

On edge-colored saturation problems

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Let \mathcal{C} be a family of edge-colored graphs. A t -edge colored graph G is (\mathcal{C}, t) -saturated if G does not contain any graph in \mathcal{C} but the addition of any edge in any color in $[t]$ creates a copy of some graph in \mathcal{C} . Similarly to classical saturation functions, define $\text{sat}_t(n, \mathcal{C})$ to be the minimum number of edges in a (\mathcal{C}, t) saturated graph. Let $\mathcal{C}_r(H)$ be the family consisting of every edge-colored copy of H which uses exactly r colors.

In this paper we consider a variety of colored saturation problems. We determine the order of magnitude for $\text{sat}_t(n, \mathcal{C}_r(K_k))$ for all r , showing a sharp change in behavior when $r \geq \binom{k-1}{2} + 2$. A particular case of this theorem proves a conjecture of Barrus, Ferrara, Vandenbussche, and Wenger. We determine $\text{sat}_t(n, \mathcal{C}_2(K_3))$ exactly and determine the extremal graphs. Additionally, we document some interesting irregularities in the colored saturation function.

KEYWORDS AND PHRASES: Saturation, edge-coloring.

1. Edge-colored saturation problems

Given a family of graphs \mathcal{F} , a graph G is \mathcal{F} -saturated if no $F \in \mathcal{F}$ is a subgraph of G , but for any $e \in E(\overline{G})$, some $F \in \mathcal{F}$ is a subgraph of $G + e$. The minimum number of edges in an n -vertex \mathcal{F} -saturated graph is the *saturation number* of \mathcal{F} and is denoted $\text{sat}(n, \mathcal{F})$. If $\mathcal{F} = \{F\}$, then we instead say that G is F -saturated, and write $\text{sat}(n, F)$. The saturation function was introduced by Erdős, Hajnal, and Moon [3] and graph and hypergraph saturation problems have received considerable attention since that time. We refer the interested reader to the dynamic survey of Faudree, Faudree, and Schmitt [5], which contains a number of results and open problems.

In this paper, we are interested in saturation problems in the setting of edge-colored graphs, which was first introduced by Hanson and Toft [8] in

1987. Let $[t] = \{1, 2, \dots, t\}$. A function $f : E(G) \rightarrow [t]$ is a t -edge-coloring of a graph G . An injective function yields a rainbow edge coloring. Given a family \mathcal{C} of edge-colored graphs, we say that a t -edge-colored graph G is (\mathcal{C}, t) -saturated if G contains no member of \mathcal{C} as a (colored) subgraph, but for any edge $e \in E(\overline{G})$ and any color $i \in [t]$, the addition of e to G in color i creates some member of \mathcal{C} . In line with classical saturation functions, we are interested in $\text{sat}_t(n, \mathcal{C})$, the minimum number of edges in a (\mathcal{C}, t) -saturated graph of order n .

In this paper, we will primarily be interested in families of edge-colored graphs. Given a graph H and a fixed palette of t colors, define $\mathcal{M}(H)$ to be the family of monochromatic edge colorings of H , $\mathcal{R}(H)$ denote the family of rainbow edge-colorings of H , and $\mathcal{C}_k(H)$ denote the set of edge colorings of H using exactly k of the t colors. Going forward, when considering $\text{sat}_t(n, \mathcal{C})$ for any of these functions, we will implicitly assume that we color these families from $[t]$. Hanson and Toft [8] determined $\text{sat}_t(n, \mathcal{F})$ where \mathcal{F} consists of monochromatic copies of K_{t_i} in color i for $1 \leq i \leq t$, and also introduced a related conjecture that we will discuss briefly in the conclusion.

1.1. Rainbow subgraphs

Barrus, Ferrara, Vandenbussche and Wenger [1] introduced $\text{sat}_t(n, \mathcal{R}(H))$ and considered several problems with a significant focus on the asymptotic behavior of this parameter for different choices of H . As discussed in [1], it is straightforward to show that for any graph H , $\text{sat}_t(n, \mathcal{M}(H)) = O(n)$. This is not the case, however, for rainbow target graphs. For instance, Barrus *et al.* show that $\text{sat}_t(n, \mathcal{R}(K_{1,r})) = \Theta(n^2)$ for $r \geq 2$, and gave two more general results that imply

$$c_1 \frac{n \log(n)}{\log \log(n)} \leq \text{sat}_t(n, \mathcal{R}(K_k)) \leq c_2 n \log(n)$$

for $k \geq 3$. They also conjectured the following.

Conjecture 1. For $k \geq 3$ and $t \geq \binom{k}{2}$,

$$\text{sat}_t(n, \mathcal{R}(K_k)) = \Theta(n \log n).$$

In Section 2, we prove some broader results about $\text{sat}_t(n, \mathcal{C}_k(H))$. As a consequence of our result there, we prove the following.

