

Enumeration of cyclic permutations in vector grid classes

KASSIE ARCHER AND L.-K. LAUDERDALE

A grid class consists of permutations whose pictorial depiction can be partitioned into increasing and decreasing parts as determined by a given matrix. In this paper, we introduce a method for enumerating cyclic permutations in vector grid classes by establishing a bijective relationship with certain necklaces. We use this method to complete the enumeration of cyclic permutations in the length 3 vector grid classes. In addition, we define an analog of Wilf-equivalence between these sets. We conclude by discussing cyclic permutations in alternating grid classes.

1. Introduction

Grid classes are well-studied classes of permutations (see [1, 15, 17] for example) that are assigned a signature in the form of a matrix M with entries in $\{-1, 0, 1\}$, which in some way determines the structure of the permutations. More specifically, permutations in a given grid class with signature M are comprised of increasing and decreasing parts laid out in a grid that is determined by M . For example, a permutation in the grid class with signature $M = \begin{bmatrix} 1 & 1 \end{bmatrix}$ is comprised of an increasing segment followed by another increasing segment, and thus has at most one descent. We are primarily concerned with the vector grid classes (i.e., those where M is a vector, sometimes referred to as *juxtaposition classes*), which received special attention in [5, 6, 4] among others. In particular, we prove a formula regarding a relationship between cyclic permutations in vector grid classes and necklaces. We then use this formula to enumerate the cyclic permutations in the length 2 and 3 vector grid classes.

Cyclic permutations in given vector grid classes have appeared in several other papers. For example, they were used to characterize the permutations realized by the periodic points of certain dynamical systems [4], appeared in the character formula for a certain representation of the symmetric group [3], and were used in the analysis of card-shuffling techniques [10]. In addition, the descent structure of cyclic permutations, which is closely related to the grid structure, was studied in various papers [14, 12, 7].

In this paper, we subscribe to the convention used in [4], which is to use a signature σ of $+$'s and $-$'s in place of the row vector matrix M of 1 's and -1 's. Our goal in this paper is two-fold: (1) to establish an enumerative relationship between cyclic permutations in grid classes with signature $\sigma \in \{+, -\}^k$ and k -ary necklaces, and (2) to use this relationship to enumerate the set of cyclic permutations in certain grid classes, illustrating its use.

In Section 2, we provide the necessary background for this paper, including definitions and propositions regarding necklaces. In [14], Gessel and Reutenauer used a bijection between necklaces and permutations to determine the number of permutations with a given cycle type and descent set. This was modified in [4] to determine the number of cyclic permutations with signature $\sigma = +^k$, $\sigma = -^k$, or $\sigma = +-$. Analysis of the cycle structure of unimodal permutations (i.e., those with signature $\sigma = +-$, can also be found in [19, 18, 13]). In Section 3, we prove the main theorem of this paper that generalizes these previous results to establish a formula relating cyclic permutations in grid classes with signature $\sigma \in \{+, -\}^k$ and k -ary necklaces.

In Section 4, we recover the enumerations of cyclic permutations in the four grid classes with signature σ when $|\sigma| = 2$. We also prove some necessary results about unimodal cyclic permutations with a given peak position, which will aid in the enumeration of cycles in length 3 vector grid classes.

In Section 5, we complete the enumeration of cyclic permutations in the eight grid classes with signature σ , where $|\sigma| = 3$. The first few terms and the OEIS reference number for each sequence can be found in Table 1.

Table 1: The signatures of the eight length 3 vector grid classes, together with the the first few terms of the enumeration of cycles in the corresponding grid classes and the OEIS entries [16]

σ	First ten terms (starting with $n = 1$)	OEIS entry
$+++$	1, 1, 2, 6, 18, 62, 186, 570, 1680, 4890	A303117
$---$	1, 1, 2, 6, 18, 58, 186, 570, 1680, 4878	A304200
$+-+$	1, 1, 2, 5, 12, 30, 78, 205, 546, 1476	A136704
$-+-$	1, 1, 2, 5, 12, 30, 78, 205, 546, 1476	A136704
$++-$	1, 1, 2, 5, 15, 42, 120, 338, 952, 2671	A303980
$+--$	1, 1, 2, 5, 15, 43, 120, 338, 952, 2672	A304201
$-++$	1, 1, 2, 5, 15, 42, 120, 338, 952, 2671	A303980
$--+$	1, 1, 2, 5, 15, 43, 120, 338, 952, 2672	A304201

In Section 5.4, we generalize the notion of Wilf-equivalence from classical pattern avoidance to this setting. We say that two grid classes are *cyc-Wilf-equivalent* if the number of cyclic permutations of length n in one grid class equals the number of cyclic permutations of length n in the other.

Additionally, we say that two grid classes are *weakly cyc-Wilf-equivalent* if their sizes are equal when $n \not\equiv 2 \pmod{4}$. The results (proven in Section 5.4) are summarized in Table 2.

Table 2: List of cyc-Wilf- and weakly cyc-Wilf-equivalence classes

cyc-Wilf-equivalence classes	weakly cyc-Wilf-equivalence classes
+++	+++ , ---
---	+-+ , -+-
+-+ , -+-	++- , +-+ , -++ , ---
++- , -++	
+-- , --+	

In Section 6, we show that the alternating grid classes of a given size are cyc-Wilf-equivalent. Finally, in Section 7, we include discussion and conjectures.

2. Background

The set of permutations of $[n] = \{1, 2, \dots, n\}$ is denoted \mathcal{S}_n and we write $\pi \in \mathcal{S}_n$ in its one-line notation as $\pi = \pi_1\pi_2 \dots \pi_n$.

2.1. Permutation statistics

We say that a permutation $\pi \in \mathcal{S}_n$ has a *descent* at position i if $\pi_i > \pi_{i+1}$. We say π has an *ascent* at position i if $\pi_i < \pi_{i+1}$. For example, the permutation 4156732 has three descents, namely those at positions 1, 5, and 6, and has three ascents, namely those at positions 2, 3, and 4.

We define a *peak* of the permutation $\pi \in \mathcal{S}_n$ to be $i \in [n]$ such that $\pi_{i-1} < \pi_i$ and $\pi_i > \pi_{i+1}$ (where $\pi_0 = \pi_{n+1} := 0$). Similarly, we define a *valley* of the permutation $\pi \in \mathcal{S}_n$ to be $i \in [n]$ such that $\pi_{i-1} > \pi_i$ and $\pi_i < \pi_{i+1}$ (where $\pi_0 = \pi_{n+1} := n + 1$). For example, the permutation 4156732 has peaks at positions 1 and 5 and has valleys at positions 2 and 7. Notice that this definition is non-standard since we allow peaks and valleys to occur at the beginning and end of the permutation.

Finally, we say a permutation is *unimodal* if it has exactly one peak. Any valleys must occur at the beginning or end of the permutation. For example, there are eight unimodal permutations of length 4:

$$1234, 1243, 1342, 2341, 1432, 2431, 3421, 4321.$$

These permutations respectively have their unique peak at position 4, 3, 3, 3, 2, 2, 2, and 1.

2.2. The σ -classes

For $m < n$, we say a permutation $\pi \in \mathcal{S}_n$ *contains* the pattern $\tau \in \mathcal{S}_m$ if there are indices $i_1 < \dots < i_m$ so that the subsequence $\pi_{i_1}\pi_{i_2} \dots \pi_{i_m}$ is in the same relative order as τ , and we say π *avoids* τ if π does not contain it. For example, the permutation $\pi = 142365$ avoids 321 since there is no length 3 subsequence of π that is decreasing. The set of permutations that avoid a given pattern τ is denoted $\text{Av}(\tau)$ and the set of permutations that avoid a set of patterns $B = \{\tau_1, \tau_2, \dots\}$ is denoted $\text{Av}(B)$. Any permutation class (i.e., a set of permutations closed under pattern containment) may be characterized in terms of pattern avoidance.

Let $\sigma = \sigma_0\sigma_1 \dots \sigma_{k-1} \in \{+, -\}^k$ and define a partition of the set $\{0, 1, \dots, k-1\}$ by $T_\sigma^+ = \{i : \sigma_i = +\}$ and $T_\sigma^- = \{i : \sigma_i = -\}$. That is, T_σ^+ is the set of locations of $+$ in σ , and T_σ^- is the set of locations of $-$ in σ . For example, if $\sigma = ++-+$, then $T_\sigma^+ = \{0, 1, 3\}$ and $T_\sigma^- = \{2\}$. We define the σ -class, denoted \mathcal{S}^σ , as in [2, 4, 6], to be the set of permutations comprised of k contiguous (possibly empty or singleton) segments, so that the i -th such segment is increasing when $i \in T_\sigma^+$ and decreasing when $i \in T_\sigma^-$. For example \mathcal{S}^{+-} is the set of unimodal permutations and \mathcal{S}^{+++} is the set of permutations with at most two descents. A σ -class is an example of a row vector grid class as defined in [1, 15, 17]. Indeed, it is exactly the length k vector grid class for the matrix $M = [M_0 M_1 \dots M_{k-1}]$, where

$$M_i = \begin{cases} +1 & \text{if } \sigma^i = + \\ -1 & \text{if } \sigma^i = -, \end{cases}$$

and can be drawn on a series of line segments each with slope ± 1 , as determined by σ .

Any $\pi \in \mathcal{S}_n^\sigma := \mathcal{S}^\sigma \cap \mathcal{S}_n$ must admit a σ -segmentation defined in [4] (and referred to as a *gridding* in [1, 8, 9]) to be a sequence $\mathbf{e} = (e_0, e_1, \dots, e_k)$ such that $0 = e_0 \leq e_1 \leq \dots \leq e_k = n$ and each segment $\pi_{e_i+1}\pi_{e_i+2} \dots \pi_{e_{i+1}}$ is increasing if $i \in T_\sigma^+$ and decreasing if $i \in T_\sigma^-$. For example, if $\sigma = ++-$, then the permutation $\tau = 268147953$ has two σ -segmentations, namely $(0, 3, 6, 9)$ and $(0, 3, 7, 9)$. If $\sigma = -+-$, then the permutation $\tau' = 862347951$ has four σ -segmentations: $(0, i, j, 9)$, where $i \in \{2, 3\}$ and $j \in \{6, 7\}$. The pictorial representations of the permutations τ and τ' are seen in Figure 1 below.

