

On enumerators of Smirnov words by descents and cyclic descents

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Dedicated to the memory of Jeff Remmel

A Smirnov word is a word over the positive integers in which adjacent letters must be different. A symmetric function enumerating these words by descent number arose in the work of Shareshian and the second named author on q -Eulerian polynomials, where a t -analog of a formula of Carlitz, Scoville, and Vaughan for enumerating Smirnov words is proved. A symmetric function enumerating a circular version of these words by cyclic descent number arose in the work of the first named author on chromatic quasisymmetric functions of directed graphs, where a t -analog of a formula of Stanley for enumerating circular Smirnov words is proved.

In this paper we obtain new t -analogs of the Carlitz–Scoville–Vaughan formula and the Stanley formula in which the roles of descent number and cyclic descent number are switched. These formulas show that the Smirnov word enumerators are polynomials in t whose coefficients are e -positive symmetric functions. We also obtain expansions in the power sum basis and the fundamental quasisymmetric function basis, complementing earlier results of Shareshian and the authors.

Our work relies on studying refinements of the Smirnov word enumerators that count certain restricted classes of Smirnov words by descent number. Applications to variations of q -Eulerian polynomials and to the chromatic quasisymmetric functions introduced by Shareshian and the second named author are also presented.

KEYWORDS AND PHRASES: Smirnov word, Eulerian polynomial, symmetric function, chromatic quasisymmetric function.

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1. Introduction

We consider words $w = w_1w_2 \dots w_n$ over the alphabet of positive integers \mathbb{P} with no adjacent repeated letters; that is $w_i \neq w_{i+1}$ for all $i \in [n-1] := \{1, \dots, n-1\}$. We refer to these words as *Smirnov words* as is often done in the literature; see e.g. [17, 14, 22, 3, 24, 15, 25].

For $n \geq 1$, let W_n be the set of Smirnov words of length n . Now define the Smirnov word enumerator

$$W_n(\mathbf{x}) := \sum_{w \in W_n} x_w,$$

where $\mathbf{x} := x_1, x_2, \dots$ is a sequence of indeterminates and $x_w := x_{w_1}x_{w_2} \dots x_{w_n}$. Clearly $W_n(\mathbf{x})$ is a symmetric function. Carlitz, Scoville, and Vaughan [7, equation (7.12)] derived the generating function formula

$$(1.1) \quad \sum_{n \geq 1} W_n(\mathbf{x})z^n = \frac{\sum_{i \geq 1} i e_i(\mathbf{x})z^i}{1 - \sum_{i \geq 2} (i-1)e_i(\mathbf{x})z^i},$$

where $e_i(\mathbf{x})$ is the elementary symmetric function of degree i . An important consequence of this formula is that $W_n(\mathbf{x})$ is e -positive, which means that when expanded in the elementary symmetric function basis for the ring of symmetric functions, the coefficients are nonnegative.

The symmetric function $W_n(\mathbf{x})$ was also considered by Stanley [35] in the context of chromatic symmetric functions and by Dollhopf, Goulden, and Greene [10] in the context of pair avoiding word enumerators. Stanley

also considered a circular version of Smirnov words, that is, Smirnov words whose first and last letter are different. Let

$$W_n^{\neq}(\mathbf{x}) := \sum_{\substack{w \in W_n \\ w_1 \neq w_n}} x_w.$$

Stanley [35, Proposition 5.4] proved

$$(1.2) \quad \sum_{n \geq 1} W_n^{\neq}(\mathbf{x}) z^n = \frac{\sum_{i \geq 2} i(i-1) e_i(\mathbf{x}) z^i}{1 - \sum_{i \geq 2} (i-1) e_i(\mathbf{x}) z^i}.$$

It follows from this formula that $W_n^{\neq}(\mathbf{x})$ is e -positive.

Given any word $w \in \mathbb{P}^n$, where $n \geq 1$, the *descent number* of w is defined by

$$\text{des}(w) := |\{i \in [n-1] : w_i > w_{i+1}\}|$$

and the *cyclic descent number* is defined by

$$(1.3) \quad \text{cdes}(w) := |\{i \in [n] : w_i > w_{i+1}\}|,$$

where $w_{n+1} := w_1$. Now define the refined Smirnov word enumerators

$$\begin{aligned} W_n(\mathbf{x}, t) &:= \sum_{w \in W_n} t^{\text{des}(w)} x_w \\ \tilde{W}_n(\mathbf{x}, t) &:= \sum_{w \in W_n} t^{\text{cdes}(w)} x_w \\ W_n^{\neq}(\mathbf{x}, t) &:= \sum_{\substack{w \in W_n \\ w_1 \neq w_n}} t^{\text{des}(w)} x_w \\ \tilde{W}_n^{\neq}(\mathbf{x}, t) &:= \sum_{\substack{w \in W_n \\ w_1 \neq w_n}} t^{\text{cdes}(w)} x_w. \end{aligned}$$

The first and fourth of these Smirnov word enumerators have been studied before. The main objective of this paper is to study the other two Smirnov word enumerators. We start with a brief review of what is known for $W_n(\mathbf{x}, t)$ and $\tilde{W}_n^{\neq}(\mathbf{x}, t)$.

1.1. Summary of known results

The refined Smirnov word enumerator $W_n(\mathbf{x}, t)$ arose in the work of Shareshian and the second named author on q -Eulerian polynomials [29]. Stanley (personal communication) observed that the $r = 1$ case of [29, Theorem 1.2] is equivalent to the following t -analog of (1.1),

$$(1.4) \quad \sum_{n \geq 1} W_n(\mathbf{x}, t) z^n = \frac{\sum_{i \geq 1} [i]_t e_i(\mathbf{x}) z^i}{1 - \sum_{i \geq 2} t [i-1]_t e_i(\mathbf{x}) z^i},$$

where

$$[n]_t := 1 + t + \cdots + t^{n-1} = \frac{t^n - 1}{t - 1}.$$

Indeed, this follows from [29, Equation (7.7) and Theorem 3.6]; see [33, Section 4.1]. (For another proof of (1.4), see Remark 5.6.) It follows from (1.4) that $W_n(\mathbf{x}, t)$ is e -positive, that is, $W_n(\mathbf{x}, t)$ is a polynomial in t whose coefficients are e -positive symmetric functions.

Equation (1.4) can be restated as

$$(1.5) \quad 1 + \sum_{n \geq 1} W_n(\mathbf{x}, t) z^n = \frac{(1-t)E(z)}{E(tz) - tE(z)},$$

where

$$E(z) := \sum_{n \geq 0} e_n(\mathbf{x}) z^n.$$

The expression on the right hand side of (1.5) (or its image under the involution ω that takes $e_n(\mathbf{x})$ to the complete homogeneous symmetric function $h_n(\mathbf{x})$) has arisen in various other contexts. Stanley obtained the expression when considering the representation of the symmetric group on the cohomology of the toric variety associated with the dual permutohedron [34]. A conjecture of Shareshian and the second named author [31, 32, 33], proved by Brosnan and Chow [5] and subsequently by Guay-Paquet [20], generalizes the connection between the Smirnov word enumerator and the toric varieties to a connection between chromatic quasisymmetric functions and Hessenberg varieties. The expression also arose in the work of Shareshian and the second named author on the representation of the symmetric group on homology of Rees products of posets [30].

By combining (1.5) with a result of Stembridge [38], one obtains an expansion of $\omega W_n(\mathbf{x}, t)$ in the power sum symmetric functions $p_\lambda(\mathbf{x})$. The expansion is given by

$$(1.6) \quad \omega W_n(\mathbf{x}, t) = \sum_{\lambda \vdash n} \left(A_{\ell(\lambda)}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right) \frac{p_\lambda(\mathbf{x})}{z_\lambda},$$

where $A_m(t)$ is the m th Eulerian polynomial, z_λ is a constant associated with the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)})$, and $\ell(\lambda)$ is the length of λ .

Recall that the Eulerian polynomials $A_n(t)$ have two well-known combinatorial interpretations, which are given by

$$(1.7) \quad A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)},$$

where \mathfrak{S}_n is the symmetric group on $[n]$, and des and exc are MacMahon’s classical equidistributed permutation statistics, descent number and excedance number, respectively.

In [28, 29] Shareshian and the second named author introduce a q -analog of the exc interpretation of $A_n(t)$ and in [32] they introduce a q -analog of the des interpretation of $A_n(t)$. These q -analogs are shown to be equal in [32, Theorem 9.7]. They are defined by

$$A_n(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq 2}(\sigma^{-1})} t^{\text{des}(\sigma)},$$

where maj is MacMahon’s classical major index and $\text{maj}_{\geq k}$ is a more general permutation statistic introduced by Rawlings in [26]. These permutation statistics are defined in Section 6. By taking the stable principal specialization of both sides of (1.5), the following q -analog of Euler’s classical formula is established in [29] for the exc interpretation of $A_n(q, t)$ and in [32] for the des interpretation:

$$(1.8) \quad 1 + \sum_{n \geq 1} A_n(q, t) \frac{z^n}{[n]_q!} = \frac{(1-t) \exp_q(z)}{\exp_q(tz) - t \exp_q(z)},$$

where

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q \quad \text{and} \quad \exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}.$$

In [27], Sagan, Shareshian and the second named author use the expansion (1.6) of $\omega W_n(\mathbf{x}, t)$ in the power sum basis to show that $A_n(q, t)$ evaluated at any n th root of unity is a polynomial in t with positive integer coefficients. For results on cycle-type refinements of the exc interpretation of $A_n(q, t)$ see [29, 21, 27].

The Smirnov word enumerator $\tilde{W}_n^\neq(\mathbf{x}, t)$ arose in the work [11, 12, 13] of the first named author on chromatic quasisymmetric functions of directed graphs. The first named author proves the t -analog of (1.2),

$$(1.9) \quad \sum_{n \geq 1} \tilde{W}_n^\neq(\mathbf{x}, t) z^n = \frac{\sum_{i \geq 2} it[i-1]_t e_i(\mathbf{x}) z^i}{1 - \sum_{i \geq 2} t[i-1]_t e_i(\mathbf{x}) z^i}$$

from which e -positivity of $\tilde{W}_n^\neq(\mathbf{x}, t)$ follows. (A subsequent alternative proof of (1.9) was given in [1].) As a consequence of a general result obtained in [11, 12, 13] on power sum expansions of chromatic quasisymmetric functions, the first named author also obtains the following expansion analogous to (1.6):

$$(1.10) \quad \omega \tilde{W}_n^\neq(\mathbf{x}, t) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) > 1}} \left(nt A_{\ell(\lambda)-1}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right) \frac{p_\lambda}{z^\lambda} + nt[n-1]_t \frac{p_n}{n}.$$

1.2. New results

In this paper we obtain results for $\tilde{W}_n(\mathbf{x}, t)$ and $W_n^\neq(\mathbf{x}, t)$, analogous to those described above. For instance, in Section 2 we prove the t -analog of (1.1) given by

$$(1.11) \quad \sum_{n \geq 1} \tilde{W}_n(\mathbf{x}, t) z^n = \frac{\sum_{i \geq 1} it^{i-1} e_i(\mathbf{x}) z^i}{1 - \sum_{i \geq 2} t[i-1]_t e_i(\mathbf{x}) z^i},$$

and the t -analog of (1.2) given by

$$(1.12) \quad \sum_{n \geq 1} W_n^\neq(\mathbf{x}, t) z^n = \frac{\sum_{i \geq 2} ([i]_t + it[i-2]_t) e_i(\mathbf{x}) z^i}{1 - \sum_{i \geq 2} t[i-1]_t e_i(\mathbf{x}) z^i}.$$

From this it follows that $\tilde{W}_n(\mathbf{x}, t)$ and $W_n^\neq(\mathbf{x}, t)$ are symmetric in \mathbf{x} and e -positive. We also obtain expansions in the power sum symmetric functions analogous to (1.6) and (1.10) in Section 4.

