A ratio of alternants formula for loop Schur functions

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Lam and Pylyavskyy introduced loop symmetric functions as a generalization of symmetric functions. They defined loop Schur functions as generating functions over semistandard tableaux with respect to a "colored weight," and they proved a Jacobi–Trudi-style determinantal formula for these generating functions. We prove that loop Schur functions can be expressed as a ratio of "loop alternants," extending the analogy with Schur functions. As an application, we give a new proof of the loop version of the Murnaghan–Nakayama rule.

KEYWORDS AND PHRASES: Loop symmetric function, Schur function, birational R-matrix, bialternant formula, Murnaghan–Nakayama rule.

1. Introduction

Motivated by their study of total positivity in loop groups, Lam and Pylyavskyy introduced a generalization of symmetric polynomials, which they called loop symmetric functions [15]. These are polynomials in m sets of n variables, which are invariant under a certain birational action of the symmetric group S_m. When $n = 1$, the birational action reduces to the permutation of the variables $x_1, \ldots, x_m$, and loop symmetric functions are the symmetric polynomials in m variables.

The ring of loop symmetric functions is generated by polynomials $e_k^{(r)}$ (resp., $h_k^{(r)}$), which are "loop analogues" of the elementary (resp., complete homogeneous) symmetric polynomials. Lam and Pylyavskyy showed that certain determinants in the $e_k^{(r)}$ (resp., $h_k^{(r)}$) are given by summing over semistandard Young tableaux of a fixed shape λ, with the weight refined by "coloring" the boxes of the Young diagram with elements of $\mathbb{Z}/n\mathbb{Z}$. They defined loop Schur functions to be these determinants. Due to the importance

of Schur polynomials in many areas of mathematics, it is tempting to look for loop generalizations of their many remarkable properties. For example, Ross and Zong [25] proved a generalization of the Gopakumar–Mariño–Vafa formula [20, 23] in Gromov–Witten/Donaldson–Thomas theory, in which the Schur polynomials in the original formula are replaced with loop Schur functions.

Schur polynomials were originally defined as the ratio of two alternating polynomials, or alternants. The main result of this article is a generalization of the ratio of alternants formula for Schur polynomials to the loop setting (Theorem 3.1). This formula was stated without proof in [13]. Because the generalized alternants are defined using the birational S_m action, they are rational functions, rather than polynomials as in the classical case; however, we derive an alternative ratio of alternants formula which expresses loop Schur functions as a ratio of two polynomials (Theorem 3.3). We use the ratio of alternants formula to deduce a loop generalization of the Murnaghan–Nakayama rule (Theorem 4.1). This result was also stated in [13], and was proved combinatorially by Ross [24].

Prior to the work of Lam and Pylyavskyy, the birational symmetric group action had been identified as a geometric R-matrix in the theory of geometric crystals [27, 12]. In other words, the action tropicalizes to a piecewise-linear formula for the combinatorial R-matrix for a tensor product of one-row Kirillov–Reshetikhin crystals of type $A_{n-1}^{(1)}$ [7]. This raises the possibility that any given function on a tensor product of one-row crystals which is invariant under the combinatorial R-matrix might be the tropicalization of a ratio of loop symmetric functions. Lam and Pylyavskyy [17] showed that the intrinsic energy [11, 9, 8], an important function in affine crystal theory which is closely related to Lascoux and Schützenberger's charge statistic [22], is in fact the tropicalization of a certain loop Schur function. This loop Schur function turns out to be related to our alternant formulas (see Remark 3.6). Additionally, tensor products of one-row crystals can be viewed as states in the (generalized) box-ball system, a well-studied cellular automaton that exhibits soliton behavior [26, 7]. Formulas for the scattering of a given state into solitons are conjecturally given by tropicalizations of a cylindric variant of loop Schur functions [18].

We note that the birational S_m action also arose in the context of the local Langlands program [2], and was studied in [4].

This paper is organized as follows. In Section 2, we review the basics of loop symmetric functions and the birational S_m action. Section 3 contains statements and proofs of the two ratio of alternants formulas (Theorems 3.1 and 3.3), and Section 4 discusses the loop Murnaghan–Nakayama rule (Theorem 4.1).

2. Loop symmetric functions

2.1. Loop elementary and homogeneous symmetric functions

Fix integers $m, n \geq 1$. For $i \in \{1, \ldots, m\}$ and $j \in \mathbb{Z}/n\mathbb{Z}$, let $x_i^{(j)}$ be an indeterminate, and let $\mathbb{Q}(x_i^{(j)})$ be the field of rational functions in these mn indeterminates. Let $\mathbf{x}_i = (x_i^{(1)}, \ldots, x_i^{(n)})$. We view the superscript as a "color."

Lam and Pylyavskyy [15] introduced *loop symmetric functions* as a subring of polynomials in the m sets of variables $\mathbf{x}_1, \ldots, \mathbf{x}_m$. We begin by recalling the definitions of two types of loop symmetric functions.