Every σ -class is a permutation class and can therefore be characterized in terms of pattern avoidance. For example, $\mathcal{S}^{+-} = \text{Av}(213, 312)$. In [5], Atkinson gives a constructive proof that all permutations of a σ -class can be characterized as avoiding a finite set of patterns.

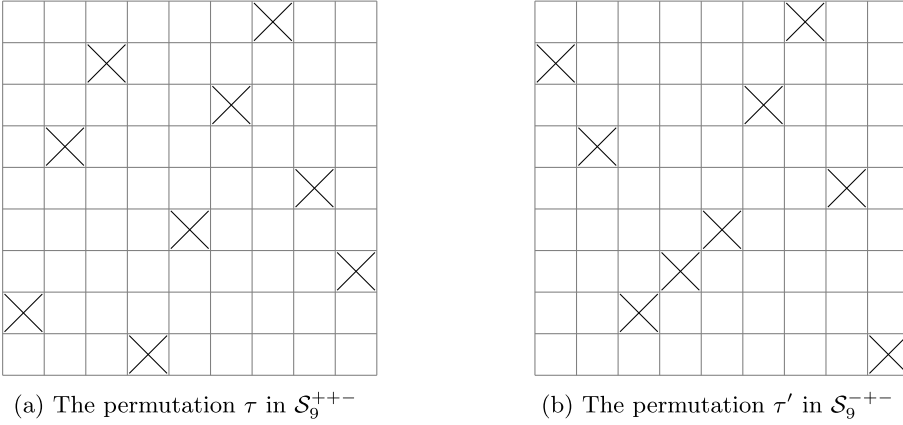


Figure 1: Pictorial representations of the permutations $\tau = 268147953$ and $\tau' = 862347951$ in \mathcal{S}_9^{++++} and \mathcal{S}_9^{-+++} respectively.

2.3. Cyclic permutations

Let \mathcal{C}_n denote the set of cyclic permutations of \mathcal{S}_n (i.e., the permutations in \mathcal{S}_n that are composed of a single cycle with length n). The map defined in [11] by

$$\begin{aligned} \theta : \mathcal{S}_n &\rightarrow \mathcal{C}_n \\ \tau &\mapsto \pi \end{aligned}$$

sends a permutation $\tau = \tau_1\tau_2 \dots \tau_n \in \mathcal{S}_n$ to the cyclic permutation

$$\pi = (\tau_1\tau_2 \dots \tau_n) = \pi_1\pi_2 \dots \pi_n \in \mathcal{C}_n.$$

For example, $\theta(24513) = (24513) = 34251$. Let $\bar{\mathcal{S}}_n$ denote the set of equivalence classes of permutations in \mathcal{S}_n up to cyclic rotation. For example, $|\bar{\mathcal{S}}_3| = 2$ since $[123] = \{123, 231, 312\}$ and $[132] = \{132, 321, 213\}$ are the only two elements. Clearly, we can define a bijection $\bar{\theta} : \bar{\mathcal{S}}_n \rightarrow \mathcal{C}_n$ by $\bar{\theta}([\tau]) = \theta(\tau)$.

Additionally, we denote by \mathcal{C}^σ the set of cyclic permutations in the σ -class (i.e., $\mathcal{C}_n^\sigma := \mathcal{C}_n \cap \mathcal{S}^\sigma$). For a signature σ , we set $c_\sigma(n) := |\mathcal{C}_n^\sigma|$. The proposition below implies that $c_{++-}(n) = c_{-++}(n)$ and that $c_{+--}(n) = c_{--+}(n)$. As a result, the signatures $++-$ and $-++$ are cyc-Wilf-equivalent, as are the signatures $+--$ and $--+$.

Proposition 2.1 ([4, Proposition 2.7]). *For $\sigma = \sigma_0\sigma_1 \dots \sigma_{k-1} \in \{+, -\}^k$, let σ^r denote the signature $\sigma_{k-1}\sigma_{k-2} \dots \sigma_0$. Then $|\mathcal{C}_n^\sigma| = |\mathcal{C}_n^{\sigma^r}|$.*

The complementary signature σ^c , defined as $\sigma_i^c = +$ if $\sigma_i = -$ and $\sigma_i^c = -$ if $\sigma_i = +$, is not necessarily cyc-Wilf-equivalent to σ . For example, if $\sigma = + - -$, then $\sigma^c = - + +$ and these are not cyc-Wilf-equivalent. We conjecture in Section 7 that σ and σ^c are always weakly cyc-Wilf-equivalent.

2.4. Words and necklaces

A *word* of length n on k letters is a sequence $s = s_1s_2\dots s_n$ with $s_i \in \{0, 1, \dots, k-1\}$ for all $i \in [n]$. We denote the set of all such words by $\mathcal{W}_k(n)$. For example,

$$\mathcal{W}_3(2) = \{00, 01, 02, 10, 11, 12, 20, 21, 22\}.$$

A word s is *r-periodic* if it can be written as the concatenation of r copies of a word q of length $\frac{n}{r}$ (i.e., if $s = q^r$), and a word is *primitive* if it is not r -periodic for any $r > 1$. In other words, we say a word is primitive if it cannot be written as the concatenation of two or more copies of a shorter word. For example, the word 0020112 is primitive and the word 00120012 is 2-periodic. As another example, all binary words of length 7 are primitive with the exceptions of the word 0000000 and the word 1111111.

We define the *evaluation* of a word to be the sequence $(a_0, a_1, \dots, a_{k-1})$, where a_i is defined as

$$a_i = |\{j \in [n] : s_j = i\}|.$$

That is, the evaluation of a word records the number of times each letter appears in the word. For example, the evaluation of the word 0012001022 is $(5, 2, 3)$ since there are five 0's, two 1's and three 2's. The evaluation of 0020020 is $(5, 0, 2)$ since there are five 0's, zero 1's, and two 2's.

We denote the cyclic rotation of the word $s = s_1s_2\dots s_n$ starting with s_i by

$$s_{[i, \rightarrow]} = s_i s_{i+1} \dots s_n s_1 \dots s_{i-1}.$$

For example, if $s = 010111$, then $s_{[4, \rightarrow]} = 111010$ and $s_{[6, \rightarrow]} = 101011$. We denote the set of cyclic rotations of s by $[s]$, so that

$$[s] = \{s_{[i, \rightarrow]} : i \in [n]\}.$$

For example,

$$[01011] = \{01011, 10110, 01101, 11010, 10101\}.$$

Notice that the relation given by “ $s \sim t$ if $s \in [t]$ ” is an equivalence relation. This equivalence relation preserves periodicity. That is, if $s \sim t$ and s is r -periodic, then t is r -periodic. In particular, if s is primitive, then so is t . We refer to a single equivalence class $[s]$ as a *necklace*.

We refer to the set $\{[s] : s \in \mathcal{W}_k(n)\}$ as the set of necklaces of length n on k letters and denote it by $\overline{\mathcal{W}}_k(n)$. A necklace $[s]$ is called *r-periodic* (respectively, *primitive*) if a representative s of the equivalence class is r -periodic (respectively, primitive). Let $\mathcal{N}_k(n)$ denote the set of primitive necklaces on k letters of length n , and let $L_k(n)$ denote the number of such necklaces. (This notation is standard as $L_k(n)$ is the number of *Lyndon words*, where a Lyndon word refers to a specific representative of a given necklace.) The following proposition is well-known and is easily shown using Möbius inversion. We let μ denote the number-theoretic Möbius function.

Proposition 2.2. *If $n \geq 1$, the number of necklaces of length n on k letters is equal to*

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu(d) k^{\frac{n}{d}}.$$

For a given $\sigma \in \{+, -\}^k$ and word $s \in \mathcal{W}_k(n)$, we define $o_\sigma(s)$ to be

$$o_\sigma(s) = |\{i \in [n] : s_i \in T_\sigma^-\}|.$$

That is, $o_\sigma(s)$ is the number of terms in s that are locations of $-$ in σ . We define $o_\sigma([s]) = o_\sigma(s)$ for any necklace $[s]$ as well.

Example 2.3. If $\sigma = +-$, then $T_\sigma^- = \{1\}$, so $o_\sigma(s)$ is the number of 1’s that appear in s . Thus, we have $o_\sigma(01001) = 2$ and $o_\sigma(01111) = 4$.

Example 2.4. If $\sigma = -+--$, then $T_\sigma^- = \{0, 2, 3\}$, so $o_\sigma(s)$ is the number of 0’s, 2’s, or 3’s that appear in s . Thus, $o_\sigma(1203123) = 5$ and $o_\sigma(0000103) = 6$.

Let $L_k^o(n; \sigma)$ denote the size of the set of primitive necklaces $[s]$ so that s is a primitive word of size n for which $o_\sigma(s)$ is odd. We define $\mathcal{N}_k^*(n; \sigma) \subseteq \overline{\mathcal{W}}_k(n)$ to be the set of primitive necklaces together with all 2-periodic necklaces $[s] = [q^2]$ such that q is a primitive word of length $\frac{n}{2}$ for which $o_\sigma(q)$ is odd. Notice that when n is odd these necklaces are exactly those from $\mathcal{N}_k(n)$ and when n is even, the size of this set is exactly $L_k(n) + L_k^o(\frac{n}{2}; \sigma)$. For ease of notation, we set $L_k^*(n; \sigma) := L_k(n) + L_k^o(\frac{n}{2}; \sigma)$ with the assumption that $L_k^o(\frac{n}{2}; \sigma) = 0$ when n is odd.

Example 2.5. Suppose $\sigma = +-$ and $n = 8$. Then $\mathcal{N}_2^*(8; +-)$ is comprised of all the primitive necklaces given by the set $\mathcal{N}_2(8)$ together with the 2-periodic necklaces $[00010001]$ and $[01110111]$.

The set $\mathcal{N}_k^*(n; \sigma)$ appears in many proofs in this paper and is a major part of the statement and proof of our main theorem, Theorem 3.1. Using Möbius inversion, it is a straightforward exercise to compute $L_k^o(n; \sigma)$ and $L_k^*(n; \sigma)$ for a given σ of length 2 or 3 (as in the proof of Theorem 5.10). Therefore, in this paper we present the enumeration of C_n^σ for each $\sigma \in \{+, -\}^3$ in terms of these numbers.