Equation (1.11) can be restated as

$$(1.13) \quad \sum_{n \geq 1} \tilde{W}_n(\mathbf{x}, t) z^n = \frac{(1-t) \frac{\partial}{\partial t} E(tz)}{E(tz) - tE(z)}.$$

By specializing (1.13), using an expansion of $\tilde{W}_n(\mathbf{x})$ in the fundamental quasisymmetric functions derived in Section 5, we obtain in Section 6, the cyclic analog of (1.8),

$$\sum_{n \geq 1} \left(\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq 2}(\sigma^{-1})} t^{\text{cdes}(\sigma)} \right) \frac{z^n}{[n]_q!} = \frac{(1-t) \frac{\partial}{\partial t} \exp_q(tz)}{\exp_q(tz) - t \exp_q(z)}.$$

Our work relies on studying certain restricted Smirnov word enumerators, which are components of all the Smirnov word enumerators discussed above. For $n \geq 1$, let

$$\begin{aligned} W_n^<(\mathbf{x}, t) &:= \sum_{\substack{w \in W_n \\ w_1 < w_n}} t^{\text{des}(w)} x_w \\ W_n^>(\mathbf{x}, t) &:= \sum_{\substack{w \in W_n \\ w_1 > w_n}} t^{\text{des}(w)} x_w \\ W_n^=(\mathbf{x}, t) &:= \sum_{\substack{w \in W_n \\ w_1 = w_n}} t^{\text{des}(w)} x_w. \end{aligned}$$

Clearly

$$(1.14) \quad W_n(\mathbf{x}, t) = W_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t) + W_n^=(\mathbf{x}, t)$$

$$(1.15) \quad \tilde{W}_n(\mathbf{x}, t) = tW_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t) + W_n^=(\mathbf{x}, t)$$

$$(1.16) \quad W_n^\neq(\mathbf{x}, t) = W_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t)$$

$$(1.17) \quad \tilde{W}_n^\neq(\mathbf{x}, t) = tW_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t).$$

It is an exercise in [18, Exercise 2.9.11] that $W_n^<(\mathbf{x}, t)$, $W_n^>(\mathbf{x}, t)$, and $W_n^=(\mathbf{x}, t)$ are symmetric in \mathbf{x} . In Section 2, we derive results for $W_n^<(\mathbf{x}, t)$, $W_n^>(\mathbf{x}, t)$, and $W_n^=(\mathbf{x}, t)$ analogous to those of $W_n(\mathbf{x}, t)$, which not only establish symmetry, but also e -positivity of $W_n^<(\mathbf{x}, t)$ and $W_n^>(\mathbf{x}, t)$. When appropriately combined they yield the above mentioned results for $\tilde{W}_n(\mathbf{x}, t)$ and $W_n^\neq(\mathbf{x}, t)$. They also enable us to recover the previous results for $W_n(\mathbf{x}, t)$ and $\tilde{W}_n^\neq(\mathbf{x}, t)$. However they do not provide new proofs of the previous results since their proofs rely on these results.

The Smirnov word enumerator $W_n(\mathbf{x}, t)$ is an example of a chromatic quasisymmetric function. (The chromatic quasisymmetric functions are a refinement of Stanley's chromatic symmetric functions, which were introduced by Shareshian and the second named author in [31, 32]). Indeed, $W_n(\mathbf{x}, t)$ is the chromatic quasisymmetric function of the naturally labeled path graph with n nodes. In Section 3 our results for $W_n^<(\mathbf{x}, t)$ and $W_n^>(\mathbf{x}, t)$ are used to obtain new results in the study of chromatic quasisymmetric functions. For instance, we use e -positivity of $W_n^<(\mathbf{x}, t)$ and $W_n^>(\mathbf{x}, t)$ to establish e -positivity of the chromatic quasisymmetric function of the naturally labeled cycle C_n , providing an example of an e -positive chromatic quasisymmetric function not covered by the refinement of the Stanley–Stembridge e -positivity conjecture appearing in [31, 32] nor by the directed graph version appearing in [11, 12, 13].

Smirnov words have been used in the literature to enumerate unconstrained words; see e.g. [14, 23, 24, 25]. In a forthcoming paper we use results discussed in this paper to obtain analogous results for unconstrained words.

The paper is organized as follows. In Section 2, we derive our e -positivity results for the Smirnov word enumerators $\tilde{W}_n(\mathbf{x}, t)$, $W_n^\neq(\mathbf{x}, t)$, $W_n^<(\mathbf{x}, t)$, and $W_n^>(\mathbf{x}, t)$. Our e -positivity results for the chromatic quasisymmetric function of the naturally labeled cycle appear in Section 3. In Section 4, we obtain the power sum expansions of the various Smirnov word enumerators. Section 5 deals with expansions in Gessel's basis of fundamental quasisymmetric functions and consequences obtained by setting $x_i = 1$ for $i \in [m]$ and $x_i = 0$ otherwise. Finally, in Section 6, we apply stable principal specialization to the results of the previous sections to obtain results on variations of q -Eulerian polynomials.

2. Expansion in the elementary symmetric functions

In this section we derive formulas that refine (1.4) and (1.9) and then use the refinements to prove (1.11) and (1.12).

Theorem 2.1. *Let*

$$\begin{aligned}
 (2.1) \quad D(\mathbf{x}, t, z) &:= 1 - \sum_{i \geq 2} t[i-1]_t e_i(\mathbf{x}) z^i, \\
 a_i(t) &:= \frac{d}{dt} [i]_t = \sum_{j=0}^{i-2} (j+1) t^j, \\
 b_i(t) &:= t^{i-1} a_i(t^{-1}) = \sum_{j=1}^{i-1} (i-j) t^j, \\
 c_i(t) &:= it[i-2]_t,
 \end{aligned}$$

for all $i \geq 2$. Then

$$(2.2) \quad \sum_{n \geq 1} W_n^<(\mathbf{x}, t) z^n = \frac{1}{D(\mathbf{x}, t, z)} \sum_{i \geq 2} a_i(t) e_i(\mathbf{x}) z^i$$

$$(2.3) \quad \sum_{n \geq 1} W_n^>(\mathbf{x}, t) z^n = \frac{1}{D(\mathbf{x}, t, z)} \sum_{i \geq 2} b_i(t) e_i(\mathbf{x}) z^i$$

$$(2.4) \quad \sum_{n \geq 1} W_n^=(\mathbf{x}, t) z^n = \frac{1}{D(\mathbf{x}, t, z)} (e_1(\mathbf{x}) z - \sum_{i \geq 2} c_i(t) e_i(\mathbf{x}) z^i).$$

Before proving the theorem, we observe that

$$\begin{aligned}
 a_i(t) + b_i(t) &= 1 + (i+1)t + (i+1)t^2 + \dots + (i+1)t^{i-2} + t^{i-1} \\
 &= [i]_t + it[i-2]_t.
 \end{aligned}$$

Hence,

$$a_i(t) + b_i(t) - c_i(t) = [i]_t,$$

which shows that Theorem 2.1 refines (1.4) since $W_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t) + W_n^=(\mathbf{x}, t) = W_n(\mathbf{x}, t)$. Also

$$ta_i(t) + b_i(t) = it[i-1]_t,$$

which shows that Theorem 2.1 also refines (1.9) since $tW_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t) = \tilde{W}_n^{\neq}(\mathbf{x}, t)$.

We prove (2.4) first. Then we use (2.4), (1.4), and (1.9) to derive (2.2). Equation (2.3) follows from (2.2). Our proof of (2.4) uses the transfer-matrix method discussed in [36, Section 4.7] and borrows ingredients from the first named author’s proof of (1.9) in [12].

Before presenting the proof of (2.4), we give a brief review of the transfer-matrix method. A *walk* of length n on a directed graph $G = ([k], E)$ is a sequence of vertices v_0, v_1, \dots, v_n such that $(v_{i-1}, v_i) \in E$ for all $i \in [n]$. A walk is *closed* if $v_0 = v_n$. We attach weights in some commutative ring R to the edges of G . Let $\text{wt} : E \rightarrow R$ be the weight function. Now define the weight $\text{wt}(w)$ of a walk $w := v_0, v_1, \dots, v_n$ to be the product $\text{wt}(v_0, v_1)\text{wt}(v_1, v_2) \dots \text{wt}(v_{n-1}, v_n)$. For each $i, j \in [k]$, define $\mathcal{W}_{i,j,n}$ to be the set of walks of length n from i to j and let

$$U_{i,j,n} := \sum_{w \in \mathcal{W}_{i,j,n}} \text{wt}(w).$$

The transfer-matrix method enables one to express the generating function $\sum_{n \geq 0} U_{i,j,n} z^n$ in terms of the adjacency matrix A for the edge-weighted digraph G . That is, A is the $k \times k$ matrix whose (i, j) -entry is

$$A_{i,j} := \begin{cases} \text{wt}(i, j) & \text{if } (i, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.7.2 of [36] states that for all $i, j \in [k]$,

$$(2.5) \quad \sum_{n \geq 0} U_{i,j,n} z^n = \frac{(-1)^{i+j} \det(I - zA : j, i)}{\det(I - zA)},$$

where $(B : i, j)$ is the matrix obtained from matrix B by removing row i and column j .

Proof of (2.4). As in [35] and [12], we view a Smirnov word $w_1 w_2 \dots w_n$ over the alphabet $[k]$ as a walk w_1, w_2, \dots, w_n of length $n - 1$ on the digraph $G = ([k], E)$, where

$$E = \{(i, j) : i, j \in [k] \text{ and } i \neq j\}.$$

We let the edge weights belong to the commutative ring $\mathbb{Z}[x_1, \dots, x_k, t]$ and for all $(i, j) \in E$, and we set

$$\text{wt}(i, j) := \begin{cases} x_j & \text{if } i < j \\ tx_j & \text{if } i > j. \end{cases}$$

Note that if w is a Smirnov word over the alphabet $[k]$ then

$$t^{\text{des}(w)} x_w = x_{w_1} \text{wt}(w),$$

where w_1 is the first letter of w . Hence

$$W_n^-(x_1, \dots, x_k, t) := W_n^-(x_1, \dots, x_k, 0, 0, \dots, t) = \sum_{i=1}^k x_i U_{i,i,n-1}.$$

It follows from this and (2.5) that

$$\begin{aligned} \sum_{n \geq 1} W_n^-(x_1, \dots, x_k, t) z^n &= z \sum_{i=1}^k x_i \sum_{n \geq 0} U_{i,i,n} z^n \\ (2.6) \qquad \qquad \qquad &= \frac{z \sum_{i=1}^k x_i \det(I - zA : i, i)}{\det(I - zA)}, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & x_2 & x_3 & \dots & x_k \\ tx_1 & 0 & x_3 & \dots & x_k \\ tx_1 & tx_2 & 0 & \dots & x_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ tx_1 & tx_2 & tx_3 & \dots & 0 \end{bmatrix}.$$

In [12, Proof of Theorem 6.1], [13, p. 71], the first named author proves that¹

$$(2.7) \qquad \det(I - zA) = 1 - \sum_{j \geq 2} e_j(x_1, \dots, x_k) t[j - 1]_t z^j.$$

It follows that

$$\det(I - zA : i, i) = 1 - \sum_{j \geq 2} e_j(x_1, \dots, \hat{x}_i, \dots, x_k) t[j - 1]_t z^j,$$

where \hat{x}_i denotes deletion of x_i . Multiplying both sides by x_i and summing over all $i \in [k]$ yields,

$$\sum_{i=1}^k x_i \det(I - zA : i, i) = \sum_{i=1}^k x_i - \sum_{j \geq 2} \sum_{i=1}^k x_i e_j(x_1, \dots, \hat{x}_i, \dots, x_k) t[j - 1]_t z^j.$$

¹This is obtained from the formula in [12, 13] by replacing t with t^{-1} and each x_i with tx_i .

One can see that

$$\sum_{i=1}^k x_i e_j(x_1, \dots, \hat{x}_i, \dots, x_k) = (j + 1)e_{j+1}(x_1, \dots, x_k),$$

since both sides enumerate $(j+1)$ -subsets of $[k]$ with a distinguished element. Hence

$$\sum_{i=1}^k x_i \det(I - zA : i, i) = e_1(x_1, \dots, x_k) - \sum_{j \geq 2} (j + 1)e_{j+1}(x_1, \dots, x_k)t[j - 1]_t z^j.$$

Upon multiplying both sides by z , we see that the numerator of the right hand side of (2.6) is

$$e_1(x_1, \dots, x_k)z - \sum_{j \geq 3} j e_j(x_1, \dots, x_k)t[j - 2]_t z^j.$$

It therefore follows from (2.6) and (2.7) that

$$\sum_{n \geq 1} W_n^=(x_1, \dots, x_k, t)z^n = \frac{e_1(x_1, \dots, x_k)z - \sum_{j \geq 3} j e_j(x_1, \dots, x_k)t[j - 2]_t z^j}{1 - \sum_{j \geq 2} e_j(x_1, \dots, x_k)t[j - 1]_t z^j}.$$

The desired result (2.4) follows by taking the limit as k goes to infinity. \square

Proof of (2.2). It follows from (1.14), (1.4), and (2.4) that

$$\begin{aligned} (2.8) \quad \sum_{n \geq 1} W_n^<(\mathbf{x}, t)z^n + \sum_{n \geq 1} W_n^>(\mathbf{x}, t)z^n &= \sum_{n \geq 1} W_n(\mathbf{x}, t) - \sum_{n \geq 1} W_n^=(\mathbf{x}, t) \\ &= \frac{B(\mathbf{x}, t, z)}{D(\mathbf{x}, t, z)}, \end{aligned}$$

where

$$\begin{aligned} (2.9) \quad B(\mathbf{x}, t, z) &= \sum_{i \geq 1} [i]_t e_i(\mathbf{x})z^i - (e_1(\mathbf{x})z - \sum_{i \geq 2} it[i - 2]_t e_i(\mathbf{x})z^i) \\ &= \sum_{i \geq 2} [i]_t e_i(\mathbf{x})z^i + \sum_{i \geq 2} it[i - 2]_t e_i(\mathbf{x})z^i \\ &= \sum_{i \geq 2} ([i]_t + it[i - 2]_t)e_i(\mathbf{x})z^i. \end{aligned}$$