Theorem 1. Let $k \geq 3$, and $t \geq c$ be fixed.

1. If $c \geq \binom{k-1}{2} + 2$, then $\text{sat}_t(n, \mathcal{C}_c(K_k)) = \Theta(n \log n)$.
2. If $c \leq \binom{k-1}{2} + 1$, then $\text{sat}_t(n, \mathcal{C}_c(K_k)) = \Theta(n)$.

Independent of our work here, Girão, Lewis and Popielarz [7] determined the asymptotics of $\text{sat}_t(n, \mathcal{R}(H))$ for every connected graph H without a pendant edge, and as a consequence also resolve Conjecture 1 in the affirmative. Furthermore, Korándi [11] recently showed that

$$\text{sat}_t(n, \mathcal{R}(K_k)) = \Theta_k \left(\frac{n \log n}{\log t} \right)$$

and gave sharp asymptotics (in t and n) for $\text{sat}_t(n, \mathcal{R}(K_3))$. The techniques utilized across all three papers are quite diverse, and provide an interesting spectrum of possible approaches to problems of this type.

1.2. Irregularities

It has been well-documented (see, for instance [8, 5]) that the classical (un-colored) saturation function is not monotone in n or with respect to subgraph and family inclusion. That is, there is a graph H such that $\text{sat}(n, H) \leq \text{sat}(n + 1, H)$ for infinitely many n , distinct graphs $H_1 \subseteq H_2$ such that $\text{sat}(n, H_2) \leq \text{sat}(n, H_1)$, and distinct families $\mathcal{F}_1 \subseteq \mathcal{F}_2$ such that $\text{sat}(n, \mathcal{F}_2) \leq \text{sat}(n, \mathcal{F}_1)$. Before continuing on to our main results, we give some examples of similar irregular behaviors for $\text{sat}_t(n, \mathcal{C}_k(H))$.

Recall that it was shown in [1] that $\text{sat}_t(n, \mathcal{R}(K_{1,k})) = \Theta(n^2)$ for all $t \geq k \geq 3$, and further that if T is any tree with $k \geq 4$ vertices that is not a star, then $\text{sat}_t(n, \mathcal{R}(T)) = O(n)$ when $t \geq \binom{k-1}{2}$. This immediately establishes that $H_1 \subseteq H_2$ does not necessarily imply that $\text{sat}_t(n, \mathcal{R}(H_1)) \leq \text{sat}_t(n, \mathcal{R}(H_2))$. More interestingly, in our opinion, is the following result, which establishes that $\text{sat}_t(n, \mathcal{C}_k(H))$ is not monotone (increasing or decreasing) in k . Recall that $\mathcal{M}(H) = \mathcal{C}_1(H)$ and $\mathcal{R}(H) = \mathcal{C}_{|E(H)|}(H)$.

Theorem 2. For $t \geq 3$ and n sufficiently large,

$$\text{sat}_t(n, \mathcal{C}_2(K_{1,3})) < \text{sat}_t(n, \mathcal{C}_1(K_{1,3})) < \text{sat}_t(n, \mathcal{C}_3(K_{1,3})).$$

First recall $\text{sat}_t(n, \mathcal{C}_1(K_{1,3})) = O(n)$ and $\text{sat}_t(n, \mathcal{C}_3(K_{1,3})) = \Theta(n^2)$ [1]. The theorem then follows from two propositions.

Proposition 3. For $n \geq 2t \geq 6$, $\text{sat}_t(n, \mathcal{C}_1(K_{1,3})) \geq tn/2$.

Proof. Suppose G is saturated for $\mathcal{C}_1(K_{1,3})$. If we have a non-edge uv in G , then adding uv to G in any color produces a monochromatic $K_{1,3}$ at u or at v . Since v can be saturated for at most $\lfloor d(v)/2 \rfloor$ colors, there are at least $t - \lfloor d(v)/2 \rfloor$ colors in which we could add edge uv without producing a monochromatic $K_{1,3}$ at v . If $d(u) < 2(t - \lfloor d(v)/2 \rfloor)$, then there is some color in which uv can be added that produces neither a monochromatic $K_{1,3}$ at u or at v . Thus if $uv \notin E(G)$, then $d(u) \geq 2(t - \lfloor d(v)/2 \rfloor)$.

