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Remarks on the recurrence and transience of non-backtracking random walks

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A short proof of the equivalence of the recurrence of a non-backtracking random walk and that of a simple random walk on regular infinite graphs is given. It is then shown how this proof can be extended in certain cases where the graph in question is not regular.

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A *non-backtracking random walk* (abbr. NBRW) is a random process defined on the vertices of a graph G by the transition probabilities

(1)
$$P(X_1 = y|X_0 = x) = \begin{cases} \frac{1}{deg(x)} & \text{if } y \sim x \\ 0 & \text{if } y \not\sim x \ , \end{cases}$$
$$P(X_{n+1} = y|X_n = x_2, X_{n-1} = x_1) = \begin{cases} \frac{1}{deg(x_2)-1} & \text{if } y \sim x_2 \text{ and } y \neq x_1 \\ 0 & \text{if } y \not\sim x_2 \text{ or } y = x_1 \ . \end{cases}$$

Recently NBRWs have received much attention in the scientific literature due to their connection to spectral methods for community detection, [KMM+13, BLM18, Abb17] and also because they mix faster than the corresponding simple random walks on various graphs G [ABLS07, LXE12, Kem16]. They have also played a role in the spectral analysis of random matrices [Sod07].

Consideration of the NBRW on $\mathbb{Z}^d$ dates back to [MS96, Section 5.3] and the process was also analyzed in [FvdH13] where the Green's function and functional central limit theorem were studied. In [Kem18], the question of recurrence of a non-backtracking random walk on the integer lattice was settled: the walk is transient on $\mathbb{Z}$ (trivially) and on $\mathbb{Z}^d$ with $d \geq 3$, and recurrent on $\mathbb{Z}^2$. Following this, in [Her, Prop 1.1], the following extension was proved.

Theorem 1. *If G is a regular infinite graph of degree $k \geq 3$, then a non-backtracking random walk is recurrent on G if, and only if, a simple random walk is recurrent on G.*

The result in [Kem18] was proved by a rather intricate counting argument, and in [Her] the universal cover of G is used to prove Theorem 1 in amongst a larger work on reversibility. Each of these works contain a number of results of independent interest, however if we are only interested in Theorem 1 there is a very short and simple probabilistic coupling-type proof available, as we now present.

Proof. We begin with an algorithm which modifies deterministic sequences of the vertices of G. Let us suppose $(x_0, x_1, x_2, \ldots)$ is such a sequence with x_i initially being in position i (but the positions may later change). Our algorithm will use a cursor #, starting in position 0, and move along the sequence and essentially remove the backtracking contained in the sequence. The algorithm is as follows:

1. If # is at position 0, move it to the right.
2. If # is at position $n > 0$, compare the numbers in positions $n - 1$ and $n + 1$, and

2a. if they are not equal, then move # to the right,

2b. if they are equal, then erase the numbers in positions n and $n + 1$ and shift the remaining part of the sequence two positions to the left to close the gap (for example, the very first time this step occurs, that portion of the sequence will then read $(\ldots, x_{n-1}, x_{n+2}, \ldots)$). Next, move # to the left.

3. Repeat.

The reason for moving # left in step (2b) is to check whether the erasure has introduced a new backtracking. Note that, if our initial sequence is such that # eventually leaves any given finite subset of positions forever, then the output of this algorithm will be a sequence which contains no backtracking.

Let us apply this algorithm to a simple random walk (abbr. SRW) on the vertices of G. If $(X_n)_n$ denotes a SRW, then we may apply our algorithm to the random sequence $(X_0, X_1, X_2, \ldots)$. It is straightforward to verify that the regularity of G implies that the output from the algorithm is a NBRW, provided only that # eventually leaves any finite subset of positions forever. Examining the (now random) movements of #, notice that they have independent increments and that at any position $n > 0$, there is probability $\frac{1}{k}$ of moving to the left and probability $\frac{k-1}{k}$ of moving to the right. These movements constitute a birth-death chain, which is well known to be transient in this case since $\frac{k-1}{k} > \frac{1}{2}$ (see for instance [Nor98]).