It follows from (1.17) and (1.9) that

$$(2.10) \quad t \sum_{n \geq 1} W_n^<(\mathbf{x}, t) z^n + \sum_{n \geq 1} W_n^>(\mathbf{x}, t) z^n = \sum_{n \geq 1} \tilde{W}_n^{\neq}(\mathbf{x}, t) z^n = \frac{C(\mathbf{x}, t, z)}{D(\mathbf{x}, t, z)},$$

where

$$C(\mathbf{x}, t, z) = \sum_{i \geq 2} it[i-1]_t e_i(\mathbf{x}) z^i.$$

By subtracting (2.8) from (2.10), we obtain

$$(t-1) \sum_{n \geq 1} W_n^<(\mathbf{x}, t) z^n = \frac{C(\mathbf{x}, t, z) - B(\mathbf{x}, t, z)}{D(\mathbf{x}, t, z)} = \frac{\sum_{i \geq 2} (it^{i-1} - [i]_t) e_i(\mathbf{x}) z^i}{D(\mathbf{x}, t, z)}.$$

Note that

$$(2.11) \quad (t-1)(1 + 2t + 3t^2 + \dots + (i-1)t^{i-2}) = (i-1)t^{i-1} - [i-1]_t = it^{i-1} - [i]_t.$$

Hence

$$\sum_{n \geq 1} W_n^<(\mathbf{x}, t) z^n = \frac{\sum_{i \geq 2} (1 + 2t + 3t^2 + \dots + (i-1)t^{i-2}) e_i(\mathbf{x}) z^i}{D(\mathbf{x}, t, z)}$$

as desired. □

Proof of (2.3). This follows immediately from (2.2), (2.8), and (2.9). For an alternative proof, one can use (2.2) and the fact that $W_n^>(\mathbf{x}, t) = t^{n-1} W_n^<(\mathbf{x}, t^{-1})$, which follows from the involution on the set of Smirnov words that reverses each word. □

Since

$$(2.12) \quad \frac{1}{D(\mathbf{x}, t, z)} = \sum_{m \geq 0} \left(\sum_{i \geq 2} t[i-1]_t e_i(\mathbf{x}) z^i \right)^m,$$

we have the following consequence of Theorem 2.1.

Corollary 2.2. *For all $n \geq 1$, the polynomials $W_n^<(\mathbf{x}, t)$ and $W_n^>(\mathbf{x}, t)$ are e -positive.*

Note that it follows from Theorem 2.1 that the coefficient of $e_n(\mathbf{x})$ in the e -basis expansion of $W_n^=(\mathbf{x}, t)$ is $-nt[n - 2]_t$ if $n \geq 2$. Hence $W_n^=(\mathbf{x}, t)$ fails to be e -positive. However, observe that the coefficient $c_\lambda(t)$ of $e_\lambda(\mathbf{x})$ is in $\mathbb{N}[t]$ if the smallest part of λ is 1, and $-c_\lambda(t) \in \mathbb{N}[t]$ otherwise.

We obtain equivalent formulations of (2.2) and of (2.3) by multiplying the numerators and denominators of the right hand sides of the equations by $1 - t$.

Corollary 2.3. *We have*

$$(2.13) \quad \sum_{n \geq 1} W_n^<(\mathbf{x}, t) z^n = \frac{(1 - t) \frac{\partial}{\partial t} \sum_{i \geq 2} [i]_t e_i(\mathbf{x}) z^i}{E(tz) - tE(z)},$$

$$(2.14) \quad \sum_{n \geq 1} W_n^>(\mathbf{x}, t) z^n = \frac{(1 - t) \sum_{i \geq 2} (\sum_{j=1}^{i-1} (i - j) t^j) e_i(\mathbf{x}) z^i}{E(tz) - tE(z)},$$

where $E(z) := \sum_{n \geq 0} e_n(\mathbf{x}) z^n$.

We now use Theorem 2.1 to prove (1.12) and (1.11), which we restate here.

Corollary 2.4. *We have*

$$\sum_{n \geq 1} W_n^\neq(\mathbf{x}, t) z^n = \frac{\sum_{i \geq 2} ([i]_t + it[i - 2]_t) e_i(\mathbf{x}) z^i}{D(\mathbf{x}, t, z)}.$$

Consequently, $W_n^\neq(\mathbf{x}, t)$ is e -positive.

Proof. We use the facts that $W_n^\neq(\mathbf{x}, t) = W_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t)$ and $a_i(t) + b_i(t) = [i]_t + it[i - 2]_t$. □

Corollary 2.5. *We have*

$$\sum_{n \geq 1} \tilde{W}_n(\mathbf{x}, t) z^n = \frac{\sum_{i \geq 1} it^{i-1} e_i(\mathbf{x}) z^i}{D(\mathbf{x}, t, z)}.$$

Consequently, $\tilde{W}_n(\mathbf{x}, t)$ is e -positive.

Proof. We use the fact that

$$(2.15) \quad \tilde{W}_n(\mathbf{x}, t) = tW_n^{<}(\mathbf{x}, t) + (W_n(\mathbf{x}, t) - W_n^{<}(\mathbf{x}, t))$$

and equations (2.2), (2.11), and (1.4). □

Corollary 2.6. *We have*

$$(2.16) \quad \sum_{n \geq 1} \tilde{W}_n(\mathbf{x}, t) z^n = \frac{(1-t) \frac{\partial}{\partial t} E(tz)}{E(tz) - tE(z)}.$$

In [29, Theorem 5.1] (see also [32, Corollary C.5]), it is observed that (1.4) implies that the Smirnov enumerator $W_n(\mathbf{x}, t)$ has a stronger property than e -positivity, namely e -unimodality, and in [11, 12] the same is observed for $\tilde{W}_n^{\neq}(\mathbf{x}, t)$ as a consequence of (1.9). Here we show that $W_n^{\neq}(\mathbf{x}, t)$ also has the stronger property, while $\tilde{W}_n(\mathbf{x}, t)$ does not.

A polynomial $A(t) = a_0 + a_1t + \dots + a_nt^n \in \mathbb{Q}[t]$ is said to be *unimodal*² if

$$0 \leq a_0 \leq a_1 \leq \dots \leq a_{c-1} \leq a_c \geq a_{c+1} \geq \dots \geq a_n \geq 0,$$

for some c . Let $\Lambda_{\mathbb{Q}}$ denote the \mathbb{Q} -algebra of symmetric functions over $\mathbf{x} := x_1, x_2, \dots$. Given $r, s \in \Lambda_{\mathbb{Q}}$, we say that $r \leq_e s$ if $s - r$ is e -positive. A polynomial $A(t) = a_0 + a_1t + \dots + a_nt^n \in \Lambda_{\mathbb{Q}}[t]$ is said to be *e -unimodal* if

$$0 \leq_e a_0 \leq_e a_1 \leq_e \dots \leq_e a_{c-1} \leq_e a_c \geq_e a_{c+1} \geq_e \dots \geq_e a_n \geq_e 0,$$

for some c . We say that $A(t)$ (over any coefficient ring) is *palindromic* with center of symmetry $\frac{n}{2}$ if $a_j = a_{n-j}$ for $0 \leq j \leq n$. (Note that the center of symmetry is unique unless $A(t)$ is the zero polynomial, in which case every number of the form $\frac{n}{2}$, where $n \in \mathbb{N}$, satisfies the definition of center of symmetry.) It is easy to see that $A(t) \in \Lambda_{\mathbb{Q}}[t]$ is e -unimodal and palindromic with center of symmetry c if and only if the coefficient of each e_{λ} in the e -expansion of $A(t)$ is a unimodal, palindromic polynomial in $\mathbb{Q}[t]$ with center of symmetry c ; see [32, Proposition B.3].

Lemma 2.7. *Let $(g_n(t))_{n \geq 2}$ be a sequence of polynomials in $\mathbb{Q}[t]$, such that each $g_n(t)$ is unimodal and palindromic with center of symmetry $\frac{n+r}{2}$, where*

²Note that our definition is nonstandard in that positivity of the coefficients is also required.

$r \geq -2$ is a fixed integer. If $(G_n(\mathbf{x}, t))_{n \geq 2}$ is a sequence of polynomials in $\Lambda_{\mathbb{Q}}[t]$ that satisfies

$$\sum_{n \geq 2} G_n(\mathbf{x}, t) z^n = \frac{\sum_{n \geq 2} g_n(t) e_n(\mathbf{x}) z^n}{D(\mathbf{x}, t, z)},$$

where $D(\mathbf{x}, t, z)$ is defined in (2.1), then each $G_n(\mathbf{x}, t)$ is e -unimodal and palindromic with center of symmetry $\frac{n+r}{2}$.

Proof. By (2.12),

$$(2.17) \quad G_n(\mathbf{x}, t) = \sum_{m \geq 1} \sum_{\substack{k_1, \dots, k_m \geq 2 \\ \sum_{i=1}^m k_i = n}} e_{k_1} \dots e_{k_m} t^{m-1} g_{k_1}(t) \prod_{i=2}^m [k_i - 1]_t.$$

Since the product of nonzero palindromic, unimodal polynomials is palindromic, unimodal with center of symmetry equal to the sum of the centers of symmetry of the factors (see e.g. [32, Proposition B.1]), for each nonzero $g_{k_1}(t)$, the polynomial $t^{m-1} g_{k_1}(t) \prod_{i=2}^m [k_i - 1]_t$ is palindromic and unimodal with center of symmetry equal to

$$m - 1 + \frac{k_1 + r}{2} + \sum_{i=2}^m \frac{k_i - 2}{2} = \frac{n + r}{2}.$$

Since the sum of palindromic, unimodal polynomials with the same center of symmetry is palindromic and unimodal with the same center of symmetry as the summands (see e.g. [32, Proposition B.3]), it follows from (2.17) that $G_n(\mathbf{x}, t)$ is palindromic and e -unimodal with center of symmetry $\frac{n+r}{2}$. \square

Corollary 2.8 (of Corollary 2.4). *For all $n \geq 2$, the Smirnov word enumerator $W_n^{\neq}(\mathbf{x}, t)$ is e -unimodal and palindromic with center of symmetry $\frac{n-1}{2}$.*

Proof. Since $[i]_t + it[i - 2]_t$ is unimodal and palindromic with center of symmetry $\frac{i-1}{2}$, the result follows from Lemma 2.7 \square

We note that although $W_n(\mathbf{x}, t)$, $W_n^{\neq}(\mathbf{x}, t)$, and $\tilde{W}^{\neq}(\mathbf{x}, t)$ are all e -unimodal and palindromic, this is not the case for $\tilde{W}_n(\mathbf{x}, t)$. Indeed, it follows from Corollary 2.5 that

$$\tilde{W}_5(\mathbf{x}, t) = e_{4,1}t + (e_{2,2,1} + e_{4,1} + 2e_{3,2})t^2 + (e_{4,1} + 5e_{3,2})t^3 + (5e_5)t^4,$$

which is neither palindromic nor e -unimodal. However, for certain partitions λ , the coefficient of e_λ is a palindromic, unimodal polynomial in t .

Corollary 2.9 (of Corollary 2.5). *For $\lambda \vdash n$, let $c_\lambda(t)$ be the coefficient of e_λ in the expansion of $\tilde{W}_n(\mathbf{x}, t)$ in the elementary symmetric functions. If $\lambda = (\lambda_1 \geq \dots \geq \lambda_{k-1} \geq 1)$ then*

$$c_\lambda(t) = t^{k-1} \prod_{i=1}^{k-1} [\lambda_i - 1]_t,$$

which is palindromic and unimodal. If $\lambda = j^k$ then

$$c_\lambda(t) = jt^{j+k-2} [j - 1]_t^{k-1},$$

which is palindromic and unimodal.

3. Chromatic quasisymmetric function of the cycle

In this section, we discuss the connection between the Smirnov word enumerators and the chromatic quasisymmetric functions introduced by Shareshian and the second named author in [31, 32]. We use results of the previous section to provide an example of an e -positive chromatic quasisymmetric function not covered by the refinement of the Stanley–Stembridge e -positivity conjecture appearing in [31, 32] or its directed graph extension appearing in [11, 12].