Lemma 2.6. *If $n \geq 1$, then $L_3^*(n; \sigma) = L_3^*(n; \sigma_r)$.*

Proof. It is enough to show that $L_3^o(\frac{n}{2}; \sigma) = L_3^o(\frac{n}{2}; \sigma_r)$. When n is odd, the equality is clear, and thus we assume n is even. We define the following bijection from necklaces $[s]$ of length $\frac{n}{2}$ with $o_\sigma(s)$ odd to necklaces $[s']$ of size $\frac{n}{2}$ with $o_{\sigma_r}(s')$ odd. Let $[s]$ be a primitive necklace of length $\frac{n}{2}$ for which $o_\sigma(s)$ is odd and take $[s']$ to be the necklace obtained by setting $s'_i = k - 1 - s_i$. Notice that $T_{\sigma_r}^- = \{k - 1 - i : i \in T_\sigma^-\}$, and thus $o_{\sigma_r}(s') = |\{i \in [\frac{n}{2}] : s'_i \in T_{\sigma_r}^-\}|$ is equal to $o_\sigma(s)$. Therefore, $o_{\sigma_r}(s')$ is odd and so $L_3^o(\frac{n}{2}; \sigma) = L_3^o(\frac{n}{2}; \sigma_r)$. \square

2.5. An ordering on words and necklaces

Let $s = s_1s_2 \dots s_n$ and $t = t_1t_2 \dots t_n$ be words. Let \prec_σ be the linear order on $\mathcal{W}_k(n)$ defined by $s \prec_\sigma t$ if one of the following holds:

- (1) $s_1 < t_1$,
- (2) $s_1 = t_1 \in T_\sigma^+$ and $s_2s_3 \dots s_n \prec_\sigma t_2t_3 \dots t_n$, or
- (3) $s_1 = t_1 \in T_\sigma^-$ and $t_2t_3 \dots t_n \prec_\sigma s_2s_3 \dots s_n$.

Equivalently, $s \prec_\sigma t$ if, letting $j \geq 1$ be the smallest such that $s_j \neq t_j$, either $o_\sigma(s_1s_2 \dots s_{j-1})$ is even and $s_j < t_j$, or $o_\sigma(s_1s_2 \dots s_{j-1})$ is odd and $s_j > t_j$.

Example 2.7. If $\sigma = +^k$, then $s \prec_\sigma t$ exactly when $s < t$ under the standard lexicographical ordering. However, if $\sigma = + - -$, then $02212 \prec_\sigma 02211$ since their first four terms agree, an odd number of these first four terms are elements of T_σ^- , and in the fifth position $2 > 1$.

For $\sigma \in \{+, -\}^k$, let us define a map $\Pi_\sigma : \mathcal{N}_k^*(n; \sigma) \rightarrow \bar{\mathcal{S}}_n$. First, suppose $s = [s_1s_2 \dots s_n]$ is a primitive necklace. Then the words $s_{[i, \rightarrow]}$ are all distinct for $i \in [n]$. Thus, there exists a unique permutation $\tau \in \mathcal{S}_n$ such that $\tau_i < \tau_j$ if and only if $s_{[i, \rightarrow]} \prec_\sigma s_{[j, \rightarrow]}$. That is, τ records the ordering under \prec_σ of the shifts of s . In this case, let $\Pi_\sigma([s]) = [\tau]$. If $[s]$ is not a primitive necklace, then $s = q^2$ for some primitive word q of length $r := \frac{n}{2}$ so that $o_\sigma(q)$ is odd. In this case, the $s_{[i, \rightarrow]}$ are all distinct for $1 \leq i \leq \frac{n}{2}$ and $s_{[i, \rightarrow]} = s_{[i-r, \rightarrow]}$ for all $\frac{n}{2} < i \leq n$. For $[s]$ non-primitive, we define $\Pi_\sigma([s]) = [\tau]$, where for a given representative $s = s_1s_2 \dots s_n$, τ is the unique permutation such that:

- $\tau_i < \tau_j$ when $s_{[i, \rightarrow]} \prec_\sigma s_{[j, \rightarrow]}$;
- $\tau_1 < \tau_{r+1}$; and
- for all $1 \leq i < \frac{n}{2}$,
 - if $\tau_i < \tau_{r+i}$ and $s_i \in T_\sigma^+$, then $\tau_{i+1} < \tau_{r+i+1}$,
 - if $\tau_i < \tau_{r+i}$ and $s_i \in T_\sigma^-$, then $\tau_{i+1} > \tau_{r+i+1}$,
 - if $\tau_i > \tau_{r+i}$ and $s_i \in T_\sigma^+$, then $\tau_{i+1} > \tau_{r+i+1}$, and
 - if $\tau_i > \tau_{r+i}$ and $s_i \in T_\sigma^-$, then $\tau_{i+1} < \tau_{r+i+1}$.

Under this definition, though taking different representatives $s^{(1)}$ and $s^{(2)}$ of the necklace $[s]$ would result in different permutations $\tau^{(1)}$ and $\tau^{(2)}$, they would be the same up to cyclic rotation. That is, we would have that $[\tau^{(1)}] = [\tau^{(2)}]$, and thus Π_σ is well-defined. First, let us see an example of Π_σ for a primitive necklace.

Example 2.8. Suppose $\sigma = +-$ and $[s] = [0010011]$. Taking $s = 0010011$, there are seven distinct words to compare under the ordering \prec_σ , namely

$$\begin{aligned}
 s_{[1, \rightarrow]} &= 0010011, \\
 s_{[2, \rightarrow]} &= 0100110, \\
 s_{[3, \rightarrow]} &= 1001100, \\
 s_{[4, \rightarrow]} &= 0011001, \\
 s_{[5, \rightarrow]} &= 0110010, \\
 s_{[6, \rightarrow]} &= 1100100, \text{ and} \\
 s_{[7, \rightarrow]} &= 1001001.
 \end{aligned}$$

Under this ordering, we clearly have that $s_{[1, \rightarrow]}, s_{[2, \rightarrow]}, s_{[4, \rightarrow]}$ and $s_{[5, \rightarrow]}$ are the four smallest words under the ordering determined by \prec_σ since in each case, their first element is 0. To compare each pairwise, we check up to the first place the two words disagree. If they disagree in the i -th position and there are an even number of ones preceding that, the word with 0 in the i -th position is smaller. If there are an odd number of ones, the word with 1 in the i -th place is smaller.

For example, $s_{[1, \rightarrow]}$ and $s_{[4, \rightarrow]}$ disagree for the first time in the 4-th position. The first three elements, 001, have an odd number of ones, so $s_{[4, \rightarrow]}$ is smaller than $s_{[1, \rightarrow]}$. Continuing this process, we obtain that $\tau_1 = 2$, $\tau_2 = 4$, $\tau_4 = 1$, and $\tau_5 = 3$. Similarly, we can compare $s_{[3, \rightarrow]}, s_{[6, \rightarrow]}$ and $s_{[7, \rightarrow]}$ and find that $\tau_3 = 7$, $\tau_6 = 5$, and $\tau_7 = 6$. Thus $\Pi_\sigma([0010011]) = [2471356]$.

Now, let us consider a non-primitive necklace as an example.

Example 2.9. Suppose that $\sigma = + - -$ and $[s] = [01210121]$. Let us take the representative $s = 01210121$ and compute the corresponding τ . Clearly, we can see that

$$s_{[1,\rightarrow]} = s_{[5,\rightarrow]} \prec_{\sigma} s_{[2,\rightarrow]} = s_{[6,\rightarrow]} \prec_{\sigma} s_{[4,\rightarrow]} = s_{[8,\rightarrow]} \prec_{\sigma} s_{[3,\rightarrow]} = s_{[7,\rightarrow]}.$$

Now, $\tau_i < \tau_j$ if $s_{[i,\rightarrow]} \prec_{\sigma} s_{[j,\rightarrow]}$. When $j = i + 4$, we must decide what occurs. We take $\tau_1 < \tau_5$. Since $s_1 = 0 \in T_{\sigma}^+$, we also have $\tau_2 < \tau_6$. Since $s_2 = 1 \in T_{\sigma}^-$, we have $\tau_3 > \tau_7$. Since $s_3 = 2 \in T_{\sigma}^-$, we obtain $\tau_4 < \tau_8$. Taken together, we get $[\tau] = [13852476]$.

3. Main theorem

This section includes the main theorem in this paper, Theorem 3.1, which states that, for $\sigma \in \{+, -\}^k$, the set of σ -segmentations of permutations in \mathcal{C}_n^{σ} is equinumerous to $\mathcal{N}_k^*(n; \sigma)$. We will use Theorem 3.1 to enumerate the set of cyclic permutations in several vector grid classes in Sections 4 and 5.

Theorem 3.1. *For $k \geq 2$, $n \geq 3$ and $\sigma \in \{+, -\}^k$,*

$$L_k^*(n; \sigma) = \sum_{\pi \in \mathcal{C}_n^{\sigma}} |\{\mathbf{e} : \mathbf{e} \text{ is a } \sigma\text{-segmentation of } \pi\}|.$$

Furthermore, the number of necklaces in $\mathcal{N}_k^*(n; \sigma)$ with evaluation $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$ is equal to the number of permutations $\pi \in \mathcal{C}_n^{\sigma}$ that admit the σ -segmentation $\mathbf{e} = (e_0, e_1, \dots, e_k)$, where $e_0 = 0$ and $e_i = \sum_{j < i} a_j$ for $i \in [k]$.

For any $\sigma \in \{+, -\}^k$ and any $n \geq 1$, let us define the set $SS(n; \sigma)$ as

$$SS(n; \sigma) = \{(\pi, \mathbf{e}) : \pi \in \mathcal{C}_n^{\sigma}, \mathbf{e} \text{ is a } \sigma\text{-segmentation of } \pi\}.$$

That is, $SS(n; \sigma)$ consists of all cyclic *gridded permutations* [1, 8, 9]. To prove Theorem 3.1, we construct a bijection from $\mathcal{N}_k^*(n; \sigma)$ to $SS(n; \sigma)$ that sends a necklace with evaluation $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$ to some (π, \mathbf{e}) , where $\pi \in \mathcal{C}_n^{\sigma}$ admits a σ -segmentation $\mathbf{e} = (e_0, e_1, \dots, e_k)$ with $e_0 = 0$ and $e_i = \sum_{j < i} a_j$ for $i \in [k]$. This bijection associates a permutation to the necklace by observing the relative order of the elements that comprise the necklace with respect to a certain ordering determined by σ . This proof requires several lemmas and a few additional definitions.