It is now immediate that a graph G transient for a SRW is also transient for a NBRW, since our algorithm can in no way turn finitely many visits to any vertex in a SRW into infinitely many for a NBRW.

To see that a G recurrent for a SRW is also recurrent for a NBRW is slightly more subtle. Fix a vertex v and let $(n_i)_{i\in\mathbb{N}}$ be the sequence of (random) times that $(X_n)_n$ visits v. Also, if the cursor # ever visits X_{n_i}, let p_i be the position that X_{n_i} is in when # visits it for the first time. If # never visits X_{n_i} (due to an erasure resulting from step (2b) of the algorithm), simpy set $p_i = \infty$.

Note that the event of a given visit to v of the SRW, $X_{n_i} = v$, not being erased by our algorithm contains the event

$$E_i := \{p_i < \infty \text{ and the cursor \# visits } p_i \text{ exactly once.}\}.$$

Note that $P(E_i) > 0$ since the birth-death chain # is transient, and that this probability does not depend on i. Note also that the events $\{E_i\}_i$ are independent, thus infinitely many visits to any vertex in G for a SRW must remain infinitely many in a NBRW. □

Remarks.

- It is satisfying to note the reason that the proof fails for $k = 2$: the resulting birth-death chain produced by the algorithm is a SRW on the integers, which is well known to be recurrent; therefore our algorithm fails and a NBRW cannot be produced in this fashion.
- The condition that G is regular is necessary for the proof (although it can be weakened somewhat; see below). To see this, note that if G is not regular then the following situation may arise. Let us suppose v is a vertex of degree 3 adjacent to x, y, and z. If a NBRW reaches v via x then it should have equal probabilities of passing next to y and z. However if the degree of y is higher than that of z then a passage of a SRW from v to z is more likely to be erased by what follows than one from v to y. When we apply our algorithm to a SRW, then, our output is a random process without backtracking, but it is not equal in distribution to a NBRW.

Incidentally, we may adjust the method of proof used in Theorem 1 in order to handle the following situation.

Proposition 1. *Suppose G is an infinite graph such that every vertex has either degree k_1 or k_2, with $k_1 > k_2 \geq 2$. Suppose further that every vertex of degree k_1 is adjacent only to vertices of degree k_2, and every vertex of degree k_2 is adjacent only to vertices of degree k_1. Then a non-backtracking random walk is recurrent on G if, and only if, a simple random walk is recurrent on G.*

Remark. A graph as described in the proposition is commonly referred to as a (k_1, k_2)-*biregular graph* or *semiregular bipartite graph.*

The only adjustment to our previous proof is to note that the birth-death chain produced by our algorithm has two different probabilities; for instance, if we start at a vertex of degree k_1 then the resulting birth-death chain moves to the right with probability $\frac{k_1-1}{k_1}$ at even positions in the sequence, and $\frac{k_2-1}{k_2}$ at odd positions. Nevertheless, this birth-death chain is still easily seen to be transient by the methods in [Nor98]; alternatively we can appeal to the theory of electric networks and their connections to random walk. In [DS84, Ch. 5] it is shown that the birth-death chain is transient precisely when the resistance to ∞ of an associated electric network is finite. If we let r_j denote the resistance between integers $j-1$ and j, then it may be checked that the resistances r_j in this case satisfy

$$r_{j+1} = \begin{cases} \frac{r_j}{k_1-1} & \text{if } j \text{ is odd} \\ \frac{r_j}{k_2-1} & \text{if } j \text{ is even ,} \end{cases} \tag{2}$$

Since our associated electric network has resistors all in one series, the resistance to ∞ in this case is simply $\sum_{j=1}^{\infty} r_j$, and it is straightforward to verify that the relationships (2) imply that this sum is finite, thus the birth-death chain is transient. The rest of the proof persists unchanged.

