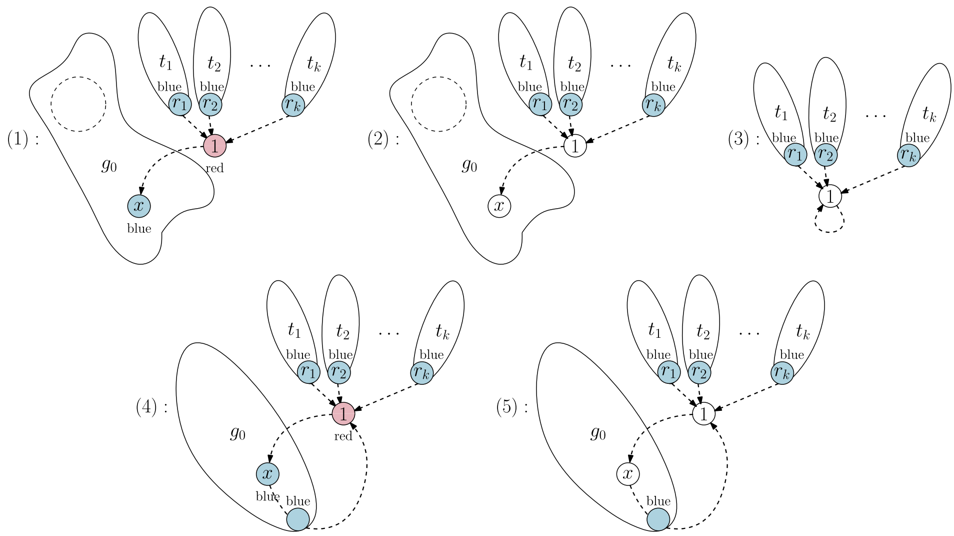

Figure 7: Pattern 132: Decomposition of a connected mapping c with respect to the node with smallest label, where the five cases described below may occur. Starting nodes of occurrences of the pattern 21 and 132 relevant to this decomposition are coloured blue and red, respectively.

3. Node 1 is part of a loop in c: here the number of red vertices in c is equal to the sum of the red vertices in $t_1, \dots, t_k$, and the number of blue vertices in c is k plus the sum of the number of blue vertices in $t_1, \dots, t_k$.
4. Node 1 is part of a non-loop cycle of c and x is in g_0 coloured blue: in this case, the link from 1 to x creates in c a new pattern 132 and node 1 will be coloured red. Furthermore, the link from the root of g_0 to node 1 completing the cycle creates a new pattern 21 and thus the root of g_0 will be coloured blue. Summarizing, the number of red vertices in c is one plus the sum of the number of red vertices in $g_0, t_1, \dots, t_k$, whereas the number of blue vertices in c is $k+1$ (for each of the nodes linked to 1) plus the sum of the number of blue vertices in $g_0, t_1, \dots, t_k$.
5. Node 1 is part of a non-loop cycle of c and x is not blue in g_0: in that case, the link from 1 to x neither creates a pattern 21 nor a pattern 132, and thus node 1 remains uncoloured. The number of red vertices in c is equal to the sum of the number of red vertices in $g_0, t_1, \dots, t_k$, and the number of blue vertices in c is $k+1$ plus the sum of the number of blue vertices in $g_0, t_1, \dots, t_k$.

Combining these cases and using the combinatorial constructions defined in Section 3 leads to the following recursive description of the family $\mathcal{C}$:

$$\begin{aligned}\mathcal{C} = {} & \mathcal{Z}^{\Box} * \Theta_B(\mathcal{C}) * \textsc{Set}(\mathcal{F} \times \{B\}) \times \{R\} \\ & + \mathcal{Z}^{\Box} * \big(\Theta_{\mathcal{Z}}(\mathcal{C}) \setminus \Theta_B(\mathcal{C})\big) * \textsc{Set}(\mathcal{F} \times \{B\}) \\ & + \mathcal{Z}^{\Box} * \textsc{Set}(\mathcal{F} \times \{B\}) + \mathcal{Z}^{\Box} * \Theta_B(\mathcal{F}) * \textsc{Set}(\mathcal{F} \times \{B\}) \times \{B, R\} \\ & + \mathcal{Z}^{\Box} * \big(\Theta_{\mathcal{Z}}(\mathcal{F}) \setminus \Theta_B(\mathcal{F})\big) * \textsc{Set}(\mathcal{F} \times \{B\}) \times \{B\}. \end{aligned} \tag{36}$$

