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Weighted variants of the Andrásfai-Erdős-Sós theorem

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A well known result due to Andrásfai, Erdős, and Sós asserts that for $r \geqslant 2$ every K_{r+1}-free graph G on n vertices with $\delta(G) > \frac{3r-4}{3r-1}n$ is r-partite. We study related questions in the context of weighted graphs, which are motivated by recent work on the Ramsey-Turán problem for cliques.

1. Introduction

1.1. Simple graphs

Extremal graph theory began with Turán's discovery [20] that for $r \geqslant 2$ every n-vertex graph G with more than $\frac{r-1}{r} \cdot \frac{n^2}{2}$ edges contains a K_{r+1}, i.e., a clique on $r+1$ vertices. The constant $\frac{r-1}{r}$ appearing here is optimal, as can be seen by looking at balanced complete r-partite graphs. Simonovits [16] proved that this extremal configuration is subject to a stability phenomenon roughly saying that a K_{r+1}-free graph with almost the maximum number of edges is "almost" r-partite.

Theorem 1.1 (Simonovits)**.** *For every $r \geqslant 2$ and $\varepsilon > 0$ there exists some $\delta > 0$ such that every K_{r+1}-free graph G on n vertices with at least $\left(\frac{r-1}{r} - \delta\right)\frac{n^2}{2}$ edges admits a partition $V(G) = W_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} W_r$ satisfying $\sum_{i=1}^{r} e(W_i) < \varepsilon n^2$.*

A standard proof of Theorem 1.1 starts with the observation that an iterative deletion of vertices with small degree allows us to reduce to the case that $\delta(G) > \left(\frac{r-1}{r} - \eta\right)n$ holds for an arbitrary constant $\eta \ll \varepsilon$ chosen in advance. One then takes a clique of order r in G, whose existence is guaranteed by Turán's theorem, and observes that the joint neighbourhoods $\widetilde{W}_1, \ldots, \widetilde{W}_r$ of the $(r-1)$-subsets of this clique are mutually disjoint independent sets, since otherwise G would contain a K_{r+1}. Finally, the minimum degree condition ensures that these sets cover all but at most $r\eta n$ vertices of G, for which reason any partition $V(G) = W_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} W_r$ with $W_i \supseteq \widetilde{W}_i$ for all $i \in [r]$ has the desired property.

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As a matter of fact, however, the second part of the argument can be omitted by appealing instead to a result of Andrásfai, Erdős, and Sós [1] telling us that an appropriate minimum degree condition of the form $\delta(G) > \left(\frac{r-1}{r} - \eta\right)n$ implies that K_{r+1}-free graphs are r-partite. For an elegant alternative proof of this fact we refer to Brandt [4].

Theorem 1.2 (Andrásfai, Erdős, and Sós)**.** *Let G be for some $r \geqslant 2$ a K_{r+1}-free graph on n vertices satisfying $\delta(G) > \frac{3r-4}{3r-1}n$. Then there is a homomorphism from G to K_r, i.e., G is r-colourable.*

The constant $\frac{3r-4}{3r-1}$ appearing here is optimal. This can be seen by constructing for some n divisible by $(3r-1)$ a graph G on a set V of n vertices having a partition

$$V = A_1 \charfusion[\mathbin]{\cup}{\cdot} \ldots \charfusion[\mathbin]{\cup}{\cdot} A_5 \charfusion[\mathbin]{\cup}{\cdot} B_1 \charfusion[\mathbin]{\cup}{\cdot} \ldots \charfusion[\mathbin]{\cup}{\cdot} B_{r-2}$$

such that

$$|A_1| = \ldots = |A_5| = \tfrac{n}{3r-1} \quad \text{and} \quad |B_1| = \ldots = |B_{r-2}| = \tfrac{3n}{3r-1}\,,$$

and whose set of edges is as follows:

- there are all edges from a vertex in A_i to a vertex in A_{i+1}, where the indices are taken modulo 5;
- all edges from A_i to B_j, where $i \in [5]$, $j \in [r-2]$;
- and all edges from B_j to $B_{j'}$, where $j, j' \in [r-2]$ are distinct.

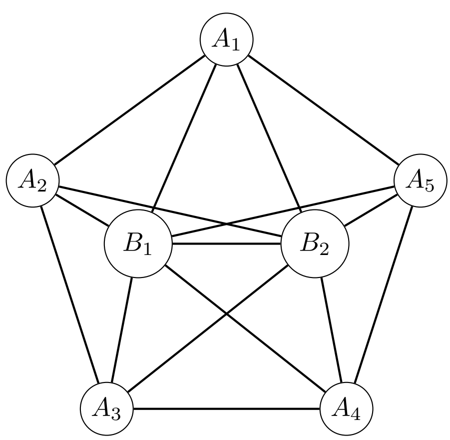


Figure 1.1: The extremal graph for the case $r = 4$ of Theorem 1.2.

1.2. Weighted graphs

The concepts and problems discussed in the previous subsection make sense in the broader context of weighted graphs as well. For the purposes of this article, these are defined as follows.

Definition 1.3. A *weighted graph* is a pair $G = (V, w)$ consisting of a finite *vertex set* V and a symmetric function $w\colon V^2 \longrightarrow \mathbb{R}_{\geqslant 0}$ such that $w(x, x) = 0$ holds for all $x \in V$.

Here the word "symmetric" means that we require $w(x, y) = w(y, x)$ for all $x, y \in V$. The notions of subgraphs, induced subgraphs, and isomorphisms extend in the following way from ordinary graphs to weighted graphs.

Definition 1.4. Let $G = (V, w)$ and $G' = (V', w')$ be two weighted graphs. We say that G' is a *subgraph* of G if $V' \subseteq V$ and, additionally, $w'(x, y) \leqslant w(x, y)$ holds for all $x, y \in V'$. If this conditions holds with equality throughout we call G' an *induced subgraph* of G. Finally, G and G' are said to be *isomorphic* if there is a bijection $\varphi\colon V \longrightarrow V'$ satisfying $w(x, y) = w'\big(\varphi(x), \varphi(y)\big)$ for all $x, y \in V$.

For two weighted graphs F and G we say that G is *F-free* if G does not possess any subgraph isomorphic to F. More generally, if $\mathscr{F}$ is a set of weighted graphs such that G is F-free for every $F \in \mathscr{F}$, then G is said to be *$\mathscr{F}$-free*. The natural analogue of the "number of edges" of a weighted graph $G = (V, w)$ is, of course, the quantity

$$e(G) = \tfrac{1}{2} \sum_{(x,y) \in V^2} w(x, y)\,.$$

Now for every such set $\mathscr{F}$ of weigthed graphs and every finite set $D \subseteq \mathbb{R}_{\geqslant 0}$ one may look at the *extremal function* $n \longmapsto \mathrm{ex}_D(n, \mathscr{F})$ sending every positive integer n to the maximum of $e(G)$ as G varies over $\mathscr{F}$-free weighted graphs of order n whose weight function only attains values in D. The natural generalisation of Turán's problem to this context asks to determine these functions for all choices of $\mathscr{F}$ and D, the classical case being $D = \{0, 1\}$.