If $\delta(G) \geq t$, then we get that $e(G) \geq tn/2$ so we may assume that $\delta(G) = \ell < t$. In this case, rather than directly compute the number of edges in G , we consider the degree sum. We will condition on whether or not a vertex is in the closed neighborhood of a fixed vertex of minimum degree. To this end, let v be a vertex with $d(v) = \ell$. For $u \notin N[v]$, we have $d(u) \geq 2(t - \lfloor \ell/2 \rfloor)$ and, for all vertices, we have $d(u) \geq \ell$.

$$\begin{aligned} \sum_{w \in V(G)} d(w) &= \sum_{w \notin N[v]} d(w) + \sum_{w \in N[v]} d(w) \\ &\geq (n - (\ell + 1)) \cdot (2t - \ell) + (\ell + 1) \cdot \ell \\ &= n(2t - \ell) - 2(\ell + 1)(t - \ell) \\ &= nt + (t - \ell)[n - 2(\ell + 1)] \end{aligned}$$

For $\ell \leq t - 1$, the degree sum is at least $nt + (n - 2t)$. Thus when $n \geq 2t$, we find $e(G) \geq nt/2$. □

Next, we give an upper bound on $\text{sat}_t(n, \mathcal{C}_2(K_{1,3}))$ by providing a more general saturated graph for $\mathcal{C}_{s-1}(K_{1,s})$. This suffices to complete the proof of Theorem 2.

Proposition 4. *For n sufficiently large with respect to t and $t \geq s - 1$, $\text{sat}_t(n, \mathcal{C}_{s-1}(K_{1,s})) < tn/2$.*

Proof. Let $n = 2k + r$ where $r \in \{1, 2\}$. Let H be a graph on $2k$ vertices produced by packing t perfect matchings onto $V(H)$, with each matching in a distinct color. Let G be the graph that is the disjoint union of H and K_r , where edges in the K_r are colored arbitrarily. Then G is a $\mathcal{C}_{s-1}(K_{1,s})$ -saturated graph; any added edge has at least one endpoint in H and results in an $(s - 1)$ -colored $K_{1,s}$ centered at that vertex. □

It would be interesting to determine if there exist other graphs exhibiting such unusual behavior. For instance, for any pattern x_1, \dots, x_{k-1} of “ \uparrow ” and “ \downarrow ” symbols, does there exist a graph H with k edges such that $\text{sat}_t(n, \mathcal{C}_i(H)) \geq \text{sat}_t(n, \mathcal{C}_{i+1}(H))$ if and only if $x_i = \uparrow$?

2. Asymptotics

In this section, we prove the following general result that implies Conjecture 1. Together with Theorem 7, this result implies Theorem 1.

Theorem 5. *Let \mathcal{H} be a family of edge-colored graphs where for each $H \in \mathcal{H}$, for each edge $uv \in E(H)$ there is a rainbow path with 2 edges connecting u to v in H . Then for any integer $t, t \geq 3$, we have*

$$\left(\frac{1}{3} - o(1)\right) \frac{n \log n}{\log t} \leq \text{sat}_t(n, \mathcal{H}).$$

Before we proceed with the proof of this theorem, we make the following simple observation that will be useful going forward.

Observation 1. *If \mathcal{H} is as in Theorem 5 and if G is (\mathcal{H}, t) -saturated, then for any nonedge uv in G there is a 2-edge path with two colors connecting u to v in G .*

We use this observation to prove Theorem 5 via a reduction to a specific covering problem. Let \mathcal{F} be a family of complete t -partite graphs with U_1^F, \dots, U_t^F the partite sets of $F \in \mathcal{F}$. We say that \mathcal{F} is a t -partite cover of a graph H if $E(H) \subseteq \bigcup_{F \in \mathcal{F}} E(F)$. Define

$$f(H) = \min_{\mathcal{F}} \sum_{F \in \mathcal{F}} (|U_1^F| + \dots + |U_t^F|),$$

where the minimum is taken over all \mathcal{F} a t -partite cover of H .

Let \mathcal{H} be a family of t -edge-colored graphs where for each $H \in \mathcal{H}$ and each edge $uv \in E(H)$ there is a 2-edge path with two colors connecting u to v in H . Assume G is (\mathcal{H}, t) -saturated. We create a t -partite cover of the complement of G . For each vertex v and $1 \leq i \leq t$, let $\Gamma_i(v)$ be the set of vertices adjacent to v in G with edge color i . For each vertex v , let G_v be the complete t -partite graph on $V(G)$ with partite sets $\Gamma_1(v), \dots, \Gamma_t(v)$. By Observation 1, if x and y are not adjacent in G , then there is a rainbow path of length 2 between them. If the vertex in the middle of this path is v , then G_v contains the edge xy . Therefore, $\bigcup_{v \in V(G)} G_v$ is a t -partite cover of the complement of G .