Let \mathbb{N}_0 denote the set of nonnegative integers and let \mathbb{N}_0^k denote the set of sequences with length k and entries from \mathbb{N}_0 . Suppose $\sigma \in \{+, -\}^k$ and let $\varphi_\sigma : \mathcal{N}_k^*(n; \sigma) \rightarrow \mathcal{S}_n \times \mathbb{N}_0^k$ be defined by $\varphi_\sigma([s]) = (\pi, \mathbf{e})$ where $\pi = \bar{\theta} \circ \Pi_\sigma([s])$ and given that the evaluation of $[s]$ is $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$, we have that $\mathbf{e} = (e_0, e_1, \dots, e_k)$ with $e_0 = 0$ and $e_i = \sum_{j < i} a_j$ for $i \in [k]$. We will show in Lemmas 3.5 through 3.8 that $\varphi_\sigma([s])$ is indeed an element of $SS(n; \sigma)$ and that the map φ_σ is a bijection. First, consider the following two examples that demonstrate that $\varphi_\sigma([s]) \in SS(n; \sigma)$.

Example 3.2. Suppose $\sigma = +-$ and consider the necklace $[s] = [0010011]$. Notice that $[s]$ has evaluation $\mathbf{a} = (4, 3)$ since it has four 0's and three 1's. In Example 2.8, we saw that $\Pi_\sigma([0010011]) = [2471356]$. Thus, $\varphi_\sigma([0010011]) = (\pi, \mathbf{e})$, where $\pi = (2471356) = 3457621$ and $\mathbf{e} = (0, 4, 7)$. Notice that $3457621 \in \mathcal{C}_7^\sigma$ since it is increasing, then decreasing, and that \mathbf{e} is a σ -segmentation of π .

Example 3.3. Suppose $\sigma = +--$ and $[s] = [01210121]$, which has evaluation $\mathbf{a} = (2, 4, 2)$. We saw in Example 2.9 that $\Pi_\sigma([01210121]) = [13852476]$. Thus, $\varphi_\sigma([01210121]) = (\pi, \mathbf{e})$, where $\pi = (13852476) = 34872165$ and $\mathbf{e} = (0, 2, 6, 8)$. Notice that $34872165 \in \mathcal{C}_8^\sigma$ since 34 is increasing, 8721 is decreasing, and 65 is decreasing, and that $\mathbf{e} = (0, 2, 6, 8)$ is a σ -segmentation of π .

In both examples above, we have that $\varphi_\sigma([s]) \in SS(n; \sigma)$. We will see in Lemma 3.5 that in general this will happen.

Given a permutation τ of length n and a vector $\mathbf{e} = (e_0, e_1, \dots, e_k)$ with $0 = e_0 \leq e_1 \leq \dots \leq e_k = n$, let us define the τ -monotone word $s = s_1 s_2 \dots s_n$ induced by \mathbf{e} by letting $s_i = t$ whenever $e_t < \tau_i \leq e_{t+1}$. For example, if $\tau = 34872165$ and $\mathbf{e} = (0, 2, 6, 8)$, then $s = 11220011$ is the τ -monotone word induced by \mathbf{e} . Notice that if the evaluation of s is $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$, then $e_i = \sum_{j < i} a_j$ for all $i \in [k]$. We say that $[s]$ is the $[\tau]$ -monotone necklace if s is a τ -monotone word for some representatives s and τ of $[s]$ and $[\tau]$, respectively.

Let $\sigma \in \{+, -\}^k$ be arbitrary and let $\psi_\sigma : SS(n; \sigma) \rightarrow \overline{\mathcal{W}}_k(n)$ be defined by $\psi_\sigma((\pi, \mathbf{e})) = [s]$, where $[s] = [s_1 s_2 \dots s_n]$ is the $[\tau]$ -monotone necklace for $[\tau] = \bar{\theta}^{-1}(\pi)$. We will show that φ_σ gives a bijection from $\mathcal{N}_k^*(n; \sigma)$ onto $SS(n; \sigma)$ and that ψ_σ is the inverse of φ_σ .

Example 3.4. Suppose $\sigma = ++-$. Let us consider the cyclic permutation $\pi = 46723581 = (14265378) \in \mathcal{C}_8^\sigma$ together with the σ -segmentation $\mathbf{e} = (0, 3, 7, 8)$. Then $\psi_\sigma((\pi, \mathbf{e}))$ will have three 0's, four 1's, and one 2. The cycle notation of π informs us where to place each letter. Thus, we obtain

$\psi_\sigma((\pi, \mathbf{e})) = [01011012]$. Notice that $\varphi_\sigma([01011012]) = (\pi, \mathbf{e})$. If we consider instead the σ -segmentation $\mathbf{f} = (0, 3, 6, 8)$ of π , then we obtain $\psi_\sigma((\pi, \mathbf{f})) = [01011022]$. Again, we get that $\varphi_\sigma([01011022]) = (\pi, \mathbf{f})$.

Lemma 3.5. *Suppose $n, k \geq 1$ and $\sigma \in \{+, -\}^k$. Let $[s] \in \mathcal{N}_k^*(n; \sigma)$. Then $\varphi_\sigma([s]) \in SS(n; \sigma)$.*

Proof. Let $\Pi_\sigma(s) = \tau$ and $\varphi_\sigma(s) = (\pi, \mathbf{e})$, where $\mathbf{e} = (e_0, e_1, \dots, e_k)$. It is clear from the definition of Π_σ that for all $a, b \in [n]$, $\tau_a < \tau_b$ implies $s_a \leq s_b$. It follows that for $0 \leq t < k$, $s_i = t$ if and only if $e_t < \tau_i \leq e_{t+1}$. Now, suppose that $e_t < \tau_i < \tau_j \leq e_{t+1}$, and so $s_{[i, \rightarrow]} \preceq_\sigma s_{[j, \rightarrow]}$ with $s_i = s_j = t$. If $t \in T_\sigma^+$, then $s_{[i+1, \rightarrow]} \prec_\sigma s_{[j+1, \rightarrow]}$, and so $\tau_{i+1} < \tau_{j+1}$. Similarly, if $t \in T_\sigma^-$, then $s_{[j+1, \rightarrow]} \prec_\sigma s_{[i+1, \rightarrow]}$, and so $\tau_{i+1} > \tau_{j+1}$.

Now let $0 \leq t < k$, and suppose that the indices j such that $s_j = t$ are j_1, \dots, j_m , ordered in such a way that $\tau_{j_1} < \tau_{j_2} < \dots < \tau_{j_m}$, where $m = e_{t+1} - e_t$. Then $\tau_{j_\ell} = e_{t-1} + \ell$ for $1 \leq \ell \leq m$, and by the previous paragraph, we also have that $\tau_{j_1+1} < \tau_{j_2+1} < \dots < \tau_{j_m+1}$ if $t \in T_\sigma^+$, and $\tau_{j_1+1} > \tau_{j_2+1} > \dots > \tau_{j_m+1}$ if $t \in T_\sigma^-$. Using that $\tau_{j_\ell+1} = \pi_{\tau_{j_\ell}} = \pi_{e_t+\ell}$, this is equivalent to $\pi_{e_t+1} < \pi_{e_t+2} < \dots < \pi_{e_t+m}$ if $t \in T_\sigma^+$, and $\pi_{e_t+1} > \pi_{e_t+2} > \dots > \pi_{e_t+m}$ if $t \in T_\sigma^-$. Note that $e_t + m = e_{t+1}$, so this condition states that \mathbf{e} is a σ -segmentation of π . Since π is a cyclic permutation, this proves that $\pi \in \mathcal{C}^\sigma$. \square

Lemma 3.6. *Let $\sigma \in \{+, -\}^k$ be arbitrary, let $\tau \in \mathcal{S}_n$ with $\pi = \theta(\tau) \in \mathcal{C}^\sigma$, and suppose that $\mathbf{e} = (e_0, e_1, \dots, e_k)$ is a σ -segmentation of π . Suppose that $e_t < \tau_i < \tau_j \leq e_{t+1}$ for some $1 \leq i, j \leq n$. Then $\tau_{i+1} < \tau_{j+1}$ if $t \in T_\sigma^+$, and $\tau_{i+1} > \tau_{j+1}$ if $t \in T_\sigma^-$, where we let $\tau_{n+1} := \tau_1$.*

Proof. Since $e_t < \tau_i < \tau_j \leq e_{t+1}$, both π_{τ_i} and π_{τ_j} lie in the segment $\pi_{e_t+1} \dots \pi_{e_{t+1}}$. If $t \in T_\sigma^+$, this segment is increasing, so $\tau_{i+1} = \pi_{\tau_i} < \pi_{\tau_j} = \tau_{j+1}$. The argument is analogous if $t \in T_\sigma^-$. \square

Lemma 3.7. *Suppose $n, k \geq 1$ and $\sigma \in \{+, -\}^k$. Let $\pi \in \mathcal{C}_n^\sigma$ and \mathbf{e} be a σ -segmentation of π . Then $\psi_\sigma((\pi, \mathbf{e})) \in \mathcal{N}_k^*(n; \sigma)$.*

Proof. Let $\psi_\sigma((\pi, \mathbf{e})) = [s]$ and $\tau = \theta^{-1}(\pi)$. Assume that $\mathbf{e} = (e_0, e_1, \dots, e_k)$. If s is primitive, we are done. Suppose that s is not primitive, so it can be written as q^m for some $m \geq 2$ and some primitive word q with $|q| = r = \frac{n}{m}$. Then, $s_i = s_{i+r}$ for all i (using addition modulo n in the index for the remainder of the proof). Fix i , and let $t = s_i = s_{i+r}$. Because of the way that s is defined, we must have $e_t < \tau_i, \tau_{i+r} \leq e_{t+1}$, so we can apply Lemma 3.6 to this pair.