Similar as in this case, the *generalised Turán densities*

$$\pi_D(\mathscr{F}) = \lim_{n \to \infty} \frac{\mathrm{ex}_D(n, \mathscr{F})}{n^2/2} \tag{1.1}$$

are easily shown to exist. Questions concerning $\mathrm{ex}_D(n, \mathscr{F})$ and $\pi_D(\mathscr{F})$ are often studied in the literature, both for their own sake (see e.g. [9, 15]) and due to their connection with other parts of extremal combinatorics.

For instance, De Caen and Füredi [5] realised that such results can be applied to determine the Turán density of the Fano plane. To this end, they needed to know the value of $\pi_D(\mathscr{F})$, where $D = \{0, 1, 2, 3, 4\}$ and $\mathscr{F}$ denotes the set of all corresponding weighted graphs F on four vertices with $e(F) \geqslant 21$. Their approach is occasionally called the *link multigraph method* and led to many further results on Turán's hypergraph problem (see also [2, 10–12, 14]).

An earlier occurrence of a Turán problem for weighted graphs appeared in the determination of the so-called *Ramsey-Turán density* of even cliques due to Erdős, Hajnal, Szemerédi, and Sós [6]. This result belongs to an area initiated by Vera T. Sós, which is called *Ramsey-Turán theory*. Given a graph F, a number n of vertices, and a real number $m > 0$ she defined the *Ramsey-Turán number* $\mathrm{RT}(n, m, F)$ to be the maximum number of edges that an F-free graph G on n vertices with $\alpha(G) < m$ can have. The problem is especially interesting if F is a clique and it is customary in this setting to pass to the *Ramsey-Turán density function* $f_t \colon (0, 1) \to \mathbb{R}$ defined by

$$f_t(\delta) = \lim_{n\to\infty} \frac{\mathrm{RT}(n, \delta n, K_t)}{n^2/2}\,.$$

A further simplification can be obtained by restricting the attention to the *Ramsey-Turán densities*

$$\varrho(K_t) = \lim_{\delta\to 0} f_t(\delta)\,.$$

Such quantities have been intensively studied in the literature (see e.g. [3, 6, 7, 18] for important milestones and [17] for a beautiful survey). Owing to all these efforts it is known that

$$\varrho(K_t) = \begin{cases} \frac{t-3}{t-1} & \text{if } t \geqslant 3 \text{ is odd,} \\ \frac{3t-10}{3t-4} & \text{if } t \geqslant 4 \text{ is even.} \end{cases} \tag{1.2}$$

The even case is much harder and in their solution Erdős, Hajnal, Szemerédi, and Sós applied a result on the Turán density of a certain $\{0, 1, 2\}$-valued collection $\mathscr{F}_t$ of weighted graphs to a reduced graph obtained by means of Szemerédi's regularity lemma [19]. These specific families $\mathscr{F}_t$ of weighted graphs are introduced in Definiton 1.5 below.

A few years ago, Fox, Loh, and Zhao [8] proved $f_4(\delta) = \frac{1}{4} + \Theta(\delta)$. In an attempt to generalise some of their arguments to larger even cliques we realised that the values of the Ramsey-Turán density function $f_t(\delta)$ can be determined explicitly for $\delta \ll t^{-1}$. Notably in [13] we proved that

$$f_t(\delta) = \begin{cases} \frac{t-3}{t-1} + \delta & \text{if } t \geqslant 3 \text{ is odd,} \\ \frac{3t-10}{3t-4} + \delta - \delta^2 & \text{if } t \geqslant 4 \text{ is even} \end{cases}$$

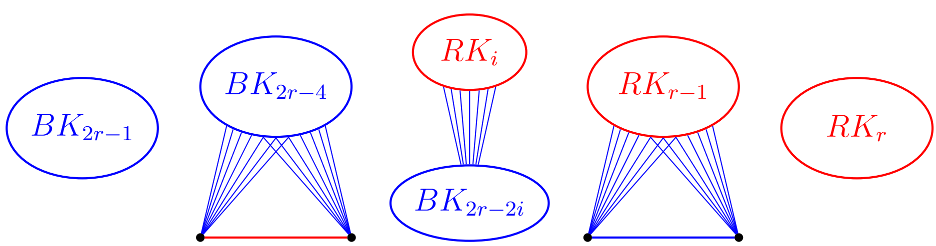
holds provided that δ is sufficiently small in a sense depending on t. In order to obtain these precise formulae we needed a stability result in the spirit of Theorem 1.1 but for the collections $\mathscr{F}_t$ of weighted graphs mentioned above (see e.g. [13, Proposition 3.5]). While working on this subject, we proved analogues of the Andrásfai-Erdős-Sós theorem as well. They form the main contribution of the present work.

1.3. Results

Throughout the rest of this article we only need to deal with weighted graphs $G = (V, w)$ satisfying $w[V^2] \subseteq \{0, 1, 2\}$. We regard such structures as *coloured* complete graphs on V by drawing a *green*, *blue*, or *red* edge between any two distinct vertices $x, y \in V$ depending on whether $w(x, y)$ attains the value 0, 1, or 2. For a nonnegative integer n the red and blue n-vertex clique are denoted by RK_n and BK_n, respectively. Moreover, for $n \geqslant 2$ we mean by RK_n^- the graph obtained from an RK_n by recolouring one of its edges blue.

Definition 1.5. For two integers $a \geqslant b \geqslant 1$ the coloured graph $G_{a+b,b}$ of order a consists of an RK_b and a BK_{a-b} that are connected to each other by blue edges. Moreover, for every positive integer t we write

$$\mathscr{F}_t = \left\{G_{t,i} \colon 1 \leqslant i \leqslant \tfrac{t}{2}\right\}.$$



(a) $G_{2r,1}$ (b) $G_{2r,2}$ (c) $G_{2r,i}$ (d) $G_{2r,r-1}$ (e) $G_{2r,r}$

Figure 1.2: The family $\mathscr{F}_{2r} = \{G_{2r,1}, G_{2r,2}, \ldots, G_{2r,r}\}$.

Erdős, Hajnal, Szemerédi, and Sós proved in [6] that

$$\pi_{\{0,1,2\}}(\mathscr{F}_t) = \begin{cases} \frac{2(t-3)}{t-1} & \text{if } t \geqslant 3 \text{ is odd,} \\ \frac{2(3t-10)}{3t-4} & \text{if } t \geqslant 4 \text{ is even,} \end{cases}$$

which in turn leads to (1.2) via the regularity method for graphs. In order to state the related results in the spirit of Theorem 1.2 one needs a notion of minimum degree for coloured graphs and it will be useful to have a notion of homomorphisms as well.

Now if $G = (V, w)$ denotes a coloured graph and $x \in V$, it is natural to call

$$d(x) = \sum_{y \in V} w(x, y)$$

the *degree* of x. Moreover, the quantity $\delta(G) = \min\{d(v) : v \in V\}$ will be referred to as the *minimum degree* of G.