Definition 2.1. For $k \geq 1$ and $r \in \mathbb{Z}/n\mathbb{Z}$, define the *loop elementary symmetric function*

$$e_k^{(r)} = e_k^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{1 \leq i_1 < \ldots < i_k \leq m} x_{i_1}^{(r)} x_{i_2}^{(r+1)} \cdots x_{i_k}^{(r+k-1)},$$

and define the *loop homogeneous symmetric function*

$$h_k^{(r)} = h_k^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq m} x_{i_1}^{(r)} x_{i_2}^{(r-1)} \cdots x_{i_k}^{(r-k+1)}.$$

Also set $e_0^{(r)} = h_0^{(r)} = 1$, and $e_k^{(r)} = h_k^{(r)} = 0$ for $k < 0$.

Note that if we ignore colors (or take $n = 1$), then the loop elementary (resp., homogeneous) symmetric functions are simply the ordinary elementary (resp., homogeneous) symmetric polynomials in m variables.

Define the *ring of loop symmetric functions* (in m variables), denoted $\mathrm{LSym}_m(n)$, to be the subring of $\mathbb{Q}(x_i^{(j)})$ generated by the loop elementary symmetric functions. It follows from Theorem 2.4 below that each $h_k^{(r)}$ can be expressed as a polynomial in the loop elementary symmetric functions (and conversely), so the loop homogeneous symmetric functions also generate $\mathrm{LSym}_m(n)$.

2.2. Loop Schur functions

Given partitions λ and μ, write $\mu \subset \lambda$ if the Young diagram of μ is contained in that of λ. If $\mu \subset \lambda$, define λ/μ to be the *skew diagram* (or *skew shape*) obtained by removing the boxes of μ from the Young diagram of λ. A *semistandard tableau of shape* λ/μ is a filling of the skew diagram λ/μ with positive integers so that each row is weakly increasing, and each col-

umn is strictly increasing. For a box (i, j) in λ/μ, define its *content* to be $c(i, j) = i - j \pmod{n}$. For $r \in \mathbb{Z}/n\mathbb{Z}$ and a semistandard tableau T of shape λ/μ, define the *r-weight* of T by

$$\mathrm{wt}_r(T) = \prod_{(i,j) \in \lambda/\mu} x_{T(i,j)}^{(r+c(i,j))}.$$

Definition 2.2. For $r \in \mathbb{Z}/n\mathbb{Z}$ and partitions $\mu \subset \lambda$, define the *loop skew Schur function*

$$s_{\lambda/\mu}^{(r)} = s_{\lambda/\mu}^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_T \mathrm{wt}_r(T)$$

where the sum is over semistandard tableaux of shape λ/μ with entries in $\{1, \ldots, m\}$. If $\mu = \emptyset$, then $s_{\lambda}^{(r)} = s_{\lambda/\emptyset}^{(r)}$ is a *loop Schur function*. Set $s_{\emptyset}^{(r)} = 1$. Note that $s_{\lambda/\mu}^{(r)} = 0$ if any column of λ/μ has more than m boxes.

Example 2.3. Let $n = 3, m = 2$, and $\lambda = (3, 2)$. There are two semistandard tableaux of shape λ with entries in $\{1, 2\}$:

Computing 2-weights, we have

$$s_{(3,2)}^{(2)} = x_1^{(2)} x_1^{(1)} x_1^{(3)} x_2^{(3)} x_2^{(2)} + x_1^{(2)} x_1^{(1)} x_2^{(3)} x_2^{(3)} x_2^{(2)}.$$

Observe that $s_{(k)}^{(r)} = h_k^{(r)}$ and $s_{(1^k)}^{(r)} = e_k^{(r)}$. These identities are special cases of Jacobi–Trudi formulas for loop skew Schur functions.

Theorem 2.4 ([15, §7.2][1]). *For two partitions $\mu \subset \lambda$, we have*

$$s_{\lambda/\mu}^{(r)} = \det(h_{\lambda_i - \mu_j - i + j}^{(r - \mu_j + j - 1)}) = \det(e_{\lambda_i' - \mu_j' - i + j}^{(r + \mu_j' - j + 1)})$$

where λ' is the transpose of λ. Thus, $s_{\lambda/\mu}^{(r)}$ is a loop symmetric function.

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[1]Two rings of loop symmetric functions (the "whirl" ring and the "curl" ring) are defined in [15]. We have chosen to use the "whirl" version, as in [13, 17, 18]. The "curl" loop Schur functions are defined by assigning to box (i, j) the content $j - i$ rather than $i - j$. In particular, what we denote by $h_k^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ is $h_k^{(r-k+1)}(\mathbf{y}_1, \ldots, \mathbf{y}_m)$, where $\mathbf{y}_i = \mathbf{x}_{m-i+1}$, in the notation of [15].

The proof is a standard application of the Lindström/Gessel–Viennot lattice path method [19, 6].

Unfortunately, loop Schur functions do not span the ring of loop symmetric functions, and they are not linearly independent! It is an open problem to find a "Schur-like" basis of $\mathrm{LSym}_m(n)$.

2.3. Whirls and curls

Elementary (resp., homogeneous) symmetric polynomials are the coefficients of powers of t in the formal power series $\prod_i(1 + x_i t)$ (resp., $\prod_i(1 - x_i t)^{-1}$). Using the correspondence between formal power series and Toeplitz matrices, these polynomials can be viewed as matrix entries of an infinite upper triangular matrix which is constant along each diagonal. Here we give an analogous description of loop elementary and homogeneous symmetric functions, following [15].