Next we note that

$$\sum_v (|\Gamma_1(v)| + \dots + |\Gamma_t(v)|) = \sum_v d(v) = 2e(G).$$

Therefore, we have that if G is (\mathcal{H}, t) -saturated, then

$$(1) \quad f(\overline{G}) \leq 2e(G).$$

We need the following lemma, which we modify from a result of Katona and Szemerédi [10].

Lemma 1.

$$f(K_n) \geq \frac{n \log n}{\log t}.$$

Proof. Let $\mathcal{F} = \{F_j\}_{j=1}^\ell$ be a t -partite cover of K_n where F_j has partite sets U_1^j, \dots, U_t^j . Create a matrix M with the rows indexed by $V(K_n)$ and the columns indexed by \mathcal{F} as follows:

$$M_{ij} = \begin{cases} k & \text{if } i \in U_k^j, \\ * & \text{if } i \text{ is not in any partite set of } F_j. \end{cases}$$

For each vertex v , let d_v be the number of entries which are not $*$ in the row corresponding to v (i.e. d_v is the number of t -partite graphs in \mathcal{F} which use v). Then in the row corresponding to v there are $|\mathcal{F}| - d_v$ entries with $*$. We create a new matrix M' where for each row v we replace it with $t^{|\mathcal{F}| - d_v}$ rows putting all possible replacements of $*$ with elements from $\{1, \dots, t\}$ and leaving all other entries the same.

We claim that each row in M' is distinct. To see this, if a pair of rows are in the $t^{|\mathcal{F}| - d_v}$ rows which correspond to the same vertex v , then the replacements of $*$ with $\{1, \dots, t\}$ will be different in at least one position. If a pair of rows in M' correspond to distinct vertices u and v , then because \mathcal{F} is a t -partite cover of K_n , there is an $F \in \mathcal{F}$ where u and v are in different partite sets of F and therefore there is a column in M' that distinguishes the two rows.

Since the total number of distinct rows is at most $t^{|\mathcal{F}|}$, we have

$$\sum_v t^{|\mathcal{F}| - d_v} \leq t^{|\mathcal{F}|}$$

and therefore

$$\sum_v \frac{1}{t^{d_v}} \leq 1.$$

Now the AM-GM inequality implies

$$\sqrt[n]{\prod_v \frac{1}{t^{d_v}}} \leq \frac{1}{n} \sum_v \frac{1}{t^{d_v}} \leq \frac{1}{n}.$$

Rearranging gives $\sum_v d_v \geq n \log_t n$. Noting that $\sum_v d_v = \sum_{j=1}^{\ell} |U_1^j| + \dots + |U_t^j|$ finishes the proof. \square

Next we need to show that covering K_n with t -partite graphs is not much different from covering the complement of a sparse graph with t -partite graphs.

Lemma 2. *Let H be a graph on n vertices. Then*

$$f(\overline{H}) \geq f(K_n) - (e(H) + n).$$

Proof. Let \mathcal{F} be an t -partite covering of \overline{H} with weight $f(\overline{H})$. We will construct a family of t -partite graphs that covers H with weight at most $e(H) + n$, which will certify that

$$f(K_n) \leq f(\overline{H}) + (e(H) + n).$$

Order the vertices of H arbitrarily as v_1, \dots, v_n . For $1 \leq i \leq n$ define a t -partite graph H_i with partite sets U_1^i, \dots, U_t^i where

$$\begin{aligned} U_1^i &= \{v_i\}, \\ U_2^i &= \{v_j : v_i \sim v_j, j > i\}, \\ U_k^i &= \emptyset \quad (\text{for } 3 \leq k \leq t). \end{aligned}$$

Then $\{H_i\}_{i=1}^n$ partitions the edge set of H into stars, and

$$\sum_i (|U_1^i| + \dots + |U_t^i|) = e(H) + n. \quad \square$$

We are now ready to complete the proof of Theorem 5

Proof of Theorem 5. Let G be (\mathcal{H}, t) -saturated and assume for a contradiction that $e(G) < \frac{n \log n}{3 \log t} - n/2$. Then $\bigcup_{v \in V(G)} G_v$ is a t -partite cover of \overline{G} , implying that $f(\overline{G}) \leq 2e(G) < \frac{2n \log n}{3 \log t} - n$. This, together with Lemma 2 implies $f(K_n) < \frac{n \log n}{\log t}$, which contradicts Lemma 1. \square

The following corollary to Theorem 4 implies Conjecture 1, and resolves Conjecture 2 in [7]. This conjecture was also affirmed in [11], where the focus was strictly the rainbow setting.