Definition 1.6. A *homomorphism* from a weighted graph $G = (V, w)$ to a weighted graph $G' = (V', w')$ is a map $\varphi : V \longrightarrow V'$ with the property that any two distinct vertices $x, y \in V$ satisfy $w(x, y) \leqslant w'\big(\varphi(x), \varphi(y)\big)$.

For odd indices, we shall obtain the following.

Theorem 1.7. *Suppose that for some $r \geqslant 2$ we have an $\mathscr{F}_{2r+1}$-free coloured graph G of order n with $\delta(G) > \frac{6r-8}{3r-1}n$. Then there is a homomorphism from G to RK_r or, explicitly, there is a partition*

$$V(G) = W_1 \cupdot \ldots \cupdot W_r$$

such that all edges within the partition classes are green.

Consider the coloured graph obtained from the extremal graph described in Subsection 1.1 by replacing the edges there by red edges and colouring all other pairs green. This coloured graph has a minimum degree of exactly $\frac{6r-8}{3r-1}n$ but, as it does not contain a BK_{r+1}, it cannot contain a member of $\mathscr{F}_{2r+1}$ either. On the other hand, it does not admit a homomorphism to RK_r and thus it shows that the constant $\frac{6r-8}{3r-1}$ appearing in Theorem 1.7 is optimal. Let us also note that the extremal graphs for $\pi_{\{0,1,2\}}(\mathscr{F}_{2r+1}) = \frac{2(r-1)}{r}$ are RK_r and its symmetric blow-ups, which is why we aimed to get a homomorphism into RK_r in the conclusion of Theorem 1.7.

As in Ramsey-Turán theory, the even case will be much harder. Let us recall that in [6] the extremal graphs for $\pi_{\{0,1,2\}}(\mathscr{F}_{2r}) = \frac{2(3r-5)}{3r-2}$ have been determined to be certain blow-ups of RK_r^-. (E.g., the two "special" vertices get blown up by a factor of 2, while the $r-2$ remaining vertices receive a factor of 3.) Therefore, our goal is to enforce, by an appropriate minimum degree condition, that an $\mathscr{F}_{2r}$-free coloured graph admits a homomorphism into RK_r^-.

Theorem 1.8. *Let $r \geqslant 3$ be an integer and let G be a $\mathscr{F}_{2r}$-free coloured graph of order n with $\delta(G) > \frac{14r-24}{7r-5}n$. Then there is a homomorphism from G to RK_r^-. In other words, there is a partition*

$$V(G) = W_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} W_r$$

such that all edges within the partition classes are green and no edge from W_1 to W_2 is red.

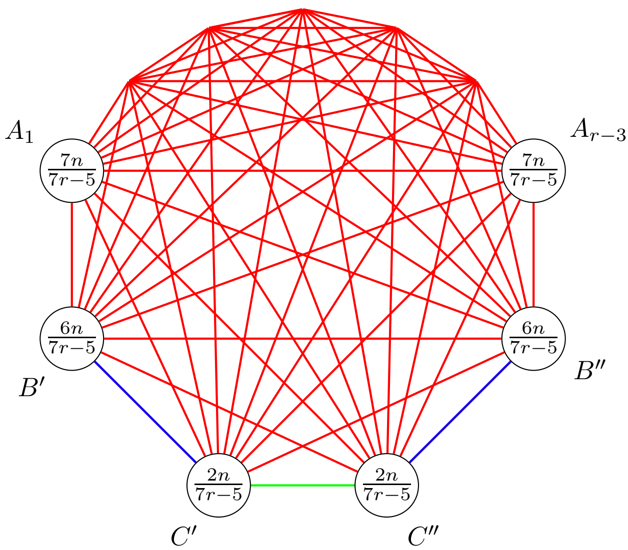


Figure 1.3: Extremal graph for Theorem 1.8.

The reason why we stated this only for $r \geqslant 3$ is that for $r = 2$ one can easily show a stronger result. This is because $\mathscr{F}_2$ consists only of a red edge and a blue triangle. Hence a direct application of the case $r = 2$ of Theorem 1.2 shows that the desired conclusion can already be obtained from the weaker minimum degree assumption that $\delta(G) > \frac{2}{5}n$. For $r \geqslant 3$, however, the constant $\frac{14r-24}{7r-5}$ appearing in Theorem 1.8 is optimal. This can be seen by taking n to be an arbitrary multiple of $7r-5$, a vertex set V of size n

with a partition

$$V = A_1 \cupdot \ldots \cupdot A_{r-3} \cupdot B' \cupdot B'' \cupdot C' \cupdot C''$$

satisfying

$$|A_1| = \ldots = |A_{r-3}| = \tfrac{7n}{7r-5}, \ |B'| = |B''| = \tfrac{6n}{7r-5}, \quad \text{and} \quad |C'| = |C''| = \tfrac{2n}{7r-5}\,,$$

and colouring

- the edges within the partition classes green,
- the edges from C' to C'' green as well,
- the edges from B' to C' and from B'' to C'' blue,
- and all remaining edges red.

We would like to remark that whenever a weighted Turán density $\pi_D(\mathscr{F})$ and the corresponding family $\mathscr{E}$ of extremal graphs have been determined one may ask, similarly, for the *Andrásfai-Erdős-Sós threshold* $\alpha_D(\mathscr{F})$, defined to be the infimal real number α with the following property: Every $\mathscr{F}$-free weighted graph (V, w) with $w[V^2] \subseteq D$ and $\delta(G) > \alpha|V|$ admits an homomorphism into a member of $\mathscr{E}$. For instance, Theorem 1.8 and the graph in Figure 1.3 show $\alpha_{\{0,1,2\}}(\mathscr{F}_{2r}) = \frac{14r-24}{7r-5}$ for $r \geqslant 3$. It would be interesting to study such thresholds $\alpha_D(\mathscr{F})$ in further cases, e.g. for the pairs $(D, \mathscr{F})$ whose Turán densities have been determined in [9].

2. Excluding blue cliques

Many intermediate steps in the proofs of our main results are of the following form: We already know that the coloured graph G under consideration is $\mathscr{F}$-free for some set $\mathscr{F}$ of coloured graphs and we would like to show that for a certain other coloured graph F it must be the case that G is F-free as well. The usual strategy for handling such a problem begins by assigning a positive integral *weight* γ_z to every $z \in V(F)$. Assuming for simplicity that F itself would be a subgraph of G we obtain

$$\sum_{x \in V} \sum_{z \in V(F)} \gamma_z w(x, z) = \sum_{z \in V(F)} \gamma_z d_G(z) \geqslant \delta(G) \sum_{z \in V(F)} \gamma_z\,.$$

Consequently there will exist some vertex $x \in V$ such that

$$\sum_{z \in V(F)} \gamma_z w(x, z) \geqslant \frac{\delta(G)}{n} \sum_{z \in V(F)} \gamma_z\,. \tag{2.1}$$

The basic plan to proceed from this point is that we try to prove by means of some case analysis that this conditions implies $V(F) \cup \{x\}$ to support some member of $\mathscr{F}$, contrary to G being $\mathscr{F}$-free. Often the situation will be a bit more complicated and we will need to iterate this argument multiple times before such a contradiction emerges. For analysing the condition (2.1) it is usually helpful to rewrite it in terms of the function $\widetilde{w}\colon V^2 \longrightarrow \{0,1,2\}$ defined by $\widetilde{w}(u,v) = 2 - w(u,v)$ for all $u, v \in V$. In fact one can easily check that (2.1) is equivalent to

$$\sum_{z \in V(F)} \gamma_z \widetilde{w}(x,z) \leqslant \left(2 - \frac{\delta(G)}{n}\right) \sum_{z \in V(F)} \gamma_z \,. \tag{2.2}$$

Our first argument of this form will establish the following lemma, which will later be used to show that a coloured graph G satisfying the assumption of either Theorem 1.7 or Theorem 1.8 cannot contain a BK_{r+1} (see Lemma 3.1 and Lemma 4.1 below).