An *n-periodic matrix* is a $\mathbb{Z} \times \mathbb{Z}$ array $(X_{ij})_{(i,j)\in\mathbb{Z}}$ such that $X_{ij} = 0$ if $i - j$ is sufficiently large, and $X_{ij} = X_{i+n,j+n}$ for all i, j. Multiplication of these matrices is defined in the usual way: if $X = (X_{ij})$ and $Y = (Y_{ij})$, then

$$(XY)_{ij} = \sum_{k\in\mathbb{Z}} X_{ik}Y_{kj}.$$

The hypothesis that $X_{ij} = 0$ for $i - j$ sufficiently large ensures that each of these sums is finite, so the product is well-defined. It is clear that the product of two n-periodic matrices is n-periodic.

Given an n-periodic matrix $X = (X_{ij})$, define $X^c = (-1)^{i+j}X_{ij}$. If X is invertible, define $X^{-c} = (X^{-1})^c$. It is easy to see that $(XY)^c = X^cY^c$, and $(X^c)^{-1} = X^{-c}$.

Definition 2.5. Let $a^{(1)}, \ldots, a^{(n)}$ be variables. The *whirl* $M(a^{(1)}, \ldots, a^{(n)})$ is the n-periodic matrix (M_{ij}) with $M_{ii} = 1$, $M_{i,i+1} = a^{(i)}$ (interpret the superscripts mod n), and all other entries zero. The *curl* $N(a^{(1)}, \ldots, a^{(n)})$ is the n-periodic matrix $N(a^{(1)}, \ldots, a^{(n)}) = M(a^{(1)}, \ldots, a^{(n)})^{-c}$.

For example, when $n = 3$,

$$M(a^{(1)}, a^{(2)}, a^{(3)}) = \begin{pmatrix} 1 & a^{(1)} & 0 & 0 & 0 & \\ 0 & 1 & a^{(2)} & 0 & 0 & \\ 0 & 0 & 1 & a^{(3)} & 0 & \cdots \\ 0 & 0 & 0 & 1 & a^{(1)} & \\ 0 & 0 & 0 & 0 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}$$

and

$$N(a^{(1)}, a^{(2)}, a^{(3)}) = \begin{pmatrix} 1 & a^{(1)} & a^{(2)}a^{(1)} & a^{(3)}a^{(2)}a^{(1)} & a^{(1)}a^{(3)}a^{(2)}a^{(1)} \\ 0 & 1 & a^{(2)} & a^{(3)}a^{(2)} & a^{(1)}a^{(3)}a^{(2)} \\ 0 & 0 & 1 & a^{(3)} & a^{(1)}a^{(3)} & \cdots \\ 0 & 0 & 0 & 1 & a^{(1)} \\ 0 & 0 & 0 & 0 & 1 \\ & & & \vdots & & & \ddots \end{pmatrix}.$$

Note that we are depicting only the quadrant of the matrix with $i, j \geq 1$.

It is straightforward to show that loop elementary (resp., homogeneous) symmetric functions are the matrix entries of a product of whirls (resp., curls).

Lemma 2.6 ([15, §7.2]). *Set*

$$A = M(\mathbf{x}_1)M(\mathbf{x}_2)\cdots M(\mathbf{x}_m), \qquad B = A^{-c} = N(\mathbf{x}_m)N(\mathbf{x}_{m-1})\cdots N(\mathbf{x}_1).$$

Then

$$A_{ij} = e^{(i)}_{j-i}(\mathbf{x}_1, \ldots, \mathbf{x}_m) \qquad and \qquad B_{ij} = h^{(j-1)}_{j-i}(\mathbf{x}_1, \ldots, \mathbf{x}_m).[2]$$

Example 2.7. When $n = 2$ and $m = 3$,

$$M(\mathbf{x}_1)M(\mathbf{x}_2)M(\mathbf{x}_3) =$$

$$\begin{pmatrix} 1 & x_1^{(1)} + x_2^{(1)} + x_3^{(1)} & x_1^{(1)}x_2^{(2)} + x_1^{(1)}x_3^{(2)} + x_2^{(1)}x_3^{(2)} & x_1^{(1)}x_2^{(2)}x_3^{(1)} \\ 0 & 1 & x_1^{(2)} + x_2^{(2)} + x_3^{(2)} & x_1^{(2)}x_2^{(1)} + x_1^{(2)}x_3^{(1)} + x_2^{(2)}x_3^{(1)} & \cdots \\ 0 & 0 & 1 & x_1^{(1)} + x_2^{(1)} + x_3^{(1)} \\ 0 & 0 & 0 & 1 \\ & & \vdots & & & \ddots \end{pmatrix}.$$

The entries in the top row of this matrix are $e^{(1)}_0, e^{(1)}_1, e^{(1)}_2, e^{(1)}_3$ (followed by $e^{(1)}_4 = e^{(1)}_5 = \cdots = 0$), in agreement with Lemma 2.6.

Remark 2.8. Theorem 2.4 (combined with Lemma 2.6) shows that loop skew Schur functions are precisely the minors of the matrix $N(\mathbf{x}_m)\cdots N(\mathbf{x}_1)$. Loop Schur functions are the minors using consecutive columns; minors using consecutive rows are the loop skew Schur functions of anti-partition shape

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[2]The discrepancy between this formula and that of [15, Lem. 7.3] is explained by the previous footnote.