Corollary 6. *If $c \geq \binom{k-1}{2} + 2$ and $t \geq c$, then*

$$\text{sat}_t(n, \mathcal{C}_c(K_k)) \geq \left(\frac{1}{3} - o(1)\right) \frac{n \log n}{\log t}.$$

Proof. Let $c \geq \binom{k-1}{2} + 2$ and consider an edge uv in G , an edge-colored K_k with exactly c colors. There are at most $\binom{k-2}{2}$ colors on the edges in $G - \{u, v\}$, and at most one additional color on uv . This leaves at least $c - \binom{k-2}{2} \geq k - 1$ colors on the edges with one endpoint in $\{u, v\}$ and one endpoint in $V(G) - \{u, v\}$, implying that there is some vertex x such that the edges of uxv receive distinct colors. We can therefore apply Theorem 5 to $\text{sat}_t(n, \mathcal{C}_c(K_k))$. □

As we demonstrate next, the bound of $c \geq \binom{k-1}{2} + 2$ is sharp.

Theorem 7. *If $c \leq \binom{k-1}{2} + 1$ and $t \geq c$ are fixed, then $\text{sat}_t(n, \mathcal{C}_c(K_k)) = O(n)$.*

Proof. For fixed $c \leq \binom{k-1}{2} + 1$ and $t \geq c$, we construct a $\mathcal{C}_c(K_k)$ -saturated graph with $O(n)$ edges. As we are not interested in determining the relevant saturation number exactly, we make no effort to optimize the number of edges in our construction.

Assume that n is sufficiently large, and consider an edge-coloring of $H' = K_{k-1}$ using exactly $c - 1$ colors. Create an edge-colored copy of $K_k - e$ by choosing some vertex v in H' , adding a new vertex v' , and connecting v' to $V(H') - \{v\}$ such that vx and $v'x$ have the same colors for each $x \in V(H') - \{v\}$. Repeat the duplication of v to create $H_{S,p} = K_{k-2} \vee I$, where I is an independent set of size p and S is the set of colors appearing on $E(H_{S,p})$. Note that $H_{S,p}$ contains no copy of K_k , but the addition of any edge to $H_{S,p}$ in a color from $[t] - S$ creates a copy of K_k with exactly c colors.

We create the edge-colored graph G' by taking the union of the graphs $H_{S,p}$ with $p = n - \binom{t}{c-1}(k-2)$ for each of the $\binom{t}{c-1}$ choices of S , under the assumption that I is common to each such graph. Note that for any u and v in I and any color $c_0 \in [t]$, adding uv in color c_0 to G' creates a copy of K_k with exactly c colors within $H_{S,p}$ for any S that does not contain c_0 . To create the desired saturated graph G , iteratively add edges to $G' - I$ in any permissible color until either $G - I$ is complete, or no colored edge can be added to $G - I$ without creating an element of $\mathcal{C}_c(K_k)$. In either case, G is $\mathcal{C}_c(K_k)$ -saturated and has at most

$$\binom{t}{c-1}(k-2) \left(n - \binom{t}{c-1}(k-2) \right) + \binom{\binom{t}{c-1}(k-2)}{2}$$

edges, which is $O(n)$ edges as desired. \square

Theorem 1 now follows as a consequence of Corollary 6 and Theorem 7.

3. 2-colored triangles

In this section, we prove the following exact result.

Theorem 8. *If $t = 2$ and $n \geq 11$ or if $t \geq 3$ and $n \geq 9$, then*

$$\text{sat}_t(n, \mathcal{C}_2(K_3)) = 2n - 4.$$

Furthermore, if $t \geq 3$, then $K_{2,(n-2)}$ is the unique saturated graph.

Before we proceed, we require a simple technical lemma.

Lemma 3. *Let x_1, x_2, \dots, x_t be integers with $x_1 \geq x_2 \geq \dots \geq x_t \geq 2$. For $1 \leq p < q \leq t$ and let $x'_i := x_i$ for $i \notin \{p, q\}$, $x'_p := x_p + 1$ and $x'_q = x_q - 1$, then*

$$\sum_{i=1}^t \binom{x_i}{2} < \sum_{i=1}^t \binom{x'_i}{2}.$$

Proof. We must show that

$$\binom{x_p + 1}{2} + \binom{x_q - 1}{2} > \binom{x_p}{2} + \binom{x_q}{2},$$

but this is equivalent to showing $x_p > x_q - 1$, which holds by assumption. \square

Proposition 9. *For all $n \geq 11$,*

$$\text{sat}_2(n, \mathcal{C}_2(K_3)) = 2n - 4.$$

Proof. Consider the edge-colored graph $K_{2,(n-2)}$ where x and y are the vertices in the partite class of size 2 and all edges incident with x are red while all edges incident with y are blue. This shows that $\text{sat}_2(n, \mathcal{C}_2(K_3)) \leq 2n - 4$ (other constructions exist). We will show $\text{sat}_2(n, \mathcal{C}_2(K_3)) \geq 2n - 4$.