As a bit of an aside, we may also give a partial answer to an intriguing question posed in [Her]. Question 1.11 in that work is as follows:

Let G be a connected graph of bounded degree such that the length of any path of vertices of degree 2 is bounded by a finite constant $L > 0$. Is it the case that a SRW on G is transient iff the NBRW on G is transient?

We will show that the answer is in the affirmative provided that the vertices of the graph have only one possible degree other than 2. In other words, we have the following proposition.

Proposition 2. *Let G be a connected graph where there is a constant $k > 2$ such that every vertex has either degree 2 or k. Suppose further that the length of any path of vertices of degree 2 is bounded by a finite constant $L > 0$. Then a non-backtracking random walk is recurrent on G if, and only if, a simple random walk is recurrent on G.*

Proof. We again make use of the theory of electric networks. Let V denote the set of all vertices of G of degree k, and let G' be a multigraph with vertices V and with edges drawn as follows. For any two vertices of G', we draw an edge between them if these two vertices are adjacent in G. Next, for any two vertices of G' we draw additional edges if there is a path of vertices of degree 2 connecting them in G; note that these additional edges may cause G' to be a multigraph rather than a graph because the additional edges may induce multiple edges between points or possibly even be loops. Assign to each edge in G' a resistance of one if the edge exists in G or a resistance equal to the length of the corresponding path of vertices of degree 2 in G otherwise (if the reader is uncomfortable with the possible loops or multiple edges, they may at this point erase all loops and collapse any set of multiple edges between two points into a single edge with resistance calculated in parallel in order to obtain a genuine graph). Suppose we start a SRW $(X_n)_n$ on G at a point in V (recurrence and transience for a SRW on a connected multigraph of bounded degree are not affected by the initial point, since there is always a positive probability of passing between any two given vertices). Define a sequence of stopping times by $\tau_0 = 0$ and

$$\tau_n = \min\{j > \tau_{n-1} : X_j \in V\} \quad \text{for } n \geq 1.$$

It then may be checked that the process $(X_{\tau_0}, X_{\tau_1}, X_{\tau_2}, \ldots)$ on V is equal in distribution to a weighted random walk (abbr. WRW) on G', as defined in [DS84]. We will prove the following set of equivalences.

$$\begin{aligned} &SRW \text{ on } G \text{ is recurrent} \overset{(1)}{\Longleftrightarrow} WRW \text{ on } G' \text{ is recurrent} \\ &\overset{(2)}{\Longleftrightarrow} SRW \text{ on } G' \text{ is recurrent} \overset{(3)}{\Longleftrightarrow} NBRW \text{ on } G' \text{ is recurrent} \\ &\overset{(4)}{\Longleftrightarrow} NBRW \text{ on } G \text{ is recurrent} \end{aligned}$$

(1) follows since X_{τ_n} is recurrent precisely when X_n is.
(2) is immediate from Theorem 2.4.3 of [DS84], which states that recurrence is equivalent for a WRW and a SRW (which can be realized as having all resistances set to 1) provided that we have a finite upper and lower bound on the resistances, as we do in this case due to the existence of the constant L.
(3) is immediate from Theorem 1.
(4) follows by noting that if we let X_n be a NBRW on G and define the stopping times τ_n as above, then the process $X_{\tau_0}, X_{\tau_1}, X_{\tau_2}, \ldots$ on V is equal in distribution to a NBRW on G', as the process X_n may not reverse directions on the paths of vertices of degree 2.

These equivalences complete the proof of the proposition. □

This method of proof can be further extended to the following situation, with Proposition 1 taking the place of Theorem 1 where required. Details are omitted.

Proposition 3. *Let G be a connected graph where there are constants $2 < k_1 < k_2$ such that every vertex has degree $2, k_1$, or k_2. Suppose further that the length of any path of vertices of degree 2 is bounded by a finite constant $L > 0$. Suppose also that the graph G' formed as in the proof of Proposition 2 is of the form required in Proposition 1. Then a non-backtracking random walk is recurrent on G if, and only if, a simple random walk is recurrent on G.*

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