(i.e., of shape λ/μ where λ is a rectangle). A similar statement can be made about minors of $M(\mathbf{x}_1) \cdots M(\mathbf{x}_m)$.[3]

2.4. The birational S_m action

Symmetric polynomials in m variables are the invariants of the natural action of S_m on the polynomial ring in m variables. Loop symmetric functions are the invariants of a more complicated S_m action on $\mathbb{Q}(x_i^{(j)})$, as we now explain.

Let $\mathbf{x} = (x^{(1)}, \ldots, x^{(n)})$ and $\mathbf{y} = (y^{(1)}, \ldots, y^{(n)})$. For $r \in \mathbb{Z}/n\mathbb{Z}$, set

$$\kappa^{(r)}(\mathbf{x}, \mathbf{y}) = h_{n-1}^{(r-1)}(\mathbf{x}, \mathbf{y}) = \sum_{s=0}^{n-1} x^{(r-1)} \cdots x^{(r-s)} y^{(r-s+1)} \cdots y^{(r-n+1)}$$

where, as usual, the superscripts live in $\mathbb{Z}/n\mathbb{Z}$.

Definition 2.9. For $i = 1, \ldots, m-1$, define $s_i : \mathbb{Q}(x_i^{(j)}) \to \mathbb{Q}(x_i^{(j)})$ to be the $\mathbb{Q}$-algebra homomorphism which fixes $x_k^{(j)}$ for $k \neq i, i+1$, and acts on $x_i^{(j)}, x_{i+1}^{(j)}$ by

$$s_i(x_i^{(j)}) = x_{i+1}^{(j+1)} \frac{\kappa^{(j+1)}(\mathbf{x}_i, \mathbf{x}_{i+1})}{\kappa^{(j)}(\mathbf{x}_i, \mathbf{x}_{i+1})}, \qquad s_i(x_{i+1}^{(j)}) = x_i^{(j-1)} \frac{\kappa^{(j-1)}(\mathbf{x}_i, \mathbf{x}_{i+1})}{\kappa^{(j)}(\mathbf{x}_i, \mathbf{x}_{i+1})}.$$

If $M = (f_{ab})$ is a matrix with entries in $\mathbb{Q}(x_i^{(j)})$, we define $s_i(M)$ to be the matrix $(s_i(f_{ab}))$.

Theorem 2.10 ([27, §2] [15, §6]).

1. *For $i = 1, \ldots, m-1$,*

$$M(\mathbf{x}_i)M(\mathbf{x}_{i+1}) = s_i(M(\mathbf{x}_i)M(\mathbf{x}_{i+1})).$$

2. *The maps s_i satisfy the relations*

$$s_i^2 = \mathrm{id}, \qquad s_i s_j = s_j s_i \quad if \ |i-j| \geq 2, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

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[3]One must reflect a submatrix of $M(\mathbf{x}_1) \cdots M(\mathbf{x}_m)$ over the anti-diagonal (or equivalently, transpose and rotate the submatrix 180°) to obtain the matrix of loop elementary symmetric functions appearing in Theorem 2.4. Thus, loop Schur functions are the minors of $M(\mathbf{x}_1) \cdots M(\mathbf{x}_m)$ using consecutive rows.

Remark 2.11. The deepest part of Theorem 2.10 is the fact that the maps s_i and s_{i+1} satisfy the braid (or Yang–Baxter) relation. Many different proofs of this result have appeared in the literature: in addition to [27, 15], see [4], [16, §6.5], [5, §5.2], and the combination of [2, §8.7] and [1, §6.2].

Theorem 2.10(2) shows that the maps s_i generate an action of S_m on $\mathbb{Q}(x_i^{(j)})$. We call this the *birational S_m action* (it is also called the birational R-matrix in the literature). Theorem 2.10(1) and Lemma 2.6 imply that the loop elementary symmetric functions (and thus all loop symmetric functions) are invariant under this action. In fact, the loop elementary symmetric functions are algebraically independent generators of the subring of polynomial invariants for this action [14], but we will not use this result.

2.5. Loop power sums

We will need one additional class of loop symmetric functions. For $i = 1, \ldots, m$, set

$$\pi_i = x_i^{(1)} x_i^{(2)} \cdots x_i^{(n)}.$$

Lemma 2.12. *For $w \in S_m$, we have $w(\pi_i) = \pi_{w(i)}$.*

Proof. Since S_m is generated by the s_i, it suffices to show that $s_i(\pi_j) = \pi_{s_i(j)}$ for each i, j. We compute

$$s_i(\pi_i) = s_i(x_i^{(1)}) s_i(x_i^{(2)}) \cdots s_i(x_i^{(n)}) = x_{i+1}^{(2)} \frac{\kappa_i^{(2)}}{\kappa_i^{(1)}} x_{i+1}^{(3)} \frac{\kappa_i^{(3)}}{\kappa_i^{(2)}} \cdots x_{i+1}^{(1)} \frac{\kappa_i^{(1)}}{\kappa_i^{(n)}} = \pi_{i+1}.$$

Similarly, $s_i(\pi_{i+1}) = \pi_i$, and clearly $s_i(\pi_j) = \pi_j$ if $j \neq i, i+1$. □

Definition 2.13. For each positive integer k, define the *loop power sum symmetric function*

$$p_k = p_k(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{i=1}^m \pi_i^k.$$

By Lemma 2.12, the polynomials p_k are invariant under the birational S_m action. The loop Murnaghan–Nakayama rule (Theorem 4.1) expresses p_k as an alternating sum of loop Schur functions.