Lemma 2.1. *Let $q > b \geqslant 1$ be integers and suppose that G is a coloured graph on n vertices with $\delta(G) > \left(2 - \frac{12}{3q+3b-5}\right)n$ containing a BK_q. Then either BK_{q+1} or $G_{q+b,b}$ is a subgraph of G.*

Proof. Assume contrariwise that G is $\{BK_{q+1}, G_{q+b,b}\}$-free. For each integer $k \in [0, b-1]$ we define

$$p_k = \max(0, k+q+1-2b)\,.$$

In view of

$$k + p_k = \max(k, 2k+q+1-2b) \leqslant \max(b-1, q-1) < q \tag{2.3}$$

there exists a coloured graph H_k of order q having a vertex partition

$$V(H_k) = A \mathbin{\dot\cup} B \mathbin{\dot\cup} C$$

satisfying

- $|A| = k$, $|B| = p_k$, $|C| = q - (k + p_k)$,
- all edges of H_k connecting a vertex in A with a vertex in $A \cup C$ are red,
- and all other edges of H_k are blue (see Fig. 2.1 on the next page).

Since all edges of H_0 are blue and $V(H_0)$ has size q, we know that G contains a copy of H_0. Now let k_* denote the largest integer in $[0, b-1]$ with the property that G contains a copy of H_{k_*} and put $p_* = p_{k_*}$.

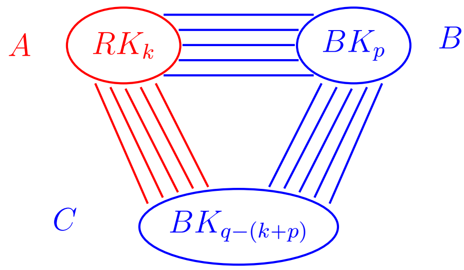


Figure 2.1: The coloured graph H_k.

Let $A \cup B \cup C \subseteq V(G)$ be the vertex set of such a copy with the notation as above. Notice that the calculation (2.3) shows $C \neq \varnothing$. So if $k_* = b-1$, then A and an arbitrary vertex in C would form an RK_b, while the remaining vertices in $B \cup C$ would form a BK_{q-b}. Due to the absence of green edges from A to $B \cup C$ this means that G would contain a $G_{q+b,b}$, which is absurd.

This consideration proves

$$k_* \leqslant b-2 \tag{2.4}$$

and our maximal choice of k_* entails that G does not contain an H_{k_*+1}.

First Case: $k_* < 2b-q-1$.

This yields $p_* = p_{k_*+1} = 0$ and $B = \varnothing$. We assign the weight 3 to the vertices in A and the weight 2 to the vertices in C. In view of $q > b$ the total weight of all vertices in $A \cup C$ is

$$3k_*+2(q-k_*) = 2q+k_* \leqslant 2b+q-2 \leqslant 2b+q-2+\tfrac{1}{2}(q-b-1) = \tfrac{1}{2}(3q+3b-5)\,.$$

Writing γ_z for the weight of every vertex $z \in A \cup C$ we find, by the argument leading to (2.2), a vertex $x \in V(G)$ satisfying

$$\sum_{z \in A \cup C} \gamma_z \widetilde{w}(x,z) < \frac{6(3q+3b-5)}{3q+3b-5} = 6\,.$$

Owing to the integrality of the left side we obtain

$$3\sum_{a \in A} \widetilde{w}(x,a) + 2\sum_{c \in C} \widetilde{w}(x,c) \leqslant 5\,. \tag{2.5}$$

Because of $BK_{q+1} \nsubseteq G$ there exists a vertex $z \in A \cup C$ with $\widetilde{w}(x,z) = 2$. In view of (2.5) this can only happen if $z \in C$ and thus we infer

$$3\sum_{a\in A}\widetilde{w}(x,a) + 2\sum_{c\in C\smallsetminus\{z\}}\widetilde{w}(x,c) \leqslant 1\,,$$

which in turn tells us that all members of $A \cup C \smallsetminus \{z\}$ are red neighbours of x. Moreover, $z \in C$ and $\widetilde{w}(x,z) = 2$ imply $x \notin A$. Consequently,

$$(A \cup \{x\}) \mathbin{\dot\cup} (C \smallsetminus \{z\})$$

is the vertex partition of an H_{k_*+1} in G and we have reached a contradiction.

Second Case: $k_* \geqslant 2b - q - 1$.

Observe that now we have $p_* = k_* + q + 1 - 2b$ and $p_{k_*+1} = p_* + 1$. This time we assign the weight $\gamma_z = 2$ to every $z \in A$ and the weight $\gamma_z = 1$ to every $z \in B \mathbin{\dot\cup} C$. As before we find a vertex $x \in V(G)$ with

$$\sum_{z\in A\cup B\cup C}\gamma_z\widetilde{w}(x,z) < \frac{12(k_*+q)}{3q+3b-5} \overset{(2.4)}{<} \frac{12(q+b-2)}{3(q+b-2)} = 4\,,$$

i.e.,

$$2\sum_{a\in A}\widetilde{w}(x,a) + \sum_{y\in B\cup C}\widetilde{w}(x,y) \leqslant 3\,.$$

Again the absence of a BK_{q+1} in G leads us to a vertex $z \in B \cup C$ with $\widetilde{w}(x,z) = 2$. Moreover, there are only red edges from x to A and at most one blue but no green edges from x to $B \cup C \smallsetminus \{z\}$. This implies, however, that $(A \cup \{x\}) \cup (B \cup C \smallsetminus \{z\})$ supports an H_{k_*+1} in G, which is again a contradiction. □

3. The proof of Theorem 1.7

An iterative application of Lemma 2.1 leads to the following result.

Lemma 3.1. *For $r \geqslant 2$ every $\mathscr{F}_{2r+1}$-free coloured graph G of order n with $\delta(G) > \frac{6r-8}{3r-1}n$ is BK_{r+1}-free.*

Proof. Let q be maximal with $BK_q \subseteq G$ and assume for the sake of contradiction that $q \geqslant r+1$. Due to $G_{2r+1,1} = BK_{2r}$ we have $q < 2r$ and, hence, the number $b = 2r + 1 - q$ satisfies $q > b \geqslant 1$. Since $\delta(G) > \big(2 - \frac{12}{3(2r+1)-5}\big)n$, it follows from Lemma 2.1 that either $BK_{q+1} \subseteq G$ or $G_{2r+1,b} \subseteq G$. The former, however, contradicts the maximality of q and the latter contradicts G being $\mathscr{F}_{2r+1}$-free. □

Now Theorem 1.7 follows by means of a simple application of the Andrásfai-Erdős-Sós theorem.