We introduce the trivariate generating function

$$\begin{aligned}\tilde{C}(z, w, v) & := \sum_{c \in \mathcal{C}} \frac{z^{|c|} \, w^{\sharp \text{ blue nodes in } c} \, v^{\sharp \text{ red nodes in } c}}{|c|!} \\ & = \sum_{n \geq 1} \sum_{\ell \geq 0} \sum_{m \geq 0} \tilde{C}_{n,\ell,m} \frac{z^n w^\ell v^m}{n!},\end{aligned}$$

where $\tilde{C}_{n,\ell,m}$ denotes the number of connected n-mappings with ℓ occurrences of 21 and m occurrences of 132. An application of the symbolic method to the formal equation (36) leads to the following first-order linear PDE for $\tilde{C} = \tilde{C}(z, w, v)$, with $\tilde{F} = \tilde{F}(z, w, v)$ the corresponding generating function for trees studied in Section 3.2:

$$\tilde{C}_z = vwe^{w\tilde{F}} \tilde{C}_w + e^{w\tilde{F}} (z\tilde{C}_z - w\tilde{C}_w) + e^{w\tilde{F}} + vw^2 e^{w\tilde{F}} \tilde{F}_w + we^{w\tilde{F}} (z\tilde{F}_z - w\tilde{F}_w).$$

Taking into account (7), slight simplifications occur yielding the following PDE together with the initial condition $\tilde{C}(0, w, v) = 0$:

$$\big(1 - ze^{w\tilde{F}}\big)\tilde{C}_z - (v-1)we^{w\tilde{F}} \tilde{C}_w = w\tilde{F}_z - (w-1)e^{w\tilde{F}}. \tag{37}$$

In order to solve this PDE one can again apply the method of characteristics, thus searching for a suitable substitution of variables, such that it can be reduced to an ordinary differential equation. Since the coefficients of the partial derivatives in the defining equations (7) and (37) of the functions $\tilde{F}$ and $\tilde{C}$, respectively, match, this suggests to choose a first integral obtained for $\tilde{F}$. Indeed, taking into account (9) and that $\tilde{F}$ is itself only given implicitly as solution of a functional equation, the following pair of substitutions works fine (where we consider v as a parameter and with $\tilde{F}$ satisfying (10)):

$$\tilde{Q} = \tilde{Q}(z, w) = we^{(v-1)\tilde{F}}, \qquad \tilde{F} = \tilde{F}(z, w).$$

Note that this gives the inverse transform

$$w = w(\tilde{F}, \tilde{Q}) = \tilde{Q} e^{-(v-1)\tilde{F}}, \quad z = z(\tilde{F}, \tilde{Q}) = e^{-\tilde{F}} \int_0^{\tilde{F}} e^{t(1-\tilde{Q}e^{-(v-1)t})} dt.$$

After somewhat lengthy computations, which are here omitted, we obtain the following solution of (37) involving the corresponding g.f. $\tilde{F}$ for trees:

$$\tilde{C}(z,w,v) = \log\left(\frac{1}{1 - we^{\tilde{F}} \int_0^{\tilde{F}} e^{-t(2-v)-w(\tilde{F}-t)e^{(v-1)t}} dt}\right) + \tilde{F} - \frac{w}{v-1}\left(e^{(v-1)\tilde{F}} - 1\right). \tag{38}$$

We remark that it is an easy task to check that $\tilde{C}(z,w,v)$ given by (38) satisfies (37) and the initial condition, thus that it is indeed the required solution.