3. Alternants

Schur polynomials were originally defined by the formula $s_\lambda(x_1, \ldots, x_m) = a_{\lambda+\delta}/a_\delta$, where a_α is the determinant of the matrix $(x_j^{\alpha_i})$, and δ is the

staircase partition $(m - 1, m - 2, \ldots, 1, 0)$. These determinants are called *alternants* because they are anti-symmetric (or alternating) with respect to permutation of the variables. We now present a generalization of alternants to the loop setting, following [13].

For a sequence $\alpha = (\alpha_1, \ldots, \alpha_m)$ of non-negative integers, define an $m \times m$ matrix $A_\alpha^{(r)}$ by

$$(A_\alpha^{(r)})_{ij} = t_{j,m}(x_m^{(r+m-1)} x_m^{(r+m-2)} \cdots x_m^{(r+m-\alpha_i)})$$

where $t_{a,b}$ is the transposition in S_m which swaps a and b, acting on $\mathbb{Q}(x_i^{(j)})$ by the birational S_m action. For example, if $n = 3$ and $m = 2$, then

$$(1) \quad A_{(4,2)}^{(2)} = \begin{pmatrix} t_{1,2}\left(x_2^{(3)} x_2^{(2)} x_2^{(1)} x_2^{(3)}\right) & x_2^{(3)} x_2^{(2)} x_2^{(1)} x_2^{(3)} \\ \\ t_{1,2}\left(x_2^{(3)} x_2^{(2)}\right) & x_2^{(3)} x_2^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} x_1^{(2)} x_1^{(1)} x_1^{(3)} x_1^{(2)} \dfrac{x_1^{(1)} x_1^{(3)} + x_1^{(1)} x_2^{(3)} + x_2^{(1)} x_2^{(3)}}{x_1^{(2)} x_1^{(1)} + x_1^{(2)} x_2^{(1)} + x_2^{(2)} x_2^{(1)}} & x_2^{(3)} x_2^{(2)} x_2^{(1)} x_2^{(3)} \\ \\ x_1^{(2)} x_1^{(1)} \dfrac{x_1^{(3)} x_1^{(2)} + x_1^{(3)} x_2^{(2)} + x_2^{(3)} x_2^{(2)}}{x_1^{(2)} x_1^{(1)} + x_1^{(2)} x_2^{(1)} + x_2^{(2)} x_2^{(1)}} & x_2^{(3)} x_2^{(2)} \end{pmatrix}.$$

Set $a_\alpha^{(r)} = \det(A_\alpha^{(r)})$. It is easy to see that $a_\alpha^{(r)}$ is anti-symmetric with respect to the birational S_m action; we call this determinant a *loop alternant*. The following result was stated without proof in [13].[4]

Theorem 3.1. *For λ a partition with at most m parts, we have*

$$(2) \qquad\qquad s_\lambda^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \frac{a_{\lambda+\delta}^{(r)}}{a_\delta^{(r)}},$$

where $\delta = (m - 1, m - 2, \ldots, 1, 0)$.

Before proving this result, we derive several corollaries. Recall that $\pi_i = x_i^{(1)} \cdots x_i^{(n)}$.

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[4]The original statement of this result ([13, Thm. 5.6]) is incorrect; using the indexing conventions of that paper, the superscript of the loop Schur function should be $r - m + 1$ rather than $r - 1$.

Corollary 3.2. *For $r \in \mathbb{Z}/n\mathbb{Z}$, we have*

$$(3) \qquad a_\delta^{(r)} = \frac{\displaystyle\prod_{1 \le i < j \le m} (\pi_i - \pi_j)}{s_{(n-1)\delta}^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m)}$$

where $k\delta = (k(m-1), k(m-2), \ldots, k, 0)$.

Proof. Taking $\lambda = (n-1)\delta$ in Theorem 3.1, we have

$$a_\delta^{(r)} = \frac{a_{n\delta}^{(r)}}{s_{(n-1)\delta}^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m)}.$$

The last column of $A_{n\delta}^{(r)}$ has entries $(A_{n\delta}^{(r)})_{i,m} = \pi_m^{m-i}$. Since the birational S_m action permutes the π_j, we have $(A_{n\delta}^{(r)})_{i,j} = \pi_j^{m-i}$, so $A_{n\delta}^{(r)}$ has determinant $\prod_{i<j}(\pi_i - \pi_j)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We now combine Theorem 3.1 and Corollary 3.2 with results of Lam and Pylyavskyy [17] to obtain a variant of Theorem 3.1 which expresses $s_\lambda^{(r)}$ as a ratio of two polynomials.