A map $\kappa : [n] \rightarrow \mathbb{P}$ is said to be a *proper coloring* of a graph $G = ([n], E)$ if $\kappa(i) \neq \kappa(j)$ for all $\{i, j\} \in E$. Now define the *chromatic quasisymmetric function*³ of G as

$$X_G(\mathbf{x}, t) := \sum_{\kappa \in \mathcal{C}(G)} t^{\text{des}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)},$$

where $\mathcal{C}(G)$ is the set of proper colorings $\kappa : [n] \rightarrow \mathbb{P}$ of G and

$$(3.1) \quad \text{des}(\kappa) := |\{\{i, j\} \in E : i < j \text{ and } \kappa(i) > \kappa(j)\}|.$$

When $t = 1$, $X_G(\mathbf{x}, t)$ is Stanley’s chromatic symmetric function $X_G(\mathbf{x})$. Note that $X_G(\mathbf{x}, t)$ is a polynomial in t whose coefficients are quasisymmetric

³In the definition given in [32] ascents are counted instead of descents, but in the case that $X_G(\mathbf{x}, t)$ is symmetric, it is shown in [32, Corollary 2.7] that the definitions are equivalent.

functions. We view G as a labeled graph and note that the definition of $X_G(\mathbf{x}, t)$ depends on the vertex labeling, and not just on the isomorphism class of G , as is the case for $X_G(\mathbf{x})$.

Since Smirnov words of length n can be viewed as proper colorings of the naturally labeled path

$$P_n := ([n], \{\{i, i+1\} : i \in [n-1]\}),$$

it follows that

$$W_n(\mathbf{x}, t) = X_{P_n}(\mathbf{x}, t).$$

The Smirnov word enumerator $\tilde{W}_n^\neq(\mathbf{x}, t)$ can also be viewed as a chromatic quasisymmetric function, but in the more general sense considered by the first named author in [11, 12, 13] and Awan–Bernardi in [4, Section 8], in which labeled graphs are replaced by directed graphs and the definition of $\text{des}(\kappa)$ given in (3.1) is replaced by

$$\text{des}(\kappa) := |\{(i, j) \in E : \kappa(i) > \kappa(j)\}|.$$

Labeled graphs can be viewed as directed graphs by orienting each edge from smaller vertex to larger vertex; so the digraph version of chromatic quasisymmetric function is more general than the labeled graph version. One can see that

$$\tilde{W}_n^\neq(\mathbf{x}, t) = X_{\vec{C}_n}(\mathbf{x}, t),$$

where \vec{C}_n is the directed cycle defined by

$$\vec{C}_n := ([n], \{\{i, i+1\} : i \in [n-1]\} \cup \{(n, 1)\}).$$

The longstanding Stanley–Stembridge conjecture [35] asserts that $X_G(\mathbf{x})$ is e -positive when G is the incomparability graph of a $(3+1)$ -free poset. In [19], Guay-Paquet proves that if the Stanley–Stembridge conjecture holds for incomparability graphs of posets that are both $(3+1)$ -free and $(2+2)$ -free (known as *unit interval graphs*) then it holds in general.

In [32] Shareshian and the second named author show that $X_G(\mathbf{x}, t)$ is symmetric when G is a unit interval graph with a certain natural labeling; these are called *natural unit interval graphs*. They also conjecture that $X_G(\mathbf{x}, t)$ is e -positive and e -unimodal when G is a natural unit interval graph. The path P_n is an example of a natural unit interval graph for which the conjecture holds since $X_{P_n}(\mathbf{x}, t) = W_n(\mathbf{x}, t)$. The symmetry result and

e -positivity conjecture of [32] are generalized in [11, 12, 13] to a class of directed graphs called *circular indifference digraphs* in [12, 13]. With the view that a labeled graph is an acyclic digraph, the natural unit interval graphs form the class of acyclic circular indifference digraphs. The directed cycle \vec{C}_n is an example of a circular indifference digraph for which the extended conjecture holds since $X_{\vec{C}_n}(\mathbf{x}, t) = \tilde{W}_n^\neq(\mathbf{x}, t)$.

Here we consider the labeled cycle

$$(3.2) \quad C_n := ([n], \{\{i, i + 1\} : i \in [n - 1]\} \cup \{\{1, n\}\}).$$

If we view C_n as a directed graph (by orienting its edges from smaller vertex to larger vertex), we get a directed graph that is identical to the directed cycle \vec{C}_n except for the edge $(n, 1)$ in \vec{C}_n , which is oriented as $(1, n)$ in C_n . For $n \geq 4$, the labeled cycle C_n is not a natural unit interval graph, nor is it a circular indifference digraph. Nevertheless, since

$$(3.3) \quad X_{C_n}(\mathbf{x}, t) = W_n^<(\mathbf{x}, t) + tW_n^>(\mathbf{x}, t),$$

it follows from Corollary 2.2 that $X_{C_n}(\mathbf{x}, t)$ is symmetric and e -positive. This shows that the class of labeled graphs with e -positive chromatic quasisymmetric function is strictly larger than the class of natural unit interval graphs, and the class of digraphs with e -positive chromatic quasisymmetric function is strictly larger than the class of indifference digraphs.

Next we address the question of e -unimodality of $X_{C_n}(\mathbf{x}, t)$. From Theorem 2.1 and equation (3.3), we obtain the next result. For $n > 0$, let

$$[-n]_t := \frac{t^{-n} - 1}{t - 1} = -t^{-n}[n]_t.$$

Corollary 3.1 (of Theorem 2.1). *We have*

$$(3.4) \quad \sum_{n \geq 2} X_{C_n}(\mathbf{x}, t)z^n = \frac{\sum_{i \geq 2} ([2]_t[i]_t + it^2[i - 3]_t)e_i(\mathbf{x})z^i}{D(\mathbf{x}, t, z)},$$

where $D(\mathbf{x}, t, z)$ is defined in (2.1).

Theorem 3.2. *Let $n \geq 2$.*

1. *If n is odd, $X_{C_n}(\mathbf{x}, t)$ is e -unimodal and palindromic with center of symmetry $\frac{n}{2}$.*
2. *If n is even,*

- (a) $X_{C_n}(\mathbf{x}, t)$ is e -positive and palindromic with center of symmetry $\frac{n}{2}$, but is not e -unimodal.
- (b) $X_{C_n}(\mathbf{x}, t) + t^{\frac{n}{2}}e_{2^{\frac{n}{2}}}(\mathbf{x})$ is e -unimodal and palindromic with center of symmetry $\frac{n}{2}$.

Proof. Let $U_n(\mathbf{x}, t)$ and $V_n(\mathbf{x}, t)$ be defined respectively by

$$\sum_{n \geq 2} U_n(\mathbf{x}, t)z^n = \frac{([2]_t[2]_t + 2t^2[2 - 3]_t)e_2(\mathbf{x})z^2}{D(\mathbf{x}, t, z)}$$

and

$$\sum_{n \geq 2} V_n(\mathbf{x}, t)z^n = \frac{\sum_{i \geq 3} ([2]_t[i]_t + it^2[i - 3]_t)e_i(\mathbf{x})z^i}{D(\mathbf{x}, t, z)}.$$

Then $X_{C_n}(\mathbf{x}, t) = U_n(\mathbf{x}, t) + V_n(\mathbf{x}, t)$.

We have

$$\sum_{n \geq 2} U_n(\mathbf{x}, t)z^n = \frac{(1 + t^2)e_2(\mathbf{x})z^2}{D(\mathbf{x}, t, z)}.$$

It follows from (2.12) that

$$(3.5) \quad U_n(\mathbf{x}, t) = \sum_{m \geq 1} \sum_{\substack{k_2, \dots, k_m \geq 2 \\ \sum_{i=2}^m k_i = n - 2}} e_{k_2} \dots e_{k_m} t^{m-1} (1 + t^2) \prod_{i=2}^m [k_i - 1]_t.$$

Note that for any $k \geq 3$

$$(1 + t^2)[k]_t = 1 + t + 2t^2 + \dots + 2t^{k-1} + t^k + t^{k+1},$$

and for $k = 2$,

$$(1 + t^2)[k]_t = 1 + t + t^2 + t^3.$$

In either case, $(1 + t^2)[k]_t$ is unimodal and palindromic with center of symmetry $\frac{k+1}{2}$.

As in the proof of Lemma 2.7, we use Propositions B.1 and B.3 of [32]. Consider the term of the right side of (3.5) corresponding to the $(m - 1)$ -tuple (k_2, \dots, k_m) . If $k_j \geq 3$ for some $j \geq 2$ then since $(1 + t^2)[k_j - 1]_t$ is unimodal and palindromic, $t^{m-1}(1 + t^2) \prod_{i=2}^m [k_i - 1]_t$ is a product of nonzero

unimodal and palindromic polynomials. Hence $t^{m-1}(1+t^2)\prod_{i=2}^m[k_i-1]_t$ is unimodal and palindromic with center of symmetry

$$m-1 + \frac{k_j}{2} + \sum_{\substack{i=2 \\ i \neq j}}^m \frac{k_i-2}{2} = \frac{n}{2}.$$

It follows that if λ has a part of size at least 3 then the coefficient of e_λ in $U_n(\mathbf{x}, t)$ is palindromic and unimodal with center of symmetry $\frac{n}{2}$. If λ does not have a part of size at least 3 then all the parts must be 2, which means that n is even. Hence if n is odd then $U_n(\mathbf{x}, t)$ is e -unimodal and palindromic with center of symmetry $\frac{n}{2}$.

Now if λ does not have a part of size at least 3 then $\lambda = 2^m$, where $n = 2m$. By (3.5), the coefficient of e_λ in $U_n(\mathbf{x}, t)$ is $t^{m-1}(1+t^2)$. Hence if λ does not have a part of size at least 3 then coefficient of e_λ in $U_n(\mathbf{x}, t) + t^m e_{2^m}$ is unimodal and palindromic with center of symmetry $m = \frac{n}{2}$. From the argument in the previous paragraph, the same is true if λ has a part of size at least 3. It follows that if n is even, $U_n(\mathbf{x}, t) + t^m e_{2^m}$ is e -unimodal and palindromic with center of symmetry $m = \frac{n}{2}$.

It follows from Lemma 2.7 that $V_n(\mathbf{x}, t)$ is also palindromic and e -unimodal with center of symmetry $\frac{n}{2}$. Since $X_{C_n}(\mathbf{x}, t) = U_n(\mathbf{x}, t) + V_n(\mathbf{x}, t)$, Parts (1) and (2b) hold. Palindromicity of $X_{C_n}(\mathbf{x}, t)$ in the even case follows from Part (2b). The assertion in Part (2a) that $X_{C_n}(\mathbf{x}, t)$ is not e -unimodal in the even case follows from the fact the coefficient of $e_{2^m}(\mathbf{x})$ is $t^{m-1}(1+t^2)$, which is not unimodal. □

4. Expansion in the power sum symmetric functions

In this section we use the e -expansion for $\tilde{W}_n(\mathbf{x}, t)$ given in Corollary 2.6 to derive a power sum expansion for $\tilde{W}_n(\mathbf{x}, t)$. From this we obtain power sum expansions for the other Smirnov word enumerators.

Let $A_n(t)$ be the Eulerian polynomial defined in (1.7) for $n \geq 1$ and let

$$A_0(t) := t^{-1}.$$

Recall that ω is the standard involution on $\Lambda_{\mathbb{Q}}$ taking the elementary symmetric function e_n to the complete homogeneous symmetric function h_n , and let

$$H(z) := \sum_{n \geq 0} h_n(\mathbf{x})z^n,$$

where $h_n(\mathbf{x})$ is the complete homogenous symmetric function of degree n . Stembridge [38, Proposition 3.3] proves that

$$(4.1) \quad 1 + \sum_{n \geq 1} \sum_{\lambda \vdash n} \left(A_{\ell(\lambda)}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right) \frac{p_\lambda(\mathbf{x})}{z_\lambda} z^n = \frac{(1-t)H(z)}{H(tz) - tH(z)}$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)})$ and

$$z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$$

if λ has m_i parts of size i for each i . By combining this with (1.5) one obtains the expansion of $\omega W_n(\mathbf{x}, t)$ in the power sum symmetric functions given in (1.6).

We will use the e -expansion for $\tilde{W}_n(\mathbf{x}, t)$ obtained in Section 2 to obtain the following analog of (1.6), which expresses the coefficients of $\frac{p_\lambda(\mathbf{x})}{z_\lambda}$ in the power sum expansion of $\omega \tilde{W}_n(\mathbf{x}, t)$ as a polynomial in t with positive integer coefficients. (A combinatorial proof that does not make use of the e -expansion is given in [13, Section 6.5].)

Theorem 4.1. *For all $n \geq 1$,*

$$(4.2) \quad \omega \tilde{W}_n(\mathbf{x}, t) = \sum_{\lambda \vdash n} \left(A_{\ell(\lambda)-1}(t) \sum_{i=1}^{\ell(\lambda)} \lambda_i t^{\lambda_i} \prod_{j \in [\ell(\lambda)] \setminus \{i\}} [\lambda_j]_t \right) \frac{p_\lambda(\mathbf{x})}{z_\lambda},$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)})$.