Proof of Theorem 1.7. Let H denote the simple graph on $V(G)$ whose edges correspond to the blue or red edges of G. The minimum degree condition on G yields

$$\delta(H) \geqslant \tfrac{1}{2}\delta(G) > \tfrac{3r-4}{3r-1}n$$

and Lemma 3.1 tells us that H is K_{r+1}-free. So by Theorem 1.2 H is r-partite and the claim follows. $\square$

4. The proof of Theorem 1.8

Again we begin by utilising Lemma 2.1.

Lemma 4.1. *For $r \geqslant 3$ every $\mathscr{F}_{2r}$-free coloured graph G of order n with $\delta(G) > \frac{14r-24}{7r-5}n$ is BK_{r+1}-free.*

Proof. As in the proof of Lemma 3.1 we look at the largest integer q with $BK_q \subseteq G$ and observe that $G_{2r,1} = BK_{2r-1}$ shows $q \leqslant 2r-1$. So assuming $q \geqslant r+1$ Lemma 2.1 would again tell us that either BK_{q+1} or $G_{2r,2r-q}$ is a subgraph of G, both of which is absurd. Actually, this argument only requires the lower bound $\delta(G) > \big(2 - \frac{12}{6r-5}\big)n$ on the minimum degree of G, which is less than what we stated. $\square$

In order to define the homomorphism demanded by Theorem 1.8 it would be tremendously helpful to know that no induced subgraph of G with three vertices has exactly one red edge. While not being true in general, this assertion will turn out to hold in the important special case that all edges of G that are not themselves red belong to the common red neigbourhood of some RK_{r-2} (see Lemma 4.7 below). This property of G can in turn be derived from a certain "edge-maximality" condition (see Lemma 4.3 and Lemma 4.6 below). The definition that follows facilitates talking about this plan.

Definition 4.2. Let $G = (V, w)$ be a coloured graph.

- (*a*) If G is $\mathscr{F}_{2r}$-free and every $\mathscr{F}_{2r}$-free coloured graph $G' = (V, w')$ having G as a subgraph (i.e., satisfying $w'(x,y) \geqslant w(x,y)$ for all $x, y \in V$) coincides with G, then we say that G is *extremal*.
- (*b*) A blue or green edge of G is called *secure* if it is contained in the common red neighbourhood of some RK_{r-2}.
- (*c*) A *wicked triangle* in G is a triple (x, y, z) of distinct vertices, such that xy is red and xz, yz are either blue or green. If in this situation both xz and yz are blue, then (x, y, z) is said to be a *blue wicked triangle*.

We shall see later that one only needs to deal with the extremal case when proving Theorem 1.8. As indicated above we will prove in this case that all blue and green edges are indeed secure and that wicked triangles do not exist. We commence with the easiest of these claims, the security of blue edges.

Lemma 4.3. *If G designates an extremal $\mathscr{F}_{2r}$-free coloured graph with n vertices and $\delta(G) > \frac{14r-24}{7r-5}n$, then all blue edges of G are secure.*

Proof. Let xy denote an arbitrary blue edge of G. By extremality, the weighted graph G' arising from G by recolouring xy red contains, for some $i \in [r]$, a subgraph isomorphic to $G_{2r,i}$. If $i \neq r$ this subgraph would have at least $r+1$ vertices no two of which are connected by a green edge in G. Consequently, G would contain a BK_{r+1}, which contradicts Lemma 4.1. So G' contains an RK_r and, as G was RK_r-free, the vertices x and y must belong to this RK_r. Its remaining $r-2$ vertices form, in G, an RK_{r-2} whose red neighbourhood contains x and y. □

At this moment we could already rule out the existence of blue wicked triangles (see part *(i)* of Lemma 4.7 below). However, the argument for doing so is very similar to the proof that, provided the green edges are secure as well, there cannot be any wicked triangles at all. For this reason we postpone this step and consider the green edges first. But it will be important to remember that we may already assume the absence of blue wicked triangles when treating the security of green edges.

As a further preparation towards this latter task we need to exclude a configuration that is closely tied to the example given at the end of the introduction demonstrating the optimality of the minimum degree condition in Theorem 1.8.

Definition 4.4. By J we mean the coloured graph of order $r+1$ with vertex set

$$A \cup \{b', b'', c', c''\},$$

where $|A| = r-3$, $c'c''$ is green, and $b'c'$, $b''c''$ are blue, while all other edges are red (see Fig. 4.1).

Lemma 4.5. *A $\{RK_r, BK_{r+1}\}$-free coloured graph G of order n satisfying $\delta(G) > \frac{14r-24}{7r-5}n$ cannot contain J as subgraph.*

Proof. Otherwise let $Q = A \cup \{b', b'', c', c''\} \subseteq V(G)$ be the vertex set of a copy of J in G, the notation being as in Definition 4.4. We assign weights to

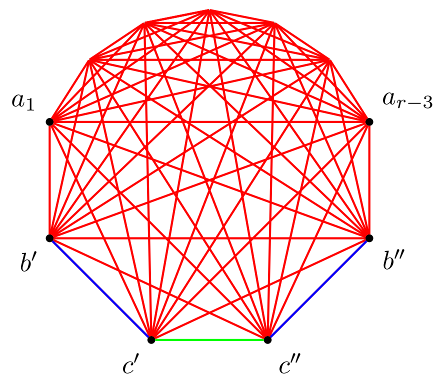


Figure 4.1: The coloured graph J with $A = \{a_1, \ldots, a_{r-3}\}$.

the members of Q according to the formula

$$\gamma_q = \begin{cases} 7 & \text{if } q \in A, \\ 6 & \text{if } q \in \{b', b''\}, \\ 2 & \text{if } q \in \{c', c''\}. \end{cases}$$

So the total weight of all vertices is $7(r-3) + 6 \cdot 2 + 2 \cdot 2 = 7r - 5$ and the standard argument leads to a vertex $x \in V(G)$ with

$$\sum_{q \in Q} \gamma_q \widetilde{w}(x, q) \leqslant 13\,. \tag{4.1}$$

This inequality allows us to analyse the set $T = \{q \in Q \colon \widetilde{w}(x, q) = 2\}$. As as immediate consequence of (4.1) we have $T \subseteq \{b', b'', c', c''\}$. Moreover, the assumption $b' \in T$ would imply that $Q \smallsetminus \{b'\}$ is contained in the red neighbourhood of x, but then $A \cup \{b'', c', x\}$ would induce an RK_r in G, which is absurd. By symmetry the same consideration applies to b'' as well and thus we have $T \subseteq \{c', c''\}$.