Actually we are interested in results for arbitrary (not only connected) mappings and thus consider the trivariate generating function

$$\tilde{G}(z,w,v) := \sum_{n\geq 0} \sum_{\ell \geq 0} \sum_{m\geq 0} \tilde{G}_{n,\ell,m} \frac{z^n w^\ell v^m}{n!},$$

where $\tilde{G}_{n,\ell,m}$ denotes the number of n-mappings with ℓ occurrences of 21 and m occurrences of 132. According to the SET-construction relating connected mappings with arbitrary mappings, it simply holds that $\tilde{G} = e^{\tilde{C}}$ for the respective generating functions, and we obtain the following solution for $\tilde{G} = \tilde{G}(z,w,v)$:

$$\tilde{G} = \frac{e^{\tilde{F}} e^{\frac{w}{v-1}(1-e^{(v-1)\tilde{F}})}}{1 - we^{\tilde{F}} \int_0^{\tilde{F}} e^{(v-2)t - w(\tilde{F}-t)e^{(v-1)t}} dt}. \tag{39}$$

We remark that by differentiating (11) with respect to z and comparing with (39) one further obtains the connection

$$\tilde{G} = e^{\frac{w}{v-1}(1-e^{(v-1)\tilde{F}})} \cdot \tilde{F}_z. \tag{40}$$

Setting $w = 1$ in (39) gives the solution for the bivariate g.f. $G(z,v) := \tilde{G}(z,1,v) = \sum_{n\geq 0} \sum_{m \geq 0} G_{n,m} \frac{z^n v^m}{n!}$ of the number $G_{n,m}$ of n-mappings with exactly m occurrences of the pattern 132 stated in Theorem 3. Furthermore, setting $v = 0$ yields the g.f. $M(z) := G(z, 0) = \sum_{n \geq 0} M_n \frac{z^n}{n!}$ of the number M_n of 132-avoiding n-mappings given in Theorem 1.

Asymptotic results for the pattern 132 presented in Theorem 2 and Theorem 4 can be deduced easily from the corresponding results for labelled trees, by using singularity analysis and the relations

$$M(z) = e^{-(1-e^{-T(z)})} \cdot T'(z) \quad \text{and} \quad G(z, v) = e^{\frac{1}{v-1}(1-e^{(v-1)F(z,v)})} \cdot F_z(z, v),$$

which follows from (40). Thus we omit these straightforward computations.

6. Outlook and open problems

The study of consecutive permutation patterns in labelled trees could be extended in various ways. We mention a few such directions for which we obtained some preliminary results via the method presented.

- Sets of patterns. There seem to be several interesting classes of sets of patterns of length 3; some (but as it seems, not all) of them could be treated by using a decomposition with respect to the largest or smallest labelled vertex.
- Patterns of length 4 or higher. Although computations quickly get quite involved, there is some hope to obtain at least partial results.
- Other tree families. There are other combinatorial tree families, most notably labelled ordered trees and labelled binary trees, where the approach presented could be applied. Again, computations are more involved, since one has to take into account the number of possible "attachment points" to reconstruct a tree after the decomposition.

Another line of research would concern to give combinatorial explanations of the results obtained, in particular, for the generating functions $T^{[\sigma]}(z)$ avoiding a consecutive pattern σ stated in Theorem 1. Namely, it is a very interesting observation of an anonymous referee that the compositional inverse functions $T^{[\sigma]}(z)^{\langle -1 \rangle}$ are, for all patterns studied, the generating functions of well-known simple integer sequences with alternating signs. More precisely, they are given as follows, as can be shown at the level of generating functions, e.g., by characterizing them via certain differential equations:

$$\bar{T}^{[\sigma]}(z) := T^{[\sigma]}(z)^{\langle -1 \rangle} = \sum_{n \geq 1} (-1)^{n-1} \bar{T}_n^{[\sigma]} \frac{z^n}{n!},$$

with $\bar{T}_n^{[\sigma]}$ given as follows.