We will need the following Lemma, which is implicit in the proof of (4.1) in [38]. We include the proof for the sake of completeness.

Lemma 4.2. *For all $k \geq 1$,*

$$(4.3) \quad \left(\frac{H(z)}{H(tz)} \right)^k = 1 + \sum_{n \geq 1} \sum_{\lambda \vdash n} \left(k^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i}) \right) \frac{p_\lambda(\mathbf{x})}{z_\lambda} z^n.$$

Proof. For each $k \geq 1$, let $\varphi_k : \Lambda_{\mathbb{Q}}[z] \rightarrow \Lambda_{\mathbb{Q}}[t, z]$ be the algebra homomorphism determined by

$$\varphi_k(p_r(\mathbf{x})) = k(1 - t^r)p_r(\mathbf{x})$$

for all $r \geq 1$. Since $H(z) = 1 + \sum_{n \geq 1} \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} z^n$,

$$\varphi_k(H(z)) = 1 + \sum_{n \geq 1} \sum_{\lambda \vdash n} \left(k^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i}) \right) \frac{p_\lambda}{z_\lambda} z^n.$$

To complete the proof we show that $\varphi_k(H(z))$ is equal to the left hand side of (4.3). We use the fact that

$$(4.4) \quad H(z) = \exp \left(\sum_{r \geq 1} \frac{p_r(\mathbf{x})}{r} z^r \right)$$

to obtain

$$\begin{aligned} \varphi_k(H(z)) &= \exp \left(\sum_{r \geq 1} \frac{\varphi_k(p_r)}{r} z^r \right) \\ &= \exp \left(\sum_{r \geq 1} \frac{k(1 - t^r)p_r}{r} z^r \right) \\ &= \left(\frac{\exp \left(\sum_{r \geq 1} \frac{p_r}{r} z^r \right)}{\exp \left(\sum_{r \geq 1} t^r \frac{p_r}{r} z^r \right)} \right)^k \\ &= \left(\frac{H(z)}{H(tz)} \right)^k. \end{aligned}$$

□

Proof of Theorem 4.1. For each $\lambda \vdash n$, set

$$c_\lambda(t) := A_{\ell(\lambda)-1}(t) \sum_{i=1}^{\ell(\lambda)} \lambda_i t^{\lambda_i} \prod_{j \in [\ell(\lambda)] \setminus \{i\}} [\lambda_j]_t.$$

We will prove that

$$(4.5) \quad \sum_{n \geq 1} \sum_{\lambda \vdash n} c_\lambda(t) \frac{p_\lambda(\mathbf{x})}{z_\lambda} z^n = \frac{(1-t) \frac{\partial}{\partial t} H(tz)}{H(tz) - tH(z)},$$

which by Corollary 2.6 is equivalent to (4.2).

We have

$$c_\lambda(t) = (t-1) \frac{A_{\ell(\lambda)-1}(t)}{(t-1)^{\ell(\lambda)}} \sum_{i=1}^{\ell(\lambda)} \lambda_i t^{\lambda_i} \prod_{j \in [\ell(\lambda)] \setminus \{i\}} (t^{\lambda_j} - 1)$$

$$\begin{aligned}
 &= (t-1)(-1)^{\ell(\lambda)} \frac{A_{\ell(\lambda)-1}(t)}{(1-t)^{\ell(\lambda)}} t \frac{d}{dt} \left(\prod_{j=1}^{\ell(\lambda)} (t^{\lambda_j} - 1) \right) \\
 &= (t-1) \frac{tA_{\ell(\lambda)-1}(t)}{(1-t)^{\ell(\lambda)}} \frac{d}{dt} \left(\prod_{j=1}^{\ell(\lambda)} (1 - t^{\lambda_j}) \right).
 \end{aligned}$$

For the case $\ell(\lambda) = 1$, we have

$$c_{(n)}(t) = nt^{n-1}.$$

For the case $\ell(\lambda) > 1$, we use the classical identity⁴ (see [36, Proposition 1.4.4 and equation (1.36)]),

$$(4.6) \quad \frac{tA_{m-1}(t)}{(1-t)^m} = \sum_{k \geq 1} k^{m-1} t^k,$$

for all $m > 1$. This yields

$$\begin{aligned}
 c_\lambda(t) &= (t-1) \sum_{k \geq 1} k^{\ell(\lambda)-1} t^k \frac{d}{dt} \left(\prod_{j=1}^{\ell(\lambda)} (1 - t^{\lambda_j}) \right) \\
 &= (t-1) \sum_{k \geq 1} \frac{t^k}{k} \frac{d}{dt} \left(k^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} (1 - t^{\lambda_j}) \right).
 \end{aligned}$$

It follows that

$$(4.7) \quad \sum_{n \geq 1} \sum_{\lambda \vdash n} c_\lambda(t) \frac{p_\lambda(\mathbf{x})}{z_\lambda} z^n = \sum_{n \geq 1} nt^{n-1} \frac{p_n(\mathbf{x})}{n} z^n + (t-1) \sum_{k \geq 1} \frac{t^k}{k} \frac{\partial}{\partial t} U_k(\mathbf{x}, t, z)$$

where

$$U_k(\mathbf{x}, t, z) := \sum_{n \geq 1} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) > 1}} k^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} (1 - t^{\lambda_j}) \frac{p_\lambda(\mathbf{x})}{z_\lambda} z^n.$$

Note that the first summation on the right hand side of (4.7) can be

⁴This is Euler’s original definition of what we now call Eulerian polynomials.

expressed as

$$\begin{aligned}
 \sum_{n \geq 1} n t^{n-1} \frac{p_n}{n} z^n &= \frac{\partial}{\partial t} \sum_{n \geq 1} t^n \frac{p_n}{n} z^n \\
 &= \frac{\partial}{\partial t} \ln H(tz) \\
 (4.8) \qquad \qquad \qquad &= \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)}
 \end{aligned}$$

with the second equality following from (4.4).

To evaluate the second summation on the right hand side of (4.7), we use Lemma 4.2 to obtain

$$\begin{aligned}
 U_k(\mathbf{x}, t, z) &= \sum_{n \geq 1} \sum_{\lambda \vdash n} k^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} (1 - t^{\lambda_j}) \frac{p_{\lambda}(\mathbf{x})}{z_{\lambda}} z^n - \sum_{n \geq 1} k(1 - t^n) \frac{p_n(\mathbf{x})}{n} z^n \\
 &= \left(\frac{H(z)}{H(tz)} \right)^k - 1 - k \sum_{n \geq 1} (1 - t^n) \frac{p_n(\mathbf{x})}{n} z^n.
 \end{aligned}$$

By (4.8),

$$\begin{aligned}
 \frac{\partial}{\partial t} U_k(\mathbf{x}, t, z) &= k \left(\frac{H(z)}{H(tz)} \right)^{k-1} \frac{\partial}{\partial t} \left(\frac{H(z)}{H(tz)} \right) + k \sum_{n \geq 1} t^{n-1} p_n(\mathbf{x}) z^n \\
 &= -k \left(\left(\frac{H(z)}{H(tz)} \right)^k \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)} - \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)} \right) \\
 &= -k \left(\left(\frac{H(z)}{H(tz)} \right)^k - 1 \right) \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)}.
 \end{aligned}$$

Hence the second summation is

$$- \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)} \sum_{k \geq 1} t^k \left(\left(\frac{H(z)}{H(tz)} \right)^k - 1 \right).$$

Plugging this and (4.8) into (4.7) yields

$$\sum_{n \geq 1} \sum_{\lambda \vdash n} c_{\lambda}(t) \frac{p_{\lambda}(\mathbf{x})}{z_{\lambda}} z^n$$

$$\begin{aligned}
 &= \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)} \\
 &\quad + (1-t) \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)} \sum_{k \geq 1} t^k \left(\left(\frac{H(z)}{H(tz)} \right)^k - 1 \right) \\
 &= \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)} \left(1 + (1-t) \sum_{k \geq 1} t^k \left(\left(\frac{H(z)}{H(tz)} \right)^k - 1 \right) \right) \\
 &= \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)} \left(1 - (1-t) \sum_{k \geq 0} t^k + (1-t) \sum_{k \geq 0} t^k \left(\frac{H(z)}{H(tz)} \right)^k \right) \\
 &= \frac{\frac{\partial}{\partial t}(H(tz))}{H(tz)} (1-t) \frac{1}{1 - t \frac{H(z)}{H(tz)}} \\
 &= \frac{(1-t) \frac{\partial}{\partial t} H(tz)}{H(tz) - tH(z)},
 \end{aligned}$$

which establishes (4.5). □

The following result expresses the coefficients of $\frac{p_\lambda(\mathbf{x})}{z_\lambda}$ in the power sum expansion of $\omega W_n^{<}(\mathbf{x}, t)$ as a polynomial in t with positive integer coefficients.

Theorem 4.3. *For all $n \geq 1$,*

$$\omega W_n^{<}(\mathbf{x}, t) = \sum_{\lambda \vdash n} \frac{d}{dt} \left(t A_{\ell(\lambda)-1}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right) \frac{p_\lambda(\mathbf{x})}{z_\lambda}.$$

Proof. Let $c_\lambda^{<}(t)$ be the coefficient of $(z_\lambda)^{-1} p_\lambda(\mathbf{x})$ in $\omega W_n^{<}(\mathbf{x}, t)$. By (1.14) and (1.15), $\omega W_n^{<}(\mathbf{x}, t) = (t-1)^{-1} (\omega \tilde{W}_n(\mathbf{x}, t) - \omega W_n(\mathbf{x}, t))$. Hence from (4.2) and (1.6), we obtain,

$$\begin{aligned}
 c_\lambda^{<}(t) &= (t-1)^{-1} \left(A_{\ell(\lambda)-1}(t) \sum_{i=1}^{\ell(\lambda)} \lambda_i t^{\lambda_i} \prod_{j \in [\lambda(t)] \setminus \{i\}} [\lambda_j]_t - A_{\ell(\lambda)}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right) \\
 &= \frac{A_{\ell(\lambda)-1}(t)}{(t-1)^{\ell(\lambda)}} \sum_{i=1}^{\ell(\lambda)} \lambda_i t^{\lambda_i} \prod_{j \in [\lambda(t)] \setminus \{i\}} (t^{\lambda_j} - 1) - \frac{A_{\ell(\lambda)}(t)}{(t-1)^{\ell(\lambda)+1}} \prod_{i=1}^{\ell(\lambda)} (t^{\lambda_i} - 1) \\
 &= \frac{t A_{\ell(\lambda)-1}(t)}{(t-1)^{\ell(\lambda)}} \frac{d}{dt} \left(\prod_{i=1}^{\ell(\lambda)} (t^{\lambda_i} - 1) \right) - \frac{A_{\ell(\lambda)}(t)}{(t-1)^{\ell(\lambda)+1}} \prod_{i=1}^{\ell(\lambda)} (t^{\lambda_i} - 1).
 \end{aligned}$$

From (4.6), one can see that

$$\frac{d}{dt} \left(\frac{tA_{\ell(\lambda)-1}(t)}{(1-t)^{\ell(\lambda)}} \right) = \frac{A_{\ell(\lambda)}(t)}{(1-t)^{\ell(\lambda)+1}}.$$

Hence,

$$\begin{aligned} c_{\lambda}^{\leq}(t) &= \frac{tA_{\ell(\lambda)-1}(t)}{(t-1)^{\ell(\lambda)}} \frac{d}{dt} \left(\prod_{i=1}^{\ell(\lambda)} (t^{\lambda_i} - 1) \right) + \frac{d}{dt} \left(\frac{tA_{\ell(\lambda)-1}(t)}{(t-1)^{\ell(\lambda)}} \right) \prod_{i=1}^{\ell(\lambda)} (t^{\lambda_i} - 1) \\ &= \frac{d}{dt} \left(\frac{tA_{\ell(\lambda)-1}(t)}{(t-1)^{\ell(\lambda)}} \prod_{i=1}^{\ell(\lambda)} (t^{\lambda_i} - 1) \right) \\ &= \frac{d}{dt} \left(tA_{\ell(\lambda)-1}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right). \end{aligned} \quad \square$$

Corollary 4.4. For $\lambda \vdash n$, let $c_{\lambda}^{\leq}(t)$ be the coefficient of $z_{\lambda}^{-1}p_{\lambda}(\mathbf{x})$ in the power sum expansion of $\omega W_n^{\leq}(\mathbf{x}, t)$ and $c_{\lambda}^{\geq}(t)$ be the coefficient of $z_{\lambda}^{-1}p_{\lambda}(\mathbf{x})$ in the power sum expansion of $\omega W_n^{\geq}(\mathbf{x}, t)$. If $(a_0, a_1, \dots, a_{n-1})$ satisfies

$$(4.9) \quad tA_{\ell(\lambda)-1}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t = \sum_{i=0}^{n-1} a_i t^i$$

then

$$c_{\lambda}^{\leq}(t) = \sum_{i=0}^{n-2} (i+1)a_{i+1}t^i \quad \text{and} \quad c_{\lambda}^{\geq}(t) = \sum_{i=1}^{n-1} (n-i)a_{n-i}t^i.$$

Consequently for $\lambda = (n)$,

$$(4.10) \quad c_{(n)}^{\leq}(t) = \sum_{i=0}^{n-2} (i+1)t^i \quad \text{and} \quad c_{(n)}^{\geq}(t) = \sum_{i=1}^{n-1} (n-i)t^i.$$

Proof. We use the fact that $W_n^{\geq}(\mathbf{x}, t) = t^{n-1}W_n^{\leq}(\mathbf{x}, t^{-1})$. □

One can use Corollary 4.4 to expand the other Smirnov word enumerators in the power sum basis. For instance, one can recover the expansion given in (1.10), which we restate now.