Suppose G is an n -vertex $(\mathcal{C}_2(K_3), 2)$ -saturated graph. We will consider cases based on the minimum degree of G .

Case 1: $\delta(G) = 1$.

Let $u \in V(G)$ be of degree 1 with neighbor v . Say that the color of uv

is blue. Then, v must be adjacent to every other vertex w in G by a red edge for otherwise we could add the blue edge uw . There is no blue edge in $G - \{u, v\}$ for this would yield a $\mathcal{C}_2(K_3)$. Thus, $G - \{u, v\}$ is a complete graph consisting of only red edges. Thus $e(G) \geq 2n - 3$.

Case 2: $\delta(G) = 2$.

Let u be a vertex with degree 2 and suppose v_1 and v_2 are the neighbors of u . If there is a vertex y which is not adjacent to u , v_1 , or v_2 , then the graph would not be saturated since we could add the edge uy in either color and not obtain a triangle. So every vertex is adjacent to u , v_1 , or v_2 .

Suppose first that v_1v_2 is an edge in G . Then, v_1, v_2 and u form a triangle and so all 3 edges are the same color, say blue. The only common neighbor of v_1 and v_2 is u since any further neighbor w would be connected to v_1 and v_2 by blue edges. But then the graph would not be saturated since we could add the edge uw in blue. Now $X := V(G) - \{u, v_1, v_2\}$ is the set of vertex which are neighbors of v_1 or v_2 . Note that every edge from $\{v_1, v_2\}$ to a vertex $x \in X$ is red, for otherwise we could add the blue edge xu . The graph induced on the vertex set X is connected since two connected components could be connected by a red edge without forming a $\mathcal{C}_2(K_3)$. Thus, we have at least $3 + |X| + |X| - 1 = 2n - 4$ edges. From now on we may assume that v_1v_2 is not an edge.

First, suppose that both uv_1 and uv_2 are blue. Let X denote the set of vertices in $V(G) - \{u, v_1, v_2\}$ which are connected to exactly one of v_1 and v_2 , and let Y denote the set of vertices in $V(G) - \{u, v_1, v_2\}$ which are connected to both v_1 and v_2 . So $X \cup Y = V(G) - \{u, v_1, v_2\}$. All edges from X to $\{v_1, v_2\}$ are red, and at least one edge from every vertex of Y to $\{v_1, v_2\}$ is red. If $X = \emptyset$ we are done, so suppose there is at least one vertex in X . The graph induced on the set X must be connected for otherwise we could add a red edge connecting two components. Thus, there are at least $|X| - 1$ edges in this induced subgraph, yielding a total of at least $2 + 2|Y| + |X| + |X| - 1 = 2n - 5$ edges. We now suppose that these $2n - 5$ edges are the only edges in G and argue to a contradiction.

Let N_1 and N_2 denote the neighborhoods of v_1 and v_2 in X respectively. Since there are no edges with one endpoint in X and the other in Y , there must be a blue edge in the induced subgraph on X for otherwise we could add a red edge from a vertex in X to either v_1 or v_2 , the one it is not connected to. This blue edge has one vertex in N_1 and one vertex in N_2 , say w_1 and w_2 , respectively. Observe that Y must be nonempty for otherwise we could add the blue edge v_1v_2 . Let $z \in Y$ and assume v_1z is red. Then we could add the edge w_1z with color red. Thus, the graph G could not be saturated with $2n - 5$ edges and so contains at least $2n - 4$.

Second suppose uv_1 is blue and wv_2 is red. Define X and Y as in the previous subcase. Again assume X is nonempty for otherwise we have $2n - 4$ edges. But now Y may be empty. Edges from v_1 to X are red, and edges from v_2 to X are blue. Again, the graph induced on X is connected. Thus, again we have at least $2n - 5$ edges. Now suppose that there are no other edges in G and we will argue to a contradiction. If N_2 was empty, then for $w \in N_1$, we could add the edge wv_2 and not create a triangle. Since $N_1, N_2 \neq \emptyset$ and the graph induced on X is connected, there is an edge w_1w_2 with $w_1 \in N_1$ and $w_2 \in N_2$. Suppose w_1w_2 is red (the blue case is similar), then we can add the edge v_1w_2 with color red. Thus $2n - 5$ edges do not suffice for G . Thus G must have at least $2n - 4$ edges.

Case 3: $\delta(G) = 3$.