For $k \ge 1$ and $r \in \mathbb{Z}/n\mathbb{Z}$, let

$$\sigma_k^{(r)}(\mathbf{x}_a, \ldots, \mathbf{x}_b) = \sum x_{i_1}^{(r)} x_{i_2}^{(r-1)} \cdots x_{i_k}^{(r-k+1)},$$

where the sum is over weakly increasing sequences $a \le i_1 \le \cdots \le i_k \le b$, such that each of the numbers $a+1, a+2, \ldots, b$ appears in the sequence at most $n-1$ times. Set $\sigma_0^{(r)} = 1$. For example, if $n = 3$, then

$$\begin{aligned} \sigma_4^{(2)}(\mathbf{x}_1, \mathbf{x}_2) &= x_1^{(2)} x_1^{(1)} x_1^{(3)} x_1^{(2)} + x_1^{(2)} x_1^{(1)} x_1^{(3)} x_2^{(2)} + x_1^{(2)} x_1^{(1)} x_2^{(3)} x_2^{(2)} \\ &= x_1^{(2)} x_1^{(1)} \sigma_2^{(3)}(\mathbf{x}_1, \mathbf{x}_2). \end{aligned}$$

For a sequence $\alpha = (\alpha_1, \ldots, \alpha_m)$ of non-negative integers, define an $m \times m$ matrix $B_\alpha^{(r)}$ by

$$(B_\alpha^{(r)})_{ij} = x_j^{(r+j-1)} x_j^{(r+j-2)} \cdots x_j^{(r+j-\alpha_i)} \sigma_{(n-1)(m-j)}^{(r+j-\alpha_i-1)}(\mathbf{x}_j, \ldots, \mathbf{x}_m).$$

Set $b_\alpha^{(r)} = \det(B_\alpha^{(r)})$.

Theorem 3.3. *For λ a partition with at most m parts, we have*

$$(4) \qquad s_\lambda^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \frac{b_{\lambda+\delta}^{(r)}}{b_\delta^{(r)}} = \frac{b_{\lambda+\delta}^{(r)}}{\displaystyle\prod_{1 \le i < j \le m} (\pi_i - \pi_j)}.$$

Example 3.4. Let $n = 3, m = 2$, and $\lambda = (3, 2)$. Then

$$B_{\lambda+\delta}^{(2)} = B_{(4,2)}^{(2)}$$
$$= \begin{pmatrix} x_1^{(2)} x_1^{(1)} x_1^{(3)} x_1^{(2)} (x_1^{(1)} x_1^{(3)} + x_1^{(1)} x_2^{(3)} + x_2^{(1)} x_2^{(3)}) & x_2^{(3)} x_2^{(2)} x_2^{(1)} x_2^{(3)} \\ x_1^{(2)} x_1^{(1)} (x_1^{(3)} x_1^{(2)} + x_1^{(3)} x_2^{(2)} + x_2^{(3)} x_2^{(2)}) & x_2^{(3)} x_2^{(2)} \end{pmatrix}$$

(cf. (1)). The determinant of this matrix is

$$b_{(4,2)}^{(2)} = x_1^{(2)} x_1^{(1)} x_2^{(3)} x_2^{(2)} \left(x_1^{(3)} x_1^{(2)} x_1^{(1)} x_1^{(3)} + x_1^{(3)} x_1^{(2)} x_1^{(1)} x_2^{(3)} \right.$$
$$\left. - x_1^{(3)} x_2^{(2)} x_2^{(1)} x_2^{(3)} - x_2^{(3)} x_2^{(2)} x_2^{(1)} x_2^{(3)} \right).$$

Similarly, one computes $b_{(1,0)}^{(2)} = x_1^{(1)} x_1^{(2)} x_1^{(3)} - x_2^{(1)} x_2^{(2)} x_3^{(3)}$, and the ratio $b_{(4,2)}^{(2)}/b_{(1,0)}^{(2)}$ is indeed equal to the loop Schur function $s_{(3,2)}^{(2)}$ from Example 2.3.

Proof of Theorem 3.3. The entries of the matrix $A_\alpha^{(r)}$ are given by

$$(A_\alpha^{(r)})_{ij} = t_{j,m}(x_m^{(r+m-1)} x_m^{(r+m-2)} \cdots x_m^{(r+m-\alpha_i)})$$
$$= s_j s_{j+1} \cdots s_{m-2} s_{m-1}(x_m^{(r+m-1)} x_m^{(r+m-2)} \cdots x_m^{(r+m-\alpha_i)}),$$

where the second equality comes from writing

$$t_{j,m} = s_j s_{j+1} \cdots s_{m-2} s_{m-1} s_{m-2} \cdots s_{j+1} s_j$$

and observing that the maps $s_j, \ldots, s_{m-2}$ do not affect the variables $x_m^{(a)}$. By [17, Lem. 3.1],

$$s_j s_{j+1} \cdots s_{m-2} s_{m-1}(x_m^{(a)}) = x_j^{(a-m+j)} \frac{\sigma_{(n-1)(m-j)}^{(a-m+j-1)}(\mathbf{x}_j, \ldots, \mathbf{x}_m)}{\sigma_{(n-1)(m-j)}^{(a-m+j)}(\mathbf{x}_j, \ldots, \mathbf{x}_m)},$$

so

$$(A_\alpha^{(r)})_{ij} = x_j^{(r+j-1)} x_j^{(r+j-2)} \cdots x_j^{(r+j-\alpha_i)} \frac{\sigma_{(n-1)(m-j)}^{(r+j-\alpha_i-1)}(\mathbf{x}_j, \ldots, \mathbf{x}_m)}{\sigma_{(n-1)(m-j)}^{(r+j-1)}(\mathbf{x}_j, \ldots, \mathbf{x}_m)}$$

$$= \frac{(B_\alpha^{(r)})_{ij}}{\sigma_{(n-1)(m-j)}^{(r+j-1)}(\mathbf{x}_j, \ldots, \mathbf{x}_m)}.$$