Corollary 4.5 (Ellzey [12]). *For $\lambda \vdash n$, let $\tilde{c}_\lambda^\neq(t)$ be the coefficient of $z_\lambda^{-1}p_\lambda(\mathbf{x})$ in the power sum expansion of $\omega\tilde{W}_n^\neq(\mathbf{x}, t)$. Then*

$$\tilde{c}_\lambda^\neq(t) = \begin{cases} ntA_{\ell(\lambda)-1}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t & \text{if } \ell(\lambda) > 1 \\ nt[n-1]_t & \text{if } \ell(\lambda) = 1. \end{cases}$$

Proof. We have

$$\begin{aligned} \tilde{c}_\lambda^\neq(t) &= tc_\lambda^<(t) + c_\lambda^>(t) \\ &= \sum_{i=1}^{n-1} ia_it^i + \sum_{i=1}^{n-1} (n-i)a_{n-i}t^i, \end{aligned}$$

where the a_i are as in (4.9).

Now let $\ell(\lambda) > 1$. We claim that $a_i = a_{n-i}$ for all $i \in [n-1]$. Indeed, it is well known that the Eulerian polynomials are palindromic and unimodal. Clearly the same is true for each $[\lambda_i]_t$. Hence $\sum_{i=0}^{n-1} a_it^i$ is the product of palindromic, unimodal polynomials, which implies by [32, Proposition B.1] that it is palindromic (and unimodal). Note that $a_0 = 0$ when $\ell(\lambda) > 1$ and $a_1, a_{n-1} \neq 0$. Hence the claim holds. It follows that

$$\begin{aligned} \tilde{c}_\lambda^\neq(t) &= \sum_{i=1}^{n-1} ia_it^i + \sum_{i=1}^{n-1} (n-i)a_it^i \\ &= \sum_{i=1}^{n-1} na_it^i \\ &= ntA_{\ell(\lambda)-1}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t. \end{aligned}$$

The case $\ell(\lambda) = 1$ follows immediately from (4.10). □

For the Smirnov word enumerator $W_n^\neq(\mathbf{x}, t)$ and the chromatic quasisymmetric function $X_{C_n}(\mathbf{x}, t)$, the formulas for the expansion coefficients in the power sum basis that follow from Corollary 4.4 do not seem to reduce to simple formulas except when $\lambda = (n)$. We have the following result in this case.

Corollary 4.6. *The coefficient of $n^{-1}p_n(\mathbf{x})$ in the power sum expansion of $\omega W_n^\neq(\mathbf{x}, t)$ is $[n]_t + nt[n-2]_t$ and in the power sum expansion of $\omega X_{C_n}(\mathbf{x}, t)$ is $[2]_t[n]_t + nt^2[n-3]_t$.*

Remark 4.7. Corollary 4.6 also follows from Corollaries 2.4 and 3.1 since the coefficient of h_n in the h -expansion of a symmetric function equals the coefficient of $n^{-1}p_n$ in the power sum expansion.

5. Expansion in the fundamental quasisymmetric functions

In [32], Shareshian and the second named author derive, for all labeled incomparability graphs, an expansion of the chromatic quasisymmetric function in Gessel’s basis of fundamental quasisymmetric functions, and the first named author does the same for all directed graphs in [12]. The expansion formula in [32, Theorem 3.1] applied to $\omega X_{P_n}(\mathbf{x}, t) = \omega W_n(\mathbf{x}, t)$ is given in (5.2) below. (A different expansion formula is obtained by applying the formula in [12].) Here we give analogous expansions for $\omega W_n^{<}(\mathbf{x}, t)$, $\omega W_n^{>}(\mathbf{x}, t)$, and $\omega \tilde{W}_n(\mathbf{x}, t)$. These expansions immediately yield expansion formulas for the chromatic quasisymmetric functions $X_{\vec{C}_n}(\mathbf{x}, t)$ and $X_{C_n}(\mathbf{x}, t)$, which are different from the ones obtained by applying the formula in [12].

For $n \geq 1$ and $S \subseteq [n - 1]$, let $D(S)$ be the set of all functions $f : [n] \rightarrow \mathbb{P}$ such that

- $f(i) \geq f(i + 1)$ for all $i \in [n - 1]$, and
- $f(i) > f(i + 1)$ for all $i \in S$.

The *fundamental quasisymmetric function* associated with $S \subseteq [n]$ is defined as⁵

$$F_{n,S}(\mathbf{x}) := \sum_{f \in D(S)} x_f,$$

where $x_f := x_{f(1)}x_{f(2)} \cdots x_{f(n)}$. In fact, the set $\{F_{n,S} : S \subseteq [n - 1]\}$ is a basis for the vector space of homogeneous quasisymmetric functions of degree n ; see [37, Proposition 7.19.1].

Let ω be the linear involution on the ring of quasisymmetric functions defined on the basis of fundamental quasisymmetric functions by

$$(5.1) \quad \omega F_{n,S} := F_{n,[n-1] \setminus S}.$$

It can be shown that ω is a ring automorphism; see [37, Exercise 7.94 a]. Since ω takes $h_n = F_{n,\emptyset}$ to $e_n = F_{n,[n-1]}$, the involution ω restricts to the usual involution ω on the ring of symmetric functions.

⁵This is nonstandard notation for Gessel’s fundamental quasisymmetric function. Our $F_{n,S}$ is equal to $L_{\alpha(S)}$ in [37], where $\alpha(S)$ is the reverse of the composition associated with S .

For $\sigma \in \mathfrak{S}_n$ and $k \in [n]$, define

$$\begin{aligned} \text{DES}(\sigma) &:= \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}, \\ \text{DES}_{\geq k}(\sigma) &:= \{i \in [n-1] : \sigma(i) - \sigma(i+1) \geq k\}, \\ \text{ASC}_{\geq k}(\sigma) &:= \{i \in [n-1] : \sigma(i+1) - \sigma(i) \geq k\}. \end{aligned}$$

Note that $\text{DES}(\sigma) = \text{DES}_{\geq 1}(\sigma)$ and $\text{des}(\sigma) = |\text{DES}(\sigma)|$.

The expansion formula in [32, Theorem 3.1] applied to $X_{P_n}(\mathbf{x}, t) = W_n(\mathbf{x}, t)$ yields

$$(5.2) \quad \omega W_n(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} F_{n, \text{DES}_{\geq 2}(\sigma^{-1})}(\mathbf{x}).$$

Now we give analogous expansions.

Theorem 5.1. *For all $n \geq 1$,*

$$(5.3) \quad \omega W_n^{<}(\mathbf{x}, t) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) < \sigma(n)}} t^{\text{des}(\sigma)} F_{n, \text{DES}_{\geq 2}(\sigma^{-1})}(\mathbf{x})$$

$$(5.4) \quad \omega W_n^{>}(\mathbf{x}, t) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) > \sigma(n)}} t^{\text{des}(\sigma)} F_{n, \text{ASC}_{\geq 2}(\sigma^{-1})}(\mathbf{x}).$$

Proof of (5.3). The first part of the proof is similar to that of [32, Theorem 3.1], which is based on the $t = 1$ case in [8, Corollary 2], and to that of [12, Theorem 3.1]. The second part diverges somewhat from these proofs.

Part 1 An acyclic orientation of a graph $G = (V, E)$ is an assignment of a direction to each $e \in E$ that produces no directed cycles. An acyclic orientation yields an acyclic directed graph on V whose edges are of the form (u, v) where $\{u, v\} \in E$. Clearly, to avoid cycles, only one of (u, v) and (v, u) can be an edge of the directed graph.

Given an acyclic orientation \bar{a} of the labeled path P_n , let $E(\bar{a})$ be the set of directed edges of P_n under the orientation \bar{a} . Let $AO_n^{>}$ be the set of all acyclic orientations \bar{a} of P_n except for the one for which $E(\bar{a}) = \{(i, i+1) : i \in [n-1]\}$.

For each $\bar{a} \in AO_n^{>}$, let $W_{\bar{a}}$ be the set of Smirnov words $w = w_1 w_2 \cdots w_n$ such that the following hold:

- $w_1 > w_n$,
- $w_i < w_{i+1}$ if $(i, i+1) \in E(\bar{a})$ and $i \in [n-1]$,

- $w_i > w_{i+1}$ if $(i + 1, i) \in E(\bar{a})$ and $i \in [n - 1]$.

For $\bar{a} \in AO_n^>$, let $\text{asc}(\bar{a})$ be the number of directed edges in $E(\bar{a})$ of the form $(i, i + 1)$ where $i \in [n - 1]$. For $w \in W_n$, let $\text{asc}(w) := |\{i \in [n - 1] : w_i < w_{i+1}\}|$. Clearly if $w \in W_{\bar{a}}$ then $\text{asc}(w) = \text{asc}(\bar{a})$. Also each $w \in W_n$ for which $w_1 > w_n$ belongs to a unique $W_{\bar{a}}$, where $\bar{a} \in AO_n^>$. Thus by the involution on W_n that reverses Smirnov words, i.e. sends $w_1 w_2 \cdots w_n$ to $w_n w_{n-1} \cdots w_1$, we can see that

$$(5.5) \quad W_n^{<}(\mathbf{x}, t) = \sum_{\substack{w \in W_n \\ w_1 > w_n}} \mathbf{x}_w t^{\text{asc}(w)} = \sum_{\bar{a} \in AO_n^>} t^{\text{asc}(\bar{a})} \sum_{w \in W_{\bar{a}}} \mathbf{x}_w.$$

Now for each acyclic orientation $\bar{a} \in AO_n^>$, define a poset $P_{\bar{a}}$ on $[n]$ by letting $i <_{P_{\bar{a}}} j$ if $(i, j) \in E(\bar{a}) \cup \{(n, 1)\}$ and taking the transitive closure of this antisymmetric relation. Let us define a *labeling* of $P_{\bar{a}}$ to be a bijection from $P_{\bar{a}}$ to $[n]$, so a labeling is just a permutation in \mathfrak{S}_n . A labeling ρ is said to be *decreasing* if $\rho(i) > \rho(j)$ for all $i <_{P_{\bar{a}}} j$. A *linear extension* of $P_{\bar{a}}$ under a labeling ρ is a listing of its labeled elements $\rho(p_1), \rho(p_2), \dots, \rho(p_n)$ such that if $p_i <_{P_{\bar{a}}} p_j$ then $i < j$. We view this list as a permutation in \mathfrak{S}_n written in one line notation. Let $L(P_{\bar{a}}, \rho)$ be the set of linear extensions of $P_{\bar{a}}$ under the labeling ρ .

Now fix a decreasing labeling $\rho_{\bar{a}}$ of $P_{\bar{a}}$ for each $\bar{a} \in AO_n^>$. For any subset $S \subseteq [n - 1]$, define $n - S = \{i \mid n - i \in S\}$. Then by the theory of P -partitions [37, Corollary 7.19.5], we have that

$$(5.6) \quad \sum_{w \in W_{\bar{a}}} \mathbf{x}_w = \sum_{\sigma \in L(P_{\bar{a}}, \rho_{\bar{a}})} F_{n, n - \text{DES}(\sigma)},$$

where $\text{DES}(\sigma)$ is the usual descent set of a permutation, i.e. $\text{DES}(\sigma) = \{i \in [n - 1] : \sigma(i) > \sigma(i + 1)\}$.