Suppose first that $g = o_\sigma(q)$ is even. If $\tau_i < \tau_{i+r}$, then applying Lemma 3.6 r times we get $\tau_{i+r} < \tau_{i+2r}$, since the inequality involving $\tau_{i+\ell}$ and $\tau_{i+r+\ell}$

switches exactly g times as ℓ increases from 0 to r . Starting with $i = 1$ and applying this argument repeatedly, we see that if $\tau_1 < \tau_{1+r}$, then

$$\tau_1 < \tau_{1+r} < \tau_{1+2r} < \cdots < \tau_{1+(m-1)r} < \tau_{1+mr} = \tau_1,$$

which is a contradiction. A symmetric argument shows that if $\tau_1 > \tau_{1+r}$, then

$$\tau_1 > \tau_{1+r} > \tau_{1+2r} > \cdots > \tau_{1+(m-1)r} > \tau_{1+mr} = \tau_1,$$

also a contradiction.

It remains to consider the case that $g = o_\sigma(q)$ is odd. If m is even and $m \geq 4$, then letting $q' = q^2$ we have $s = (q')^{\frac{m}{2}}$. Letting $r' = |q'| = 2r$ and $g' = o_\sigma(q') = 2g$, the same argument as above using r' and g' yields a contradiction. If m is odd, suppose without loss of generality that $\tau_1 < \tau_{1+r}$. Note that applying Lemma 3.6 r times to the inequality $\tau_i < \tau_{i+r}$ (respectively $\tau_i > \tau_{i+r}$) yields $\tau_{i+r} > \tau_{i+2r}$ (respectively $\tau_{i+r} < \tau_{i+2r}$) in this case, since the inequality involving $\tau_{i+\ell}$ and $\tau_{i+r+\ell}$ switches an odd number of times. Consider two cases:

- If $\tau_1 < \tau_{1+2r}$, then Lemma 3.6 applied repeatedly in blocks of $2r$ times yields $\tau_1 < \tau_{1+2r} < \tau_{1+4r} < \cdots < \tau_{1+(m-1)r}$. Applying now Lemma 3.6 r times starting with $\tau_1 < \tau_{1+(m-1)r}$ gives $\tau_{1+r} > \tau_{1+mr} = \tau_1$, which contradicts the assumption $\tau_1 < \tau_{1+r}$.
- If $\tau_1 > \tau_{1+2r}$, applying Lemma 3.6 r times we get $\tau_{1+r} < \tau_{1+3r}$, and by repeated application of the lemma in blocks of $2r$ times it follows that $\tau_{1+r} < \tau_{1+3r} < \tau_{1+5r} < \cdots < \tau_{1+(m-2)r} < \tau_{1+mr} = \tau_1$, contradicting again the assumption $\tau_1 < \tau_{1+r}$.

The only case left is when g is odd and $m = 2$, that is, when $s_1 s_2 \dots s_n = q^2$ and q has an odd number of letters in T_σ^- . Thus, we have shown that $[s] \in \mathcal{N}_k^*(n; \sigma)$. □

Lemma 3.8. *Suppose $k \geq 1$ and $\sigma \in \{+, -\}^k$. Then φ_σ is a bijection from $\mathcal{N}_k^*(n; \sigma)$ onto $SS(n; \sigma)$, and its inverse is ψ_σ .*

Proof. Fix $n \geq 1$. We will first show that for a given $\pi \in \mathcal{C}_n^\sigma$ and a σ -segmentation \mathbf{e} of π , we have that $(\pi, \mathbf{e}) = \varphi_\sigma(\psi_\sigma((\pi, \mathbf{e})))$. Let $\tau \in \mathcal{S}_n$ so that $\theta(\tau) = \pi$ and let $s = s_1 s_2 \dots s_n$ be the τ -monotone word induced by the σ -segmentation \mathbf{e} . We need to show that $\Pi_\sigma([s]) = [\tau]$.

Suppose that $\tau_i < \tau_j$. If $s_{[i, \rightarrow]} \neq s_{[j, \rightarrow]}$, let $a \geq 0$ be the smallest such that $s_{i+a} \neq s_{j+a}$, and let $h = |\{0 \leq \ell \leq a-1 : s_{i+\ell} \in T_\sigma^-\}|$. If h is even, then Lemma 3.6 applied a times shows that $\tau_{i+a} < \tau_{j+a}$. Since $s_{i+a} \neq s_{j+a}$, we must then have $s_{i+a} < s_{j+a}$ because of τ -monotonicity. Thus, $s_{[i, \rightarrow]} \prec_\sigma s_{[j, \rightarrow]}$

by definition of \prec_σ , since the word $s_i s_{i+1} \dots s_{i+a-1} = s_j s_{j+1} \dots s_{j+a-1}$ has an even number of letters in T_σ^- . Similarly, if h is odd, then Lemma 3.6 shows that $\tau_{i+a} > \tau_{j+a}$. Since $s_{i+a} \neq s_{j+a}$, we must have $s_{i+a} > s_{j+a}$, and thus $s_{[i,\rightarrow]} \prec_\sigma s_{[j,\rightarrow]}$ by definition of \prec_σ . If s is primitive, the case $s_{[i,\rightarrow]} = s_{[j,\rightarrow]}$ can never occur when $i \neq j$, and so $\pi_i < \pi_j$ if and only if $s_{[i,\rightarrow]} \prec_\sigma s_{[j,\rightarrow]}$. It follows that $\Pi_\sigma([s]) = [\tau]$.

Let $r := \frac{n}{2}$. If s is not primitive, we can only have $s_{[i,\rightarrow]} = s_{[j,\rightarrow]}$ if $i = j \pm r$. Suppose $\rho = \Pi_\sigma([s])$. In this case, we have that $\rho_1 < \rho_{1+r}$, and for $1 < i \leq r$, $\rho_i < \rho_{i+r}$ exactly when either $\rho_{i-1} < \rho_{i+r-1}$ with $s_{i-1} \in T_\sigma^+$ or $\rho_{i-1} > \rho_{i+r-1}$ with $s_{i-1} \in T_\sigma^-$. Thus, if $\tau_1 < \tau_{1+r}$, we get $\rho = \tau$. If $\tau_1 > \tau_{1+r}$, we get $\rho = \tau_{r+1} \dots \tau_n \tau_1 \dots \tau_r$, a cyclic rotation of τ . Either way, $[\rho] = [\tau]$ and so $\Pi_\sigma([s]) = [\tau]$.

Let $[s] \in \mathcal{N}_k^*(n; \sigma)$ with evaluation $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$. We will see that $\psi_\sigma(\varphi_\sigma([s])) = [s]$. Let $[\tau] = \Pi_\sigma([s])$ and let $\mathbf{e} = (e_0, e_1, \dots, e_k)$ be the partial sums given by $e_0 = 0$ and $e_i = \sum_{j < i} a_j$ for $i \in [k]$. It is enough to show that the $[\tau]$ -monotone necklace induced by \mathbf{e} is $[s]$. However, this is clear from the first paragraph of the proof of Lemma 3.5. □

Proof of Theorem 3.1. Since the left hand side of the formula is the size of the set $\mathcal{N}_k^*(n; \sigma)$ and the right hand side is the size of the set $SS(n; \sigma)$, Lemma 3.8 implies the equality. Since φ_σ sends necklaces with evaluation $\mathbf{a} = (a_0, a_1, \dots, a_{k-1})$ to permutations $\pi \in \mathcal{C}_n^\sigma$ and the associated σ -segmentation $\mathbf{e} = (e_0, e_1, \dots, e_k)$ with $e_0 = 0$ and $e_i = \sum_{j < i} a_j$ for $i \in [k]$, we are done. □

4. Cycles in length 2 vector grid classes

Here, we provide the enumeration of \mathcal{C}_n^{++} , \mathcal{C}_n^{+-} , \mathcal{C}_n^{-+} , and \mathcal{C}_n^{--} as corollaries of Theorem 3.1. These theorems also appear in [14, 4]. We also enumerate the number of permutations in \mathcal{C}_n^{+-} with a peak at i , which will be useful for enumerating \mathcal{C}_n^σ for $\sigma \in \{+, -\}^3$ in Section 5.

4.1. Enumerating \mathcal{C}_n^{++} , \mathcal{C}_n^{+-} , \mathcal{C}_n^{-+} , and \mathcal{C}_n^{--}

Theorem 4.1. *If $n \geq 2$, then $|\mathcal{C}_n^{++}| = L_2(n)$ and $|\mathcal{C}_n^{--}| = L_2^*(n; --)$.*

Proof. First suppose $\sigma = ++$ and $\pi \in \mathcal{C}_n^\sigma$. Then π is a cyclic permutation with at most one descent. Since π is cyclic and $n \geq 2$, π must have exactly one descent. Suppose the descent of π occurs at position i . Then there is

only one σ -segmentation of π , namely $(0, i, n)$. Thus $|\mathcal{C}_n^{++}| = L_2^*(n; \sigma)$. Since $T_\sigma^- = \emptyset$ in this case, we obtain $|\mathcal{C}_n^{++}| = L_2(n)$.

Similarly, if $\pi \in \mathcal{C}_n^{--}$, then π has exactly one ascent and thus one σ -segmentation. Thus the result follows. □

Theorem 4.2. *If $n \geq 2$, then*

$$|\mathcal{C}_n^{+-}| = |\mathcal{C}_n^{-+}| = \frac{L_2^*(n; +-)}{2}.$$

Proof. By Proposition 2.1, it is enough to consider $|\mathcal{C}_n^{+-}|$ only. If $\pi \in \mathcal{C}_n^{+-}$ and π has a peak at position i , then there are two σ -segmentations of π , namely $(0, i - 1, n)$ and $(0, i, n)$. Since this is true for all $\pi \in \mathcal{C}_n^{+-}$, the result follows. □

4.2. Unimodal cycles with a given peak

In this section, we enumerate the number of cyclic unimodal permutations (i.e., those permutations in \mathcal{C}_n^{+-}) with its peak in a given position. We let $\bar{\sigma} = +-$ and denote by $\Lambda(n)$ the size of the set $\mathcal{C}_n^{\bar{\sigma}}$.

Recall that we say there is a peak at i if $\pi_i > \pi_{i-1}$ and $\pi_i > \pi_{i+1}$, or if $i = 1$ and $\pi_1 > \pi_2$, or if $i = n$ and $\pi_{n-1} < \pi_n$. Let $\Lambda(n, i)$ denote the number of cyclic permutations in $\mathcal{C}_n^{\bar{\sigma}}$ with a peak at position i (i.e., the number of permutations $\pi \in \mathcal{C}_n^{\bar{\sigma}}$ with $\pi_i = n$). In order to enumerate the cyclic permutations in the $++--$ and $+---$ -classes in Section 5, we will find $\Lambda(n, i)$, whose formula is given in Lemma 4.6.