Taking determinants, we have

$$(5) \qquad\qquad a_\alpha^{(r)} = \frac{b_\alpha^{(r)}}{\prod_{j=1}^m \sigma_{(n-1)(m-j)}^{(r+j-1)}(\mathbf{x}_j, \ldots, \mathbf{x}_m)}.$$

This identity holds for all $\alpha \in (\mathbb{Z}_{\geq 0})^m$, so

$$\frac{b_{\lambda+\delta}^{(r)}}{b_\delta^{(r)}} = \frac{a_{\lambda+\delta}^{(r)}}{a_\delta^{(r)}} = s_\lambda^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$$

by Theorem 3.1, proving the first equality in (4). The denominator of (5) is equal to $s_{(n-1)\delta}^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ by [17, Thm. 2.5], so the second equality follows from Corollary 3.2. □

Proof of Theorem 3.1. This proof is adapted from an argument in [21, §I.3]. Unless otherwise noted, all loop symmetric functions are in the variables $\mathbf{x}_1, \ldots, \mathbf{x}_m$.

Set $\alpha = \lambda + \delta$. Define an $m \times m$ matrix $H_\alpha^{(r)}$ by

$$(H_\alpha^{(r)})_{ij} = h_{\alpha_i-m+j}^{(r+j-1)} = h_{\lambda_i-i+j}^{(r+j-1)}.$$

Note that $\det(H_\alpha^{(r)}) = s_\lambda^{(r)}$ by Theorem 2.4. Define an $m \times m$ matrix $E^{(r)}$ by

$$(E^{(r)})_{ij} = (-1)^{m-i} t_{j,m}(e_{m-i}^{(r+i)}[\widehat{m}])$$

where $e_k^{(r)}[\widehat{m}] = e_k^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_{m-1})$. We will show that

$$(6) \qquad\qquad H_\alpha^{(r)} E^{(r)} = A_\alpha^{(r)}.$$

To this end, set $X = N(\mathbf{x}_m) \cdots N(\mathbf{x}_1)$ and $Y = M(\mathbf{x}_1)^c \cdots M(\mathbf{x}_{m-1})^c$. Clearly $XY = N(\mathbf{x}_m)$, so

$$(7) \qquad (XY)_{uv} = \sum_k X_{uk} Y_{kv} = h_{v-u}^{(v-1)}(\mathbf{x}_m)$$

by Lemma 2.6. Applying $t_{j,m}$ to (7) and using the fact that every matrix entry of X is fixed by the birational S_m action (by Theorem 2.10(1)), we obtain

$$(8) \qquad \sum_k X_{uk} t_{j,m}(Y_{kv}) = t_{j,m}(h_{v-u}^{(v-1)}(\mathbf{x}_m)).$$

By Lemma 2.6, (8) is the identity

$$(9) \qquad \sum_k h_{k-u}^{(k-1)}(-1)^{k+v} t_{j,m}(e_{v-k}^{(k)}[\widehat{m}]) = t_{j,m}(h_{v-u}^{(v-1)}(\mathbf{x}_m)).$$

Setting $u = r + m - \alpha_i$, $v = r + m$, and $k = r + s$, (9) becomes

$$(10) \qquad \sum_s h_{\alpha_i-m+s}^{(r+s-1)}(-1)^{s+m} t_{j,m}(e_{m-s}^{(r+s)}[\widehat{m}]) = t_{j,m}(h_{\alpha_i}^{(r+m-1)}(\mathbf{x}_m)).$$

Since $e_{m-s}^{(r+s)}[\widehat{m}] = 0$ unless $0 \leq m - s \leq m - 1$, we only need to sum over values of s between 1 and m. Thus, the two sides of (10) are precisely the i, j entries of the two sides of (6).

To complete the proof, observe that $H_\delta^{(r)}$ is upper triangular with 1's on the diagonal, so it has determinant 1, and thus (6) with $\alpha = \delta$ implies

$$\det(E^{(r)}) = \det(A_\delta^{(r)}) = a_\delta^{(r)}.$$

Taking determinants of (6), we obtain

$$s_\lambda^{(r)} a_\delta^{(r)} = a_{\lambda+\delta}^{(r)}. \qquad \square$$

Remark 3.5. If colors are identified (i.e., if $x_i^{(j)}$ is specialized to y_i for all j) then the birational S_m action reduces to the permutation of the variables y_i, so $(A_\alpha^{(r)})_{ij} = y_j^{\alpha_i}$, and Theorem 3.1 specializes to the original definition of the Schur polynomials. Under this specialization, we also have $(B_\alpha^{(r)})_{ij} = y_j^{\alpha_i} \sigma_{(n-1)(m-j)}(y_j, \ldots, y_m)$, where $\sigma_i(y_j, \ldots, y_m)$ is the sum of monomials of

degree i in $y_j, \ldots, y_m$ such that the exponents of $y_{j+1}, \ldots, y_m$ are at most $n-1$. Thus, $b_\alpha^{(r)}$ becomes

$$(11) \qquad a_\alpha \prod_{j=1}^{m} \sigma_{(n-1)(m-j)}(y_j, \ldots, y_m),$$

and Theorem 3.3 is trivially equivalent to Theorem 3.1 in this case.