We need to show that $\sum_v d(v) \geq 4n - 8$. Suppose by way of contradiction that $\sum_v d(v) \leq 4n - 10$. For every edge which is not in G , there must exist a path of length 2 between its endpoints. It follows that

$$(2) \quad \sum_{v \in V(G)} \binom{d(v)}{2} \geq \binom{n}{2} - \frac{1}{2} \sum_{v \in V(G)} d(v) \geq \binom{n}{2} - \frac{1}{2}(4n - 10) = \frac{n^2}{2} - \frac{5n}{2} + 5.$$

On the other hand, by Lemma 3 we have

$$(3) \quad \sum_{v \in V(G)} \binom{d(v)}{2} \leq \binom{n-7}{2} + (n-1)\binom{3}{2} = \frac{n^2}{2} - \frac{9n}{2} + 25.$$

It follows from (2) and (3) that $n \leq 10$, a contradiction. □

Proposition 10. For all $t \geq 3$ and $n \geq 9$,

$$\text{sat}_t(n, \mathcal{C}_2(K_3)) = 2n - 4.$$

Moreover, every $(\mathcal{C}_2(K_3), t)$ -saturated graph is a coloring of $K_{2, n-2}$.

Proof of Proposition 10. A construction is given by the following. Take two vertices u and v and a collection of vertices u_1, u_2, \dots, u_{n-2} . Take red edges from u to u_1, \dots, u_{n-2} and blue edges from v to u_1, \dots, u_{n-3} and a red edge from v to u_{n-2} . Therefore $\text{sat}_t(n, \mathcal{C}_2(K_3)) \leq 2n - 4$.

Now we will establish the lower bound. Let G be an n -vertex $(\mathcal{C}_2(K_3), t)$ -saturated graph with $t \geq 3$ with $e(G)$ as small as possible. First, we show that the minimum degree of G is at least 2. Suppose $u \in V(G)$ is a vertex of degree 1 with neighbor v and let w be any other vertex. If vw is not an edge, then we can add uw without creating a triangle. If vw is an edge with

the same color as uv , then we may add uw with the same color. If vw is an edge with a different color than uv , then we may add vw with an arbitrary distinct third color (since $t \geq 3$).

We need to show that $\sum_{v \in V(G)} d(v) \geq 4n - 8$. Observe that if G is $(\mathcal{C}_2(K_3), t)$ -saturated, then for every edge $e = \{x, y\} \in E(\overline{G})$ there must be at least two paths of length 2 between x and y . Since the number of paths of length 2 in G is $\sum_{v \in V(G)} \binom{d(v)}{2}$, it follows that

$$(4) \quad \sum_{v \in V(G)} \binom{d(v)}{2} \geq 2 \left(\binom{n}{2} - e(G) \right) = n^2 - n - \sum_{v \in V(G)} d(v).$$

Under the assumption that $\sum_{v \in V(G)} d(v) \leq 4n - 10$, the right hand side of (4) is at least $n^2 - 5n + 10$. By Lemma 3, if $\sum_{v \in V(G)} d(v) \leq 4n - 10$, then

$$\sum_{v \in V(G)} \binom{d(v)}{2} \leq \binom{n-1}{2} + \binom{n-5}{2} + (n-2) \binom{2}{2} = n^2 - 6n + 14.$$

This is a contradiction for $n \geq 5$.

Since $\sum_{v \in V(G)} d(v)$ cannot be odd, it remains to show that if G is $(\mathcal{C}_2(K_3), t)$ -saturated with $\sum_{v \in V(G)} d(v) = 4n - 8$, then G is a coloring of $K_{2, n-2}$. In this case the right hand side of (4) is $n^2 - 5n + 8$.

First, we observe that it is not possible for the maximum degree to be at most $n - 3$, for then (by Lemma 3) we would have

$$\sum_{v \in V(G)} \binom{d(v)}{2} \leq \binom{n-3}{2} + \binom{n-3}{2} + \binom{4}{2} + (n-3) = n^2 - 6n + 15,$$

which is too small for $n \geq 8$.

Suppose the maximum degree of G is $n - 2$. We see that the second largest degree is at least $n - 3$ for otherwise we have

$$\sum_{v \in V(G)} \binom{d(v)}{2} \leq \binom{n-2}{2} + \binom{n-4}{2} + \binom{4}{2} + (n-3) = n^2 - 6n + 16,$$

which is too small for $n \geq 9$. Thus, the remaining possible degree sequences starting with $n - 2$ are $(n - 2, n - 2, 2, \dots, 2)$ and $(n - 2, n - 3, 3, 2, \dots, 2)$. Note that $(n - 2, n - 2, 2, \dots, 2)$ yields a colored $K_{2, n-2}$.