Let $L_2(n, i)$ denote the number binary Lyndon words of length n with i 1's, whose formula is well-known. Using Möbius inversion, one can find that

$$L_2(n, i) = \frac{1}{n} \sum_{d|\gcd(n,i)} \mu(d) \binom{n/d}{i/d}.$$

Lemma 4.3. *If $n \geq 1$ and $0 \leq i \leq n$, then $L_2(n, i) = L_2(n, n - i)$.*

Proof. By interchanging the 0's and 1's in each word, we obtain the result. □

Lemma 4.4. *If $n \geq 1$ and $i \in [n - 1]$, then*

$$\Lambda(n, i) + \Lambda(n, i + 1) = L_2(n, i)$$

except in the case when n is even and $n + i \equiv 2 \pmod{4}$; in this case,

$$\Lambda(n, i) + \Lambda(n, i + 1) = L_2(n, i) + L_2\left(\frac{n}{2}, \frac{i}{2}\right).$$

Proof. Consider the second statement in Theorem 3.1 for $\bar{\sigma} = +-.$ It implies that the number of necklaces in $\mathcal{N}_2^*(n; \bar{\sigma})$ with i zeros and $n - i$ ones is equal to the number of cyclic permutations in $\mathcal{C}_n^{\bar{\sigma}}$ that admit the $\bar{\sigma}$ -segmentation $(0, i, n)$. These are exactly the permutations with a peak at i or $i + 1$. In the case when either n is odd or n is even and $n + i \not\equiv 2 \pmod{4}$, the number of necklaces in $\mathcal{N}_2^*(n; \bar{\sigma})$ with i zeros and $n - i$ ones is exactly $L_2(n, n - i)$. When n is even and $n + i \equiv 2 \pmod{4}$, the number of such necklaces is $L_2(n, n - i) + L_2\left(\frac{n}{2}, \frac{n-i}{2}\right)$. Since $L_2(n, n - i) = L_2(n, i)$ and $L_2\left(\frac{n}{2}, \frac{n-i}{2}\right) = L_2\left(\frac{n}{2}, \frac{i}{2}\right)$ by Lemma 4.3, the result follows. \square

Lemma 4.5. *If $n \geq 1$ and $n \not\equiv 2 \pmod{4}$, then for $i \in [n]$,*

$$\Lambda(n, i) = \Lambda(n, n - i + 1).$$

Proof. The cases when $n \leq 2$ are easily checked. Now let $n > 2$. If $i = 1$, then $\Lambda(n, 1) = 0 = \Lambda(n, n)$. We proceed by induction on i . Assume $\Lambda(n, i - 1) = \Lambda(n, n - i + 2)$ for some $i \in \{2, 3, \dots, n - 1\}$. First suppose n is odd. Since $L_2(n, i - 1) = L_2(n, n - i + 1)$, applying Lemma 4.4 twice and Lemma 4.3 we obtain

$$\begin{aligned} \Lambda(n, i - 1) + \Lambda(n, i) &= L_2(n, i - 1) \\ &= L_2(n, n - i + 1) \\ &= \Lambda(n, n - i + 1) + \Lambda(n, n - i + 2). \end{aligned}$$

The result now follows from our inductive hypothesis.

Now suppose $n \equiv 0 \pmod{4}$. If $i - 1 \not\equiv 2 \pmod{4}$, then the above argument can be made by again using Lemma 4.4 twice and Lemma 4.3 once. If $i - 1 \equiv 2 \pmod{4}$, notice that $n - i + 1 \equiv 2 \pmod{4}$ as well. Therefore, using the same lemmas,

$$\begin{aligned} \Lambda(n, i - 1) + \Lambda(n, i) &= L_2(n, i - 1) + L_2\left(\frac{n}{2}, \frac{i - 1}{2}\right) \\ &= L_2(n, n - i + 1) + L\left(\frac{n}{2}, \frac{n - i + 1}{2}\right) \\ &= \Lambda(n, n - i + 1) + \Lambda(n, n - i + 2). \end{aligned}$$

The result follows from our inductive hypothesis. \square

Lemma 4.6. *Let $n \geq 3$ and $i \in [n]$. If n is odd, then*

$$\Lambda(n, i) = \sum_{j=1}^{i-1} (-1)^{i+j+1} L_2(n, j),$$

and if n is even, then

$$\Lambda(n, i) = \sum_{j=1}^{i-1} (-1)^{i+j+1} L_2(n, j) + (-1)^{i+1} \sum_{\substack{k < i \\ 4|(n+k+2)}} L_2\left(\frac{n}{2}, \frac{k}{2}\right).$$

Proof. Observe that $\Lambda(n, i)$ is only defined for $i \in [n]$. Since $\Lambda(n, 1) = 0$ the result is clear for $i = 1$. Thus, we assume $i > 1$ and induct on i with $i \in \{2, 3, \dots, n - 1\}$. If $i = 2$, then from Lemma 4.4 we obtain the equality

$$\Lambda(n, 1) + \Lambda(n, 2) = L_2(n, 1)$$

and the result holds. Thus, we assume the result holds true for all permissible integers less than i . First suppose n is an odd integer. It follows that

$$\Lambda(n, i - 1) + \Lambda(n, i) = L_2(n, i - 1)$$

by Lemma 4.4; therefore

$$\begin{aligned} \Lambda(n, i) &= L_2(n, i - 1) - \Lambda(n, i - 1) \\ &= L_2(n, i - 1) - \sum_{j=1}^{i-2} (-1)^{i+j} L_2(n, j) \\ &= L_2(n, i - 1) + \sum_{j=1}^{i-2} (-1)^{i+j+1} L_2(n, j) \\ &= \sum_{j=1}^{i-1} (-1)^{i+j+1} L_2(n, j), \end{aligned}$$

where the second equality holds by inductive hypothesis.

Now assume that n is even and i satisfies $n + i - 1 \equiv 2 \pmod{4}$. Then

$$\Lambda(n, i - 1) + \Lambda(n, i) = L_2(n, i - 1) + L_2\left(\frac{n}{2}, \frac{i}{2}\right)$$

by Lemma 4.4. Following a similar argument as above,

$$\begin{aligned}
 \Lambda(n, i) &= L_2(n, i - 1) - \Lambda(n, i - 1) + L_2\left(\frac{n}{2}, \frac{i - 1}{2}\right) \\
 &= L_2(n, i - 1) - \sum_{j=1}^{i-2} (-1)^{i+j} L_2(n, j) \\
 &\quad - (-1)^i \sum_{\substack{k < i-1 \\ 4|(n+k+2)}} L_2\left(\frac{n}{2}, \frac{k}{2}\right) + L_2\left(\frac{n}{2}, \frac{i - 1}{2}\right) \\
 &= L_2(n, i - 1) + \sum_{j=1}^{i-2} (-1)^{i+j+1} L_2(n, j) \\
 &\quad + (-1)^{i+1} \sum_{\substack{k < i \\ 4|(n+k+2)}} L_2\left(\frac{n}{2}, \frac{k}{2}\right) \\
 &= \sum_{j=1}^{i-1} (-1)^{i+j+1} L_2(n, j) + (-1)^{i+1} \sum_{\substack{k < i \\ 4|(n+k+2)}} L_2\left(\frac{n}{2}, \frac{k}{2}\right).
 \end{aligned}$$

A similar argument holds when n is even and $n + i - 1 \not\equiv 2 \pmod{4}$. In this case, the rightmost sum does not acquire an extra term when rewriting it as a sum over $k < i$. □

5. Cycles in length 3 vector grid classes

In this section, we enumerate cyclic permutations in each vector grid class of length 3. In Section 5.1, we enumerate \mathcal{C}_n^{+++} and \mathcal{C}_n^{---} , in Section 5.2, we enumerate \mathcal{C}_n^{+-+} and \mathcal{C}_n^{-+-} , and in Section 5.3, we enumerate \mathcal{C}_n^{+--} , \mathcal{C}_n^{-+-} , \mathcal{C}_n^{-++} , and \mathcal{C}_n^{--+} . Finally, in Section 5.4, we determine Wilf-equivalence classes for the length 3 vector grid classes.

5.1. Enumerating \mathcal{C}_n^{+++} and \mathcal{C}_n^{---}

In Theorem 5.1, we enumerate both \mathcal{C}_n^{+++} and \mathcal{C}_n^{---} .

Theorem 5.1. *If $n \geq 1$, then*

$$c_{+++}(n) = L_3(n) - nL_2(n),$$

and if $n \geq 3$, then

$$c_{---}(n) = L_3^*(n; ---) - nL_2^*(n; --).$$

Proof. First, let us find $c_{+++}(n)$. The cases when n is 1 or 2 are easily checked. Suppose $n \geq 3$. Notice that $c_{+++}(n) = |\mathcal{C}_n^{+++} \setminus \mathcal{C}_n^{++}| + |\mathcal{C}_n^{++}|$. We know also from Theorem 4.1 that $|\mathcal{C}_n^{++}| = L_2(n)$. We also know that permutations in \mathcal{C}_n^{++} have exactly one descent, since the only permutation with fewer descents is the increasing permutation, which is not cyclic when $n \geq 2$.

If $\pi \in \mathcal{C}_n^{+++} \setminus \mathcal{C}_n^{++}$, then π must have exactly 2 descents. Suppose the descent set is $\{i, j\}$, with $i < j$. Then we must have that the $+++$ -segmentation of π is $(0, i, j, n)$. Thus, there is only one $+++$ -segmentation. If $\pi \in \mathcal{C}_n^{++}$, then π has exactly one descent. If the descent is at position i , then the $+++$ -segmentation is $(0, e_1, e_2, n)$, where $\{e_1, e_2\} = \{i, j\}$ for any $j \in \{0, 1, 2, \dots, n\}$. Thus, by Theorem 3.1, we obtain that

$$L_3^*(n; +++) = |\mathcal{C}_n^{+++} \setminus \mathcal{C}_n^{++}| + (n + 1)|\mathcal{C}_n^{++}|.$$

Taken with the formula for $c_{+++}(n)$ in the second sentence of this proof and the fact that $L_3^*(n; \sigma) = L_3(n)$, we obtain the result $c_{+++}(n) = L_3(n) - nL_2(n)$. The formula for $c_{---}(n)$ is found similarly. □

As stated in the introduction, the theorem above is a special case of theorems that appear in [14] and [4]. The general theorem is stated below. Let $n \geq 1$ and $C_n(t) = \sum_{\pi \in \mathcal{C}_n} t^{\text{des}(\pi)}$, where $\text{des}(\pi)$ is the number of descents of π .