Let $e : P_{\bar{a}} \rightarrow [n]$ be the identity labeling of $P_{\bar{a}}$. Hence $L(P_{\bar{a}}, e)$ is the set of linear extensions of $P_{\bar{a}}$ with its original labeling. Note that $\sigma \in L(P_{\bar{a}}, e)$ if and only if $\rho_{\bar{a}}\sigma \in L(P_{\bar{a}}, \rho_{\bar{a}})$, where $\rho_{\bar{a}}\sigma$ denotes the product of $\rho_{\bar{a}}$ and σ in \mathfrak{S}_n . Hence from (5.6), we have

$$(5.7) \quad \sum_{w \in W_{\bar{a}}} \mathbf{x}_w = \sum_{\sigma \in L(P_{\bar{a}}, e)} F_{n, n - \text{DES}(\rho_{\bar{a}}\sigma)}.$$

Note that if $\bar{a} \in AO_n^>$ and $\sigma \in L(P_{\bar{a}}, e)$ then $\sigma^{-1}(1) > \sigma^{-1}(n)$. Conversely, every permutation $\sigma \in \mathfrak{S}_n$ with $\sigma^{-1}(1) > \sigma^{-1}(n)$ is a linear extension in $L(P_{\bar{a}}, e)$ for a unique $\bar{a} \in AO_n^>$. Let $\bar{a}(\sigma)$ denote this unique element

of $AO_n^>$ associated with σ . Now combining (5.7) with (5.5) yields

$$\begin{aligned} W_n^<(\mathbf{x}, t) &= \sum_{\bar{a} \in AO_n^>} t^{\text{asc}(\bar{a})} \sum_{\sigma \in L(P_{\bar{a}}, e)} F_{n, n-\text{DES}(\rho_{\bar{a}}\sigma)} \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^{-1}(1) > \sigma^{-1}(n)}} t^{\text{asc}(\bar{a}(\sigma))} F_{n, n-\text{DES}(\rho_{\bar{a}(\sigma)}\sigma)}, \end{aligned}$$

where recall $\rho_{\bar{a}(\sigma)}$ is a decreasing labeling of $P_{\bar{a}(\sigma)}$. Note that $\text{asc}(\bar{a}(\sigma)) = \text{des}((\sigma^R)^{-1})$, where σ^R is the reverse of σ . Hence

$$(5.8) \quad W_n^<(\mathbf{x}, t) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^{-1}(1) > \sigma^{-1}(n)}} t^{\text{des}((\sigma^R)^{-1})} F_{n, n-\text{DES}(\rho_{\bar{a}(\sigma)}\sigma)}.$$

Part 2 As in the proof of [32, Theorem 3.1], our next step is to construct a particular decreasing labeling $\tilde{\rho}_{\bar{a}}$ of $P_{\bar{a}}$ for each $\bar{a} \in AO_n^>$. However the construction in the proof of [32, Theorem 3.1] does not work in this case because $E(\bar{a}) \cup \{(n, 1)\}$ is not the edge set of an acyclic orientation of an incomparability graph. The construction used here is also quite different from that of [12, Theorem 3.1].

Let P be the poset whose covering relations are those of $P_{\bar{a}}$ minus the relation $n <_{P_{\bar{a}}} 1$. One can see that the covering relations of P are given by one of the following:

1. $1 >_P 2 >_P \cdots >_P n$
2. $1 >_P 2 >_P \cdots >_P i_1 <_P i_1 + 1 <_P \cdots <_P i_2 >_P i_2 + 1 >_P \cdots$
3. $1 <_P 2 <_P \cdots <_P i_1 >_P i_1 + 1 >_P \cdots >_P i_2 <_P i_2 + 1 <_P \cdots$

where $1 < i_1 < i_2 < \cdots < i_k < n$ and $k \geq 1$. In the first case let

$$\tilde{\rho}_{\bar{a}}(1) = 1, \tilde{\rho}_{\bar{a}}(2) = 2, \dots, \tilde{\rho}_{\bar{a}}(n) = n.$$

In the second case let

$$\begin{aligned} \tilde{\rho}_{\bar{a}}(1) &= 1, \tilde{\rho}_{\bar{a}}(2) = 2, \dots, \tilde{\rho}_{\bar{a}}(i_1 - 1) = i_1 - 1, \\ \tilde{\rho}_{\bar{a}}(i_2) &= i_1, \tilde{\rho}_{\bar{a}}(i_2 - 1) = i_1 + 1 \dots, \tilde{\rho}_{\bar{a}}(i_1) = i_2, \\ \tilde{\rho}_{\bar{a}}(i_2 + 1) &= i_2 + 1, \tilde{\rho}_{\bar{a}}(i_2 + 2) = i_2 + 2, \dots, \tilde{\rho}_{\bar{a}}(i_3 - 1) = i_3 - 1, \\ &\dots \end{aligned}$$

In the third case let

$$\tilde{\rho}_{\bar{a}}(i_1) = 1, \tilde{\rho}_{\bar{a}}(i_1 - 1) = 2, \dots, \tilde{\rho}_{\bar{a}}(1) = i_1,$$

$$\begin{aligned} \tilde{\rho}_{\bar{a}}(i_1 + 1) &= i_1 + 1, \tilde{\rho}_{\bar{a}}(i_1 + 2) = i_1 + 2, \dots, \tilde{\rho}_{\bar{a}}(i_2 - 1) = i_2 - 1, \\ \tilde{\rho}_{\bar{a}}(i_3) &= i_2, \tilde{\rho}_{\bar{a}}(i_3 - 1) = i_2 + 1, \dots, \tilde{\rho}_{\bar{a}}(i_2) = i_3, \\ &\dots \end{aligned}$$

One can see that in all three cases $\tilde{\rho}_{\bar{a}}$ is a decreasing labeling of $P_{\bar{a}}$. Also it is easy to see that in all three cases the following claim holds.

Claim. For each $\bar{a} \in AO_n^>$, if x and y are incomparable in $P_{\bar{a}}$ and $x < y$ then $\tilde{\rho}_{\bar{a}}(x) < \tilde{\rho}_{\bar{a}}(y)$.

We use the claim to show that

$$(5.9) \quad \text{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma) = [n - 1] \setminus \text{ASC}_{\geq 2}(\sigma),$$

for all $\sigma \in \mathfrak{S}_n$. If $i \in \text{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma)$ then $\tilde{\rho}_{\bar{a}(\sigma)}\sigma(i) > \tilde{\rho}_{\bar{a}(\sigma)}\sigma(i + 1)$. It thus follows from the claim that if $\sigma(i)$ and $\sigma(i + 1)$ are incomparable in $P_{\bar{a}(\sigma)}$ then $\sigma(i) > \sigma(i + 1)$, which implies $i \notin \text{ASC}_{\geq 2}(\sigma)$. On the other hand if $\sigma(i)$ and $\sigma(i + 1)$ are comparable in $P_{\bar{a}(\sigma)}$ then $\sigma(i + 1)$ covers $\sigma(i)$ since $\sigma \in L(P_{\bar{a}(\sigma)}, e)$. This implies that one of the following holds

- $\sigma(i + 1) = \sigma(i) + 1$,
- $\sigma(i + 1) = \sigma(i) - 1$,
- $\sigma(i + 1) = 1$ and $\sigma(i) = n$.

In all cases, $i \notin \text{ASC}_{\geq 2}(\sigma)$. Thus

$$\text{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma) \subseteq [n - 1] \setminus \text{ASC}_{\geq 2}(\sigma).$$

Conversely, if $i \notin \text{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma)$ then $\tilde{\rho}_{\bar{a}(\sigma)}\sigma(i) < \tilde{\rho}_{\bar{a}(\sigma)}\sigma(i + 1)$. It thus follows from the claim that if $\sigma(i)$ and $\sigma(i + 1)$ are incomparable in $P_{\bar{a}(\sigma)}$ then $\sigma(i) < \sigma(i + 1)$. Since j and $j + 1$ are comparable in $P_{\bar{a}(\sigma)}$ for all $j \in [n - 1]$, we have $\sigma(i + 1) - \sigma(i) \geq 2$. Thus $i \in \text{ASC}_{\geq 2}(\sigma)$. On the other hand, if $\sigma(i)$ and $\sigma(i + 1)$ are comparable in $P_{\bar{a}(\sigma)}$ then $\sigma(i) <_{P_{\bar{a}(\sigma)}} \sigma(i + 1)$ since $\sigma \in L(P_{\bar{a}(\sigma)}, e)$. But since $\tilde{\rho}_{\bar{a}(\sigma)}$ is a decreasing labeling, $\tilde{\rho}_{\bar{a}(\sigma)}\sigma(i) > \tilde{\rho}_{\bar{a}(\sigma)}\sigma(i + 1)$, which contradicts our assumption that $i \notin \text{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma)$. Hence this case is impossible. We have shown

$$\text{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma) \supseteq [n - 1] \setminus \text{ASC}_{\geq 2}(\sigma),$$

which completes the proof of (5.9).

Hence by (5.1) and (5.9), equation (5.8) becomes

$$\begin{aligned} \omega W_n^<(\mathbf{x}, t) &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^{-1}(1) > \sigma^{-1}(n)}} t^{\text{des}((\sigma^R)^{-1})} F_{n, n-\text{ASC}_{\geq 2}(\sigma)} \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^{-1}(1) < \sigma^{-1}(n)}} t^{\text{des}(\sigma^{-1})} F_{n, \text{DES}_{\geq 2}(\sigma)}. \end{aligned} \quad \square$$

Remark 5.2. There is an alternative proof of Theorem 5.1 involving standardization, which will be given in a forthcoming paper.

Proof of (5.4). A similar proof can be given here. One can also use (5.3) to prove this. Indeed, by the involution on W_n that reverses Smirnov words, we obtain

$$W_n^>(\mathbf{x}, t) = t^{n-1} W_n^<(\mathbf{x}, t^{-1}).$$

By the involution on \mathfrak{S}_n that reverses permutations,

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) > \sigma(n)}} t^{\text{des}(\sigma)} F_{n, \text{ASC}_{\geq 2}(\sigma^{-1})} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) < \sigma(n)}} t^{n-1-\text{des}(\sigma)} F_{n, \text{DES}_{\geq 2}(\sigma^{-1})}.$$

The result now follows from (5.3). □

By combining (5.2), (5.3), and (5.4), one gets fundamental quasisymmetric function expansions of the other Smirnov word enumerators $W_n^=(\mathbf{x}, t)$, $W_n^\neq(\mathbf{x}, t)$, $\tilde{W}_n(\mathbf{x}, t)$, $\tilde{W}_n^\neq(\mathbf{x}, t)$ and of the chromatic quasisymmetric function $X_{C_n}(\mathbf{x}, t)$. The resulting expansion for $\tilde{W}_n(\mathbf{x}, t)$ has a particularly nice form.

Corollary 5.3. *For all $n \geq 1$,*

$$(5.10) \quad \omega \tilde{W}_n(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{cdes}(\sigma)} F_{n, \text{DES}_{\geq 2}(\sigma^{-1})}(\mathbf{x}).$$

Proof. We use the fact that $\tilde{W}_n(\mathbf{x}, t) = t W_n^<(\mathbf{x}, t) + (W_n(\mathbf{x}, t) - W_n^<(\mathbf{x}, t))$. By (5.2) and (5.3),

$$\omega W_n(\mathbf{x}, t) - \omega W_n^<(\mathbf{x}, t) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) > \sigma(n)}} t^{\text{des}(\sigma)} F_{n, \text{DES}_{\geq 2}(\sigma^{-1})}.$$

It follows from this and (5.3) that

$$\begin{aligned} \omega \tilde{W}_n(\mathbf{x}, t) &= \sum_{\substack{\sigma \in \tilde{\mathfrak{S}}_n \\ \sigma(1) < \sigma(n)}} t^{\text{des}(\sigma)+1} F_{n, \text{DES}_{\geq 2}(\sigma^{-1})} + \sum_{\substack{\sigma \in \tilde{\mathfrak{S}}_n \\ \sigma(1) > \sigma(n)}} t^{\text{des}(\sigma)} F_{n, \text{DES}_{\geq 2}(\sigma^{-1})} \\ &= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{cdes}(\sigma)} F_{n, \text{DES}_{\geq 2}(\sigma^{-1})}. \end{aligned} \quad \square$$

There are various ways to specialize expansions in the fundamental quasisymmetric functions to obtain enumerative results. One way is by setting $x_i = 1$ if $i \in [m]$ and $x_i = 0$ otherwise, in a formal power series $f(\mathbf{x})$. We denote this specialization by $f(1^m)$. (Another way is discussed in the next section.) It is not difficult to show that (see [37, Section 7.19]),

$$F_{n,S}(1^m) = \binom{m+n-1-|S|}{n},$$

for all $S \subseteq [n-1]$. It is clear that

$$W_n(1^m, t) = \sum_{w \in W_n \cap [m]^n} t^{\text{des}(w)}.$$

Hence by (5.2) and the fact that $\omega F_{n,S} = F_{n,[n-1] \setminus S}$,

$$(5.11) \quad \sum_{w \in W_n \cap [m]^n} t^{\text{des}(w)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} \binom{m + |\text{DES}_{\geq 2}(\sigma^{-1})|}{n},$$

for all $m, n \in \mathbb{P}$. Analogous formulas can be obtained by applying the same specialization to the expansions (5.3), (5.4), and (5.10). The expansions (5.3) and (5.10) yield the following result.