Remark 3.6. There is some interesting combinatorics associated to the product in (11). It is easy to see that

$$\sigma_{(n-1)(m-j)}(y_j, \ldots, y_m) = \prod_{k=j+1}^{m} (y_j^{n-1} + y_j^{n-2} y_k + \cdots + y_j y_k^{n-2} + y_k^{n-1}),$$

so

$$\prod_{j=1}^{m} \sigma_{(n-1)(m-j)}(y_j, \ldots, y_m) = \prod_{1 \le j < k \le m} \frac{y_j^n - y_k^n}{y_j - y_k} = \frac{a_{n\delta}}{a_\delta} = s_{(n-1)\delta}(y_1, \ldots, y_m).$$

Jucis [10] gave a bijective proof of this identity using Schensted insertion. As mentioned in the proof of Theorem 3.3, Lam and Pylyavskyy [17] proved a "colored refinement" of this identity:

$$\prod_{j=1}^{m} \sigma_{(n-1)(m-j)}^{(r+j-1)}(\mathbf{x}_j, \ldots, \mathbf{x}_m) = s_{(n-1)\delta}^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m).$$

(They used this formula to show that the loop Schur function $s_{(n-1)\delta}^{(n)}$ tropicalizes to the intrinsic energy function for tensor products of one-row crystals of type $A_{n-1}^{(1)}$.) Their proof is algebraic; we would like to have a combinatorial proof, perhaps using a "colored refinement" of Schensted insertion. It would also be nice to have a combinatorial proof of Theorem 3.3, perhaps along the lines of the argument in [3].

4. The loop Murnaghan–Nakayama rule

The Murnaghan–Nakayama rule gives the Schur expansion of the product of a power sum symmetric function and a Schur function. A loop generalization of this rule was stated in [13], and proved combinatorially by Ross [24]. Here we give a short proof based on Theorem 3.1.

Recall the loop power sums $p_k = \sum_{i=1}^m \pi_i^k$, where $\pi_i = x_i^{(1)} x_i^{(2)} \cdots x_i^{(n)}$. Recall also that a *ribbon* (or *border strip*, or *rim hook*) is a connected[5] skew Young diagram that does not contain any 2×2 squares. The *size* of a ribbon is the number of boxes it contains, and the *height* of a ribbon (denoted ht) is one less than the number of rows it contains.

Theorem 4.1. *Let λ be a partition with at most m parts, and k a positive integer. Then*

$$p_k(\mathbf{x}_1, \ldots, \mathbf{x}_m) s_\lambda^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m),$$

where the sum is over all partitions μ (with at most m parts) such that μ/λ is a ribbon of size kn.

Proof. All loop symmetric functions are in the variables $\mathbf{x}_1, \ldots, \mathbf{x}_m$. Since the birational S_m action permutes the π_i, and each color appears the same number of times in π_i, we have

$$p_k a_{\lambda+\delta}^{(r)}$$

$$= \sum_{i=1}^m \pi_i^k \sum_{w \in S_m} \mathrm{sgn}(w) \prod_{j=1}^m t_{w(j),m}(x_m^{(r+m-1)} x_m^{(r+m-2)} \cdots x_m^{(r+m-(\lambda+\delta)_j)})$$

$$= \sum_{w \in S_m} \mathrm{sgn}(w) \sum_{i=1}^m \pi_{w(i)}^k \prod_{j=1}^m t_{w(j),m}(x_m^{(r+m-1)} x_m^{(r+m-2)} \cdots x_m^{(r+m-(\lambda+\delta)_j)})$$

$$= \sum_{w \in S_m} \mathrm{sgn}(w) \sum_{i=1}^m \prod_{j=1}^m t_{w(j),m}(x_m^{(r+m-1)} x_m^{(r+m-2)} \cdots x_m^{(r+m-(\lambda+\delta)_j - kn\rho_{i,j})})$$

$$= \sum_{i=1}^m a_{\lambda+\delta+kn\epsilon_i}^{(r)},$$

where $\rho_{i,j}$ is 1 if $i = j$ and 0 otherwise, and ϵ_i is the ith standard basis vector in $\mathbb{Z}^m$. Since $a_\alpha^{(r)}$ is anti-symmetric with respect to permuting parts of α, a bit of bookkeeping (as in [21, §I.3, Ex. 11]) shows that

$$\sum_{i=1}^m a_{\lambda+\delta+kn\epsilon_i}^{(r)} = \sum (-1)^{\mathrm{ht}(\mu/\lambda)} a_{\mu+\delta}^{(r)},$$

---

[5]Two boxes that share only a vertex are not considered to be connected; thus, the diagram $(2,1)/(1)$ is not connected.

where the sum is over all partitions μ (with at most m parts) such that μ/λ is a ribbon of size kn. Now divide by $a_\delta^{(r)}$ and apply Theorem 3.1. $\qquad\square$

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