Now it follows from $A \cup \{b', b'', c', x\}$ not spanning a BK_{r+1} in G that $c' \in T$ and, similarly, we get $c'' \in T$ as well. By plugging $T = \{c', c''\}$ into (4.1) we learn

$$6 \sum_{q \in A \cup \{b', b''\}} \widetilde{w}(x, q) \leqslant 5\,,$$

and for this reason $A \cup \{b', b''\}$ is part of the red neighbourhood of x. But this means that $A \cup \{b', b'', x\}$ forms an RK_r in G, which is absurd. $\square$

Now we proceed with the security of green edges. The argument starts in a similar way as the proof of Lemma 4.3, but there will be more cases to investigate.

Lemma 4.6. *Suppose that G is an extremal $\mathscr{F}_{2r}$-free coloured graph with $\delta(G) > \frac{14r-24}{7r-5}n$. If G contains no blue wicked triangle, then all green edges of G are secure.*

Proof. Recall that G has to be $\{BK_{r+1}, J\}$-free by Lemma 4.1 and Lemma 4.5. Now consider any green edge xy of G and denote the coloured graph that one obtains from G when one recolours xy to become blue by G'. Due to the extremality of G we know that G' cannot be $\mathscr{F}_{2r}$-free. Exploiting that G is BK_{r+1}-free it is easily seen that G' must contain a $G_{2r,r-1}$ with x and y among its vertices. This $G_{2r,r-1}$ is, of course, only known to be a subgraph of G' that does not need to be induced. In fact, the absence of blue wicked triangles in G entails that "many" edges of this subgraph that "in general" would only be known to be either blue or red must actually be red. To get an overview over the possible cases, we observe that due to the symmetry between x and y one may assume that for the "distinguished" blue edge of the $G_{2r,r-1}$ one of the following three cases occurs.

(*a*) It is xy.
(*b*) It is of the form xb and genuinely blue, where b is in the RK_{r-1}.
(*c*) It is of the form xc and red, where c is in the RK_{r-1}.

In case (*a*) there may be at most one blue edge xa_x from x into the RK_{r-1}, since otherwise G would contain a blue wicked triangle. For the same reason, there can be at most one blue edge ya_y from y into the RK_{r-1}. If both blue edges exist, then $J \nsubseteq G$ implies $a_x = a_y$ and we get the configuration shown in Figure 4.2a. Similarly, the above cases (*b*) and (*c*) lead to one of the situations in Figure 4.2. Observe that it might still be the case that some of the edges drawn blue in these pictures are actually red in G.

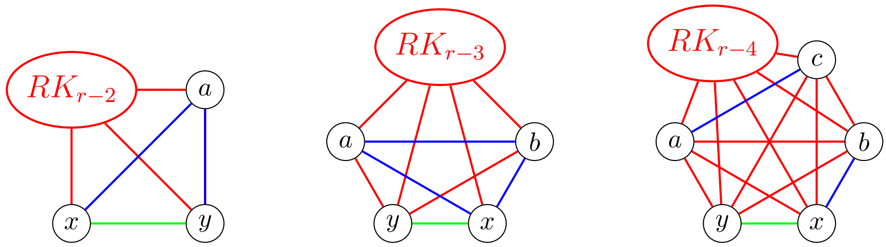
From now on we treat these three cases separately. If the configuration depicted in Figure 4.2a occurs the edge xy is secure due to the RK_{r-2} shown there.

Suppose next that we are in the case shown in Figure 4.2b and let Q be the vertex set of the RK_{r-3}. Assign

- the weight 1 to a, b, x, y,
- and the weight 2 to the members of Q.

So the total weight is $2(r-1)$ and thus there is a vertex v with

$$\sum_{z\in\{a,b,x,y\}} \widetilde{w}(v,z) + 2\sum_{q\in Q} \widetilde{w}(v,q) \leqslant 3\,.$$



(a) First case (b) Second case (c) Third case

Figure 4.2: Possibilities for the edge xy.

In combination with neither $Q \cup \{a, b, v, x\}$ nor $Q \cup \{a, b, v, y\}$ forming a BK_{r+1} this implies that either $\widetilde{w}(a, v) = 2$ or $\widetilde{w}(b, v) = 2$. By symmetry we may suppose that the latter holds, thus getting

$$\sum_{z \in \{a,x,y\}} \widetilde{w}(v, z) + 2 \sum_{q \in Q} \widetilde{w}(v, q) \leqslant 1\,.$$

It follows that all vertices in Q and at least two of a, x, and y are red neighbours of v. Moreover, $v \notin \{a, x, y\}$ and none of the edges va, vx, and vy is green. Now if va and vy are red, then $Q \cup \{a, v, y\}$ forms an RK_r in G, which is absurd. Furthermore, if va and vx are red, then $Q \cup \{a, v, x, y\}$ forms a copy of J in G, which is not possible either. So the only remaining case is that vx and vy are red and then $Q \cup \{v\}$ forms an RK_{r-2} exemplifying the security of xy.

It remains to discuss the configuration shown in Figure 4.2c, which can only arise if $r \geqslant 4$. This time we let Q denote the vertex set of the RK_{r-4}. Assigning

- the weight 4 to x, y,
- the weight 5 to a, b, c,
- and the weight 7 to the members of Q

we have distributed a total weight of $7r - 5$ and in the usual manner we find a vertex v with

$$4 \sum_{z \in \{x,y\}} \widetilde{w}(v, z) + 5 \sum_{z \in \{a,b,c\}} \widetilde{w}(v, z) + 7 \sum_{q \in Q} \widetilde{w}(v, q) \leqslant 13\,.$$

Exploiting that neither $Q \cup \{a, b, c, x\}$ nor $Q \cup \{a, b, c, y\}$ induces a BK_{r+1} we infer that $\widetilde{w}(v, \ell) = 2$ holds for some $\ell \in \{a, b, c\}$. Together with the above inequality this shows that all vertices in $Q \cup \{a, b, c, x, y\}$ except for ℓ are red neighbours of v. Due to the symmetry between a and c we may suppose that $\ell \neq a$. Now $Q \cup \{a, v\}$ is the desired RK_{r-2} with xy in its neighbourhood. $\square$

Finally, we deal with the alleged absence of wicked triangles.