Corollary 5.4. *For all $m, n \geq 1$,*

$$(5.12) \quad \sum_{\substack{w \in W_n \cap [m]^n \\ w_1 < w_n}} t^{\text{des}(w)} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) < \sigma(n)}} t^{\text{des}(\sigma)} \binom{m + |\text{DES}_{\geq 2}(\sigma^{-1})|}{n}$$

and

$$(5.13) \quad \sum_{w \in W_n \cap [m]^n} t^{\text{cdes}(w)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{cdes}(\sigma)} \binom{m + |\text{DES}_{\geq 2}(\sigma^{-1})|}{n}.$$

Remark 5.5. In [23, 24], LoBue Tiefenbruck and Remmel study the distribution of a pair of interesting statistics on Smirnov words in $[m]^n$ different from des and cdes. They use the fact that their statistics are preserved by a contraction map from unconstrained words to Smirnov words to transfer their results from Smirnov words to unconstrained words. Since des and cdes are also preserved by the contraction map, we can also transfer our results to unconstrained words.

Remark 5.6. We now describe a proof of (1.4) that is different from the proof in [29] discussed in the introduction. Theorem 3.1 of [32] gives a fundamental quasisymmetric function expansion of the chromatic quasisymmetric function $X_G(\mathbf{x}, t)$ when G is an incomparability graph. (This reduces to (5.2) when G is the path P_n .) In [2], Athanasiadis proves that the fundamental quasisymmetric function expansion implies the conjectured formula (7.14) of [32], which gives a power sum symmetric function expansion of $X_G(\mathbf{x}, t)$ when G is a natural unit interval graph. It is shown in [32, Proof of Proposition 7.9] that when $G = P_n$, the power sum symmetric function expansion reduces to (7.15) of [32], which is

$$\omega X_{P_n}(\mathbf{x}, t) = \sum_{\lambda \vdash n} \left(A_{\ell(\lambda)}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right) \frac{p_\lambda(\mathbf{x})}{z_\lambda}.$$

Hence since $W_n(\mathbf{x}, t) = X_{P_n}(\mathbf{x}, t)$, Stembridge's formula (4.1) implies (1.5), which is equivalent to (1.4).

6. Variations of q -Eulerian polynomials

Recall that the Eulerian polynomials $A_n(t)$ have two well-known combinatorial interpretations, which are given by

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)},$$

where $\text{des}(\sigma) = |\{i \in [n-1] : \sigma(i) > \sigma(i+1)\}|$ and $\text{exc}(\sigma) = |\{i \in [n-1] : \sigma(i) > i\}|$. Also recall that Euler's exponential generating function for the Eulerian polynomials is given by

$$1 + \sum_{n \geq 1} A_n(t) \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{tz} - te^z}.$$

In [29] and [32], Shareshian and the second named author obtained combinatorial interpretations of the q -Eulerian polynomials $A_n(q, t)$ that satisfy the q -exponential generating function formula

$$(6.1) \quad 1 + \sum_{n \geq 1} A_n(q, t) \frac{z^n}{[n]_q!} = \frac{(1-t) \exp_q(z)}{\exp_q(tz) - t \exp_q(z)} = 1 + \frac{(1-t) \sum_{i \geq 2} [i]_t \frac{z^i}{[i]_q!}}{\exp_q(tz) - t \exp_q(z)},$$

where

$$[n]_q! := [n]_q [n-1]_q \dots [1]_q \quad \text{and} \quad \exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}.$$

The interpretation in [29] is given by

$$(6.2) \quad A_n(q, t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)}$$

and the interpretation in [32] is given by

$$(6.3) \quad A_n(q, t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq 2}(\sigma^{-1})} t^{\text{des}(\sigma)}$$

where

$$\text{maj}(\sigma) = \sum_{\substack{i \in [n-1] \\ \sigma(i+1) > \sigma(i)}} i \quad \text{and} \quad \text{maj}_{\geq k}(\sigma) := \sum_{\substack{i \in [n-1] \\ \sigma(i+1) - \sigma(i) \geq k}} i,$$

for all $k \geq 1$. (The permutation statistic $\text{maj}_{\geq k} + \text{inv}_{<k}$, where $\text{inv}_{<k}(\sigma) = |\{(i, j) : i < j, 0 < \sigma(i) - \sigma(j) < k\}|$, was introduced by Rawlings in [26] who showed it was equidistributed with maj .) Both q -analogs of $A_n(t)$ were obtained by expanding $\omega W_n(\mathbf{x}, t)$ in the fundamental quasisymmetric functions and then taking the stable principal specialization. A formulation of the expansion obtained in [29] yields (6.2), while the formulation (5.2) obtained in [32] yields (6.3); see [32, Proof of Theorem 9.7]. From this it follows that the two q -analogs are equal. (A subsequent bijective proof was obtained by Bigeni in [6].)

In this section, we use results of the previous sections to obtain analogs of (6.1) for variations of the interpretation of $A_n(q, t)$ given by (6.3). The

variations are defined by

$$A_n^<(q, t) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) < \sigma(n)}} q^{\text{maj}_{\geq 2}(\sigma^{-1})} t^{\text{des}(\sigma)}$$

and

$$\tilde{A}_n(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq 2}(\sigma^{-1})} t^{\text{cdes}(\sigma)}.$$

We also obtain nice formulas for $A_n^<(q, t)$ and $\tilde{A}_n(q, t)$ evaluated at n th roots of unity.

The *stable principal specialization* $\text{ps}(G(\mathbf{x}))$ of a quasisymmetric function $G(\mathbf{x})$ is obtained from $G(\mathbf{x})$ by substituting q^{i-1} for x_i for all $i \geq 1$. By [16, Lemma 5.2],

$$\text{ps}(F_{n,S}(x)) = \frac{\sum_{i \in S} q^i}{(1-q)(1-q^2) \cdots (1-q^n)}$$

for all $S \subseteq [n-1]$. Hence by (5.2), (5.3), and (5.10), respectively,

$$(6.4) \quad \text{ps}(\omega W_n(\mathbf{x}, t)) = \frac{A_n(q, t)}{(1-q)(1-q^2) \cdots (1-q^n)}$$

$$(6.5) \quad \text{ps}(\omega W_n^<(\mathbf{x}, t)) = \frac{A_n^<(q, t)}{(1-q)(1-q^2) \cdots (1-q^n)}$$

$$(6.6) \quad \text{ps}(\omega \tilde{W}_n(\mathbf{x}, t)) = \frac{\tilde{A}_n(q, t)}{(1-q)(1-q^2) \cdots (1-q^n)}.$$

In [29, 32], first ω is applied to both sides of (1.5), then the stable principal specialization is taken using (6.4), and finally z is replaced by $(1-q)z$ resulting in (6.1). By doing the same to (2.13) and (2.16), using (6.5) and (6.6), respectively, we obtain the following result.

Theorem 6.1. *We have*

$$\sum_{n \geq 1} A_n^<(q, t) \frac{z^n}{[n]_q!} = \frac{(1-t) \frac{\partial}{\partial t} \sum_{i \geq 2} [i]_t \frac{z^i}{[i]_q!}}{\exp_q(tz) - t \exp_q(z)}$$

$$\sum_{n \geq 1} \tilde{A}_n(q, t) \frac{z^n}{[n]_q!} = \frac{(1-t) \frac{\partial}{\partial t} \exp_q(tz)}{\exp_q(tz) - t \exp_q(z)}.$$

In [27, Corollary 6.2], Sagan, Shareshian and the second named author show that for every n th root of unity ξ , the coefficients of the polynomial

$A_n(\xi, t)$ are positive integers. More precisely, they show that if $k|n$ and ξ_k is any primitive k th root of unity then

$$(6.7) \quad A_n(\xi_k, t) = A_{\frac{n}{k}}(t) [k]_t^{\frac{n}{k}}.$$

Consequently, $A_n(\xi_k, t)$ is a palindromic, unimodal polynomial in $\mathbb{N}[t]$. Here we prove analogous results for other Smirnov word enumerators.

A key tool in the proof of (6.7) is the following result, which is implicit in [9] and stated explicitly in [27].

Lemma 6.2 (see [27, Proposition 3.1]). *Let R be a commutative ring. Suppose $u(q) \in R[q]$ and there exists a homogeneous symmetric function $U(\mathbf{x})$ of degree n with coefficients in R such that*

$$u(q) = (1 - q)(1 - q^2) \dots (1 - q^n) \text{ps}(U(\mathbf{x})).$$

If $k|n$ then $u(\xi_k)$ is the coefficient of $z^{-1}_{\binom{n}{k}} p_{\binom{n}{k}}$ in the expansion of $U(\mathbf{x})$ in the power sum basis.

In [27], (6.7) is proved by setting $R = \mathbb{Q}[t]$ and $U(\mathbf{x}) = \omega W_n(\mathbf{x}, t)$ in Lemma 6.2. By (6.4), $u(q) = A_n(q, t)$. Hence it follows from Lemma 6.2 that $A_n(\xi_k, t)$ equals the coefficient of $z^{-1}_{\binom{n}{k}} p_{\binom{n}{k}}$ in the expansion of $\omega W_n(\mathbf{x}, t)$ in the power sum basis, which by (1.6) equals $A_{\frac{n}{k}}(t) [k]_t^{\frac{n}{k}}$. We use a similar argument to obtain the following result. Indeed, to prove (6.8) below, we set $U(\mathbf{x}) = \omega W_n^{<}(\mathbf{x}, t)$ and use (6.5) and Theorem 4.3. To prove (6.9) below, we set $U(\mathbf{x}) = \omega \tilde{W}_n(\mathbf{x}, t)$ and use (6.6) and Theorem 4.1.

Theorem 6.3. *Let $n \geq 2$ and $k|n$. If ξ_k is any primitive k th root of unity then*

$$(6.8) \quad A_n^{<}(\xi_k, t) = \frac{d}{dt} (t A_{\frac{n}{k}-1}(t) [k]_t^{\frac{n}{k}})$$

and

$$(6.9) \quad \tilde{A}_n(\xi_k, t) = nt^k A_{\frac{n}{k}-1}(t) [k]_t^{\frac{n}{k}-1}.$$

Consequently, $A_n^{<}(\xi_k, t), \tilde{A}_n(\xi_k, t) \in \mathbb{N}[t]$ and $\tilde{A}_n(\xi_k, t)$ is palindromic and unimodal.

Remark 6.4. Equation (6.7) arises in connection with the cyclic sieving phenomenon in [27]. It would be interesting to find cyclic sieving interpretations of (6.8) and (6.9).

Corollary 6.5. *For all $n \geq 2$,*

$$(6.10) \quad A_n^<(1, t) = \frac{d}{dt}(tA_{n-1}(t))$$

and

$$(6.11) \quad \tilde{A}_n(1, t) = ntA_{n-1}(t).$$

Equations (6.10) and (6.11) have elementary bijective proofs. Indeed, for each $\sigma \in \mathfrak{S}_n$, such that $\sigma(n) = n$, let \mathcal{C}_σ be the set of circular rearrangements of σ . Clearly, $|\mathcal{C}_\sigma| = n$ and for each $\tau \in \mathcal{C}_\sigma$, we have $\text{cdes}(\tau) = \text{des}(\sigma) + 1$. Hence,

$$\begin{aligned} \tilde{A}_n(1, t) &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n)=n}} \sum_{\tau \in \mathcal{C}_\sigma} t^{\text{cdes}(\tau)} \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n)=n}} nt^{\text{des}(\sigma)+1} \\ &= n \sum_{\sigma \in \mathfrak{S}_{n-1}} t^{\text{des}(\sigma)+1} \\ &= ntA_{n-1}(t). \end{aligned}$$

Now for each $\sigma \in \mathfrak{S}_n$, such that $\sigma(n) = n$, let

$$\mathcal{C}_\sigma^< := \{\tau \in \mathcal{C}_\sigma : \tau(1) < \tau(n)\}.$$

Clearly, $|\mathcal{C}_\sigma^<| = \text{des}(\sigma) + 1$ and for each $\tau \in \mathcal{C}_\sigma^<$, we have $\text{des}(\tau) = \text{des}(\sigma)$. Hence

$$\begin{aligned} A_n^<(1, t) &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n)=n}} \sum_{\tau \in \mathcal{C}_\sigma^<} t^{\text{des}(\tau)} \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n)=n}} (\text{des}(\sigma) + 1)t^{\text{des}(\sigma)} \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}} (\text{des}(\sigma) + 1)t^{\text{des}(\sigma)} \\ &= \frac{d}{dt}(tA_{n-1}(t)). \end{aligned}$$

By combining (6.7) with (6.8) and with (6.9), we obtain the following generalization of the previous corollary.

Corollary 6.6. *Let $n \geq 2$ and $k|n$. If ξ_k is any primitive k th root of unity then*

$$A_n^<(\xi_k, t) = \frac{d}{dt}(t[k]_t A_{n-k}(\xi_k, t))$$

and

$$\tilde{A}_n(\xi_k, t) = nt^k A_{n-k}(\xi_k, t).$$

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