Theorem 5.2 ([14, Theorem 6.1]). *If $n \geq 1$, then*

$$\frac{C_n(t)}{(1-t)^{n+1}} = \frac{1}{n} \sum_{d|n} \mu(d) \sum_{m=0}^{\infty} m^{n/d} t^m.$$

Notice that Theorem 5.1 implies that the signatures $+++$ and $---$ are weakly cyc-Wilf-equivalent since when $n \not\equiv 2 \pmod{4}$, we must have $L_2^*(n; --) = L_2(n)$ and $L_3^*(n; ---) = L_3(n)$. Clearly, the second statement of Theorem 5.1 is equivalent to

$$c_{---}(n) = \begin{cases} L_3(n) - nL_2(n) & \text{when } n \not\equiv 2 \pmod{4} \\ L_3(n) + L_3(\frac{n}{2}) - n(L_2(n) + L_2(\frac{n}{2})) & \text{when } n \equiv 2 \pmod{4}. \end{cases}$$

It is straightforward to see that in the case when $n \equiv 2 \pmod{4}$, we have that

$$d_3(n) := c_{----}(n) - c_{+++}(n) = L_3\left(\frac{n}{2}\right) - nL_2\left(\frac{n}{2}\right).$$

A table of these values can be seen in Table 3.

Table 3: A table of the first eight values of $d_3 := c_{----}(n) - c_{+++}(n)$ for n satisfying $n > 2$ and $n \equiv 2 \pmod{4}$

n	6	10	14	18	22	26	30	34
$d_3(n)$	-4	-12	60	1176	12012	106260	891116	7334340

5.2. Enumerating \mathcal{C}_n^{+-+} and \mathcal{C}_n^{-+-}

In this section, we enumerate the set of cyclic permutations in the $+-+$ -class and the $-+-$ -class. Recall that we define a *peak* of the permutation $\pi \in \mathcal{S}_n$ to be $i \in [n]$ so that $\pi_{i-1} < \pi_i$ and $\pi_i > \pi_{i+1}$ (where $\pi_0 = \pi_{n+1} := 0$). Similarly, we define a *valley* of the permutation $\pi \in \mathcal{S}_n$ to be $i \in [n]$ so that $\pi_{i-1} > \pi_i$ and $\pi_i < \pi_{i+1}$ (where $\pi_0 = \pi_{n+1} := n + 1$). For example, the permutation 4156732 has peaks at positions 1 and 5 and has valleys at positions 2 and 7.

Lemma 5.3. *Let $n \geq 3$. For any permutation $\pi \in \mathcal{C}_n^{+-+}$, there is a unique choice of $i, j \in [n]$ with $i < j$ so that i is a peak and j is a valley.*

Proof. Suppose that $\pi \in \mathcal{C}_n^{+-+}$. If $\mathbf{e} = (0, e_1, e_2, n)$ is a $+-+$ -segmentation of π , then $\pi_1\pi_2 \dots \pi_{e_1}$ is increasing, $\pi_{e_1+1} \dots \pi_{e_2}$ is decreasing, and $\pi_{e_2+1} \dots \pi_n$ is increasing. If we have strict inequalities $0 < e_1 < e_2 < n$, then clearly, $i = e_1$ if $\pi_{e_1} > \pi_{e_1+1}$ and $i = e_1 + 1$ otherwise, and $j = e_2$ if $\pi_{e_2} > \pi_{e_2+1}$ and $j = e_2 + 1$ otherwise. In this case, we do not have $i = j$ since that would imply that π is the increasing permutation, which is not cyclic. Similarly, at least two of the inequalities $0 \leq e_1 \leq e_2 \leq n$ are strict inequalities, otherwise π would be the increasing or decreasing permutation.

If we have that $e_1 = 0$, then $\pi \in \mathcal{C}_n^{-+-}$. In this case, we can take $i = 1$ and j as above. Again, we do not have $i = j$ as that would imply π is strictly increasing. If we have that $e_2 = n$, we can take $j = n$ and i as above. We similarly do not have $i = j$. Finally, if $e_1 = e_2$, then $i = e_1$ and $j = e_1 + 1$. \square

For example, consider the permutation $\pi = 356894127 \in \mathcal{C}_9^{+-+}$. There are two peaks (namely 5 and 9) and two valleys (namely 1 and 7), but there is a unique pair $i = 5$ and $j = 7$ satisfying the condition that $i < j$, where i is a peak and j is a valley.

Theorem 5.4. For $n \geq 3$, $c_{+-+}(n) = \frac{L_3^*(n; +-+)}{4}$.

Proof. Let $\sigma := +-+$ for the duration of this proof. By Theorem 3.1, we have

$$L_3^*(n; \sigma) = \sum_{\pi \in \mathcal{C}_n^{+-+}} |\{\mathbf{e} : \mathbf{e} \text{ is a } \sigma\text{-segmentation of } \pi\}|.$$

If $\pi \in \mathcal{C}_n^\sigma$, then there exists some unique $i, j \in [n]$ with $i < j$ such that i is a peak and j is a valley. Suppose $\mathbf{e} = (0, e_1, e_2, n)$ is a σ -segmentation of π . Certainly, $e_1 \leq i$. If $e_1 < i-1$, then $e_2 \in \{e_1, e_1+1\}$ and thus $e_2 < i$. Since i is a peak, it is followed by a descent and thus the σ -segmentation constructed is not valid. Therefore, we must have that $e_1 \in \{i-1, i\}$. For similar reasons, $e_2 \in \{j-1, j\}$ and so the following list of four σ -segmentations of π is complete:

- (a) $(0, i-1, j-1, n)$,
- (b) $(0, i-1, j, n)$,
- (c) $(0, i, j-1, n)$,
- (d) $(0, i, j, n)$.

Therefore, $L_3^*(n; \sigma) = 4c_\sigma(n)$ and the result follows. □

Notice that for a permutation $\pi \in \mathcal{C}_n^{-+-}$, there is a unique $i, j \in [n]$ with $i < j$ so that i is a valley and j is a peak. Thus, we omit the proof of the following theorem due to its similarities to the proof of Theorem 5.4.

Theorem 5.5. For $n \geq 3$, $c_{-+-}(n) = \frac{L_3^*(n; -+-)}{4}$.

5.3. Enumerating \mathcal{C}_n^{++-} , \mathcal{C}_n^{+--} , \mathcal{C}_n^{-++} , and \mathcal{C}_n^{--+}

In this section, we enumerate the set \mathcal{C}_n^σ for the four remaining signatures $\sigma \in \{+, -\}^3$.

Theorem 5.6. Let $\Lambda(n, i)$ be the value given in Lemma 4.6. If $n \geq 2$, then

$$c_{++-}(n) = \frac{L_3^*(n; ++-)}{2} - \sum_{i=2}^{n-1} i \cdot \Lambda(n, i).$$

Proof. Let $\sigma := ++-$ for the duration of this proof. If a permutation is an element of $\mathcal{C}_n^{++-} \setminus \mathcal{C}_n^{+-}$, then it has two σ -segmentations. As a result, the equality

$$c_\sigma(n) = |\mathcal{C}_n^{++-} \setminus \mathcal{C}_n^{+-}| + \Lambda(n)$$

and Theorem 3.1 imply

$$\begin{aligned} L_3^*(n; \sigma) &= \sum_{\pi \in \mathcal{C}_n^\sigma} |\{\mathbf{e} : \mathbf{e} \text{ is a } \sigma\text{-segmentation of } \pi\}| \\ &= 2(c_\sigma(n) - \Lambda(n)) + \sum_{\pi \in \mathcal{C}_n^{+-}} |\{\mathbf{e} : \mathbf{e} \text{ is a } \sigma\text{-segmentation of } \pi\}|. \end{aligned}$$

Notice there are $2(i + 1)$ σ -segmentations of each $\pi \in \mathcal{C}_n^{+-}$ with a peak at i . Specifically, each of these σ -segmentations will have one of the following forms:

- (a) $(0, j, i - 1, n)$ for each $j \in \{0, 1, \dots, i - 1\}$,
- (b) $(0, j, i, n)$ for each $j \in \{0, 1, \dots, i\}$,
- (c) $(0, i, i + 1, n)$.

A similar argument to the one in the proof of Theorem 5.4 shows that these are the only σ -segmentations of π .

Let $\mathcal{C}_{n,i}^{+-}$ denote the set of permutations $\pi \in \mathcal{C}_n^{+-}$ with a peak in position i . Then

$$\begin{aligned} \sum_{\pi \in \mathcal{C}_n^{+-}} |\{\mathbf{e} : \mathbf{e} \text{ is a } \sigma\text{-segmentation of } \pi\}| &= \sum_{i=2}^{n-1} \sum_{\pi \in \mathcal{C}_{n,i}^{+-}} 2(i + 1) \\ &= \sum_{i=2}^{n-1} 2(i + 1)\Lambda(n, i) \end{aligned}$$

and

$$L_3^*(n; +- -) = 2c_{+- -}(n) - 2\Lambda(n) + \sum_{i=2}^{n-1} 2(i + 1)\Lambda(n, i).$$

Since $\Lambda(n) = \sum \Lambda(n, i)$, the theorem follows from the equality above. □

Theorem 5.7. *Let $\Lambda(n, i)$ be the value given in Lemma 4.6. If $n \geq 2$, then*

$$c_{+--}(n) = \frac{L_3^*(n; +--)}{2} - \sum_{i=2}^{n-1} (n - i + 1)\Lambda(n, i).$$

Proof. Let $\sigma := +--$ for the duration of this proof. As in the proof of Theorem 5.6, there are exactly two σ -segmentations of π whenever $\pi \in \mathcal{C}_n^{+--} \setminus \mathcal{C}_n^{+-}$. Furthermore, for $\pi \in \mathcal{C}_n^{+-}$ with a peak at i , a similar argument to the one in the proof above shows that the σ -segmentations of each of these permutations will have one of the following forms:

- (a) $(0, i - 1, j, n)$ for each $j \in \{i - 1, i, \dots, n\}$,
- (b) $(0, i, j, n)$ for each $j \in \{i, i + 1, \dots, n\}$,
- (c) $(0, i - 2, i - 1, n)$.