Lemma 4.7. *Let G denote a $\{RK_r, BK_{r+1}\}$-free coloured graph of order n such that $\delta(G) > \frac{14r-24}{7r-5}n$.*

- (i) *If all blue edges of G are secure, then every wicked triangle of G possesses a green edge.*
- (ii) *If moreover the green edges of G are secure as well, then G contains no wicked triangles.*

Proof. Let V and w be the vertex set and weight function of G. Arguing indirectly we let (x, y, z) be a wicked triangle contradicting either of these two statements and such that subject to this $w(x, z) + w(y, z)$ is as large as possible. Set $\alpha = w(x, z)$ and $\beta = w(y, z)$. Notice that $\alpha, \beta \in \{0, 1\}$ and xz, yz are secure. Consequently, there are two $(r-2)$-sets $A, B \subseteq V$ inducing red cliques such that x, z belong to the common red neighbourhood of A while y, z belong to the common red neighbourhood of B. Let us select these sets A and B in such a way that $k = |A \cap B|$ is maximal. Since $(A \cap B) \cup \{x, y\}$ is a red clique and G is RK_r-free, we have

$$k \leqslant r - 3\,. \tag{4.2}$$

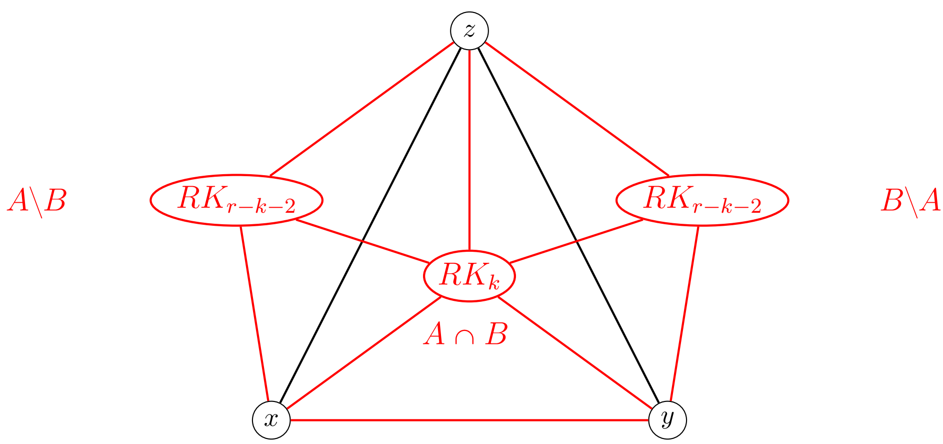


Figure 4.3: The sets A and B. The black pairs are either blue or green.

Notice that $x, y, z \notin A \cup B$. Set $Q = (A \cup B) \cup \{x, y, z\}$, and assign weights to the vertices in Q by defining

$$\gamma_q = \begin{cases} 3 + \alpha - \beta & \text{if } q \in A \smallsetminus B \text{ or } q = x, \\ 3 + \beta - \alpha & \text{if } q \in B \smallsetminus A \text{ or } q = y, \\ 7 & \text{if } q \in A \cap B, \\ r - k + 1 & \text{if } q = z \end{cases}$$

for $q \in Q$. So the total weight of the vertices in Q is

$$6(r - k - 1) + 7k + (r - k + 1) = 7r - 5$$

and by our standard argument there exists a vertex $v \in V$ with

$$\sum_{q \in Q} \gamma_q \widetilde{w}(v, q) \leqslant 13\,. \tag{4.3}$$

It follows that

$$v \notin (A \cap B) \text{ and there is no green edge from } v \text{ to } A \cap B\,. \tag{4.4}$$

Put

$$a = \sum_{q \in A \cup \{x\}} \widetilde{w}(v, q) \quad \text{and} \quad b = \sum_{q \in B \cup \{y\}} \widetilde{w}(v, q)$$

and notice that (4.3) yields

$$(3 + \alpha - \beta)a + (3 + \beta - \alpha)b + 4\widetilde{w}(v, z) \leqslant 13\,, \tag{4.5}$$

because (4.2) implies $\gamma_z \geqslant 4$.

Since $A \cup \{x, v\}$ is not an RK_r, we have $a \geqslant 1$ and, similarly, $b \geqslant 1$. So (4.5) yields that

$$vz \text{ is either blue or red.} \tag{4.6}$$

If $\alpha = 1$, i.e., if xz is blue, then the fact that $A \cup \{v, x, z\}$ is not a BK_{r+1} entails $a \geqslant 2$. Performing the same argument for b we infer

$$a \geqslant 1 + \alpha \quad \text{and} \quad b \geqslant 1 + \beta\,. \tag{4.7}$$

Since $\alpha^2 = \alpha$ and $\beta^2 = \beta$, we have

$$13 < 14 + (1 - \alpha)(1 - \beta) + \alpha\beta = (3 + \alpha - \beta)(4 - \alpha - \beta + \alpha\beta) + (3 + \beta - \alpha)(1 + \beta)$$

and by (4.5) and (4.7) this leads to

$$a \leqslant 3 - \alpha - \beta + \alpha\beta\,. \tag{4.8}$$

Now assume there would exist two distinct vertices in $A \cup \{x\}$, say s and t, that fail to be red neighbours of v. Then $v \notin A \cup \{x\}$ and, in particular, $v \notin \{s, t\}$. So the maximality of $\alpha + \beta$ gives $w(v, s) + w(v, t) \leqslant \alpha + \beta$, whence

$$a \geqslant \widetilde{w}(v, s) + \widetilde{w}(w, t) \geqslant 4 - \alpha - \beta\,.$$

In view of (4.8) this is only possible if $\alpha = \beta = 1$ and the foregoing estimate on a holds with equality. But then $A \cup \{v, x, z\}$ is a BK_{r+1} in G, which is a contradiction. Therefore all but at most one vertex in $A \cup \{x\}$ are red neighbours of v.

On the other hand, $A \cup \{v, x\}$ is not an RK_r in G, so altogether we can conclude that there is a unique $a^* \in A \cup \{x\}$ such that va^* is not red. Similarly, there is a unique $b^* \in B$ such that vb^* is not red.

Next we suppose that vz would be blue. Then, in particular, $v \notin (A \cup B)$ and the combination of (4.5) and (4.7) yields

$$13 \geqslant 4 + (1 + \alpha)(3 + \alpha - \beta) + (1 + \beta)(3 + \beta - \alpha) = 10 + 3(\alpha + \beta) + (\alpha - \beta)^2\,,$$

i.e., $\alpha = \beta = 0$. So there is no wicked triangle with a blue edge and consequently there are only red edges from v to $A \cup B$. Thus $a^* = x$ and $b^* = y$, for which reason (x, y, v) is a wicked triangle. By the maximality of $\alpha + \beta$ it follows that vx and vy are green, i.e., that $a, b \geqslant 2$. But now we get a contradiction to (4.5), which together with (4.6) proves that

$$vz \text{ is red.} \tag{4.9}$$

Since $A \cup \{v, z\}$ cannot be an RK_r, it follows that $a^* \in A$ and, similarly, we have $b^* \in B$. Owing to the uniqueness of a^* and b^* there are only two possibilities, namely $a^* \in A \smallsetminus B$ and $b^* \in B \smallsetminus A$, or $a^* = b^* \in A \cap B$. If the former alternative would hold, then the sets $A \cup \{v\} \smallsetminus \{a^*\}$ and $B \cup \{v\} \smallsetminus \{b^*\}$ would contradict the maximality of k. So the only remaining case is that there is a member $u = a^* = b^*$ of $A \cap B$ such that $Q \smallsetminus \{u\}$ is in the red neighbourhood of v. By (4.4) the edge uv is blue. Since neither $A \cup \{v, x, z\}$ nor $B \cup \{v, y, z\}$ forms a BK_{r+1}, the edges xz and yz are green, i.e., $\alpha = \beta = 0$. Let us recall that this means that there is no wicked triangle with a blue edge.