Pattern σ	$\bar{T}_n^{[\sigma]}$	OEIS sequence
123	$\sum_{k=0}^{n-1} k!$	*A003422*
132	$\sum_{k=0}^{n-1}(n-k)^k$	*A026898*
231	$\sum_{k=0}^{n-1} B_k$, with B_k the k-th Bell number	*A005001*

As pointed out by the referee a possible approach could be based on the inversion theorem shown in the PhD thesis of Brian Drake [5] (the main result has been obtained independently in [4]), which in many cases, where the coefficients of the compositional inverse of an exponential generating function have alternating signs, leads to a combinatorial explanation in terms of rooted trees with labelled leaves and unlabelled internal nodes. However, it seems that an application of this theorem to the problem studied, where the labelling is on all vertices, is not completely straightforward; the author has to leave the task of finding a combinatorial explanation for the connection between the generating functions $T^{[\sigma]}(z)$ and the sequence of numbers $\bar{T}_n^{[\sigma]}$ as an open problem.

Appendix A. Bijection φ between marked trees and mappings

Theorem 6 ([16]). *For each $n \geq 1$, there exists a bijection φ from the set of pairs (t, y), with t a rooted labelled tree of size n and $y \in t$ a node of t, to the set of n-mappings f.*

Proof. Given a pair (t, y), we consider the unique path $y \rightsquigarrow \mathsf{root}(t)$ from the node y to the root of t. It consists of the nodes $x_1 = y$, $x_2 = t(x_1), \ldots, x_{i+1} = t(x_i), \ldots, x_r = \mathsf{root}(t)$, for some $r \geq 1$. We denote by $I = (i_1, \ldots, i_k)$, with $i_1 < i_2 < \cdots < i_k$, for some $k \geq 1$, the indices of the right-to-left maxima in the sequence $x_1, x_2, \ldots, x_r$, i.e.,

$$i \in I \iff x_i > x_j, \quad \text{for all } j > i.$$

The corresponding set of nodes in the path $y \rightsquigarrow \mathsf{root}(t)$ will be denoted by $V_I := \{x_i : i \in I\}$. It follows from the definition that the root node is always contained in V_I, i.e., $x_r \in V_I$.

We can now describe the function φ by constructing an n-mapping f. The k right-to-left maxima in the sequence $x_1, x_2, \ldots, x_r$ will give rise to k connected components in the functional digraph G_f. Moreover, the nodes on the path $y \rightsquigarrow \mathsf{root}(t)$ in t will correspond to the cyclic nodes in G_f. We describe f by defining $f(x)$ for all $x \in [n]$, where we distinguish whether $x \in V_I$ or not.

(a) Case $x \notin V_I$: We set $f(x) := t(x)$.
(b) Case $x \in V_I$: We set $f(x_{i_1}) := x_1$ and $f(x_{i_j}) := t\left(x_{i_{j-1}}\right)$, for $j > 1$. This means that the nodes on the path $y \rightsquigarrow \mathsf{root}(t)$ in t form k cycles $C_1 := (x_1, \ldots, x_{i_1})$, ..., $C_k := (t(x_{i_{k-1}}), \ldots, x_r = x_{i_k})$ in G_f.

It is now easy to describe the inverse function φ^{-1}. Given a mapping f, we sort the connected components of G_f in decreasing order of their largest cyclic elements. That is, if G_f consists of k connected components and c_i denotes the largest cyclic element in the i-th component, we have $c_1 > c_2 > \ldots > c_k$. Then, for every $1 \le i \le k$, we remove the edge (c_i, d_i) where $d_i = f(c_i)$. Next we reattach the components to each other by establishing the edges (c_i, d_{i+1}), for every $1 \le i \le k-1$. This leads to the tree t. Note that the node c_k is attached nowhere since it constitutes the root of t. Setting $y = d_1$, we obtain the preimage (t, y) of f. □

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