Thus

$$\begin{aligned} \sum_{\pi \in \mathcal{C}_n^{+-}} |\{\mathbf{e} : \mathbf{e} \text{ is a } \sigma\text{-segmentation of } \pi\}| &= \sum_{i=2}^{n-1} \sum_{\pi \in \mathcal{C}_{n,i}^{+-}} 2(n + i + 2) \\ &= \sum_{i=2}^{n-1} 2(n + i + 2)\Lambda(n, i) \end{aligned}$$

and the result now follows. □

The following corollaries are immediate results of Proposition 2.1 and Lemma 2.6.

Corollary 5.8. *The signatures $++-$ and $-++$ are cyc-Wilf-equivalent, and thus if $n \geq 2$,*

$$c_{-++}(n) = \frac{L_3^*(n; -++)}{2} - \sum_{i=2}^{n-1} i \cdot \Lambda(n, i).$$

Corollary 5.9. *The signatures $+- -$ and $- - +$ are cyc-Wilf-equivalent, and thus if $n \geq 2$,*

$$c_{--+}(n) = \frac{L_3^*(n; --+)}{2} - \sum_{i=2}^{n-1} (n - i + 1)\Lambda(n, i).$$

5.4. cyc-Wilf-equivalence

In this section, we discuss the equivalence classes of the cyclic permutations in the length 3 vector grid classes. Recall two classes are cyc-Wilf-equivalent if the sets of cyclic permutations in each class are equinumerous, and we say two classes are weakly cyc-Wilf-equivalent if they are equinumerous when $n \not\equiv 2 \pmod{4}$.

Theorem 5.10. *For $n \geq 1$, $c_{+-+}(n) = c_{-+-}(n)$ and thus the signature $+-+$ is cyc-Wilf-equivalent to the signature $-+-$.*

Proof. For $n = 1$ and $n = 2$, this is easily checked and thus we assume $n \geq 3$. By Theorems 5.4 and 5.5, it is enough to show $L_3^o(\frac{n}{2}; + - +) = L_3^o(\frac{n}{2}; - + -)$. Since these are both 0 when n is odd, we need only consider the cases when n is even. It suffices to show that the number of necklaces $[s]$ of length $\frac{n}{2}$ when $o_{+ - +}(s)$ is odd is equal to the number of necklaces $[s']$ of length $\frac{n}{2}$ when $o_{- + -}(s)$ is odd. Note that for a necklace $[s]$ with $s_i \in \{0, 1, 2\}$ of length $\frac{n}{2}$, we have $o_{+ - +}(s) = |\{i \in [n] : s_i = 1\}|$ and $o_{- + -}(s) = |\{i \in [n] : s_i \neq 1\}|$ by definition. We proceed by cases.

Case 1. When $n \equiv 0 \pmod{4}$, $\frac{n}{2}$ is even, and thus for any necklace $[s]$ of length $\frac{n}{2}$, $o_{+ - +}(s)$ is odd if and only if $o_{- + -}(s)$ is odd. Therefore, $L_3^o(\frac{n}{2}; + - +) = L_3^o(\frac{n}{2}; - + -)$.

Case 2. When $n \equiv 2 \pmod{4}$, $\frac{n}{2}$ is odd and we will show using Möbius inversion that exactly half of necklaces of length $\frac{n}{2}$ have $o_{+ - +}(s)$ odd. This in turn implies that there are equally many necklaces with $o_{+ - +}(s)$ odd as there are with $o_{+ - +}(s)$ even (or equivalently, $o_{- + -}(s)$ odd).

Take $m := \frac{n}{2}$ for simplicity of notation. Let $a(m)$ be the number of ternary primitive words with $o_{+ - +}(s)$ odd and let $b(m)$ be the number of ternary words (which are not necessarily primitive) with $o_{+ - +}(s)$ odd. It is easily checked that $b(m) = \frac{3^m - 1}{2}$. Also, for any m ,

$$b(m) = \sum_{\substack{d|m \\ d \text{ odd}}} a(m/d).$$

Since m is odd, all its divisors are also odd, so we can write

$$b(m) = \sum_{d|m} a(m/d) = \sum_{d|m} a(d).$$

Using Möbius inversion, we rewrite this sum as

$$a(m) = \sum_{d|m} \mu(d)b(m/d).$$

This sum becomes

$$a(m) = \frac{1}{2} \sum_{d|m} \mu(d)(3^{m/d} - 1) = \frac{1}{2} \sum_{d|m} \mu(d)3^{m/d},$$

where the equality follows from the well-known fact that $\sum \mu(d) = 0$ when

the sum is taken over all divisors $d \mid m$ for any integer $m \geq 2$. It follows that

$$L_3^o(m; + - +) = \frac{1}{m} a(m) = \frac{1}{2m} \sum_{d \mid m} \mu(d) 3^{n/d}$$

which is exactly half of $L_3(m)$. The result follows. □

Theorem 5.11. *If $n \geq 1$ and $n \not\equiv 2 \pmod{4}$, then $c_{+ + -}(n) = c_{+ - -}(n)$ and thus the signatures $+ + -$ and $+ - -$ are weakly cyc-Wilf-equivalent.*

Proof. It is enough to show that when $n \not\equiv 2 \pmod{4}$,

$$\sum_{i=2}^{n-1} i \cdot \Lambda(n, i) = \sum_{i=2}^{n-1} (n - i + 1) \Lambda(n, i) \quad \text{and} \quad L_3^*(n; + + -) = L_3^*(n; + - -).$$

Setting $i := n - j + 1$ and using Lemma 4.5, we obtain the equalities

$$\sum_{i=2}^{n-1} i \cdot \Lambda(n, i) = \sum_{j=2}^{n-1} (n - j + 1) \Lambda(n, n - j - 1) = \sum_{j=2}^{n-1} (n - j + 1) \Lambda(n, j).$$

If n is odd, it is clear that $L_3^*(n; + + -) = L_3^*(n; + - -)$. When $n \equiv 0 \pmod{4}$, it is enough to show that $L_3^o(\frac{n}{2}; + + -) = L_3^o(\frac{n}{2}; + - -)$. Observe $L_3^o(\frac{n}{2}; + + -)$ is the size of the set of primitive necklaces of length $\frac{n}{2}$ which have an odd number of 2's. Since $\frac{n}{2}$ is even, the number of 0's and 1's must be odd as well. Therefore, $L_3^o(\frac{n}{2}; + + -) = L_3^o(\frac{n}{2}; - - +)$ and thus the equality $L_3^o(\frac{n}{2}; + + -) = L_3^o(\frac{n}{2}; + - -)$ follows from Lemma 2.6. □

6. Alternating grid classes

In this section, we prove cyc-Wilf-equivalence for a more general set of grid classes. Define the k -th up-down alternating grid class to have signature $\sigma_{alt}^{+k} = + - + - \dots \in \{+, -\}^k$ and the k -th down-up alternating grid class to have signature $\sigma_{alt}^{-k} = - + - + \dots \in \{+, -\}^k$. In this section, we show that the number of cyclic permutations in the up-down alternating grid class is equal to the number of cyclic permutations in the down-up alternating grid class (i.e., they are cyc-Wilf-equivalent). When k is even, this is automatically true by Proposition 2.1, so it remains to show this is true when k is odd.

Lemma 6.1. *For all odd $k \geq 3$ and $n \geq 3$, $L_k^*(n; \sigma_{alt}^{+k}) = L_k^*(n; \sigma_{alt}^{-k})$.*

The proof of this lemma is exactly the proof of Theorem 5.10, modified to enumerate k -ary necklaces of length $\frac{n}{2}$ with $o_{\sigma_{alt}^{+k}}(s)$ odd. Again, one may use Möbius inversion to show that when $\frac{n}{2}$ is odd, exactly half of the primitive k -ary necklaces of length $\frac{n}{2}$ have $o_{\sigma_{alt}^{+k}}(s)$ odd.

Theorem 6.2. *For all $k \geq 2$, the signatures σ_{alt}^{+k} and σ_{alt}^{-k} are cyc-Wilf-equivalent.*

Proof. As stated above, this is automatically true if the length of the signatures is even. Assume the length k is odd. The theorem follows inductively from Lemma 6.1 and Theorem 3.1. \square

7. Discussion and conjectures

Recall that for $\sigma \in \{+, -\}^k$, we define the complementary signature as $\sigma_c = \sigma_c^0 \sigma_c^1 \dots \sigma_c^{k-1}$ with $\sigma_c^i = +$ if $\sigma^i = -$ and $\sigma_c^i = -$ if $\sigma^i = +$.

Conjecture 7.1. For all $k \geq 2$ and $\sigma \in \{+, -\}^k$, σ is weakly cyc-Wilf-equivalent to σ_c .

A signature σ has k corners if there are k changes in sign. For example, $\sigma = +++$ has 0 corners, $\sigma = +-+-+-$ has 5 corners, and $\sigma = ++-- --$ has 2 corners. The following conjecture subsumes the one above since a signature and its complement have the same number of corners.

Conjecture 7.2. For all $k \geq 2$ and $\sigma \in \{+, -\}^k$, signatures with k corners are all weakly cyc-Wilf-equivalent.

More generally, one could use Theorem 3.1 to enumerate the set of cyclic permutations in other families of grid classes. Furthermore, one could generalize this theorem to a bijection between σ -segmentations of permutations with other cycle types and multisets of necklaces in order to enumerate permutations in grid classes by their cycle type. Some progress towards this for $\sigma = +-$ and $\sigma = +^k$ can be found in [18] and [14].

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KASSIE ARCHER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TEXAS AT TYLER
USA
E-mail address: karcher@uttyler.edu

L.-K. LAUDERDALE
DEPARTMENT OF MATHEMATICS
TOWSON UNIVERSITY
USA
E-mail address: llauderdale@towson.edu

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