Suppose the maximum degree is $n - 1$. If the second largest degree is at most $n - 5$, then we have

$$\sum_{v \in V(G)} \binom{d(v)}{2} \leq \binom{n-1}{2} + \binom{n-5}{2} + \binom{4}{2} + (n-3) = n^2 - 6n + 19,$$

which is too small $n \geq 9$. The remaining possible degree sequences are $(n - 1, n - 4, 3, 2, \dots, 2)$ and $(n - 1, n - 3, 2, \dots, 2)$. To finish the proof we have to check the 3 degree sequences which satisfied (4). We do this in Claim 1 below. \square

Claim 1. For $t \geq 3$, there is no $(\mathcal{C}_2(K_3), t)$ -saturated graph with any of the following degree sequences:

- $(n - 2, n - 3, 3, 2, \dots, 2)$,
- $(n - 1, n - 3, 2, \dots, 2)$,
- $(n - 1, n - 4, 3, 2, \dots, 2)$.

Proof. First, it is important to note that if G is $(\mathcal{C}_2(K_3), t)$ -saturated, then for any $uv \notin E(G)$, there must be at least two paths of length 2 from u to v because $t \geq 3$.

Consider the degree sequence $(n - 2, n - 3, 3, 2, \dots, 2)$ and let x and y be the vertices of degree $n - 2$ and $n - 3$, respectively. If x and y are not adjacent let z be the other vertex y is not adjacent to. Then z has degree at least 2 so it is adjacent to some other vertex. This defines a unique graph up to isomorphism and it is clear that for the nonedge yz there is only one path of length 2 from y to z .

Now, assume x and y are adjacent. If there exists a common vertex z such that xz and yz are both nonedges, then the remainder of the graph is forced. Namely, there is an edge from z to the other non-neighbor of y and an edge from z to one of the common neighbors of x and y . For the nonedge yz there are not two paths of length 2 between y and z . Finally, assume x and y are adjacent and the set of non-neighbors of x and the set of non-neighbors of y are disjoint. In this case there are multiple non-isomorphic graphs but a nonedge from one of the non-neighbor sets to a vertex in the set of common neighbors of degree 2 will not have two paths of length 2 between its endpoints. (Indeed, any such path of length 2 would involve x or y but one of the vertices in the nonedge is a non-neighbor of x or y .)

Consider the degree sequence $(n - 1, n - 3, 2, \dots, 2)$. Let x and y be the degree $n - 1$ and $n - 3$ vertex, respectively. It is easy to see the two non-neighbors of y must be adjacent and this defines a unique graph. Then a nonedge from y does not have two paths between its endpoints.

Consider, finally, the degree sequence $(n-1, n-4, 3, 2, \dots, 2)$. Let x and y be the degree $n-1$ and $n-4$ vertex, respectively. Let u, v and w be the three non-neighbors of y . Either these three vertices form a path of length 2, or we may assume u and v are adjacent and w is adjacent to a common neighbor of x and y . In either case the nonedge yu does not have two paths of length 2 between its vertices. \square

4. Conclusion

In this paper, we consider several existing and new problems in the realm of edge-colored saturation problems. There remain a number of potential directions of inquiry. Even given the excellent results in [7] and [11], the general problem of determining $\text{sat}_t(n, \mathcal{R}(H))$ is open in a number of cases. Of particular interest would be to determine the asymptotic behavior for general trees, or to consider the behavior of the function for disconnected graphs. For instance, it is not difficult to show that if p is even, $n \geq 5p$, and t is large, then $\text{sat}_t(n, \mathcal{R}((p+1)K_2)) \leq 5p$. The extremal graph is a rainbow copy of $\frac{p}{2}K_5$ together with $n - \frac{5p}{2}$ isolated vertices. However, it seems surprisingly difficult to show that equality holds.

We also point out that the families considered here, $\mathcal{M}(H)$, $\mathcal{R}(H)$ and $\mathcal{C}_k(H)$ are invariant up to the permutation of the palette of t colors. What if this was not the case? Suppose, for instance, that we wished to determine $\text{sat}_3(n, \mathcal{F})$, where \mathcal{F} consisted of two graphs: a triangle with two edges colored 1 and one edge colored 2, and a monochromatic triangle with all edges colored 3. In this case, not all colored edges are created equal, opening the door to a number of (delightfully) aberrant possibilities.

Finally, in [8], Hanson and Toft also introduced the related problem of determining the saturation number of the family of graphs that are Ramsey-minimal for some (H_1, \dots, H_t) . This is equivalent to determining the minimum number of edges in a graph of order n that has a t -edge-coloring with no copy of H_i in color i , such that the addition of any missing edge creates a graph wherein every t -edge-coloring contains some H_i in color i . While not our focus here, we want to highlight this general problem, which has only been considered for a limited collection of target graphs [2, 6, 10] and remains open.

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