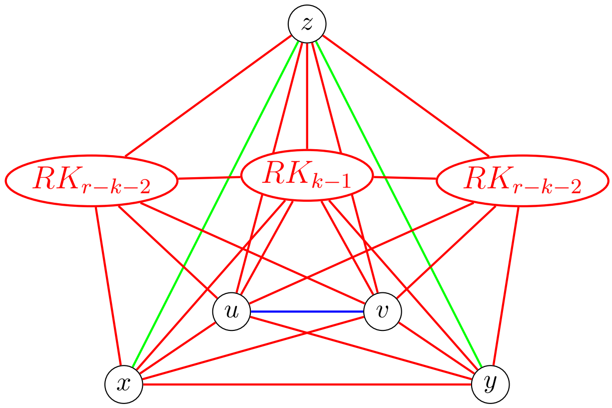


Figure 4.4: Current situation.

At this moment the weights γ_q have done for us whatever they could do, and we proceed by assigning new weights to the vertices in Q and to v. To this end, we define

$$\eta_q = \begin{cases} 1 & \text{if } q \in (A \triangle B) \cup \{x, y, z\}, \\ 2 & \text{if } q \in \{u, v\}, \\ 3 & \text{if } q \in (A \cap B) \smallsetminus \{u\} \end{cases}$$

for $q \in Q \cup \{v\}$. By (4.2) the total weight $2r + k$ is at most $3(r-1)$ and thus there is a vertex t with

$$\sum_{q \in Q \cup \{v\}} \eta_q \widetilde{w}(t, q) \leqslant 5\,. \tag{4.10}$$

We will now analyse the set $T = \big\{q \in Q \cup \{v\} \colon \widetilde{w}(q, t) = 2\big\}$. By (4.10) it needs to be disjoint to $A \cap B \smallsetminus \{u\}$. Suppose now that $v \in T$. Since the triangle (u, t, v) cannot be wicked, it is not the case that ut is a red edge, which in turn yields $\widetilde{w}(u, t) + \widetilde{w}(v, t) \geqslant 3$, contrary to (4.10). This proves that $v \notin T$ and by symmetry $u \notin T$ holds as well.

Now it follows from G being BK_{r+1}-free that each of the four sets

$$(A \smallsetminus B) \cup \{x\},\ (A \smallsetminus B) \cup \{z\},\ (B \smallsetminus A) \cup \{y\}, \ \text{ and } \ (B \smallsetminus A) \cup \{z\}$$

contains a member of T. On the other hand (4.10) yields $|T| \leqslant 2$. For these reasons, we have $T = \{a^*, b^*\}$ for two vertices $a^* \in A \smallsetminus B$ and $b^* \in B \smallsetminus A$.

Next we contend that $S = (Q \cup \{v\}) \smallsetminus T$ contains only red neighbours of t. To see this, consider an arbitrary $s \in S$. In view of $s \notin T$ the edge st is either red or blue. Moreover, at least one of a^* or b^* is a red neighbour of s, so suppose that sa^* is red. Since (s, a^*, t) cannot be a wicked triangle with a blue edge, it follows that st is indeed red.

Now the sets $A \cup \{t\} \smallsetminus \{a^*\}$ and $B \cup \{t\} \smallsetminus \{b^*\}$ contradict the maximality of k. $\square$

We conclude this section by giving the proof of our second main result.

Proof of Theorem 1.8. Let $G' = (V, w')$ be an $\mathscr{F}_{2r}$-free coloured graph with the property that $w'(x,y) \geqslant w(x,y)$ holds for all $x, y \in V$ and such that subject to this condition $e(G')$ is maximal. Then G' is extremal and satisfies $\delta(G') \geqslant \delta(G) > \frac{14r-24}{7r-5}n$. Since every homomorphism from G' to RK_r^- is also a homomorphism from G to RK_r^-, we may suppose for notational simplicity that $G' = G$, i.e., that G itself is extremal.

Now by Lemma 4.3 the blue edges of G are secure and Lemma 4.7*(i)* informs us that G contains no blue wicked triangle. This in turn implies in view of Lemma 4.6 that the green edges of G are secure as well and, hence, Lemma 4.7*(ii)* is applicable, showing that G contains no wicked triangles at all. This fact can be reformulated by saying that the reflexive and symmetric relation "$w(x,y) \in \{0,1\}$" is also transitive, i.e., an equivalence relation. Denote its (nonempty) equivalence classes by $A_1, \dots, A_m$. We will suppose moreover that this indexing has been arranged in such a way that for some integer $s \in [0, m]$ each of the sets $A_1, \dots, A_s$ spans at least one blue edge in G, whilst each of $A_{s+1}, \dots, A_m$ forms a green clique.

For every $i \in [m]$ we denote the minimum degree of the blue graph G induces on A_i by α_i. Notice that

$$\tfrac{14r-24}{7r-5}n < \delta(G) \leqslant 2(n - |A_i|) + \alpha_i$$

holds for every $i \in [m]$, whence

$$2|A_i| - \alpha_i < \tfrac{14}{7r-5}n\,. \tag{4.11}$$

For $i \in [s+1, m]$ we have $\alpha_i = 0$ and the previous inequality simplifies to $|A_i| < \frac{7}{7r-5}n$. If, however, $i \in [s]$, then the trivial bound $\alpha_i < |A_i|$ leads to $|A_i| < \frac{14}{7r-5}n$. By adding these estimates up we obtain

$$n = \sum_{i=1}^{m} |A_i| < \frac{14s + 7(m-s)}{7r-5}n < \frac{m+s}{r-1}n\,,$$

wherefore $m+s \geqslant r$. On the other hand, by taking arbitrary blue edges from each of $A_1, \dots, A_s$ as well as arbitrary vertices from each of $A_{s+1}, \dots, A_m$ we can construct a BK_{m+s} in G. So in view of Lemma 4.1 we must have $m+s=r$. Similar arguments shows that the blue graphs induced by G on $A_1, \dots, A_s$ are triangle-free. Moreover, one has $s \geqslant 1$, for otherwise G would contain an RK_r.

Now for each $i \in [s]$ we find

$$\begin{aligned} n &= |A_i| + \sum_{j \neq i} |A_j| < |A_i| + \frac{14(s-1)+7(m-s)}{7r-5} n \\ &= |A_i| + \frac{7(m+s-2)}{7r-5} n = |A_i| + \frac{7r-14}{7r-5} n \end{aligned}$$

and, consequently, $|A_i| > \frac{9}{7r-5} n$. In combination with (4.11) this leads to $2|A_i| - \alpha_i < \frac{14}{9}|A_i|$, i.e., $\alpha_i > \frac{4}{9}|A_i|$. Since the blue graph G induces on A_i is triangle-free and $\frac{4}{9} > \frac{2}{5}$, the case $r=2$ of Theorem 1.2 entails that this blue graph is bipartite.

Thus for each $i \in [s]$ there is a partition $A_i = B_i \cup C_i$ such that B_i and C_i are green cliques in G. The structure we have thereby found in G may be regarded as a homomorphism from G to a coloured graph of order $m+s=r$ having a blue matching of size s and otherwise red edges only. Due to $s \geqslant 1$ this proves Theorem 1.8. ◻

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